Hyperbolic Systems on an Interval with Dynamic Boundary Conditions

Dissertation

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Abstract

In this thesis, the well-posedness of hyperbolic systems with dynamic boundary conditions is studied. Such systems occur naturally when the dynamics on the boundary interact with the waves in the interior. By using a priori estimates and the method of Friedrichs, the L^2-well-posedness of linear systems is established. It is shown that weak solutions have a hidden regularity property, namely the L^2-trace regularity at the boundary. The a priori estimates are derived through symmetrizers and paradifferential calculus. Regularity and compatibility of the data enhances the regularity of the solutions. We also deal with a model describing the flow of fluid in an elastic tube whose ends are attached to tanks. The stability and boundary controllability of the linearized model are analyzed using semigroup theory and nonharmonic Fourier analysis. Numerical solutions of the linear model are computed using Legendre tau approximations. Next, local in time well-posedness of a class of PDE-ODE systems is established by Picard iteration. Furthermore, the existence and uniqueness of global in time smooth solutions of the two-tank model is proved for smooth data sufficiently close to the equilibrium. The proof is based on energy estimates. It is shown that solutions of the nonlinear model converge exponentially fast to the steady state. The lower order energy estimate is derived using the relative entropy, while the higher order estimates are obtained using appropriate entropy-entropy flux pairs.

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Declaration

I hereby declare that the results of this thesis, except those with citations, are the outcome of the original work done by the author. This thesis has not been previously submitted in any form for a degree at this or any other institution or university.

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Graz, 2013

Hyperbolische Systeme auf einem Intervall mit Dynamischen Randbedingungen

Gilbert Peralta

Kurzzusammenfassung

In dieser Dissertation wird die Gut-Gestelltheit hyperbolischer Systeme mit dynamischen Randbedingungen untersucht. Solche Systeme ergeben sich, wenn die Dynamik an den Rändern mit den Wellen im Inneren interagiert. Durch a priori Abschätzungen und die Methode von Friedrichs wird die L^2-Gut-Gestelltheit linearer Systeme etabliert. Es wird gezeigt, dass schwache Lösungen die Eigenschaft der versteckten Regularität haben, nämlich die L^2-Spur Regularität am Rand. Die a priori Abschätzungen werden über Symmetrisierer und paradifferentiellen Kalkül hergeleitet. Regularität und Kompatibilität der Daten erhöht die Regularität der Lösungen. Wir beschäftigen uns auch mit einem Modell, das die Strömung einer Flüssigkeit in einem elastischen Schlauch, dessen Enden mit Tanks verbunden sind, beschreibt. Die Stabilität und Rand-Steuerbarkeit des linearisierten Modells wird mit Hilfe von Halbgruppentheorie und nicht-harmonischer Fourier Analyse untersucht. Nummerische Lösungen des linearen Modells werden durch Legendre tau Approximationen berechnet. Sodann wird - durch Picard Iteration - die Gut-Gestelltheit lokal in der Zeit einer Klasse von PDE-ODE Systemen nachgewiesen. Weiters wird die Existenz und Eindeutigkeit global in der Zeit glatter Lösungen des Zwei-Tank Modells bewiesen, und zwar für glatte Anfangswerte, die hinreichend nahe dem Gleichgewicht sind. Der Beweis basiert auf Energie Abschätzungen. Es wird gezeigt, dass die Lösungen des nicht-linearen Modells exponentiell schnell gegen das Gleichgewicht konvergieren. Die Energie Abschätzung niedriger Ordnung wird anhand der relativen Entropie abgeleitet, während die Abschätzungen höherer Ordnung mit Hilfe geeigneter Entropie-Entropiefluss Paare erhalten wird.

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1

INTRODUCTION

Hyperbolic partial differential equations are recognized mathematical models in areas such as fluid dynamics, acoustics, electromagnetics, scattering theory and the general theory of relativity. The methods used and developed to understand these equations range from abstract functional analytic tools, e.g. pseudodifferential calculus and microlocal analysis, to more intuitive geometric methods, e.g. the method of characteristics and geometric optics. The prototype of hyperbolic partial differential equations is the second order wave equation modeling the vibrations of a string. Other models arise in the theory of conservation laws. The inviscid Burgers equation, the Euler equations of compressible fluid flow and Maxwell's equations of electromagnetism are some well-known examples of conservation laws in continuum physics. A historical account for the developments of conservation laws arising in continuum physics is given in Dafermos [**21**].

Essential properties of hyperbolic equations are well-posed Cauchy problems, finite speed of propagation and wave-like solutions. This means that for a given finite time, local disturbances on the initial data have effects only on parts of the domain, called the region of influence. Because information travels along characteristic curves, discontinuities and oscillations propagate through time and space. Therefore, in general, one might expect the same regularity for the initial data and the solution. This is in contrast to parabolic partial differential equations, where perturbations of the initial data have effects on the entire domain and solutions have more regularity than the initial data. In other words, parabolic differential equations exhibit infinite speed of propagation and smoothing.

The focus of this thesis is a class of hyperbolic systems of first order partial differential equations on a bounded interval that are coupled with ordinary differential equations at the boundary. These include linear systems with either constant or variable coefficients and quasilinear systems. Such systems occur naturally when the dynamics on the boundary interact with the waves in the interior.

The wave equation with oscillator boundary conditions in [**6, 39**] is one of the examples according to the literature. Suppose that a fluid is contained in a bounded domain, the evolution of the velocity potential is modeled by a second order linear wave equation. Assuming that each point of the boundary reacts like a harmonic oscillator forced by interior pressure, the normal displacement of the boundary can be described by a second order differential equation. In this way, the boundary conditions for the fluid are coupled to ordinary differential equations.

Another example is taken from multiscale blood flow models [**27, 65, 66**]. Starting from the incompressible Navier-Stokes equations and assuming that the flow is axisymmetric, hyperbolic models can be derived to study blood flow in the human cardiovascular system. The hyperbolic equations have the same form as the Euler continuity and momentum equations in gas dynamics. Important parts of the cardiovascular system such as the vessels can be described by patching several components modeled by hyperbolic equations. However, a realistic description cannot be described solely by these equations, and several authors introduced lumped parameters. These parameters can be expressed by a system of ordinary differential equations describing the mass and flow rate in a specific terminal compartment of the circulatory system. They can be derived from the hyperbolic models by integration in space and linearization. The boundary conditions for the hyperbolic PDEs are coupled to the ODEs by imposing continuity of pressure and flow rate. In the two examples given above, the differential equations at the boundary are explicitly given.

Finally, let us consider the dynamics of the sound in a compressible fluid whose surface is made of a viscoelastic material [**22, 62**]. The acoustic pressure can be modeled again by a second order wave equation, while the boundary condition is of memory-type. One has to keep track of the memory by introducing an auxiliary state. Under suitable conditions on the memory kernel, this state satisfies a differential equation on the boundary. Here, the differential equation at the boundary is introduced in the analysis and not explicitly given by the model.

Well-posedness in the Hadamard sense, i.e. existence and uniqueness of solutions and continuous dependence on the data, will be studied in appropriate function spaces. With additional smoothness and compatibility of the data, the regularity of solutions will be considered as well. Our well-posedness results are stated in Lebesgue and Sobolev spaces. Other different function spaces such as the space of continuously differentiable functions and the space of functions of bounded variation that include discontinuous solutions have been used by several authors in the literature. The advantage of using Lebesgue and Sobolev spaces is that they are Hilbert spaces. In this setting, more functional analytic methods are available in the analysis. Although we have more tools at our disposal, this limits the range of applicability of the results. Nevertheless, the results still cover a large variety of systems and the ideas presented here may be used to treat other problems.

The coupled PDE-ODE systems we consider is an initial-boundary value problem (IBVP) of the form

$$\begin{cases} \dot{u}(t) = f(u(t), s_u(t)), \quad t > 0, \\ b(u(t), h(t), r(t)) = 0, \quad t > 0, \\ \dot{h}(t) = k(h(t), u(t), s_h(t)), \quad t > 0, \\ u(0) = u_0, \\ h(0) = h_0 \end{cases} \tag{1.1}$$

where u is the state component in the domain and h is the state component on the boundary. In (1.1), f is a differential operator, b and k are trace operators with respect to u, and s_u, s_h and r are external sources.

Before looking at general systems, we study a specific physical system modeling the flow of an incompressible fluid contained in an elastic tube where each end is connected to a tank. In this particular set-up, the state component u consists of the velocity of the fluid and the cross-sectional area of tube, and the state component h consists of the level heights of the fluid in the tanks. Using mass conservation and Newton's second law, we will derive this system in detail in Chapter 2. From now on, we shall refer to this model as the *two-tank model*, see (2.6.5). Similar models have been considered in the literature in the context of valveless pumping [**13, 60**] and in multiscale blood flow models [**27, 68**].

Denoting by $z = (u, h)$ the combined state components and ignoring external sources for the moment, the system (1.1) can be written in the form

$$\begin{cases} \dot{z}(t) = F(z(t)), \quad t > 0, \\ B(z(t)) = 0, \quad t > 0, \\ z(0) = z_0. \end{cases} \tag{1.2}$$

In the two-tank model, F and B are nonlinear. In studying systems of the form (1.2) one may consider the dynamics near an equilibrium state as a first step. A state z_e is called an *equilibrium* or *steady state* of (1.2) if the equations $F(z_e) = 0$ and $B(z_e) = 0$ hold. The steady state in the two-tank model depends on the material properties of the tube as well as those of the fluid. In general, the steady state is not unique. But with an additional constraint, the system has a unique steady state. The physical quantity to be conserved is the overall volume of the fluid.

To study the behavior of the system (1.2) near the steady state z_e, we linearize it about z_e by using a Taylor approximation and neglecting higher order terms. Introducing the variable $w = z - z_e$, which is the deviation of the state from the steady state, one has the linear system

$$\begin{cases} \dot{w}(t) = Lw(t), \qquad t > 0, \\ Gw(t) = 0, \qquad t > 0, \\ w(0) = w_0. \end{cases} \tag{1.3}$$

In this system, L and G tell us how the state w evolves in time and behaves at the boundary, respectively, and w_0 is the deviation of the initial state from the equilibrium. The well-posedness of (1.3) will be established using the theory of strongly continuous semigroups of bounded linear operators. This will be accomplished by Lumer-Phillips' Theorem.

Having the well-posedness of the linearized system, we are interested in the long-time behavior of the solutions. Does the state converge to the equilibrium, or equivalently, do the deviations tend to zero in some sense? If yes, what is the rate of convergence? Using a spectral method it will be shown that indeed the state converges exponentially fast to the equilibrium as long as damping is present and the initial data lies in a factor space. The latter condition is needed since the linearization (1.3) induces a one-dimensional linear manifold of equilibria, the span of the zero eigenvector. By mass conservation, this factor space is the appropriate state space for the deviations.

The spectrum of the generator is determined first in the absence of damping and this information is used to see how the spectrum changes as the damping factor increases. The generator stays spectral and thus the only elements of its spectrum are generalized eigenvalues. It will be shown that all of the eigenvalues lie on a single line determined by the damping coefficient, except for a finite number. If there is no damping then the normalized eigenvectors form an orthonormal basis. If there is damping then the eigenvectors are not orthogonal anymore, however, they still form a Riesz basis. Except for a countable number, the eigenvalues are simple.

Riesz bases and orthonormal bases are related through bounded invertible linear transformations. The Riesz basis approach has been successfully used by Guo and collaborators to prove the stability of certain beam equations [**29, 30, 31, 32**]. The basic idea of Riesz basis generation in these papers is the application of a result similar to Bari's Theorem [**81, Theorem 15**], i.e. to prove that a sequence of generalized eigenvectors is quadratically close to a given Riesz basis. Unlike beam equations, which have increasing spectral gap (distance between consecutive eigenvalues), wave equations have an asymptotically constant spectral gap. A refinement of the Riesz basis generation theorem of Guo [**29, Theorem 6.3**] was given recently by Xu and Weiss [**79, Theorem 2.4**]. The latter result will be used in proving that the infinitesimal generator of our system is Riesz spectral, i.e. has a Riesz basis consisting of generalized eigenvectors.

With Riesz bases at our disposal, we can express every element of the state space as a nonharmonic Fourier series and in turn also for the semigroup. As a result, we have a Fourier series representation of the solution that enables us to obtain a tight decay rate. As the damping coefficient increases, the number of eigenvalues approaching zero also increases. Thus increasing the damping coefficient will not necessarily increase the decay rate. This should be expected because if the fluid is viscous then it takes time to return to the steady state.

After studying the stability of the linearized two-tank model, a boundary control system will be considered. By applying pressures on the top of each tank, the system has the form

$$\begin{cases} \dot{w}(t) = Lw(t), & t > 0, \\ Gw(t) = q(t), & t > 0, \\ w(0) = w_0. \end{cases} \tag{1.4}$$

where q is the input. Is it possible to steer an arbitrary initial data to a desired final state at finite time? The answer is yes provided that the controllability time is sufficiently large. This reflects the finite propagation property for hyperbolic partial differential equations.

In multidimensions, boundary controllability of the wave equation is not always possible even for a large controllability time. It depends not only on the time but on the region where the input is applied as well. This region should satisfy the Geometric Control Condition stating that every ray of geometric optics should meet the control region during the control period (see Bardos, Lebeau and Rauch [**5**]). For linear symmetric hyperbolic systems in a bounded interval that generate groups, boundary controllability can be achieved for sufficiently large times. This follows from the fact that every characteristics will reach the boundary where the control is applied after a finite number of reflections (see Russell [**69**]).

Because the control acts on the boundary, we have an *unbounded input.* The main idea to prove the boundary controllability of (1.4) is to reformulate the abstract IBVP as a pure initial-value problem in an extended space [**70, 77**]. We use the Riesz basis approach to prove the exact controllability of the system. To do this, we modify the arguments in Tucsnak and Weiss [**77, Proposition 8.1.3**] which work with orthonormal bases. The spectrum is divided into lower and higher frequencies. For the higher frequencies, the restricted system is controllable thanks to Ingham's Theorem. The uniform gap property of the spectrum plays an important role here. For the lower frequencies, the restricted system is finite-dimensional and the Hautus test is applied to show its controllability. By applying the simultaneous controllability theorem in [**76**], the whole system is shown to be controllable.

A minimal time of controllability for single input controls will be given. However, Ingham's Theorem will not be applicable in this problem and we need to use other perturbation results in non-harmonic Fourier analysis. In order to solve this, we separate the lower and higher frequencies and replace the non-harmonic Fourier basis elements corresponding to the lower frequencies by some harmonic ones. With this on hand, the problem will be solved by applying a generalized Kadec's $\frac{1}{4}$-Theorem, see e.g. [**81, Corollary 2, p. 196**].

An additional result obtained from the control problem is that the velocity admits L^2-traces at the boundary. This cannot be obtained directly from semigroup methods and often called a *hidden regularity property* [**44, 46, 50**]. We will revisit this hidden regularity property together with additional results later with a different perspective. The results regarding the linearized two-tank model are given in Chapter 3.

The next step is to prove the well-posedness of the nonlinear two-tank model. One of the classical methods in proving the existence of solutions of nonlinear partial differential equations is to linearize the system by freezing some of the state variables and then proceed with either a fixed-point argument or an iteration scheme. Instead of working with the specific two-tank model, we draw our attention to a more general system that includes the two-tank model. These systems are given by

$$\begin{cases} u_t(t,x) + A(u(t,x))u_x(t,x) = f(u(t,x)), & t>0,\ 0<x<1, \\ B_0 u(t,0) = b_0(p_0(t), h(t)), & t>0, \\ B_1 u(t,1) = b_1(p_1(t), h(t)), & t>0, \\ \dot{h}(t) = H(h(t), q(t), u(t,0), u(t,1)), & t>0, \\ u(0,x) = u_0(x), & 0<x<1, \\ h(0) = h_0. \end{cases} \tag{1.5}$$

In principle, there are several ways to linearize systems of the form (1.5). The one we use here is by freezing u and h in A, f and H. More precisely, we consider the linearized system

$$\begin{cases} u_t(t,x) + A(v(t,x))u_x(t,x) = f(v(t,x)), & t>0,\ 0<x<1, \\ B_0 u(t,0) = b_0(p_0(t), h(t)), & t>0, \\ B_1 u(t,1) = b_1(p_1(t), h(t)), & t>0, \\ \dot{h}(t) = H(g(t), q(t), v(t,0), v(t,1)), & t>0, \\ u(0,x) = u_0(x), & 0<x<1, \\ h(0) = h_0. \end{cases} \tag{1.6}$$

for given frozen coefficients v and g. System (1.6) is semi-decoupled in the sense that u depends in h but h does not depend on u.

The linerization of (1.5) into (1.6) leads us to linear hyperbolic systems with variable coefficients. To analyze this, we follow the frameworks and methods in Benzoni-Gavage and Serre [**9**], Chazarain and Piriou [**15**], Métivier [**55**] and Coulombel [**17**]. Most of the results in this part parallel those in multidimensions given in [**9**] and many ideas of the proofs are borrowed from this reference. However, we deviate the presentation and state further remarks. This is useful not only for the nonlinear analysis but also in studying a linear hyperbolic system with linear ODE boundary conditions for which the linearized two-tank model is a particular example. The thesis will also serve as a venue to realize that the theory originally developed to treat multidimensional problems simplifies in the case of one space dimension. We hope that the extra details will be helpful in understanding these problems.

For IBVPs, one needs to determine what are the appropriate boundary conditions. In the case of hyperbolic equations, because information propagate along characteristic, care should be taken in imposing boundary conditions in order for the problem not to be underdetermined or overdetermined. Let us consider a simple transport equation moving with unit speed

$$\begin{cases} u_t(t,x) - u_x(t,x) = 0, \quad t > 0, \quad 0 < x < 1, \\ u(0,x) = u_0(x), \quad 0 < x < 1. \end{cases} \tag{1.7}$$

The characteristics of this equation are the straight lines $x + t =$ constant. Hence, information move from right to left. This observation tells us that a boundary condition at $x = 1$ should be imposed while there is none at $x = 0$. For (1.7) to be well-posed, the appropriate boundary condition is given by

$$u(t,1) = g(t), \qquad t > 0. \tag{1.8}$$

For diagonal systems, the number of boundary conditions should be equal to the number of incoming characteristics in that boundary. For systems that are not diagonal, the Uniform Kreiss-Lopatinskiĭ (UKL) condition gives the appropriate type of boundary conditions. In the case of half-space, the UKL condition implies the decay at infinity of solutions for linear hyperbolic systems of the form $e^{\lambda t}U(x)$ with $\Re\lambda > 0$, see [**15**].

We are interested in the well-posedness of the IBVPs with variable coefficients in L^2. The weak solutions in L^2 satisfiy a variational equation that takes the form

$$(u, \Lambda w)_X = (f, w)_X + (g, \Psi w)_Z, \qquad \forall\ w \in W. \tag{1.9}$$

for some spaces X, W, Z and operators Λ, Ψ. This equation is obtained by multiplying the differential equation by appropriate test functions, integrating by parts and using the boundary and initial conditions. With an abstract a priori estimate, the variational equation (1.9) has a solution $u \in X$. Its proof is based on the Hahn-Banach and Riesz Representation Theorems. The idea of the proof can be traced back to the work of Friedrichs [**28**] for symmetric systems. Therefore, proving an a priori estimate is the first step in proving the existence of weak solutions.

Strong solutions of the intial-boundary value problems are also introduced. As with weak solutions, they also belong to L^2, however, they are limits of smooth functions that satisfy a system that is an approximation or regularization of the original problem. According to its definition, every strong solution is a weak solution. Strong solutions satisfy an energy estimate which implies the uniqueness of strong solutions. It will be shown using the so-called *weak equals strong argument* that every weak solution is also a strong solution. Consequently, weak solutions are unique.

How does the weak solution satisfy the initial-boundary value problem? To answer this, we need to consider the space of functions $u \in L^2$ such $Lu := \partial_t u + A\partial_x u \in L^2$, where A is at least Lipschitz. This space is similar to the space $\{u \in L^2 : \text{div } u \in L^2\}$ used in studying the Navier-Stokes equation. These spaces are called *graph spaces*. The usual trace operator in H^1 can be extended to define a generalized trace operator for the graph space $\{u \in L^2 : Lu \in L^2\}$, but the traces are now in $H^{-\frac{1}{2}}$. To treat IBVPs, we will also restrict the trace to the edges of the time-space domain. With these considerations, it will be seen that weak solutions satisfy the partial differential equation in the sense of distributions and the boundary conditions and initial condition are satisfied in the sense of (generalized) traces.

The well-posedness of the IBVPs is based on the well-posedness of pure boundary value problems (BVPs). It will be seen that an IBVP with homogeneous data can be solved by extending the boundary data by zero and considering the associated

BVP. For this reason, we need a well-posedness theory for BVPs. A weak solution for the BVP satisfies a variational equation that has the form (1.9) as well. Thus, deriving a priori estimates for the BVP is a crucial step.

If the system admits a *functional boundary symmetrizer* then a suitable a priori estimate can be shown. Symmetrizable systems with dissipative boundary conditions have a natural functional boundary symmetrizer. There are also systems which admit functional boundary symmetrizers without the dissipativity condition. This was initiated by Kreiss [**45**] for the case of constant coefficients and then later for variable coefficients in [**15**]. The construction of the boundary symmetrizers is based on *Kreiss symmetrizers*. With the help of the UKL condition, they can be first defined locally. In the systems that we considered, the local symmetrizers can be taken in diagonal form. The local Kreiss symmetrizers serve as building blocks in deriving a global Kreiss symmetrizer. This is done by homogeneity and compactness arguments. The passage from global Kreiss symmetrizers to functional boundary symmetrizers relies on pseudodifferential calculus for smooth coefficients and paradifferential calculus for Lipschitz coefficients. In particular, a functional boundary symmetizer can be obtained by symmetrizing the operator having the global Kreiss symmetrizer as its symbol. A short survey on pseudodifferential calculus and paradifferential calculus is provided in Appendix C.

The weak solutions for the IBVPs have L^2-traces on the boundary even though they are only in L^2 in the time-space domain. This can be attributed again to the fact that information propagate along characteristics. To illustrate this, let us consider the simple system (1.7)$-$(1.8). The solution of this problem given by the method of characteristics is

$$u(t,x) = \begin{cases} u_0(x+t), & \text{if } 0 < x < 1,\ 0 < t < 1-x, \\ g(t+x-1), & \text{if } 0 < x < 1,\ 1-x < t. \end{cases} \tag{1.10}$$

Suppose that u_0 and g are both L^2. Due to the boundary condition at $x = 1$ it is clear that u has an L^2-trace at this boundary. Likewise, from (1.10) it can be seen that the profile at the boundary $x = 0$ is given by the initial data if $0 < t < 1$ and by the boundary data g if $t > 1$. Hence, u admits an L^2-trace at the boundary $x = 0$ as well. This resembles the hidden regularity property that we have mentioned earlier for the linearized two-tank model.

In this work, we are also interested in smooth solutions of the system (1.5). We will prove well-posedness in the Sobolev space H^m for integers $m \geq 3$. Because we will do this using an iteration scheme through the linearization (1.6), we need to prove the regularity of the weak solutions for the PDE part. It is not enough to have smooth boundary and initial data, one also requires compatibility conditions. To see this, let us again consider the system (1.7)$-$(1.8). Suppose that the boundary data g and the initial data u_0 are both continuous. The the solution u, given by (1.10), is continuous on $(0,\infty) \times (0,1)$ except possibly at those points on the line $x+t = 1$. This line is the characteristic emanating at the boundary $x = 1$. To have a continuous solution, g and u_0 must satisfy the compatibility condition $g(0) = u_0(1)$. In order to have more regularity, one needs more regularity on u_0 and g and higher order compatibility conditions.

The regularity theorems for the IBVPs are based on H^m for integers $m \geq 3$ in the case of variable coefficients and for integers $m \geq 1$ in the constant coefficient case. Again, these are obtained using a priori estimates in Sobolev spaces. We follow the derivations in [**9**] and [**55**] with some modifications. As in the L^2 case, the regularity of the solutions at the boundary will be inherited from the regularity of the boundary and initial data. Regularity theorems for hyperbolic systems with smooth coefficients can be found in the paper of Rauch and Massey [**64**].

Before proceeding with the nonlinear system (1.5), we prove the well-posedness of the linear hyperbolic system with variable coefficients coupled with linear ordinary differential equations at the boundary

$$\begin{cases} (\partial_t + A(v(t,x))\partial_x + R(t,x))u(t,x) = f(t,x), \quad 0 < t < T,\ 0 < x < 1, \\ B_0 u(t,0) = g_0(t) + Q_0(t)h(t), \quad 0 < t < T, \\ B_1 u(t,1) = g_1(t) + Q_1(t)h(t), \quad 0 < t < T, \\ h'(t) = H(t)h(t) + G_0(t)u(t,0) + G_1(t)u(t,1) + S(t), \quad 0 < t < T, \\ u(0,x) = u_0(x), \quad 0 < x < 1, \\ h(0) = h_0 \end{cases} \tag{1.11}$$

The usual energy estimates imply well-posedness is used to prove well-posedness in L^2 of this system. It will be shown that u satisfies a hidden regularity property, i.e., it has L^2-trace at the boundary. This property implies that the ODE component h does not lie only in L^2 but in H^1.

In the constant coefficient case, this well-posedness result implies that the weak solution generates a $\mathcal{C}_0$-semigroup. As a result, the weak solution is the same as the solution given by the semigroup approach. In particular, we obtain the additional regularity of u at the boundary and the regularity of h. Let us have a detour with the linearized two-tank model. The linearized two-tank model is a particular example of the constant coefficient case of (1.11). Thus, the remarks stated above can be applied. Both the cross-section and the velocity have L^2-traces at the boundary and the state corresponding to the level heights lie in H^1. This is an improvement of the result that we mentioned earlier since by semigroup methods we only knew the boundary trace for the velocity.

After dealing with the linear systems in Chapter 4, the nonlinear system (1.5) will be discussed in Chapter 5. One way to prove the well-posedness of (1.5) is to prove that the map $(v, g) \mapsto (u, h)$ on a suitable function space, where (u, h) is the solution of (1.6) for a given pair of frozen coefficients (v, g), has a fixed point. However, this task is difficult to handle. Instead of using a fixed point theorem, we shall instead utilize the contraction mapping principle, that is, using a Picard iteration scheme. We start with an admissible initial pair of frozen coefficients (u^0, h^0) and define (u^1, h^1) to be the solution of the system (1.6) where (v, g) is replaced by (u^0, h^0). Then we define (u^2, h^2) to be the solution of (1.6) with the pair of frozen coefficients (u^1, h^1). We continue this procedure to obtain a sequence $((u^j, h^j))_{j \geq 1}$ and we hope that this sequence converges in some sense and that the limit satisfies the system (1.5).

The Picard iteration described above has a disadvantage, we need the time of existence to be sufficiently small due to some absorption arguments used in deriving energy estimates. This means that we are only able to prove a local-in-time well posedness of (1.5). But this is the best we can expect for quasilinear systems due to the nonlinearity. Smooth solutions may blow-up in finite time or break-up creating shocks.

Let us illustrate the discontinuities developing due to nonlinearity. We start with the simple ODE with quadratic nonlinearity

$$\begin{cases} \dot{w}(t) + w(t)^2 = 0, \quad t > 0, \\ w(0) = w_0, \end{cases} \tag{1.12}$$

where $w_0 \in \mathbb{R}$. The solution of (1.12) is given by $w(t) = w_0(w_0 t + 1)^{-1}$. Note that the steady state $w = 0$ of (1.12) is unstable. Indeed, if $w_0 < 0$ then $|w(t)| \to \infty$ as $t \to -1/w_0$. Therefore, no matter how the initial data is close to the steady state, if it is negative then the solution will blow-up in finite time.

Our next example is the well-known (inviscid) Burgers' equation

$$\begin{cases} u_t(t,x) + u(t,x)u_x(t,x) = 0, \quad t > 0, \ -\infty < x < \infty, \\ u(0) = u_0. \end{cases} \tag{1.13}$$

The characteristics for (1.13) are solutions of the differential equation

$$x'(t) = u(t, x(t)), \qquad x(0) = x_0. \tag{1.14}$$

Along characteristics we have

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t, x(t)) = u_t(t, x(t)) + u_x(t, x(t))x'(t) = 0,$$

which implies that u is constant along characteristics. Thus $x'(t) = \text{constant}$ and therefore the characteristics are the straight lines $x = x_0 + u_0(x_0)t$.

Consider the nonlinear equation

$$F(x_0; t, x) = x - x_0 - u_0(x_0)t = 0 \tag{1.15}$$

in the unknown x_0 for fixed (t, x). Suppose that u_0 is continuously differentiable. Note that $\partial_{x_0} F(x_0; t, x) = -1 - u_0'(x_0)t$. When t is small enough so that $-1 - u_0'(x_0)t \neq 0$, we can use the implicit function theorem to conclude that the equation (1.15) is solvable for $x_0 = x_0(t, x)$ given (t, x). Thus tracking back the characteristic passing from (t, x) through its intersection at x_0 on the x-axis we conclude that $u(t, x) = u_0(x_0(t, x))$.

If $u_0' \geq 0$ then (1.15) is always uniquely solvable and we have a global-in-time smooth solution provided that u_0 is smooth. The problem occurs if $u_0'(x_0) < 0$ at some point x_0 and then the system (1.15) is not solvable anymore. In this case the state u will be multivalued and the first time where such situation occurs the state u will have an infinite slope. This phenomenon is called *shock formation*. Suppose that u is a smooth solution, say continuously twice differentiable, of the Burgers' equation and let $w(t) = u_x(t, x(t))$ where x satisfies (1.14). Taking the derivative of w and using $u_{xt} + uu_{xx} + u_x^2 = 0$, which is obtained by differentiating the Burgers' equation with respect to x, it can be seen that w satisfies (1.12) with $w_0 = u_0'(x_0)$. According to our preceding discussions, w blows-up in finite time if $u'(x_0) < 0$. Therefore if the initial data is decreasing at some point x_0 then the slope of the solution at some point increases without bound in finite time.

Knowing that in general a global-in-time solution does not necessarily exist for (1.5), what phenomena occur if the maximal time of smooth solution is finite? The answer is already given by the two illustrations provided above. If the maximal time of existence is finite then the state leaves every compact subset of the hyperbolicity region or its first order derivatives blow-up. The local-in-time existence and blow-up criterion for our systems will be shown in Chapter 5.

It is known that the presence of a linear damping term can prevent shock formation at least for small and smooth initial data, see Dafermos [**21**]. Let us illustrate this in the case where there is damping for (1.12), that is,

$$\begin{cases} \dot{w}(t) + w(t)^2 + w(t) = 0, \quad t > 0, \\ w(0) = w_0, \end{cases} \tag{1.16}$$

The equation (1.16) has two steady states $w = 0$ and $w = -1$. A standard phase plane analysis shows that $w = -1$ is unstable while $w = 0$ is stable. If $w_0 > -1$ then (1.16) has a global solution w converging to the stable steady state exponentially.

Consider the Burgers' equation with damping

$$\begin{cases} u_t(t,x) + u(t,x)u_x(t,x) + u(t,x) = 0, \quad t > 0, \ -\infty < x < \infty \\ u(0) = u_0. \end{cases} \tag{1.17}$$

Define $w(t) = u_x(t, x(t))$, where $x(t)$ are the characteristics of the Burgers' equation with no damping passing through $(0, x_0)$. Then a straightforward computation shows that w satisfies (1.16). Hence, if $u' \geq -1$ then we have a global solution for (1.17). Now, if we multiply the partial differential equation in (1.17) by u, integrate by parts and assume that the solutions decay at infinity, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\mathbb{R})} + \|u(t)\|^2_{L^2(\mathbb{R})} = 0, \quad t > 0.$$

Thus $\|u(t)\|_{L^2(\mathbb{R})} = e^{-t}\|u_0\|_{L^2(\mathbb{R})}$ so that the solution decays to the steady state $u = 0$ of (1.17) exponentially fast

These two examples show that smooth data close to the steady state together with damping imply global existence of solutions and its convergence to the steady state. We will use these ideas to prove the same results for the nonlinear two-tank model.

Necessary and sufficient conditions for the existence of global solutions both for general and physical quasilinear hyperbolic systems have been developed in the past years, see [**18, 33, 47, 49, 68**]. However, there are only a few works dealing with bounded domains. In one-space dimension, Ruan et al. [**68**] investigated the global existence of smooth solutions of a network of 2×2 systems of balance laws in bounded intervals under a *dissipative condition* on the boundary conditions. This condition is similar to what has been considered in [**47, Chapter 5**]. However, the dissipative condition is not satisfied for instance by the isentropic Euler system, systems with relaxation, for boundary conditions arising in blood flow models, nor by the two-tank model.

In this thesis, two main tools were used to prove the global existence of solutions, namely, the entropy and energy methods. Any smooth solution u of a system of conservation laws satisfies an additional conservation law of the form $\eta(u)_t + q(u)_x = 0$, called *companion laws*. The function η is called an *entropy* and the function q is the corresponding *entropy flux*. For the isothermal Euler equations this additional conservation law is the conservation of mechanical energy. In general, one cannot guarantee the existence of nontrivial companion laws.

The energy method was used by Nishida [**59**] and Kawashima [**42**] for hyperbolic and hyperbolic-parabolic equations. This was then used by several authors for isothermal Euler equations [**18**], partially dissipative systems with convex entropies

[**7, 33, 80**], relaxation models with nonconvex flux [**52**], systems arising in blood flow models [**68**] and others. The main idea is to define an energy functional and to derive an estimate for this functional. Lower order estimates can be obtained using the relative entropy method [**33**]. The relative entropy associated with a strictly convex entropy, loosely speaking, can serve as a *distance* between solutions, e.g., classical, strong, weak, of conservation laws or balance laws, cf. [**21**]. For higher order estimates involving terms that do not have a dissipative term one useful criterion, at least for Cauchy problems, is the Shizuta-Kawashima condition which was formulated in [**72**]. However on a bounded interval, a different method was used in [**68**], namely the construction of entropy-entropy flux pairs for the Riemann invariants in deriving higher order estimates. In the case of bounded domains, boundary terms arise and this causes some difficulty in obtaining the necessary estimates. The dissipative condition plays a crucial role in the proof of the estimates in [**68**]. Most of the existence results use the smallness assumptions on the initial data. Even with this restriction the proofs are not trivial.

Here, we will also use the relative entropy method to obtain lower order estimates for the energy functionals and use appropriate entropy-entropy flux pairs for higher order estimates. The main idea is to construct entropy-entropy flux pairs (η, q) such that

$$\eta_t + q_x = M$$

for some source term M which is, roughly speaking, dominated by the damping term, which is the velocity in the two-tank model, or its derivatives. We will not assume the dissipative condition as in [**68**] but we use the special structure of the boundary conditions of the two-tank model.

The energy estimates imply immediately that the global solution of the nonlinear two-tank model with smooth and small data converges to the steady with respect to the norm of $H^1 \times H^1 \times \mathbb{R}^2$. The rate of convergence is exponential if one uses the norm of $L^2 \times L^2 \times \mathbb{R}^2$. To prove this we use some interpolation estimates, a Growall-type lemma and the linear stability of Chapter 3.

1.1 NOTATIONS

The sets of positive integers, integers, real numbers and complex numbers are denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. We denote by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the set of natural numbers. The same notation $|\cdot|$ for the Euclidean norms in $\mathbb{R}$ and $\mathbb{C}$ is used throughout the text. Given $z \in \mathbb{C}$, the real and imaginary parts of z are denoted by $\Re z$ and $\Im z$, respectively.

Let $O \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be an open set and $k \in \mathbb{N}_0$. The space of functions defined in O that are continuously differentiable up to order k is denoted by $\mathscr{C}^k(O)$. We let $\mathscr{C}(O) = \mathscr{C}^0(O)$ and $\mathscr{C}^\infty(O) = \bigcap_{k\in\mathbb{N}_0} \mathscr{C}^k(O)$. The elements in $\mathscr{C}^\infty(O)$ having compact support in O is denoted by $\mathscr{D}(O)$. The subset of $\mathscr{C}^\infty(O)$ that has bounded derivatives of any order is denoted by $\mathscr{C}_b^\infty(O)$. Given a subset $S \subset \mathbb{R}^d$ with nonempty interior, we define $\mathscr{D}(S) = \{u_{|S} : u \in \mathscr{D}(\mathbb{R}^d)\}$.

The Schwartz class of rapidly decreasing functions is denoted by $\mathscr{S}(\mathbb{R}^d)$. The notations $\mathscr{D}(O)'$, $\mathscr{S}'(\mathbb{R}^d)$, $\mathscr{E}'(\mathbb{R}^d)$ represent the space of distributions in O, tempered distributions and distributions with compact support, respectively. The usual notations for the Sobolev spaces $W^{s,p}(O)$, $H^s(O) = W^{s,2}(O)$ and $L^p(O) = W^{0,p}(O)$ for $s \in \mathbb{R}$ and $1 \le p \le \infty$ are used. The product of m copies of $W^{s,p}(O)$ is denoted by

$W^{s,p}(O)^m$. However, if the number of components is clear in the context, we will remove the superscript m.

Given an open set $O \subset \mathbb{R}^2 = \{(t,x) : t, x \in \mathbb{R}\}$, $\gamma \geq 1$ and a nonnegative integer m, the space $H^m_\gamma(O)$ is defined to be the usual Sobolev space with γ-depending norm

$$\|u\|_{H^m_\gamma(O)} := \sum_{|\alpha| \leq m} \gamma^{m-|\alpha|} \|\partial^\alpha u\|_{L^2(O)} < \infty.$$

It follows from the definition that

$$\gamma^{m-k}\|w\|_{H^k} \leq \|w\|_{H^m_\gamma}, \qquad 0 \leq k \leq m, \ w \in H^m. \tag{1.1.18}$$

It can be shown that there exist constants $0 < c < C$ independent of both u and γ such that

$$c \sum_{|\alpha| \leq m} \gamma^{m-|\alpha|} \|e^{-\gamma t}\partial^\alpha u\|_{L^2(O)} \leq \|e^{-\gamma t}u\|_{H^m_\gamma(O)} \leq C \sum_{|\alpha| \leq m} \gamma^{m-|\alpha|} \|e^{-\gamma t}\partial^\alpha u\|_{L^2(O)}$$

whenever $e^{-\gamma t}u \in H^m(O)$. The norm $\|u\|_{H^m_\gamma(\mathbb{R}^2)}$ is equivalent to $\| \operatorname{Op}(\lambda^{m,\lambda})u\|_{L^2(\mathbb{R}^2)}$, where $\operatorname{Op}(\lambda^{m,\lambda})$ is the pseudo-differential operator with symbol $\lambda^{m,\gamma}(\delta,\xi) = (\gamma^2 + \delta^2 + \xi^2)^{m/2}$, see Appendix C.

Let $O \subset \mathbb{R}^d$ be open and let $CH^m([0,T] \times O) = \bigcap_{p=0}^m C^p([0,T]; H^{m-p}(O))$ for $m \in \mathbb{N}_0$ be equipped with the norm

$$\|u\|_{CH^m([0,T]\times O)} = \left(\sum_{j=0}^m \sup_{\tau \in [0,T]} \|\partial_t^j u(\tau)\|^2_{H^{m-j}(O)} \right)^{\frac{1}{2}}.$$

We write $CL^2([0,T]\times O)$ instead of $CH^0([0,T]\times O)$. For each $m \in \mathbb{N}_0$, $CH^m([0,T]\times O)$ equipped with the norm $\| \cdot \|_{CH^m([0,T]\times O)}$ is a Banach space.

If X is a Hilbert space consisting of functions and $\gamma \in \mathbb{R}$, we define the weighted space $e^{\gamma t}X = \{e^{\gamma t}u : u \in X\}$. With the inner product $(w,z)_{e^{\gamma t}X} = (e^{-\gamma t}w, e^{-\gamma t}z)_X$, $w, z \in e^{\gamma t}X$, the space $e^{\gamma t}X$ becomes a Hilbert space.

2

A MODEL OF FLOW IN AN ELASTIC TUBE

The goal of this chapter is to derive a model for the flow of an incompressible fluid contained in an elastic tube. A tank or basin is connected at each end of the tube, see Figure 2.1. All throughout, the variables t and x designate for time and location, respectively. Let $A(t,x)$ be the cross section of a circular elastic tube of length ℓ that is filled with incompressible fluid of constant density ρ. The reference cross section at reference pressure p_0 is denoted by $A_0 = \pi r_0^2$. Denote by $u(t,x)$ the velocity of the fluid, and a positive velocity means flow in the positive x direction. In modeling the flow of the fluid in the tube, we apply the law of conservation of mass and Newton's second law. In the derivation, it is assumed that the fluid is a *continuum*, that is, physical properties associated with the fluid such as density, pressure and velocity are defined at every point on a given domain. This hypothesis idealizes the property that fluids are composed of discrete molecules.

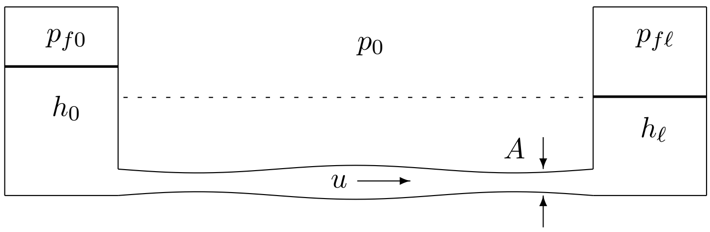

Figure 2.1.: An elastic tube connected to two tanks.

2.1 EULER'S CONTINUITY EQUATION

First we derive the continuity equation from the law of conservation of mass. The left end of the tube is located at the origin $x = 0$. The mass of the fluid in $[x_1, x_2] \subset (0, \ell)$ at time t is given by

$$M(t) = \int_{x_1}^{x_2} \rho A(t,x)\ \mathrm{d}x.$$

At time t, the flux of mass at position x is $\rho A(t,x)u(t,x)$. The rate of change of mass in $[x_1, x_2]$ is given by

$$M'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} \rho A(t,x)\ \mathrm{d}x.$$

However, this change corresponds to fluid flow across the boundary. Hence the rate of change is the difference of the fluxes at x_1 and x_2, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} \rho A(t,x)\ \mathrm{d}x = \rho A(t,x_1)u(t,x_1) - \rho A(t,x_2)u(t,x_2).$$

Integrating over a time interval $[t_1, t_2]$ tells us that the mass within $[x_1, x_2]$ at time t_2 is equal to the mass within $[x_1, x_2]$ at the previous time t_1 plus the integrated fluxes of mass across the boundaries x_1 and x_2, that is,

$$\begin{aligned}\int_{x_1}^{x_2} \rho A(t_2, x)\ \mathrm{d}x &= \int_{x_1}^{x_2} \rho A(t_1, x)\ \mathrm{d}x + \int_{t_1}^{t_2} \rho A(t, x_1) u(t, x_1)\ \mathrm{d}t \\ &\quad - \int_{t_1}^{t_2} \rho A(t, x_2) u(t, x_2)\ \mathrm{d}t.\end{aligned}$$

This is the *integral form of the law of conservation of mass.* Cancelling ρ and assuming that A and u are smooth, we can rewrite the integral form as

$$\int_{x_1}^{x_2} \int_{t_1}^{t_2} \partial_t A(t, x)\ \mathrm{d}t\ \mathrm{d}x = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \partial_x (A(t, x) u(t, x))\ \mathrm{d}x\ \mathrm{d}t.$$

Assume that the functions are well-behaved so that we can reverse the order of the integration in the right hand side. As t_1, t_2, x_1, x_2 are arbitrary, this gives the *differential form of the conservation law*

$$\partial_t A(t, x) + \partial_x (A(t, x) u(t, x)) = 0, \qquad t > 0,\ 0 < x < \ell. \tag{2.1.1}$$

also known as *Euler's continuity equation.*

2.2 LAW OF BALANCE OF MOMENTUM

Let $x(t)$ denote the position of a fluid particle at time t so that the velocity is given by

$$u(t, x(t)) = x'(t).$$

Applying the chain rule, the acceleration at time t of the fluid particle is therefore given by

$$a(t) = \frac{\mathrm{d}}{\mathrm{d}t} u(t, x(t)) = \partial_t u(t, x(t)) + \partial_x u(t, x(t)) x'(t) = \partial_t u + u \partial_x u = Du.$$

Here $D = \partial_t + u \cdot \partial_x$ denotes the *material derivative.*

To use Newton's second law, we need to consider internal and external forces acting on the fluid. The first one is stress (force per unit area). This is due to internal forces that act on a part of the fluid across its surface by other parts of the fluid. Assuming that the fluid inside the tube is *ideal*, there is a function $p(t, x)$ called the *pressure* such that if S is a surface in the fluid then the force across S per unit area is given by $p(t, x)\nu$, where ν is the unit vector normal to S. Thus, forces only act orthogonally to the surface and hence tangential forces are neglected. Intuitively, this means that rotation in the fluid is not taken into account. The stress acting on $[x_1, x_2]$ due to pressure at time t is given by

$$F_s = p(t, x_1) - p(t, x_2) = -\int_{x_1}^{x_2} \partial_x p(t, x)\ \mathrm{d}x.$$

External forces may include gravity and frictional forces. If $b(t, x)$ denotes the given body forces per unit mass and $F(t, x)$ is the friction (force per unit volume) then the net external force acting on $[x_1, x_2]$ is

$$F_e = \int_{x_1}^{x_2} \rho b(t, x) + F(t, x)\ \mathrm{d}x.$$

The net force per unit area is therefore

$$F_{\text{net}} = F_s + F_e = \int_{x_1}^{x_2} \left(-\partial_x p(t,x) + \rho b(t,x) + F(t,x)\right) \, \mathrm{d}x.$$

Thus, on any part of the fluid, the net force per unit volume is $-\partial_x p + \rho b + F$. According to Newton's second law, the net force per unit volume is equal to the product of the density of the fluid and its acceleration and so

$$\rho\left(\partial_t u + u\partial_x u\right) = -\partial_x p + \rho b + F. \tag{2.2.1}$$

This is the *differential form of the law of balance of momentum.*

2.3 FRICTIONAL FORCE

The frictional force is modeled by Hagen-Poiseuille's law for stationary laminar flow. Loosely speaking, this law states that there is a decrease in pressure due to friction. Using the model in Rath and Teipel [**63**] we have

$$F(t,x) = -\frac{8\pi\mu_0}{A_0} u(t,x)$$

where μ_0 is the viscosity of the fluid. With this model, we can see from (2.2.1) that the acceleration decreases if $\mu_0 > 0$. For example, if the fluid is flowing in the positive direction then $F(t,x) < 0$ and so the acceleration is decreased. We can think of F as dissipation.

2.4 EQUATION OF STATE

If the tube rests on solid ground, then the force due to gravity is cancelled by the opposing force, the ground reaction force. This means that $b = 0$. From the continuity equation (2.1.1) and the balance equation (2.2.1) we have the system

$$\begin{cases} \partial_t A + \partial_x(Au) = 0 \\ \partial_t u + u\partial_x u + \dfrac{1}{\rho}\partial_x p = -\dfrac{8\pi\mu_0}{\rho A_0} u \end{cases} \tag{2.4.1}$$

In (2.4.1), there are three unknown variables u, p and A. To close the system, we need an additional equation. This will be done by writing the pressure as a function of the cross sectional area, that is, $p = p(A)$. Such equation is called an *equation of state* (EOS).

To obtain the equation of the state we follow [**63**]. The two main ingredients of the derivation of the EOS are Laplace's law of cylinders and Hooke's law. Due to the difference of pressures inside and outside of the tube, the wall of the tube is stretched or compressed. If the pressure inside the tube is greater than the one outside the tube, then the tube's wall is stretched, otherwise it is compressed. We assume that the deformed tube also has circular cross section. Laplace's law of cylinders relates this difference of pressure to the radius and thickness of the tube material. The larger the difference of the inner pressure and from the outside pressure the larger the tension is. If the inner radius is large then the tension is higher, however, if the tube is thick then the tension is lower. Taking these considerations into account, if

σ_τ and s denote the tension in the wall and the thickness of the tube, respectively, then

$$\sigma_\tau(t,x) = \frac{r_0 \Delta p(t,x)}{s} = \frac{r_0(p(t,x)-p_0)}{s}. \tag{2.4.2}$$

To obtain another equation involving the tension, we can view the tube as a spring and apply Hooke's law. This law states that the extension or strain of a spring is proportional to the load applied to it. In this case, the strain is the ratio of the change in radii to the reference radius r_0. Hence

$$\sigma_\tau(t,x) = E\frac{\Delta r(t,x)}{r_0} = E\frac{r(t,x)-r_0}{r_0}. \tag{2.4.3}$$

where E is the proportionality constant, called the Young's modulus of the tube material. Suppose that the material is homogeneous so that E is constant. Solving for $r(t,x)$ in (2.4.3) and using the formula (2.4.2) we have

$$r(t,x) = \frac{r_0}{E}\sigma_\tau(t,x) + r_0 = r_0\left(1 + \frac{r_0}{Es}(p(t,x)-p_0)\right).$$

Therefore the tube's wall is stretched if $p > p_0$ while it is compressed if $p < p_0$. Because the cross section remains circular, we have

$$A(t,x) = \pi r(t,x)^2 = A_0\left(1 + \frac{r_0}{Es}(p(t,x)-p_0)\right)^2. \tag{2.4.4}$$

Assuming that $A(t,x) > 0$, solving for the pressure in (2.4.4) gives us an EOS

$$p(t,x) = \frac{sE}{r_0}\left(\sqrt{\frac{A(t,x)}{A_0}} - 1\right) + p_0. \tag{2.4.5}$$

Using the equation of state (2.4.5) in (2.4.1) we obtain the system

$$\begin{cases} \partial_t A + \partial_x(Au) = 0 \\ \partial_t u + u\partial_x u + \dfrac{sE}{\rho r_0\sqrt{A_0}}\partial_x(A(t,x)^{\frac{1}{2}}) + \dfrac{8\pi\mu_0}{\rho A_0}u = 0. \end{cases} \tag{2.4.6}$$

2.5 CONSERVATIVE AND NONCONSERVATIVE FORMS

The system (2.4.6) is called a hyperbolic system of partial differential equations. It can be written in the form $U_t + F(U)_x = G(U)$ where $U = (A,u)$ is the state vector,

$$F(U) = \begin{pmatrix} Au \\ \dfrac{1}{2}u^2 + \dfrac{sE}{\rho r_0}\left(\dfrac{A(t,x)}{A_0}\right)^{\frac{1}{2}} \end{pmatrix}$$

and

$$G(U) = \begin{pmatrix} 0 \\ -\dfrac{8\pi\mu_0}{\rho A_0}u \end{pmatrix}.$$

Such systems are called *balance laws* and the terms F and G are called the *flux* and the *source term*, respectively. If there is no source term, that is, $G = 0$, then they are called *conservation laws*. The equation $U_t + F(U)_x = G(U)$ is said to be in *conservative form* while the form $U_t + F_N(U)U_x = G(U)$ is said to be in *nonconservative form.* Assuming that the state vector U and the flux F are smooth, every system in conservative form can be written in nonconservative form by setting $F_N = DF$ where DF is the Jacobian of F. In the case of (2.4.6), it can be written in nonconservative form with

$$DF(U) = \begin{pmatrix} u & A \\ \dfrac{sE}{2\rho r_0\sqrt{A_0 A}} & u \end{pmatrix}$$

For smooth solutions with smooth fluxes the conservative and nonconservative forms are equivalent.

The eigenvalues of the Jacobian are given by

$$\lambda_{\pm}(A,u) = u \pm \left(\frac{sE\sqrt{A}}{2\rho r_0\sqrt{A_0}}\right)^{\frac{1}{2}}.$$

The speed of propagation in the nonlinear model is then given by

$$a = \left(\frac{sEr}{2\rho r_0^2}\right)^{\frac{1}{2}},$$

as in [**63**]. For small disturbances, $r \approx r_0$ so that the speed of sound of the linearized system is $a \approx \left(\frac{sE}{2\rho r_0}\right)^{1/2}$. In [**63**] it is approximately equal to 12 meters per second.

2.6 INITIAL AND BOUNDARY CONDITIONS

In order for the system (2.4.6) to be well-posed, it should be supplied by initial and boundary conditions. Denote by u^0 and A^0 the initial velocity of the fluid inside the tube and the profile of the tube, respectively. If $A > 0$ then $\lambda_- < 0 < \lambda_+$. Thus, one wave propagates from left to right and one in the opposite direction. Therefore there should be two boundary conditions, one at the left end and one at the right end. For example, when the tube lies on a table and the ends are closed by rigid lids, one could consider Dirichlet conditions for the velocity and cross section at both ends, namely,

$$\begin{aligned} u(t,0) &= u(t,\ell) = 0, \\ A(t,0) &= A(t,\ell) = A_0. \end{aligned}$$

This seems to be an overdetermination. Introducing the characteristic variables $w = -u + 4\kappa A^{1/4}$ and $z = u + 4\kappa A^{1/4}$, where $\kappa = (sE/2\rho_0\sqrt{A_0})^{1/2}$ the system (2.4.6) can be diagonalized. Assigning both u and A to each endpoints will give us values for w and z at each of the endpoints, which clearly is an overdetermination. Looking for net flow through the tube one would leave the ends open, but one could fix the cross section, for example, $A(t,0) = A(t,\ell) = A_0$. Or one could let the ends open and elastic, but enforce flow, for example $u(t,0) = u_0(t)$ and $u(t,\ell) = u_\ell(t)$. In [**75**], boundary conditions are derived from the in-stationary Bernoulli equation. The experimental setup in [**63**] seems to leave the ends open and elastic but have conditions on $p(t,0)$ and $p(t,\ell)$, which is a function of $A(t,0)$ and $A(t,\ell)$, respectively.

In the configuration of Figure 2.1, the ends of the tube are at the bottom of basins that contain water, so there is hydrostatic pressure. The pressure at the left end is then given by

$$p(t,0) = p_0 + p_{f0}(t) + \rho g h_0(t),$$

where p_{f0} is a control pressure applied to the surface of the water in the left tank, $h_0(t)$ is the level height of water in the left tank and the term $\rho g h_0(t)$ is the hydrostatic pressure. Similarly, the pressure at the right end is given by

$$p(t,\ell) = p_0 + p_{f\ell}(t) + \rho g h_\ell(t),$$

where $p_{f\ell}$ and h are the control pressure and level height of the right tank. Using the equation of state (2.4.5) we arrive at the following boundary conditions for the cross section at the left and right ends of the tube

$$\begin{aligned} A(t,0) &= A_0\left(1 + \frac{r_0}{sE}(\rho g h_0(t) + p_{f0}(t))\right)^2, && (2.6.1)\\ A(t,\ell) &= A_0\left(1 + \frac{r_0}{sE}(\rho g h_\ell(t) + p_f(t))\right)^2. && (2.6.2)\end{aligned}$$

These boundary conditions imply that the cross section is not fixed at the ends.

The rate of change of the level height of the fluid in the tank should be equal to the flux at the boundary. Thus the rate of change for the level height h_ℓ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}(A_T h_\ell(t)) = A(t,\ell)u(t,\ell), \tag{2.6.3}$$

where A_T is the cross section of the containers. Analogously, the evolution of the level height of water in the left tank is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}(A_T h_0(t)) = -A(t,0)u(t,0). \tag{2.6.4}$$

The initial level heights are denoted by $h_0(0) = h_0^0$ and $h_\ell(0) = h_\ell^0$.

From the nonconservative form of (2.4.6) together with the boundary conditions (2.6.1)−(2.6.4), we have the coupled system of hyperbolic PDEs and nonlinear ODEs

$$\begin{cases} \partial_t A(t,x) + u(t,x)\partial_x A(t,x) + A(t,x)\partial_x u(t,x) = 0 \\ \partial_t u(t,x) + u(t,x)\partial_x u(t,x) + \dfrac{sE}{2\rho r_0}(A_0 A(t,x))^{-\frac{1}{2}}\partial_x A(t,x) + \dfrac{8\pi\mu_0}{\rho A_0}u(t,x) = 0 \\ h_0'(t) = -\dfrac{1}{A_T}A(t,0)u(t,0) \\ h_\ell'(t) = \dfrac{1}{A_T}A(t,\ell)u(t,\ell) \\ A(t,0) = A_0\left(1 + \dfrac{r_0}{sE}(\rho g h_0(t) + p_{f0}(t))\right)^2 \\ A(t,\ell) = A_0\left(1 + \dfrac{r_0}{sE}(\rho g h_\ell(t) + p_{f\ell}(t))\right)^2 \\ A(0,x) = A^0(x), \qquad u(0,x) = u^0(x) \\ h_0(0) = h_0^0, \qquad h_\ell(0) = h_\ell^0. \end{cases} \tag{2.6.5}$$

for $t > 0$ and $0 < x < \ell$.

We can think of the tanks as a source term at the boundary, but this source term depends on the state. The differential equations for h_0 and h_ℓ in (2.6.5) can be integrated and substituted into the boundary conditions for A to obtain the nonlocal-in-time boundary conditions

$$\begin{aligned} A(t,0) &= A_0 \left(1 - \frac{r_0 \rho g}{s E A_T} \int_0^t A(s,0) u(s,0)\ \mathrm{d}s + \frac{r_0}{sE} p_{f0}(t)\right)^2 \\ A(t,\ell) &= A_0 \left(1 + \frac{r_0 \rho g}{s E A_T} \int_0^t A(s,\ell) u(s,\ell)\ \mathrm{d}s + \frac{r_0}{sE} p_{f\ell}(t)\right)^2 . \end{aligned}$$

However, we will not dwell on this perspective but instead we will study the PDE-ODE system (2.6.5).

Part I

LINEAR SYSTEMS

3

STABILITY AND CONTROLLABILITY OF THE LINEARIZED MODEL

Before dealing with the nonlinear system (2.6.5), its linearization about a steady state will be studied in the present chapter. The well-posedness of the linearized model is established in Section 3.2 using $\mathcal{C}_0$-semigroups. For convenience, a short summary of results regarding $\mathcal{C}_0$-semigroups is provided in Appendix A. The connection between the semigroup solution and the linearized system will be discussed in Section 3.3. In Section 3.4, the spectrum of the generator is determined and in particular it is shown that the generator is Riesz spectral, that is, it has a Riesz basis consisting of generalized eigenvectors. A boundary control system is considered in Section 3.5 and it is shown using tools in Fourier analysis that the linearized model is boundary exact controllable for sufficiently large times. For single input controls, a minimal time of controllability will be given as well. As for the wave equation with either Dirichlet or Neumann boundary control, the control can be characterize by minimizing a cost functional with PDE constraints. Finally, numerical solutions of the linear model are computed using Legendre tau approximations.

3.1 LINEARIZATION

Let us determine the equilibria of the system (2.6.5) when p_{f0} and $p_{f\ell}$ do not depend on t. Setting the derivative with respect to time to zero in (2.6.5), the first equation will give $\partial(Au)/\partial x = 0$ and so Au is constant on $[0, \ell]$. However, the third and fourth equations will give $A(t,0)u(t,0) = A(t,\ell)u(t,\ell) = 0$ and assuming that A remains positive for all $t \geq 0$ it follows that u must be identically zero on $[0,\ell]$. Using this information in the second equation we obtain that $\partial A/\partial x = 0$ and so A must be constant on the domain, say $A = A_e$. Because $\mathrm{d}h_0/\mathrm{d}t = 0$ and $\mathrm{d}h_\ell/\mathrm{d}t = 0$ then $h_0 = h_{0e}$ and $h_\ell = h_{\ell e}$ for some constants h_{0e} and $h_{\ell e}$. Thus we have

$$A_e = A_0\left(1 + \frac{r_0}{sE}(\rho g h_{0e} + p_{f0})\right)^2 = A_0\left(1 + \frac{r_0}{sE}(\rho g h_{\ell e} + p_{f\ell})\right)^2.$$

and it follows that $h_e - h_{0e} = \frac{1}{\rho g}(p_{f0} - p_f)$. We ignore the other possibility $h_{0e} + h_{\ell e} = -\frac{1}{\rho r_0 g}(2sE + r_0 p_{f0} + r_0 p_{f\ell})$ since we are interested in the case where the level heights in the tanks are both positive. If $p_{f0} = p_{f\ell}$ then the former equality coincides with the fact that the level heights in the two tanks must be the same. Note also that this is true even when the two tanks have different horizontal cross sections. If V denotes the volume of fluid in the tube and in the tanks, then $V = A_e\ell + A_T(h_{0e} + h_{\ell e})$. Therefore $p_{f0}, p_{f\ell}$ and V uniquely determine the equilibrium point. Furthermore, it is easy to see that we can choose p_{f0} and $p_{f\ell}$ such that h_{0e} and $h_{\ell e}$ are both positive.

To linearize the above system about the equilibrium point $z_e = (A_e, u_e, (A_e)_x, (u_e)_x, h_{0e}, h_{\ell e})$, where $u_e = (A_e)_x = (u_e)_x = 0$, we use Taylor series expansions about the equilibrium z_e and neglect the terms of order higher than one. From these equations we let $A = A_e + \tilde{A}$, $u = \tilde{u}$, $h_0 = h_{0e} + \tilde{h}_0$ and $h_\ell = h_{\ell e} + \tilde{h}_\ell$, which are the small deviations from the equilibrium, to obtain the linearized system

$$\begin{cases} \dfrac{\partial \tilde{A}}{\partial t} = -A_e \dfrac{\partial \tilde{u}}{\partial x}, & t > 0,\ 0 < x < \ell, \\ \dfrac{\partial \tilde{u}}{\partial t} = -\alpha \dfrac{\partial \tilde{A}}{\partial x} - \beta \tilde{u}, & t > 0,\ 0 < x < \ell, \\ \dfrac{d\tilde{h}_0}{dt} = -\dfrac{A_e}{A_T} \tilde{u}(t,0), & t > 0, \\ \dfrac{d\tilde{h}_\ell}{dt} = \dfrac{A_e}{A_T} \tilde{u}(t,\ell), & t > 0, \end{cases} \tag{3.1.1}$$

with boundary conditions

$$\tilde{A}(t,0) = \gamma \tilde{h}_0(t), \qquad \tilde{A}(t,\ell) = \gamma \tilde{h}_\ell(t), \qquad t > 0, \tag{3.1.2}$$

and initial conditions

$$\begin{cases} \tilde{A}(0,x) = \tilde{A}^0(x), & \tilde{u}(0,x) = \tilde{u}^0(x), \qquad 0 \le x \le \ell, \\ \tilde{h}_0(0) = \tilde{h}_0^0, & \tilde{h}_\ell(0) = \tilde{h}_\ell^0 \end{cases} \tag{3.1.3}$$

In the above system we used the following notations

$$\begin{aligned} \alpha &= \frac{sE}{2\rho r_0 \sqrt{A_0 A_e}}, & \beta &= \frac{8\pi\mu}{\rho A_0}, \\ \gamma &= \frac{2\rho A_0 g r_0}{sE}\left(1 + \frac{r_0}{sE}(\rho g h_{0e} + p_{f0})\right) &=& \frac{2\rho A_0 g r_0}{sE}\left(1 + \frac{r_0}{sE}(\rho g h_{\ell e} + p_{f\ell})\right), \end{aligned}$$

since, for the linearization, we assume that p_{f0} and $p_{f\ell}$ are constants. We remark that all the parameters r_0, s, A_0, A_e, E are positive while μ is nonnegative. As a result, $\alpha > 0$ and $\beta \ge 0$. The constants p_{f0} and $p_{f\ell}$ can also be chosen to be small, so that $\gamma > 0$. The resulting linear system is the coupling of PDEs in one space dimension with ODEs and sometimes such systems are referred to as *hybrid systems*. By differentiation, a second order linear model, which is a wave equation with viscous damping and Robin boundary conditions, was formulated and discussed by Bredow [**78**].

3.2 WELL-POSEDNESS OF THE LINEAR SYSTEM

In this section we prove the well-posedness of the linear system (3.1.1)–(3.1.3). For convenience, we will denote the state variables and the initial conditions for the linearized system without the tildas. Our approach utilizes the theory of strongly continuous semigroups. We will recast the system as a differential equation in an infinite-dimensional state space. Consider the Hilbert space $\mathcal{X} = L^2((0,\ell),\mathbb{C})^2 \times \mathbb{C}^2$ equipped with the inner product

$$\langle(\varphi_1,\psi_1,a_1,b_1),(\varphi_2,\psi_2,a_2,b_2)\rangle_{\mathcal{X}} = \frac{1}{A_e}\langle\varphi_1,\varphi_2\rangle_{L^2} + \frac{1}{\alpha}\langle\psi_1,\psi_2\rangle_{L^2} + \frac{\gamma A_T}{A_e}(a_1 a_2 + b_1 b_2).$$

Notice that the norm induced by the above inner product is equivalent to the usual product norm of $\mathcal{X}$.

Define the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{X}$ with domain $\mathcal{D}(\mathcal{A}) = \{(A, u, h_0, h_\ell) \in \mathcal{X} : A, u \in H^1(0, \ell), A(0) = \gamma h_0, A(\ell) = \gamma h_\ell\}$ by

$$\mathcal{A} \begin{pmatrix} A \\ u \\ h_0 \\ h_\ell \end{pmatrix} = \begin{pmatrix} -A_e u_x \\ -\alpha A_x - \beta u \\ -\frac{A_e}{A_T} u(0) \\ \frac{A_e}{A_T} u(\ell) \end{pmatrix}.$$

Observe that the last two components of the state appear only in the domain of $\mathcal{A}$. The coupled system (3.1.1) can now be phrased as an abstract Cauchy problem

$$\text{(ACP)} \quad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(A, u, h_0, h_\ell)(t) &= \mathcal{A}(A, u, h_0, h_\ell)(t), \qquad t > 0, \\ (A, u, h_0, h_\ell)(0) &= (A^0, u^0, h_0^0, h_\ell^0) \end{cases}$$

on the state space $\mathcal{X}$.

There are several ways to prove the well-posedness of (ACP). One possible approach is to split the PDE and the ODE. The PDE can be considered as a port-Hamiltonian system and hence it is well-posed [**40, Chapter 13**], and the ODE, being finite-dimensional, is also well-posed. Then one shows that the feedback interconnection of the two systems is well-posed [**73, Section 7.2**].

However, the approach presented here is based on the following lemma. It is a recap of the proof of Theorem 3 in [**22**]. In the following, X' denotes the dual space of X and $\mathcal{L}(X, Y)$ is the space of bounded linear operators from X to Y.

Lemma 3.2.1 (Lax-Milgram-Fredholm)**.** *Let V and H be Hilbert spaces such that the embedding $V \subset H$ is compact and dense. Suppose that $a_1 : V \times V \to \mathbb{C}$ and $a_2 : H \times H \to \mathbb{C}$ are two bounded sesquilinear forms such that a_1 is V-coercive and $F : V \to \mathbb{C}$ is a continuous conjugate linear form. The equation*

$$a_1(v, u) + a_2(v, u) = F(u), \qquad \forall v \in V \tag{3.2.1}$$

has either a unique solution $u \in V$ for all $F \in V'$ or has a nontrivial solution for $F = 0$.

Proof. Since a_1 is bounded, the operator $T : V \to V'$ defined by $\langle T\varphi, \psi\rangle_{V' \times V} = a_1(\varphi, \psi)$ for all $\varphi, \psi \in V$ is bounded. Furthermore, by the Lax-Milgram Lemma we have $T^{-1} \in \mathcal{L}(V', V)$. Define $S : H \to V'$ by $\langle S\varphi, \psi\rangle_{V' \times V} = a_2(\varphi, \psi)$. Since for each $\varphi \in H$

$$\|S\varphi\|_{V'} = \sup_{\|\psi\|_V = 1} |\langle S\varphi, \psi\rangle_{V' \times V}| \leq \sup_{\|\psi\|_V = 1} C\|\varphi\|_H \|\psi\|_H \leq \tilde{C}\|\varphi\|_H$$

it holds that $S \in \mathcal{L}(H, V')$ and in particular $S \in \mathcal{L}(V, V')$ is compact. The equation (3.2.1) is equivalent to $(1 + T^{-1}S)v = T^{-1}F$ in V. Since $T^{-1}S$ is compact the Fredholm alternative implies that either $-1 \in \rho(T^{-1}S)$ or $-1 \in \sigma_p(T^{-1}S)$, where $\rho(A)$ and $\sigma_p(A)$ denote the resolvent set and point spectrum of a closed operator A. □

Theorem 3.2.2. *The operator $\mathcal{A}$ generates a strongly continuous semigroup of contractions on $\mathcal{X}$, and in particular, for every $(A^0, u^0, h_0^0, h_\ell^0) \in \mathcal{D}(\mathcal{A})$ there exist unique functions $A, u \in C^1([0, \infty); L^2(0, \ell))$ and $h_0, h_\ell \in C^1[0, \infty)$ such that* (ACP) *is satisfied.*

Proof. We will use the Lumer-Phillips Theorem in reflexive Banach spaces, see Theorem A.1.1. Integrating by parts and using the boundary conditions we have

$$\langle \mathcal{A}(A,u,h_0,h_\ell),(A,u,h_0,h_\ell)\rangle_{\mathcal{X}} = -\frac{\beta}{\alpha}\|u\|_{L^2}^2 + 2i\Im\langle u, A_x\rangle_{L^2}$$

for all $(A,u,h_0,h)\in\mathcal{D}(\mathcal{A})$. Taking the real part shows that $\mathcal{A}$ is dissipative. Next we are going to show the range condition. Fix $\lambda>0$ and $(B,v,g_0,g_\ell)\in\mathcal{X}$ and define $a_1 : H^1(0,\ell)\times H^1(0,\ell)\to\mathbb{C}$, $a_2 : L^2(0,\ell)\times L^2(0,\ell)\to\mathbb{C}$ and $F : H^1(0,\ell)\to\mathbb{C}$ by

$$\begin{aligned}
a_1(\varphi,\psi) &= \frac{\alpha\gamma A_e}{A_T(\lambda+\beta)}\langle\varphi,\psi\rangle_{H^1} + \lambda\varphi(0)\overline{\psi(0)} + \lambda\varphi(\ell)\overline{\psi(\ell)}\\
a_2(\varphi,\psi) &= \frac{\gamma}{A_T}\left(\lambda - \frac{\alpha A_e}{\lambda+\beta}\right)\langle\varphi,\psi\rangle_{L^2}\\
F(\psi) &= \frac{\gamma}{A_T}\int_0^\ell B(x)\overline{\psi(x)}\,\mathrm{d}x + \frac{\gamma A_e}{A_T(\lambda+\beta)}\int_0^\ell v(x)\overline{\psi_x(x)}\,\mathrm{d}x\\
&\quad + \gamma g_0\overline{\psi(0)} + \gamma g_\ell\overline{\psi(\ell)},
\end{aligned}$$

respectively. Note that the sesquilinear forms a_1, a_2 and the conjugate linear form F satisfy the conditions of Lemma 3.2.1.

We claim that

$$(\lambda I - \mathcal{A})(A,u,h_0,h_\ell) = (B,v,g_0,g_\ell) \tag{3.2.2}$$

has a solution $(A,u,h_0,h_\ell)\in\mathcal{D}(\mathcal{A})$ if and only if there is an $A\in H^1(0,\ell)$ that satisfies

$$a_1(A,\psi) + a_2(A,\psi) = F(\psi), \qquad \forall\psi\in H^1(0,\ell). \tag{3.2.3}$$

Notice that (3.2.2) is the system of equations

$$\left\{\begin{aligned}
\lambda A + A_e u_x &= B\\
(\lambda+\beta)u + \alpha A_x &= v\\
\lambda h_0 + \tfrac{A_e}{A_T}u(0) &= g_0\\
\lambda h_\ell - \tfrac{A_e}{A_T}u(\ell) &= g_\ell.
\end{aligned}\right. \tag{3.2.4}$$

Suppose that (3.2.2), and hence (3.2.4), has a solution (A,u,h_0,h_ℓ) in $\mathcal{D}(\mathcal{A})$. Multiplying the first equation in (3.2.4) by $\frac{\gamma}{A_T}\overline{\psi}$ for $\psi\in H^1(0,\ell)$, integrating by parts, solving for u in the second equation of (3.2.4) and using the boundary conditions we obtain (3.2.3).

Conversely, let $A\in H^1(0,\ell)$ satisfy (3.2.3) for all $\psi\in H^1(0,\ell)$. Define

$$u = \frac{1}{\lambda+\beta}(v-\alpha A_x), \tag{3.2.5}$$

$$h_0 = \frac{1}{\lambda}\left(g_0 - \frac{A_e}{A_T}u(0)\right), \tag{3.2.6}$$

$$h_\ell = \frac{1}{\lambda}\left(g_\ell + \frac{A_e}{A_T}u(\ell)\right). \tag{3.2.7}$$

Notice that u solves the second equation of (3.2.4) and (3.2.6) and (3.2.7) are the third and fourth. From (3.2.3) and (3.2.5) we have

$$\begin{aligned}
\frac{\gamma}{A_T}\int_0^\ell(\lambda A(x) - B(x))\overline{\psi(x)}\,\mathrm{d}x &= \frac{\gamma}{A_T}\int_0^\ell A_e u(x)\overline{\psi_x(x)}\,\mathrm{d}x + (\gamma g_0 - \lambda A(0))\overline{\psi(0)}\\
&\quad + (\gamma g_\ell - \lambda A(\ell))\overline{\psi(\ell)}.
\end{aligned} \tag{3.2.8}$$

Since the above equation is true for all $\psi \in H^1(0,\ell)$, it also holds in particular for all test functions $\psi \in C_0^\infty(0,\ell)$, and so the above equation gives us

$$\int_0^\ell (B(x) - \lambda A(x))\overline{\psi(x)}\,\mathrm{d}x = -\int_0^\ell A_e u(x)\overline{\psi_x(x)}\,\mathrm{d}x, \qquad \forall\,\psi \in C_0^\infty(0,\ell),$$

which implies that $B - \lambda A = (A_e u)_x$ or $u_x = \frac{1}{A_e}(B - \lambda A) \in L^2(0,\ell)$. As a consequence, $u \in H^1(0,\ell)$ and the first equation in (3.2.4) holds. It remains to verify the boundary conditions $A(0) = \gamma h_0$ and $A(\ell) = \gamma h_\ell$. The left hand side of (3.2.8) can be written as

$$\begin{aligned}
\frac{\gamma}{A_T}\int_0^\ell (\lambda A(x) - B(x))\overline{\psi(x)}\,\mathrm{d}x \;&=\; -\frac{\gamma A_e}{A_T}\int_0^\ell u_x(x)\overline{\psi(x)}\,\mathrm{d}x \\
&= \frac{\gamma A_e}{A_T}(u(0)\overline{\psi(0)} - u(\ell)\overline{\psi(\ell)}) + \frac{\gamma A_e}{A_T}\int_0^\ell u(x)\overline{\psi_x(x)}\,\mathrm{d}x,
\end{aligned}$$

and therefore, upon using (3.2.6), (3.2.7) and (3.2.8) and the fact that $\lambda > 0$,

$$(\gamma h_0 - A(0))\overline{\psi(0)} + (\gamma h_\ell - A(\ell))\overline{\psi(\ell)} = 0 \tag{3.2.9}$$

for all $\psi \in H^1(0,\ell)$. Choosing appropriate functions ψ, this equation implies that $A(0) = \gamma h_0$ and $A(\ell) = \gamma h_\ell$. Therefore $(A, u, h_0, h_\ell) \in \mathcal{D}(\mathcal{A})$ and (3.2.4) holds.

We prove that the second case in Lemma 3.2.1 does not hold. Suppose that $a_1(A,\psi) + a_2(A,\psi) = 0$ for all $\psi \in H^1(0,\ell)$. This condition is equivalent to the system (3.2.2) with $(B, v, g_0, g_\ell) = 0$. From the first equation we get $A = -\frac{A_e}{\lambda}u_x$. The rest of the equations will give us

$$\begin{aligned}
\langle A, A\rangle_{L^2} \;&=\; -\frac{A_e}{\lambda}\int_0^\ell u_x(x)\overline{A(x)}\,\mathrm{d}x \\
&= -\frac{A_e}{\lambda}(u(\ell)\overline{A(\ell)} - u(0)\overline{A(0)}) + \frac{A_e}{\lambda}\int_0^\ell u(x)\overline{A_x(x)}\,\mathrm{d}x \\
&= -\frac{A_T}{\gamma}(|A(\ell)|^2 + |A(0)|^2) - \frac{\alpha A_e}{\lambda(\lambda+\beta)}\int_0^\ell |A_x(x)|^2\,\mathrm{d}x \;\le\; 0.
\end{aligned}$$

Hence $A = 0$. This proves the range condition and hence completes the proof of the theorem. ☐

3.3 THE SEMIGROUP SOLUTION AND THE PDE

Let A, u, h_0 and h_ℓ be the components of the semigroup solution $z(t) = e^{\mathcal{A}t}z^0$ to the abstract Cauchy problem and let $z^0 = (A^0, u^0, h_0^0, h_\ell^0)$. We are interested how the semigroup solution $z = (A, u, h_0, h_\ell)$ satisfies the system of partial differential equations (3.1.1)−(3.1.3). We will follow the discussion in Liu and Zheng [**53**]. First, if $A^0, u^0 \in H^1(0,\ell)$ and $h_0^0, h_\ell^0 \in \mathbb{C}$ satisfy the compatibility conditions $A^0(0) = \gamma h_0^0$ and $A^0(\ell) = \gamma h_\ell^0$ then Theorem 3.2.2 already tells us that $A, u \in C([0,\infty); H^1(0,\ell)) \cap C^1([0,\infty); L^2(0,\ell))$, $h_0, h_\ell \in C^1[0,\infty)$, the differential equations (3.1.1) are satisfied in $\mathcal{X}$ while the boundary conditions (3.1.2) are satisfied in the sense of traces. This type of solution is sometimes referred as *strong solutions*.

For data that are merely in the state space $\mathcal{X}$, the following notion of solution can be used. Given $A^0, u^0 \in L^2(0,\ell)$ and $h_0^0, h_\ell^0 \in \mathbb{C}$, the quadruple $(A, u, h_0, h_\ell) \in C([0,\infty); L^2(0,\ell)^2 \times \mathbb{C}^2)$ is called a *weak solution* of (3.1.1)–(3.1.3) if the equations

$$\begin{cases} \langle A(t), \varphi\rangle_{L^2} + \gamma A_T(h_0(t), \eta_0) + \gamma A_T(h(t), \eta_\ell) \\ = \langle A^0, \varphi\rangle_{L^2} + \gamma A_T(h_0^0, \eta_0) + \gamma A_T(h_\ell^0, \eta_\ell) + A_e \displaystyle\int_0^t \langle u(\sigma), \varphi_x\rangle_{L^2}\, \mathrm{d}\sigma \\ \langle u(t), \psi\rangle_{L^2} = \langle u^0, \psi\rangle_{L^2} + \alpha \displaystyle\int_0^t \langle A(\sigma), \psi_x\rangle_{L^2}\, \mathrm{d}\sigma - \beta \int_0^t \langle u(\sigma), \psi\rangle_{L^2}\, \mathrm{d}\sigma \\ + \alpha \displaystyle\int_0^t (\gamma h_0(\sigma), \psi(0)) - (\gamma h_\ell(\sigma), \psi(\ell))\, \mathrm{d}\sigma \end{cases} \tag{3.3.1}$$

hold for every $t \geq 0$, $\varphi, \psi \in H^1(0,\ell)$ and $\eta_0, \eta_\ell \in \mathbb{C}$ such that $\varphi(0) = \gamma\eta_0$ and $\varphi(\ell) = \gamma\eta_\ell$. See Sections 4.20 and 4.21 for an equivalent definition.

We will show that the components of the semigroup solution z comprise a weak solution of (3.1.1)−(3.1.3). To prove this, first we recall that since $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{X}$ there exists a sequence $(z_n^0)_n \subset \mathcal{D}(\mathcal{A})$ such that $z_n^0 \to z^0$ in $\mathcal{X}$. Let $z_n = (A_n, u_n, h_{0n}, h_{\ell n})$ be the strong solution corresponding to $z_n^0 = (A_n^0, u_n^0, h_{0n}^0, h_{\ell n}^0)$. For some $M \geq 1$ and $\alpha \geq 0$ we have

$$\|z - z_n\|_{C([0,T];\mathcal{X})} \leq M e^{\alpha T} \|z^0 - z_n^0\|_{\mathcal{X}}. \tag{3.3.2}$$

Multiplying the equations in (3.1.1) by φ, ψ, η_0 and η_ℓ in the respective order, integrating by parts and using the boundary conditions (3.1.2) and initial conditions (3.1.3), the equations in (3.3.1) with (A, u, h_0, h_ℓ) and $(A^0, u^0, h_0^0, h_\ell^0)$ replaced by $(A_n, u_n, h_{0n}, h_{\ell n})$ and $(A_n^0, u_n^0, h_{0n}^0, h_{\ell n}^0)$, respectively, can be obtained. Thanks to (3.3.2) we have (3.3.1) after passing to the limit $n \to \infty$. For the uniqueness of weak solutions as well as the continuous dependence of the solution on the initial data, see Theorem 3.5.7 below.

If the initial data $A^0, u^0 \in H^2(0,\ell)$ and $h_0^0, h_\ell^0 \in \mathbb{C}$ satisfy the compatibility conditions up to order one $A^0(0) = \gamma h_0^0, A(\ell) = \gamma h_\ell^0, A_T u_x^0(0) = \gamma u^0(0)$ and $-A_T u_x^0(\ell) = \gamma u^0(\ell)$ then we have a *classical solution* $A, u \in C^1([0,\infty) \times [0,\ell])$ and $h_0, h \in C^2[0,\infty)$.

In this section, we have shown the existence of weak solutions for (3.1.1)–(3.1.3) using $\mathcal{C}_0$-semigroups. This existence will be demonstrated in Chapter 4 at a different perspective and in a more general setting. In the latter approach, it will be shown further that the weak solution satisfies additional regularity other the one given in Theorem 3.2.2, namely, the L^2-trace regularity of A and u at the boundary and hence the H^1-regularity of h_0 and h_ℓ.

3.4 SPECTRAL PROPERTIES AND UNIFORM EXPONENTIAL STABILITY

At this point, we already know that $\sigma(\mathcal{A}) \subset \{z \in \mathbb{C} : \Re z \leq 0\}$ since $\mathcal{A}$ generates a contractive $\mathcal{C}_0$-semigroup on $\mathcal{X}$. Furthermore, the adjoint operator $\mathcal{A}^*$ also generates a contraction $\mathcal{C}_0$-semigroup, which is the adjoint semigroup, in other words, $(e^{\mathcal{A}t})^* = e^{\mathcal{A}^*t}$ for all $t \geq 0$.

Let us determine the $\mathcal{X}$-adjoint of $\mathcal{A}$. Define $\tilde{\mathcal{A}} : \mathcal{D}(\mathcal{A}) \to \mathcal{X}$ by

$$\tilde{\mathcal{A}} \begin{pmatrix} B \\ v \\ g_0 \\ g_\ell \end{pmatrix} = \begin{pmatrix} A_e v_x \\ \alpha B_x - \beta v \\ \frac{A_e}{A_T} v(0) \\ -\frac{A_e}{A_T} v(\ell) \end{pmatrix},$$

For each $(A, u, h_0, h_\ell), (B, v, g_0, g_\ell) \in \mathcal{D}(\mathcal{A})$, a straight forward computation yields

$$\langle \mathcal{A}(A, u, h_0, h_\ell), (B, v, g_0, g_\ell)\rangle_{\mathcal{X}} = \langle (A, u, h_0, h_\ell), \tilde{\mathcal{A}}(B, v, g_0, g_\ell)\rangle_{\mathcal{X}}$$

which implies that $(B, v, g_0, g_\ell) \in \mathcal{D}(\mathcal{A}^*)$, and this proves that $\mathcal{A}^*$ is an extension of $\tilde{\mathcal{A}}$. Using a similar argument as in the proof of Theorem 3.2.2, we can also show that $\tilde{\mathcal{A}}$ generates a $\mathcal{C}_0$-semigroup of contractions on $\mathcal{X}$, and hence $(0, \infty) \subset \rho(\mathcal{A}^*) \cap \rho(\tilde{\mathcal{A}})$. Applying [**58, Lemma 1.6.14**], we can see that $\mathcal{A}^* = \tilde{\mathcal{A}}$ and in particular $\mathcal{D}(\mathcal{A}^*) = \mathcal{D}(\tilde{\mathcal{A}}) = \mathcal{D}(\mathcal{A})$.

In the absence of friction, i.e. $\beta = 0$, we have $\mathcal{A}^* = -\mathcal{A}$ and so $\mathcal{A}$ is skew-adjoint and from Stone's Theorem, see Theorem A.1.2, the operator $\mathcal{A}$ generates a unitary $\mathcal{C}_0$-group. This will be used in the succeeding section. The operator $\mathcal{A}$ and $\mathcal{A}^*$ also generate $\mathcal{C}_0$-groups even for $\beta > 0$. To see this, let us define $\mathcal{C} \in \mathcal{L}(\mathcal{X})$ by $\mathcal{C}(A, u, h_0, h) = (0, u, 0, 0)$. Then $-\mathcal{A} = \mathcal{A}^* + 2\beta\mathcal{C}$ and $-\mathcal{A}$ generates a $\mathcal{C}_0$-semigroup satisfying $\|e^{-\mathcal{A}t}\| \leq e^{2\beta t}$ for all $t \geq 0$ (see, e.g. [**25, Theorem III.1.3**]). From Theorem 3.2.2 and [**25, p. 79**], $\mathcal{A}$ generates a $\mathcal{C}_0$-group on $\mathcal{X}$ satisfying $\|e^{\mathcal{A}t}\| \leq e^{2\beta|t|}$ for all $t \in \mathbb{R}$. The case of $\mathcal{A}^*$ is analogous. Tight decay rates will be given after we have described the spectra of the generators.

The operators $\mathcal{A}$ and $\mathcal{A}^*$ have compact resolvents and therefore their spectra consist of eigenvalues only. This is a consequence of the compactness of the embedding $H^1(0, \ell) \hookrightarrow L^2(0, \ell)$. We can now characterize the spectrum of $\mathcal{A}$ and its adjoint. Due to the differential boundary conditions, namely the third and fourth lines in (3.1.1), the eigenvalues appear on the boundary conditions of a two-point boundary value problem, see (3.4.5) for instance. To describe the spectrum of the differential operator for $\beta \geq 0$, we first describe the special case where $\beta = 0$ and use this to investigate for the case $\beta > 0$. First, we state a lemma needed for the asymptotic description of the eigenvalues.

Lemma 3.4.1. *Let $a, b, c > 0$ and $H(x) = x \cos ax - (bx^2 - c) \sin ax$ and let $(\mu_n)_{n\in\mathbb{N}}$, listed in strictly increasing order, be the positive zeros of H. Then $\mu_n = \frac{(n-1)\pi}{a} + \mathcal{O}(n^{-1})$ as $n \to \infty$.*

Proof. Define $H_1(x) = \tan ax$ and $H_2(x) = x/(bx^2 - c)$. If $(2n+1)\pi/2a = \sqrt{c}/\sqrt{b}$ for some $n \geq 0$ then $\pm\sqrt{c}/\sqrt{b}$ are zeros of H. The other zeros of H are precisely the abscissas of the points of intersection of the graphs of H_1 and H_2. If $(2n+1)\pi/2a \neq \sqrt{c}/\sqrt{b}$ for all $n \geq 0$ then the zeros of H are just the abscissas of the intersection of H_1 and H_2. By looking at the graphs of H_1 and H_2 it can be seen that for large indices n, we have $\mu_n = (n-1)\pi/a + e_n$ where $e_n \to 0$. Multiplying by a and taking the sine of both sides yields

$$\sin ae_n = (-1)^{n-1}\frac{\mu_n \cos a\mu_n}{b\mu_n^2 - c} = (-1)^{n-1}\frac{((n-1)\pi/a + \mathcal{O}(1))\cos a\mu_n}{b((n-1)\pi/a + \mathcal{O}(1))^2 - c}, \; n \to \infty.$$

Taking the inverse sine and noting that $\sin^{-1} x = \mathcal{O}(x)$ as $x \to 0$ we obtain that $e_n = \mathcal{O}(n^{-1})$. □

Theorem 3.4.2. *Let $(\mu_n)_{n\in\mathbb{Z}}$, listed in strictly increasing order, be the real solutions of the equation,*

$$\frac{2A_T}{\gamma A_e}\mu \cos\frac{\mu\ell}{\sqrt{\alpha A_e}} - \frac{\sqrt{A_e}}{\sqrt{\alpha}}\left(\frac{A_T^2}{\gamma^2 A_e^2}\mu^2 - \frac{\alpha}{A_e}\right)\sin\frac{\mu\ell}{\sqrt{\alpha A_e}} = 0, \qquad (3.4.1)$$

where $\mu_0 = 0$ and $\mu_{-n} = -\mu_n$. Then the spectrum of $\mathcal{A}$ is given by $\sigma(\mathcal{A}) = (\lambda_n)_{n\in\mathbb{Z}}$, where

$$\lambda_n = -\frac{\beta}{2} + \frac{1}{2}\operatorname{sgn}(n)\sqrt{\beta^2 - 4\mu_n^2}, \qquad n \in \mathbb{Z}^* := \mathbb{Z}\setminus\{0\}, \tag{3.4.2}$$

and $\lambda_0 = 0$, and the eigenvalues λ_n satisfy the asymptotic growth

$$\lambda_n = -\frac{\beta}{2} + \left(\frac{\sqrt{\alpha A_e}(n-1)\pi}{\ell} + \mathcal{O}(n^{-1})\right) i, \quad n \to \infty. \tag{3.4.3}$$

In particular, $\sigma(\mathcal{A}) = \sigma(\mathcal{A}^*)$. An eigenvector z_n of $\mathcal{A}$ associated with the eigenvalue λ_n is given by

$$z_n = \begin{pmatrix} \varphi_n \\ \psi_n \\ \eta_{0n} \\ \eta_{\ell n} \end{pmatrix} = \begin{pmatrix} \cos\frac{\mu_n x}{\sqrt{\alpha A_e}} - \frac{A_T\mu_n}{\gamma\sqrt{\alpha A_e}}\sin\frac{\mu_n x}{\sqrt{\alpha A_e}} \\ -\left(\frac{A_T\lambda_n}{\gamma A_e}\cos\frac{\mu_n x}{\sqrt{\alpha A_e}} + \frac{\sqrt{\alpha}\lambda_n}{\sqrt{A_e}\mu_n}\sin\frac{\mu_n x}{\sqrt{\alpha A_e}}\right) \\ \frac{1}{\gamma} \\ \frac{1}{\gamma}\left(\cos\frac{\mu_n \ell}{\sqrt{\alpha A_e}} - \frac{A_T\mu_n}{\gamma\sqrt{\alpha A_e}}\sin\frac{\mu_n \ell}{\sqrt{\alpha A_e}}\right) \end{pmatrix}, \qquad n \in \mathbb{Z}. \tag{3.4.4}$$

Similarly, an eigenvector z_n^* of $\mathcal{A}^*$ associated to the eigenvalue λ_n is given by $z_n^* = (\varphi_n, -\psi_n, \eta_{0n}, \eta_{\ell n})$ for every $n \in \mathbb{Z}$.

Proof. Note that $\lambda \in \sigma(\mathcal{A})$ if and only if there exists $(A, u, h_0, h_\ell) \in \mathcal{D}(\mathcal{A}) \setminus \{0\}$ satisfying the boundary value problem

$$\begin{cases} \frac{\partial}{\partial x}\begin{pmatrix} A \\ u \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\lambda+\beta}{\alpha} \\ -\frac{\lambda}{A_e} & 0 \end{pmatrix}\begin{pmatrix} A \\ u \end{pmatrix} \\ -\frac{A_e}{A_T}u(0) = \frac{\lambda}{\gamma}A(0), \qquad \frac{A_e}{A_T}u(\ell) = \frac{\lambda}{\gamma}A(\ell) \end{cases} \tag{3.4.5}$$

Consider the equation $\mathcal{A}(A, u, h_0, h_\ell) = 0$ where $(A, u, h_0, h_\ell) \in \mathcal{D}(\mathcal{A})$. Then we have $u = 0$ and A is constant. Since $A(0) = \gamma h_0$ and $A(\ell) = \gamma h_\ell$ we have $h_0 = h_\ell$ and so $(A, u, h_0, h_\ell) = c(\gamma, 0, 1, 1)$ for some $c \in \mathbb{C}$. Hence $0 \in \sigma(\mathcal{A})$. One can check that $-\beta \in \sigma(\mathcal{A})$ with $\beta > 0$ if and only if $\ell = -2A_T/\gamma$, hence we exclude this case under the physically relevant assumption $\ell > 0$.

Suppose that $\lambda \neq 0$ and $\lambda \neq -\beta$. By diagonalizing the 2×2 matrix in (3.4.5) we can obtain that the solution of the ODE is given by

$$A(x) = c_1 \cosh\frac{\sqrt{\lambda(\lambda+\beta)}x}{\sqrt{\alpha A_e}} - c_2\frac{\sqrt{A_e}}{\sqrt{\alpha}}\frac{\sqrt{\lambda(\lambda+\beta)}}{\lambda}\sinh\frac{\sqrt{\lambda(\lambda+\beta)}x}{\sqrt{\alpha A_e}} \tag{3.4.6}$$

$$u(x) = -c_1\frac{\sqrt{\alpha}}{\sqrt{A_e}}\frac{\lambda}{\sqrt{\lambda(\lambda+\beta)}}\sinh\frac{\sqrt{\lambda(\lambda+\beta)}x}{\sqrt{\alpha A_e}} + c_2\cosh\frac{\sqrt{\lambda(\lambda+\beta)}x}{\sqrt{\alpha A_e}} \tag{3.4.7}$$

for some $(c_1, c_2) \in \mathbb{C}^2$, where the square root denotes any fixed branch of the complex square root; for definiteness we choose the principal branch where the nonpositive real axis is the chosen branch cut.

This and the boundary conditions yield the following homogeneous system of equations

$$
\begin{aligned}
-\left(\frac{A_T\lambda}{A_e\gamma}\cosh\frac{\sqrt{\lambda(\lambda+\beta)}\ell}{\sqrt{\alpha A_e}}+\frac{\sqrt{\alpha}}{\sqrt{A_e}}\frac{\lambda}{\sqrt{\lambda(\lambda+\beta)}}\sinh\frac{\sqrt{\lambda(\lambda+\beta)}\ell}{\sqrt{\alpha A_e}}\right)c_1 & \\
+\left(\cosh\frac{\sqrt{\lambda(\lambda+\beta)}\ell}{\sqrt{\alpha A_e}}+\frac{A_T}{A_e\gamma}\frac{\sqrt{A_e}}{\sqrt{\alpha}}\sqrt{\lambda(\lambda+\beta)}\sinh\frac{\sqrt{\lambda(\lambda+\beta)}\ell}{\sqrt{\alpha A_e}}\right)c_2 &= 0 \\
\frac{A_T\lambda}{A_e\gamma}c_1+c_2 &= 0
\end{aligned}
$$

The above system in the unknowns c_1 and c_2 has a nontrivial solution if and only if the determinant of the corresponding matrix is zero and this is equivalent to the equation

$$
F(w):=\frac{2A_T}{A_e\gamma}w\cosh\frac{w\ell}{\sqrt{\alpha A_e}}+\frac{\sqrt{A_e}}{\sqrt{\alpha}}\left(\frac{A_T^2}{A_e^2\gamma^2}w^2+\frac{\alpha}{A_e}\right)\sinh\frac{w\ell}{\sqrt{\alpha A_e}}=0, \tag{3.4.8}
$$

where we put $w=\sqrt{\lambda(\lambda+\beta)}$.

Let us consider the special case where $\beta=0$. In this case, $\lambda\in\sigma(\mathcal{A})$ if and only if $F(\lambda)=0$. However, since $\mathcal{A}$ is skew-adjoint, its spectrum must lie on the imaginary axis. This implies that all zeros of F are purely imaginary. Letting $\lambda=i\mu$, where $\mu\in\mathbb{R}$, we can see that $F(i\mu)=0$ is equivalent to the equation (3.4.1). Using this for the case $\beta\geq 0$, we can see from (3.4.8) that $\lambda\in\sigma(\mathcal{A})$ if and only if $\sqrt{\lambda(\lambda+\beta)}=i\mu$ for some $\mu\in\mathbb{R}$ that satisfies (3.4.1). The asymptotic behavior (3.4.3) of the eigenvalues follows from the asymptotic behavior of the solutions of (3.4.1) given by the previous lemma

$$
\mu_n=\frac{\sqrt{\alpha A_e}(n-1)\pi}{\ell}+\mathcal{O}(n^{-1}),\quad n\to\infty. \tag{3.4.9}
$$

The fact that the spectra of $\mathcal{A}$ and $\mathcal{A}^*$ coincide comes from the symmetry of the spectrum of $\mathcal{A}$ with respect to the real axis.

Choosing $c_1=1$ and $c_2=-\frac{A_T\lambda_n}{\gamma A_e}$ in (3.4.6) and (3.4.7) gives the first and second components of the eigenvector z_n. The third and fourth components are due to the boundary conditions $\eta_{0n}=\frac{1}{\gamma}\varphi_n(0)$ and $\eta_{\ell n}=\frac{1}{\gamma}\varphi_n(\ell)$. Finally, since z_n is an eigenvector of $\mathcal{A}$ corresponding to λ_n we have

$$
(\lambda_n I-\mathcal{A}^*)z_n^*=\begin{pmatrix}\lambda_n\varphi_n+A_e(\psi_n)_x\\ -\lambda_n\psi_n-\alpha(\varphi_n)_x-\beta\psi_n\\ \lambda_n\eta_{0n}+\frac{A_e}{A_T}\psi_n(0)\\ \lambda_n\eta_{\ell n}-\frac{A_e}{A_T}\psi_n(\ell)\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},
$$

and so z_n^* is an eigenvector of $\mathcal{A}^*$ corresponding to the eigenvalue λ_n. □

If $\beta>2\mu_n$, then the eigenvalue λ_n is real and negative. This implies that $z(t,x)=(e^{\mathcal{A}t})z_n(x)=e^{\lambda_n t}z_n(x)$ monotonically decays to zero as time goes to infinity. This means that when the fluid inside the tube and tanks is sufficiently viscous, then there are solutions which decay to the equilibrium state without oscillations.

With or without viscosity, we have seen in Theorem 3.4.2 that $0\in\sigma(\mathcal{A})$ and this means that the system is not stable in $\mathcal{X}$. The eigenspace associated with the eigenvalue 0 is the one-dimensional subspace $\mathcal{X}_0:=\{c(\gamma,0,1,1):c\in\mathbb{C}\}$. Moreover, $\mathcal{N}(\mathcal{A}^*)=\mathcal{X}_0$ and so $\mathcal{A}$ and its adjoint have the same kernel. The state

$(A_e, 0, h_{0e}, h_{\ell e}) + c(\gamma, 0, 1, 1)$ for $c \neq 0$ is also an equilibrium of the nonlinear system but corresponds to different parameters V, p_{f0} and $p_{f\ell}$.

It is easy to check that the space $\mathcal{X}_0$ is invariant under the action of the group $(e^{\mathcal{A}t})_{t\in\mathbb{R}}$ and its adjoint group. If $z \in \mathcal{X}_0^{\perp}$ and $w \in \mathcal{X}_0$ then $\langle e^{\mathcal{A}t}z, w\rangle_{\mathcal{X}} = \langle z, e^{\mathcal{A}^*t}w\rangle_{\mathcal{X}} = 0$ by the invariance of $\mathcal{X}_0$ under $(e^{\mathcal{A}^*t})_{t\in\mathbb{R}}$. Hence $\mathcal{X}_0^{\perp}$ is invariant under $(e^{\mathcal{A}t})_{t\in\mathbb{R}}$, i.e., $e^{\mathcal{A}t}\mathcal{X}_0^{\perp} \subset \mathcal{X}_0^{\perp}$ for all $t \in \mathbb{R}$. From Theorem A.2.4 the restricted group $(e^{\mathcal{A}t}|_{\mathcal{X}_0^{\perp}})_{t\in\mathbb{R}}$ is a $\mathcal{C}_0$-group on $\mathcal{X}_0^{\perp}$ whose generator is given by the part of $\mathcal{A}$ in $\mathcal{X}_0^{\perp}$, denoted by $\mathcal{A}_p$. A similar argument shows that $e^{\mathcal{A}^*t}|_{\mathcal{X}_0^{\perp}} = e^{\mathcal{A}_p^*t}$ for all $t \in \mathbb{R}$ where $\mathcal{A}_p^*$ is the part of $\mathcal{A}^*$ in $\mathcal{X}_0^{\perp}$. It is easily seen that $\mathcal{A}_p^* = (\mathcal{A}_p)^*$, which means that taking the part in a closed invariant subspace and taking the adjoint commute, see Theorem A.2.3. Since $\mathcal{A}_p$ and $\mathcal{A}_p^*$ also have compact resolvents we have $\sigma(\mathcal{A}_p) = \sigma(\mathcal{A}_p^*) = \sigma(\mathcal{A}) \setminus \{0\}$, $\rho(\mathcal{A}_p) = \rho(\mathcal{A}_p^*) = \rho(\mathcal{A}) \cup \{0\}$. The semigroup generated by $\mathcal{A}_p$ will be used in the next section.

Let us characterize the nonzero eigenvalues of $\mathcal{A}$ as the viscosity β increases. In the following discussions the equality (3.4.2) is used. If $\beta \in [0, 2\mu_1)$ then all the nonzero eigenvalues have nonzero imaginary parts and $|\lambda_n| = |\mu_n|$ for all $n \in \mathbb{Z}$. Therefore as β increases on this interval, the eigenvalues are rotated positively (that is, counterclockwise) around the origin and all the eigenvalues lie on the line $\Re z = -\beta/2$. If $\beta = 2\mu_1$ then the two eigenvalues $\lambda_{\pm 1}$ coincide and both are equal to $-\beta/2$. Suppose that $\beta \in [2\mu_1, 2\mu_2)$. As β increases in this interval, λ_n for $|n| > 2$ is again rotated in the same manner as before. However, the eigenvalue λ_1 now goes to the right along the real axis while λ_{-1} goes to the left along the real axis, faster than $\Re\lambda_{\pm 2}$. When $\beta = 2\mu_2$, the eigenvalues $\lambda_{\pm 2}$ coincide while λ_{-1} is on the left of $\lambda_{\pm 2}$. The same behavior holds for the other intervals $[2\mu_n, 2\mu_{n+1})$, $n \geq 2$. Thus the larger β, the more eigenvalues there are on the interval $(-\beta/2, 0)$ and there are eigenfunctions which decay slower than those for smaller β. The eigenvalues that move to the right approach 0 as β increases.

If we denote by ζ_n the eigenvector of $\mathcal{A}$ corresponding to $\lambda_n = i\mu_n$ when $\beta = 0$ and by z_n the eigenvector of $\mathcal{A}$ when $\beta > 0$ then for all n such that $\beta < 2|\mu_n|$ we have $|\lambda_n| = |\mu_n|$,

$$\|z_n\|_{\mathcal{X}} = \|z_n^*\|_{\mathcal{X}} = \|\zeta_n\|_{\mathcal{X}} = \|\zeta_{-n}\|_{\mathcal{X}}, \tag{3.4.10}$$

and a straightforward calculation gives

$$\|\zeta_n\|_{\mathcal{X}}^2 = \frac{A_T^2 \ell \mu_n^2}{\gamma^2 \alpha A_e^2} + \frac{\ell}{A_e} + \frac{A_T}{\gamma A_e} + \frac{A_T}{\gamma A_e}\left(\cos\frac{\mu_n \ell}{\sqrt{\alpha A_e}} - \frac{A_T \mu_n}{\gamma\sqrt{\alpha A_e}}\sin\frac{\mu_n \ell}{\sqrt{\alpha A_e}}\right)^2. \tag{3.4.11}$$

Theorem 3.4.3. *Let $\beta \geq 0$. Then we have the following:*

1. *If $\beta \neq 2\mu_n$ for all $n \in \mathbb{N}$ then the normalized eigenvectors $(z_n/\|z_n\|_{\mathcal{X}})_{n\in\mathbb{Z}}$ of $\mathcal{A}$ form a Riesz basis for $\mathcal{X}$. If $\beta = 0$ then this Riesz basis is in fact an orthonormal basis.*

2. *If $\beta = 2\mu_n$ for some $n \in \mathbb{N}$ then the sequence $(z_n/\|z_n\|_{\mathcal{X}})_{n\in\mathbb{Z}\setminus\{-n\}} \cup \{Z/\|Z\|_{\mathcal{X}}\}$, where $Z = (0, -\lambda_n^{-1}\psi_n, 0, 0)^\top$ is a generalized eigenvector of $\mathcal{A}$ corresponding to λ_n satisfying $(\lambda_n I - \mathcal{A})Z = z_n$, forms a Riesz basis for $\mathcal{X}$.*

Similar results for the generator $\mathcal{A}^*$ hold, however, in (2) the vector Z should be replaced by the generalized eigenvector $Z^* := -Z$ of $\mathcal{A}^*$ corresponding to λ_n.

Proof. First consider the case where $\beta = 0$. Applying [**8, Proposition III.6.1**] to the operator $\mathcal{A}_p$, the normalized eigenvectors $(z_n/\|z_n\|_{\mathcal{X}})_{n\in\mathbb{Z}^*}$ forms an orthonormal basis for $\mathcal{X}_0^\perp$. Therefore combined with the normalized eigenvector associated with the eigenvalue 0, the sequence $(z_n/\|z_n\|_\mathcal{X})_{n\in\mathbb{Z}}$ form an orthonormal basis for $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}_0^\perp$. Now suppose that $\beta > 0$ and $\beta \neq 2\mu_n$ for all $n \in \mathbb{N}$. Again, let ζ_n be the eigenvector of $\mathcal{A}$ corresponding to the eigenvalue $\lambda_n = i\mu_n$ for the case where there is no viscosity, i.e., $\beta = 0$. The first part of (1) follows from Theorem A.3.1 once we have shown that $(z_n/\|z_n\|_\mathcal{X})_{n\in\mathbb{Z}}$ and $(\zeta_n/\|\zeta_n\|_\mathcal{X})_{n\in\mathbb{Z}}$ are quadratically close in the sense that

$$\sum_{n\in\mathbb{Z}} \left\| \frac{z_n}{\|z_n\|_\mathcal{X}} - \frac{\zeta_n}{\|\zeta_n\|_\mathcal{X}} \right\|_\mathcal{X}^2 < \infty. \tag{3.4.12}$$

Let N be the largest integer such that $\beta > 2\mu_N$. From (3.4.4) and (3.4.10)

$$\frac{z_n}{\|z_n\|_\mathcal{X}} - \frac{\zeta_n}{\|\zeta_n\|_\mathcal{X}} = \left(0, \frac{\lambda_n - i\mu_n}{i\mu_n}\frac{\zeta_{n2}}{\|\zeta_n\|_\mathcal{X}}, 0, 0\right), \qquad |n| > N. \tag{3.4.13}$$

where ζ_{n2} is the second component of ζ_n. It can be seen from (3.4.2) that $|\lambda_n - i\mu_n| \to \beta/2$ as $|n| \to \infty$ and in particular the sequence $(\lambda_n - i\mu_n)_{n\in\mathbb{Z}}$ is bounded. Because $\|\zeta_{n2}\|_{L^2(0,\ell)} \leq \sqrt{\alpha}\|\zeta_n\|_\mathcal{X}$ it follows from (3.4.13) that

$$\sum_{|n|>N} \left\| \frac{z_n}{\|z_n\|_\mathcal{X}} - \frac{\zeta_n}{\|\zeta_n\|_\mathcal{X}} \right\|_\mathcal{X}^2 \leq \sum_{|n|>N} \frac{C}{\mu_n^2}$$

for some constant $C > 0$. The last sum is finite because of (3.4.9). As a consequence, (3.4.12) is satisfied.

Finally, consider the case where $\beta = 2\mu_n$ for some $n \in \mathbb{N}$. Let us verify that Z satisfies $(\lambda_n I - \mathcal{A})Z = z_n$. Indeed,

$$\begin{aligned}(\lambda_n I - \mathcal{A})Z - z_n &= \begin{pmatrix} A_e(-\lambda_n^{-1}\psi_n)_x - \varphi_n \\ \lambda_n(-\lambda_n^{-1}\psi) - \beta\lambda_n^{-1}\psi_n - \psi_n \\ \frac{A_e}{A_T}(-\lambda_n^{-1}\psi_n)(0) - \eta_0 \\ -\frac{A_e}{A_T}(-\lambda_n^{-1}\psi_n)(\ell) - \eta_{\ell n} \end{pmatrix} \\ &= \begin{pmatrix} -\lambda_n^{-1}(\lambda_n\varphi_n + A_e(\psi_n)_x) \\ -\psi_n + 2\psi_n - \psi_n \\ -\lambda_n^{-1}(\lambda_n\eta_{0n} + \frac{A_e}{A_T}\psi_n(0)) \\ -\lambda_n^{-1}(\lambda_n\eta_{\ell n} - \frac{A_e}{A_T}\psi_n(\ell)) \end{pmatrix}\end{aligned}$$

and this is zero because z_n is an eigenvector of $\mathcal{A}$ corresponding to $\lambda_n = -\beta/2$. The same argument as in the previous case shows that the sequences $(z_n/\|z_n\|_\mathcal{X})_{n\in\mathbb{Z}\setminus\{-n\}} \cup \{Z/\|Z\|_\mathcal{X}\}$ and $(\zeta_n/\|\zeta_n\|_\mathcal{X})_{n\in\mathbb{Z}}$ are quadratically close and hence part (2) also follows from the Riesz basis generation result Theorem A.3.1. $\square$

Let $(\tilde{z}_n)_{n\in\mathbb{Z}}$ be the sequence biorthogonal to the Riesz basis $(z_n^*/\|z_n^*\|_\mathcal{X})_{n\in\mathbb{Z}}$ if $\beta \neq 2\mu_n$ for all $n \in \mathbb{N}$ or to the Riesz basis $(z_n^*/\|z_n^*\|_\mathcal{X})_{n\in\mathbb{Z}\setminus\{-n\}} \cup \{Z^*/\|Z^*\|_\mathcal{X}\}$ if $\beta = 2\mu_n$ for some $n \in \mathbb{N}$. The result we have just proved implies that every $z \in \mathcal{X}$ can be expressed uniquely as a Fourier series

$$z = \sum_{n\in\mathbb{Z}} \langle z, \tilde{z}_n\rangle_\mathcal{X} \frac{z_n^*}{\|z_n^*\|_\mathcal{X}}, \tag{3.4.14}$$

whenever $\beta \neq 2\mu_n$ for all $n \in \mathbb{N}$ and a similar equation holds for the other case. For all square-summable sequences $(a_n)_{n\in\mathbb{Z}}$ we have

$$c\Big(\sum_{n\in\mathbb{Z}} |a_n|^2\Big)^{1/2} \leq \Big\|\sum_{n\in\mathbb{Z}} \frac{a_n z_n^*}{\|z_n^*\|_\mathcal{X}}\Big\|_\mathcal{X} \leq C\Big(\sum_{n\in\mathbb{Z}} |a_n|^2\Big)^{1/2} \tag{3.4.15}$$

for some $c, C > 0$ independent of $(a_n)_{n\in\mathbb{Z}}$. Furthermore, the sequence $(\langle z, \tilde{z}_n\rangle_{\mathcal{X}})_{n\in\mathbb{Z}^*}$ is square-summable for each $z \in \mathcal{X}$. Since $e^{\mathcal{A}^*t}z_n^* = e^{\lambda_n t}z_n^*$ for all $n \in \mathbb{Z}$ it follows from (3.4.14) and the continuity of $e^{\mathcal{A}^*t}$ that when $\beta \neq 2\mu_n$ for all $n \in \mathbb{N}$ the group generated by $\mathcal{A}^*$ can be written as

$$e^{\mathcal{A}^*t}z = \sum_{n\in\mathbb{Z}^*} e^{\lambda_n t}\langle z, \tilde{z}_n\rangle_{\mathcal{X}} \frac{z_n^*}{\|z_n^*\|_{\mathcal{X}}} \tag{3.4.16}$$

for every $z \in \mathcal{X}$ and $t \in \mathbb{R}$. If $\beta = 2\mu_n$ for some $n \in \mathbb{N}$ then the group is given by

$$e^{\mathcal{A}^*t}z = \langle z, \tilde{z}_{-n}\rangle(e^{\lambda_n t}Z^* - te^{\lambda_n t}z_n^*) + \sum_{n\in\mathbb{Z}\setminus\{-n\}} e^{\lambda_n t}\langle z, \tilde{z}_n\rangle_{\mathcal{X}} \frac{z_n^*}{\|z_n^*\|_{\mathcal{X}}} \tag{3.4.17}$$

for every $z \in \mathcal{X}$ and $t \in \mathbb{R}$. Similar characterizations for the group generated by $\mathcal{A}$ hold. The reason why we choose to expand the adjoint semigroup is that we will use a duality argument in the proof of Theorem 3.5.6. Now we have the following stability result.

Theorem 3.4.4. *Let $\beta > 0$. The C_0-semigroups $(e^{\mathcal{A}_p t})_{t\geq 0}$ and $(e^{\mathcal{A}_p^* t})_{t\geq 0}$ generated by $\mathcal{A}_p$ and $\mathcal{A}_p^*$ are uniformly exponentially stable, i.e., there exist constants $M \geq 1$ and $\omega > 0$ such that*

$$\|e^{\mathcal{A}_p t}\|_{\mathcal{L}(\mathcal{X}_0^\perp)} = \|e^{\mathcal{A}_p^* t}\|_{\mathcal{L}(\mathcal{X}_0^\perp)} \leq Me^{-\omega t}, \qquad t \geq 0. \tag{3.4.18}$$

Furthermore, $\omega(\mathcal{A}_p) = s(\mathcal{A}_p)$ where $s(\mathcal{A}_p)$ is the spectral bound of $\mathcal{A}_p$ and $\omega(\mathcal{A}_p)$ is the growth bound of the semigroup generated by $\mathcal{A}_p$. For every $z_0 \in \mathcal{X}$, if $z = e^{\mathcal{A}t}z_0 \in C([0,\infty), \mathcal{X})$ is the mild solution of (ACP) *corresponding to the initial data z_0 then $\|z(t) - Pz_0\|_{\mathcal{X}} \leq M\|z_0\|_{\mathcal{X}}e^{-\omega t}$ for all $t \geq 0$, where P is the orthogonal projection of $\mathcal{X}$ onto $\mathcal{X}_0$.*

Proof. The first and second parts follow immediately from (3.4.16) and (3.4.17). For the last part, let Q be the orthogonal projection of $\mathcal{X}$ onto $\mathcal{X}_0^\perp$ so that every $z_0 \in \mathcal{X}$ can be written uniquely as $z_0 = Pz_0 + Qz_0$. Since the restriction of $e^{\mathcal{A}t}$ to $\mathcal{X}_0$ is just the identity operator on $\mathcal{X}_0$ we have $z(t) = Pz_0 + e^{\mathcal{A}_p t}Qz_0$, and the required estimate follows from (3.4.18) and the fact that $\|Q\| \leq 1$. □

The eigenvalue 0 is removed by restricting the state space to the orthogonal complement of the eigenspace corresponding to the eigenvalue zero. Define the *volume functional* $\mathcal{V} : \mathcal{X} \to \mathbb{C}$ by

$$\mathcal{V}(A, u, h_0, h) = \int_0^\ell A(x)\,\mathrm{d}x + A_T h_0 + A_T h_\ell.$$

It is clear that $\mathcal{V}$ is a bounded linear functional on $\mathcal{X}$. Recall that $z = z_e + \tilde{z}$ where $z = (A, u, h_0, h)$, $z_e = (A_e, 0, h_{0e}, h_{\ell e})$ and $\tilde{z} = (\tilde{A}, \tilde{u}, \tilde{h}_0, \tilde{h})$ are the state, the equilibrium state and the deviation of the state from the equilibrium, respectively. By the conservation of mass we must have $\mathcal{V}(z) = \mathcal{V}(z_e) = V$ and this is equivalent to $\mathcal{V}(\tilde{z}) = 0$, i.e., $\tilde{z} \in \mathcal{N}(\mathcal{V})$. One can check that $\mathcal{N}(\mathcal{V}) = \mathcal{X}_0^\perp$. This means that $\mathcal{X}_0^\perp$ is the *natural* state space for the deviations. Also, if $z(t, x) = e^{\mathcal{A}_p t}z^0(x)$ is the solution of the system then $\mathcal{V}(z(t, \cdot)) = 0$ for every $t \geq 0$ whenever $z^0 \in \mathcal{X}_0^\perp$. For this reason, we consider $\mathcal{X}_0^\perp$ to be the state space in the next section.

3.5 A BOUNDARY CONTROL SYSTEM

Consider time varying control pressures $p_{f0}(t)$ and $p_{f\ell}(t)$ applied to the left and the right tank, respectively. Linearizing about the numbers p_{f0}^* and $p_{f\ell}^*$ we have

$$\begin{aligned}\tilde{A}(t,0) &= \gamma\tilde{h}_0(t) + \frac{\gamma}{\rho g}\tilde{p}_{f0}(t),\\ \tilde{A}(t,\ell) &= \gamma\tilde{h}_\ell(t) + \frac{\gamma}{\rho g}\tilde{p}_{f\ell}(t)\end{aligned}$$

where $\tilde{p}_{f0}(t) = p_{f0}(t) - p_{f0}^*$ and $\tilde{p}_{f\ell}(t) = p_{f\ell}(t) - p_{f\ell}^*$. Again for simplicity, we ignore the tildas and we let $p_0 = \frac{\gamma}{\rho g}p_{f0}$ and $p_1 = \frac{\gamma}{\rho g}p_{f\ell}$. In this scenario, we have the system (3.1.1) with the boundary conditions

$$A(t,0) = \gamma h_0(t) + p_0(t), \qquad A(t,\ell) = \gamma h_\ell(t) + p_1(t), \qquad t > 0. \tag{3.5.1}$$

Definition 3.5.1. For $A^0, u^0 \in L^2(0,\ell)$, $h_0^0, h^0 \in \mathbb{C}$ and $p_0, p_1 \in L^2_{\text{loc}}([0,\infty),\mathbb{C})$, the tuple (A,u,h_0,h) such that $A,u \in C([0,\infty),L^2(0,\ell))$ and $h_0, h_\ell \in C([0,\infty),\mathbb{C})$ is called a *weak solution* of the system (3.1.1) with initial conditions (3.1.3) and boundary conditions (3.5.1) if

$$\begin{cases} \langle A(t),\varphi\rangle_{L^2} + \gamma A_T(h_0(t),\eta_0) + \gamma A_T(h_\ell(t),\eta_\ell) \\ \qquad = \langle A^0,\varphi\rangle_{L^2} + \gamma A_T(h_0^0,\eta_0) + \gamma A_T(h_\ell^0,\eta_\ell) + A_e\displaystyle\int_0^t \langle u(\sigma),\varphi_x\rangle_{L^2}\,\mathrm{d}\sigma \\ \langle u(t),\psi\rangle_{L^2} = \langle u^0,\psi\rangle_{L^2} + \alpha\displaystyle\int_0^t \langle A(\sigma),\psi_x\rangle_{L^2}\,\mathrm{d}\sigma - \beta\int_0^t \langle u(\sigma),\psi\rangle_{L^2}\,\mathrm{d}\sigma \\ \qquad + \alpha\displaystyle\int_0^t (\gamma h_0(\sigma) + p_0(\sigma),\psi(0)) - (\gamma h_\ell(\sigma) + p_1(\sigma),\psi(\ell))\,\mathrm{d}\sigma \end{cases}$$

for every $t \ge 0$, $\varphi,\psi \in H^1(0,\ell)$ and $\eta_0,\eta_\ell \in \mathbb{C}$ such that $\varphi(0) = \gamma\eta_0$ and $\varphi(\ell) = \gamma\eta_\ell$.

To prove the existence of such weak solutions, the system will be expressed as a boundary control system using well-known results in control theory. Because the velocity component of the eigenvector corresponding to the eigenvalue 0 vanishes, the system is not approximately controllable in $\mathcal{X}$, cf. the observation operator $\mathcal{B}^*$ in Theorem 3.5.2 below. For this reason the system is restricted to the state space $\mathcal{X}_0^\perp$.

Denote by $\mathcal{Z} = (H^1(0,\ell) \times H^1(0,\ell) \times \mathbb{C}^2) \cap \mathcal{X}_0^\perp$ the solution space endowed with the product norm of $H^1(0,\ell) \times H^1(0,\ell) \times \mathbb{C}^2$. Our input space is $\mathbb{C}^2$ and the state space is $\mathcal{X}_0^\perp$. Note that $\mathcal{Z}$ is continuously embedded in $\mathcal{X}_0^\perp$. Let $\mathcal{D}(\mathcal{A}_p^*)$ be endowed with the graph norm. Then $\mathcal{D}(\mathcal{A}_p^*) \subset \mathcal{X}_0^\perp \subset \mathcal{D}(\mathcal{A}_p^*)'$ with continuous and dense embeddings and we have

$$\langle z,\zeta\rangle_{\mathcal{D}(\mathcal{A}_p^*)'\times\mathcal{D}(\mathcal{A}_p^*)} = \langle z,\zeta\rangle_{\mathcal{X}}, \qquad \forall z \in \mathcal{X}_0^\perp,\ \zeta \in \mathcal{D}(\mathcal{A}_p^*). \tag{3.5.2}$$

Furthermore, we can see that $\mathcal{A}_p^* \in \mathcal{L}(\mathcal{D}(\mathcal{A}_p^*),\mathcal{X}_0^\perp)$ so that $(\mathcal{A}_p^*)^* \in \mathcal{L}(\mathcal{X}_0^\perp,\mathcal{D}(\mathcal{A}_p^*)')$, where the state space $\mathcal{X}_0^\perp$ is identified with its dual. The operator $(\mathcal{A}_p^*)^*$ can be viewed as an extension of $\mathcal{A}_p$ to $\mathcal{X}_0^\perp$. For more details on the interpolation and extrapolation spaces for semigroups the reader may consult [**25, p. 123–127**].

Let $\mathcal{F} : \mathcal{Z} \to \mathcal{X}_0^\perp$ and $\mathcal{G} : \mathcal{Z} \to \mathbb{C}^2$ be given by

$$\mathcal{F}(A,u,h_0,h_\ell) = \begin{pmatrix} -A_e u_x \\ -\alpha A_x - \beta u \\ -\frac{A_e}{A_T}u(0) \\ \frac{A_e}{A_T}u(\ell) \end{pmatrix}$$

and

$$\mathcal{G}(A,u,h_0,h_\ell) = \begin{pmatrix} A(0)-\gamma h_0 \\ A(\ell)-\gamma h_\ell \end{pmatrix}.$$

Note that $\mathcal{F} \in \mathcal{L}(\mathcal{Z}, \mathcal{X}_0^\perp)$, $\mathcal{G} \in \mathcal{L}(\mathcal{Z}, \mathbb{C}^2)$, $\mathcal{N}(\mathcal{G}) = \mathcal{D}(\mathcal{A}_p)$, $\mathcal{R}(\mathcal{G}) = \mathbb{C}^2$ and $\mathcal{F}|_{\mathcal{D}(\mathcal{A}_p)} = \mathcal{A}_p$. As a consequence, $(\mathcal{F},\mathcal{G})$ is a boundary control system. Then according to Theorem B.4.2, there exists a unique operator $\mathcal{B} \in \mathcal{L}(\mathbb{C}^2, \mathcal{D}(\mathcal{A}_p^*)')$, called the *control operator*, such that $\mathcal{F}z = ((\mathcal{A}_p^*)^* + \mathcal{B}\mathcal{G})z$ for all $z \in \mathcal{Z}$. A characterization of this control operator is given in the following theorem.

Theorem 3.5.2. *The input control operator $\mathcal{B} \in \mathcal{L}(\mathbb{C}^2, \mathcal{D}(\mathcal{A}_p^*)')$ is given by*

$$\mathcal{B}(c_1,c_2) = -(\mathcal{A}_p^*)^* \left(\begin{pmatrix} \kappa \\ 0 \\ \frac{1}{\gamma}(\kappa-1) \\ \frac{1}{\gamma}\kappa \end{pmatrix} c_1 + \begin{pmatrix} \kappa \\ 0 \\ \frac{1}{\gamma}\kappa \\ \frac{1}{\gamma}(\kappa-1) \end{pmatrix} c_2 \right), \tag{3.5.3}$$

where $(c_1,c_2) \in \mathbb{C}^2$ and $\kappa = \frac{A_T}{\gamma\ell + 2A_T}$. Its adjoint $\mathcal{B}^* \in \mathcal{L}(\mathcal{D}(\mathcal{A}_p^*), \mathbb{C}^2)$ is given by

$$\mathcal{B}^*(B,v,g_0,g_\ell) = \begin{pmatrix} v(0) \\ -v(\ell) \end{pmatrix}, \qquad (B,v,g_0,g_\ell) \in \mathcal{D}(\mathcal{A}_p^*). \tag{3.5.4}$$

Proof. Given $(c_1,c_2) \in \mathbb{C}^2$, consider the problem $\mathcal{F}(A,u,h_0,h_\ell) = 0$ with boundary conditions $\mathcal{G}(A,u,h_0,h_\ell) = (c_1,c_2)$. This implies $u = 0$, A is constant, $A - \gamma h_0 = c_1$ and $A - \gamma h_\ell = c_2$. Since $(A,u,h_0,h_\ell) \in \mathcal{X}_0^\perp = \mathcal{N}(\mathcal{V})$, $A\ell + A_T h_0 + A_T h_\ell = 0$. Note that the unique solution of $\mathcal{F}z = 0$ with $\mathcal{G}z = (c_1,c_2)$ is given by $-(\mathcal{A}_p^*)^* z = \mathcal{B}(c_1,c_2)$. Solving the linear system for $(A,h_0,h_\ell) \in \mathbb{C}^3$ in terms of c_1 and c_2 we can see that $\mathcal{B} : \mathbb{C}^2 \to \mathcal{D}(\mathcal{A}_p^*)'$ is given by (3.5.3).

For $z = (A,u,h_0,h_\ell) \in \mathcal{Z}$ and $\zeta = (B,v,g_0,g_\ell) \in \mathcal{D}(\mathcal{A}_p^*)$ we obtain from (3.5.2) that

$$\begin{aligned} \langle \mathcal{G}z, \mathcal{B}^*\zeta \rangle_{\mathbb{C}^2} &= \langle \mathcal{B}\mathcal{G}z, \zeta \rangle_{\mathcal{D}(\mathcal{A}_p^*)' \times \mathcal{D}(\mathcal{A}_p^*)} = \left\langle \mathcal{F}z - (\mathcal{A}_p^*)^* z, \zeta \right\rangle_{\mathcal{D}(\mathcal{A}_p^*)' \times \mathcal{D}(\mathcal{A}_p^*)} \\ &= \langle \mathcal{F}z, \zeta \rangle_{\mathcal{X}} - \langle z, \mathcal{A}_p^* \zeta \rangle_{\mathcal{X}}. \end{aligned}$$

Integrating by parts and using the surjectivity of $\mathcal{G}$ we obtain (3.5.4). ☐

In the above theorem, we have a representation of the control operator $\mathcal{B}$ in terms of the extension of the operator $\mathcal{A}_p$. However, the more important item to use in the controllability of the boundary control system, at least in our case, is the adjoint $\mathcal{B}^*$ of the control operator.

We shall make use of the Riesz basis approach to study the above boundary control system. We refer to [**43**] for various examples in this direction. From Theorem 3.4.2, the eigenvalues of $\mathcal{A}_p^*$ satisfy

$$\liminf_{|n|\to\infty} |\Im\lambda_n - \Im\lambda_{n+1}| = \frac{\sqrt{\alpha A_e}\pi}{\ell} \tag{3.5.5}$$

and, in particular the uniform gap property

$$\gamma_0 := \inf_{\substack{\lambda,\mu \in \sigma_p(\mathcal{A}_p^*) \\ \lambda \neq \mu}} |\lambda - \mu| > 0. \tag{3.5.6}$$

The following theorems in non-harmonic Fourier analysis [**37, 43**], which is a generalization of Parseval's equality, plays a crucial role in the proof of the exact controllability of our boundary control system.

Theorem 3.5.3 (Ingham). *Let $J \subset \mathbb{Z}$ and $(\lambda_m)_{m\in J}$ be a family of real numbers satisfying the gap condition*

$$\gamma_0 := \inf_{\substack{n,m\in J\\ n\neq m}} |\lambda_n - \lambda_m| > 0,$$

and let I be a bounded interval in $\mathbb{R}$. Then, there exists $C = C(\gamma_0, I)$ such that for every sequence $(a_m) \in \ell^2(J, \mathbb{C})$ we have

$$\int_I \left| \sum_{m\in J} a_m e^{i\lambda_m t} \right|^2 \mathrm{d}t \leq C \sum_{m\in J} |a_m|^2.$$

In addition, if the the length of I satisfies $|I| > \frac{2\pi}{\gamma}$ then there exists $c = c(\gamma_0, I)$ such that

$$c \sum_{m\in J} |a_m|^2 \leq \int_I \left| \sum_{m\in J} a_m e^{i\lambda_m t} \right|^2 \mathrm{d}t.$$

Ingham's Theorem is used when the Riesz basis consists of only eigenvectors. In the case when the Riesz basis contains a generalized eigenvector, the following generalization of Ingham's Theorem will be used. For the proof we refer to [**43, Theorem 4.5**].

Theorem 3.5.4 (Haraux). *Let $(\lambda_m)_{m\in J}$, $J \subset \mathbb{Z}$, be a family of complex numbers such that $\sup_{m\in J} |\Re\lambda_m| < \infty$ and for some $m_0 \in J$ the gap condition*

$$\inf_{m\neq m_0} |\lambda_m - \lambda_{m_0}| > 0$$

is satisfied. If for some interval I_0 we have

$$c \sum_{m\in J\setminus\{m_0\}} |x_m|^2 \leq \int_{I_0} \left| \sum_{m\in J\setminus\{m_0\}} x_m e^{\lambda_m t} \right|^2 \mathrm{d}t \leq C \sum_{m\in J\setminus\{m_0\}} |x_m|^2$$

for some $C \geq c > 0$ and for all $(x_m)_{m\in J\setminus\{m_0\}} \in \ell^2(\mathbb{C})$ then for all interval I with length $|I| > |I_0|$ we also have

$$\tilde{c}\left(|\tilde{x}|^2 + \sum_{m\in J} |x_m|^2 \right) \leq \int_I \left| \tilde{x} t e^{\lambda_m t} + \sum_{m\in J} x_m e^{\lambda_m t} \right|^2 \mathrm{d}t \leq \tilde{C}\left(|\tilde{x}|^2 + \sum_{m\in J} |x_m|^2 \right)$$

for all $\tilde{x} \in \mathbb{C}$ and for all $(x_m)_{m\in J} \in \ell^2(\mathbb{C})$, for some constants $\tilde{C} \geq \tilde{c} > 0$.

For single input controls the critical time of controllability will be establish with the help of the following generalization of the Kadec's $\frac{1}{4}$-Theorem in [**81, Corollary 2, p. 196**].

Theorem 3.5.5 (Generalized Kadec $\frac{1}{4}$-Theorem). *If $(\rho_n)_{n\in\mathbb{Z}}$ is a sequence of complex numbers for which*

$$\sup_{n\in\mathbb{Z}} \left| \frac{T\Im\rho_n}{2\pi} - n \right| < \frac{1}{4} \qquad \text{and} \qquad \sup_{n\in\mathbb{Z}} |\Re\rho_n| < \infty,$$

then the system $(e^{i\rho_n t})_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,T)$.

Now we are ready to state and prove the main result of this section. A direct application of Ingham's Theorem yields the exact controllability of the boundary control system for any time $\tau > \frac{2\pi}{\gamma_0}$ where γ_0 is the gap of the eigenvalues of $\mathcal{A}_p^*$ given by (3.5.6). However, in general this gap is less than that of the asymptotic distance between consecutive eigenvalues. To provide a smaller lower bound for the time of exact controllability we will separate the low and high frequencies as in [**77, Proposition 8.1.3**]. However, we need to modify the arguments in the said proposition since the eigenvectors are not orthogonal anymore, i.e., we generalize the proposition in such a way that it is still true for the case where the orthonormal basis is replaced by a Riesz basis.

Theorem 3.5.6. *Suppose that* $\beta \geq 0$. *Then the boundary control system* $(\mathcal{F}, \mathcal{G})$ *is exactly controllable in time* τ, *if* $\tau > \tau^* := \frac{2\ell}{\sqrt{\alpha A_e}}$. *That means, for any* $z^0, z^1 \in \mathcal{X}_0^\perp$ *there exists* $(p_0, p_1) \in L^2([0,\tau], \mathbb{C}^2)$ *such that the weak solution* $z \in C([0,\tau], \mathcal{X}_0^\perp)$ *of the system* (3.1.1) *with initial conditions* (3.1.3) *and boundary conditions* (3.5.1) *satisfies* $z(\tau) = z^1$.

Proof. The proof of the existence, uniqueness and regularity of the weak solution will be provided later (see Theorem 3.5.7 below). We divide the proof into several steps for ease of reading. Moreover, we first assume that $\beta \neq 2\mu_n$ for all $n \in \mathbb{N}$.

Step 1. Let us prove that $\mathcal{B}$ is an admissible control operator for $(e^{\mathcal{A}_p t})_{t\geq 0}$, or equivalently, $\mathcal{B}^*$ is an admissible observation operator for the adjoint semigroup $(e^{\mathcal{A}_p^* t})_{t\geq 0}$. The latter means that for each $t \geq 0$ there exists $C_t > 0$ such that $\int_0^t |\mathcal{B}^* e^{\mathcal{A}_p^* s} z|^2 \,\mathrm{d}s \leq C_t \|z\|_{\mathcal{X}}^2$ for all $z \in \mathcal{D}(\mathcal{A}_p^*)$. According to (3.4.10), (3.4.11) and the asymptotic behavior of μ_n given in (3.4.9) we have $\frac{1}{\mu_n^2}\|z_n^*\|_{\mathcal{X}}^2 \to \frac{A_T^2 \ell}{\gamma^2 \alpha A_e^2}$ as $|n| \to \infty$. Using this, we can see that

$$0 < d_y := \inf_{n\in\mathbb{Z}^*} \frac{|\psi_n(y)|^2}{\|z_n^*\|_{\mathcal{X}}^2} \leq \sup_{n\in\mathbb{Z}^*} \frac{|\psi_n(y)|^2}{\|z_n^*\|_{\mathcal{X}}^2} =: D_y < \infty, \qquad y = 0, \ell. \tag{3.5.7}$$

Let $M = M(\beta)$ the largest integer such that $\beta > 2\mu_M$. Thus λ_n is real whenever $|n| \leq M$. From (3.4.16), Ingham's Theorem, (3.5.7) and (3.4.15) we have

$$\begin{aligned}
\int_0^t |\mathcal{B}^* e^{\mathcal{A}_p^* s} z|^2 \,\mathrm{d}s &= \sum_{y=0,\ell} \int_0^t \left| \sum_{n\in\mathbb{Z}^*} e^{\lambda_n s} \langle z, \tilde{z}_n\rangle_{\mathcal{X}} \frac{\psi_n(y)}{\|z_n^*\|_{\mathcal{X}}} \right|^2 \mathrm{d}s \\
&\leq 2 \sum_{y=0,\ell} \Bigg(\int_0^t \sum_{0<|n|\leq M} e^{2\lambda_n s} |\langle z, \tilde{z}_n\rangle_{\mathcal{X}}|^2 \frac{|\psi_n(y)|^2}{\|z_n^*\|_{\mathcal{X}}^2} \\
&\qquad + \left| \sum_{|n|>M} e^{(-\beta/2 + i\Im\lambda_n)s} \langle z, \tilde{z}_n\rangle_{\mathcal{X}} \frac{\psi_n(y)}{\|z_n^*\|_{\mathcal{X}}} \right|^2 \mathrm{d}s \Bigg) \\
&\leq C_t \sum_{y=0,\ell} \sum_{n\in\mathbb{Z}^*} |\langle z, \tilde{z}_n\rangle_{\mathcal{X}}|^2 \frac{|\psi_n(y)|^2}{\|z_n^*\|_{\mathcal{X}}^2} \;\leq\; C_t (D_0 + D_\ell) c^{-2} \|z\|_{\mathcal{X}}^2
\end{aligned}$$

for all $z \in \mathcal{D}(\mathcal{A}_p^*)$ and $t \geq 0$. Hence $\mathcal{B}^*$ is an admissible observation operator for $(e^{\mathcal{A}_p^* t})_{t\geq 0}$.

Step 2. Now we separate the eigenfunctions into two parts. From (3.5.5), for arbitrary $\epsilon > 0$, there exists a positive integer $N = N(\epsilon)$ such that $\Re\lambda_n = -\beta/2$ for all $|n| > N$ and

$$\inf_{\substack{|m|,|n|>N \\ m\neq n}} |\Im\lambda_n - \Im\lambda_m| \geq \frac{\sqrt{\alpha A_e}}{\ell}\pi - \epsilon. \tag{3.5.8}$$

Consider the subspace $\mathcal{X}_1 = \operatorname{span}(z_n^*)_{0<|n|\le N}$ of $\mathcal{X}_0^\perp$. It is clear that $(z_n^*)_{0<|n|\le N}$ is linearly independent and so it forms a basis for $\mathcal{X}_1$. For each $t \in \mathbb{R}$, let $T(t)^*$ be the restriction of $e^{\mathcal{A}_p^* t}$ to $\mathcal{X}_1$. We note that by construction $\sigma(\mathcal{A}_p^*|_{\mathcal{X}_1}) = (\lambda_n)_{0<|n|\le N}$.

Let $\mathcal{X}_2 = \operatorname{clos}\operatorname{span}(z_n^*/\|z_n^*\|_{\mathcal{X}})_{|n|>N}$ and $\mathcal{A}_q^*$ be the part of $\mathcal{A}_p^*$ in $\mathcal{X}_2$. Notice that $\mathcal{A}_q^*$ have also compact resolvent and $\sigma(\mathcal{A}_q^*) = \sigma(\mathcal{A}_p^*) \setminus (\lambda_n)_{0<|n|\le N}$. Since $\operatorname{span}(z_n^*/\|z_n^*\|_{\mathcal{X}})_{|n|>N}$ is invariant under the $\mathcal{C}_0$-group generated by $\mathcal{A}_p^*$, its closure is also invariant under this $\mathcal{C}_0$-group. Thus $\mathcal{A}_q^*$ also generates a $\mathcal{C}_0$-group on $\mathcal{X}_2$ and $e^{\mathcal{A}_p^* t}|_{\mathcal{X}_2} = e^{\mathcal{A}_q^* t}$ for all $t \in \mathbb{R}$.

Step 3. From the discussions in the previous step, we can see that the normalized eigenvectors $(z_n^*/\|z_n^*\|_{\mathcal{X}})_{|n|>N}$ of $\mathcal{A}_q^*$ form a Riesz basis for $\mathcal{X}_2$. Let $\mathcal{B}_q^*$ be the restriction of $\mathcal{B}^*$ to $\mathcal{D}(\mathcal{A}_q^*)$. A similar application of Ingham's Theorem as above shows that $\mathcal{B}_q^*$ is an admissible observation operator for the semigroup generated by $\mathcal{A}_q^*$. Moreover, from Ingham's Theorem and (3.4.15) we have the inverse estimate

$$\begin{aligned}\int_0^\tau |\mathcal{B}_q^* e^{\mathcal{A}_q^* t} z|^2\,\mathrm{d}t \;&\geq\; \sum_{y=0,\ell} e^{-\beta\tau}\int_0^\tau \left|\sum_{|n|>N} e^{i\Im\lambda_n s}\langle z,\tilde{z}_n\rangle_{\mathcal{X}}\frac{\psi_n(y)}{\|z_n^*\|_{\mathcal{X}}}\right|^2 \mathrm{d}s\\ &\geq\; c_\tau \sum_{y=0,\ell}\sum_{|n|>N} |\langle z,\tilde{z}_n\rangle_{\mathcal{X}}|^2 \frac{|\psi_n(y)|^2}{\|z_n^*\|_{\mathcal{X}}^2} \;\geq\; c_\tau (d_0+d_\ell) C^{-2}\|z\|_{\mathcal{X}}^2\end{aligned}$$

for every $z \in \mathcal{D}(\mathcal{A}_q^*)$ and $\tau > \frac{2\pi\ell}{\sqrt{\alpha A_e \pi - \epsilon\ell}}$. Thus, the pair $(\mathcal{A}_q^*, \mathcal{B}_q^*)$ is exactly observable in time $\tau > \frac{2\pi\ell}{\sqrt{\alpha A_e \pi - \epsilon\ell}}$.

Step 4. Because $\mathcal{A}_p^*|_{\mathcal{X}_1} \in \mathcal{L}(\mathcal{X}_1) \simeq \mathcal{L}(\mathbb{C}^{2N})$, $\mathcal{B}^*|_{\mathcal{X}_1} \in \mathcal{L}(\mathcal{X}_1, \mathbb{C}^2) \simeq \mathcal{L}(\mathbb{C}^{2N}, \mathbb{C}^2)$ and $\mathcal{B}^*|_{\mathcal{X}_1} z_n^* \neq 0$ for every $0 < |n| \le N$, the Hautus test for finite-dimensional systems implies that $(\mathcal{A}_p^*|_{\mathcal{X}_1}, \mathcal{B}^*|_{\mathcal{X}_1})$ is observable. Since $\sigma(\mathcal{A}_p^*|_{\mathcal{X}_1}) \cap \sigma(\mathcal{A}_q^*) = \emptyset$, according to [**77, Proposition 6.4.2**] (see also [**76**]), the pairs $(\mathcal{A}_p^*|_{\mathcal{X}_1}, \mathcal{B}^*|_{\mathcal{X}_1})$ and $(\mathcal{A}_q^*, \mathcal{B}_q^*)$ are simultaneously exactly observable, in other words, there exists a constant $\tilde{c}_\tau > 0$ such that for all $(v, w) \in \mathcal{X}_1 \times \mathcal{D}(\mathcal{A}_q^*)$ it holds that

$$\int_0^\tau |\mathcal{B}^*|_{\mathcal{X}_1} T(t)^* v + \mathcal{B}_q^* e^{\mathcal{A}_q^* t} w|^2\,\mathrm{d}t \geq \tilde{c}_\tau (\|v\|_{\mathcal{X}}^2 + \|w\|_{\mathcal{X}}^2) \tag{3.5.9}$$

for every $\tau > \frac{2\pi\ell}{\sqrt{\alpha A_e \pi - \epsilon\ell}}$. For $k \ge N$ define the kth truncation of $z \in \mathcal{D}(\mathcal{A}_p^*)$ by

$$z^k = \sum_{0<|n|\le k} \langle z, \tilde{z}_n\rangle_{\mathcal{X}} \frac{z_n^*}{\|z_n^*\|_{\mathcal{X}}}$$

Then $z^k \to z$ in $\mathcal{X}$. Since $z^N \in \mathcal{X}_1$ and $z^k - z^N \in \operatorname{span}(\frac{z_n^*}{\|z_n^*\|_{\mathcal{X}}})_{|n|>N} \subset \mathcal{D}(\mathcal{A}_q^*)$ it follows from (3.4.15) and (3.5.9) that for any $k > N$ we have

$$\begin{aligned}&\int_0^\tau |\mathcal{B}^* e^{\mathcal{A}_p^* t} z^k|^2\,\mathrm{d}t = \int_0^\tau |\mathcal{B}^*|_{\mathcal{X}_1} T(t)^* z^N + \mathcal{B}_q^* e^{\mathcal{A}_q^* t}(z^k - z^N)|^2\,\mathrm{d}t\\ &\geq\; \tilde{c}_\tau (\|z^N\|_{\mathcal{X}}^2 + \|z^k - z^N\|_{\mathcal{X}}^2) \;\geq\; \tilde{c}_\tau c^2 \sum_{0<|n|\le k} |\langle z^k, \tilde{z}_n^*\rangle_{\mathcal{X}}|^2 \;\geq\; \tilde{c}_\tau c^2 C^{-2}\|z^k\|_{\mathcal{X}}^2.\end{aligned}$$

Because $\mathcal{B}^*$ is an admissible observation operator for the semigroup generated by $\mathcal{A}_p^*$, letting $k \to \infty$ in the last inequality we obtain the inverse estimate

$$\int_0^\tau |\mathcal{B}^* e^{\mathcal{A}_p^* t} z|^2\,\mathrm{d}t \geq \tilde{c}_\tau c^2 C^{-2}\|z\|_{\mathcal{X}}^2.$$

Therefore $(\mathcal{A}_p^*, \mathcal{B}^*)$ is exactly observable in time $\tau > \frac{2\pi\ell}{\sqrt{\alpha A_e}\pi - \epsilon\ell}$, and since $\epsilon > 0$ is arbitrary, this pair is exactly observable in time $\tau > \frac{2\ell}{\sqrt{\alpha A_e}}$.

If $\beta = 2\mu_n$ for some $n \in \mathbb{N}$ then one applies the above argument to the closure of the span of the normalized eigenvectors of $\mathcal{A}_p^*$. Notice that $\mathcal{B}^* Z^* \neq 0$. Then the series representation (3.4.17) together with Haraux's Theorem imply the exact observability in the state space $\mathcal{X}_0^\perp$ for any time $\tau > \tau^*$. In any case, the conclusion of the theorem now follows from the well known duality of exact controllability and exact observability, see Theorem B.3.5. □

Now we address the existence and uniqueness of weak solutions of (3.1.1) under the boundary conditions (3.5.1). Let $p_0, p_1 \in L^2_{\text{loc}}([0,\infty), \mathbb{C})$ and $z_0 \in \mathcal{X}_0^\perp$. Since $\mathcal{B}$ is an admissible control operator for the semigroup generated by $\mathcal{A}_p$, then using the variation of parameters formula, the function

$$z(t) = e^{\mathcal{A}_p t} z^0 + \int_0^t e^{(\mathcal{A}_p^*)^*(t-s)} \mathcal{B}(p_0(s), p_1(s))\,\mathrm{d}s \quad \text{in} \quad \mathcal{D}(\mathcal{A}_p^*)' \qquad (3.5.10)$$

is the unique function that satisfies $z \in C([0,\infty), \mathcal{X}_0^\perp)$ and

$$z(t) - z(0) = \int_0^t ((\mathcal{A}_p^*)^* z(s) + \mathcal{B}(p_0(s), p_1(s)))\,\mathrm{d}s$$

for all $t \geq 0$ (cf. [**77, Remark 4.2.6**]). The integral is computed in $\mathcal{D}(\mathcal{A}_p^*)'$. Therefore, for each $\zeta \in \mathcal{D}(\mathcal{A}_p^*)$ we have from (3.5.2)

$$\langle z(t) - z(0), \zeta\rangle_{\mathcal{X}} = \int_0^t (\langle z(s), \mathcal{A}_p^* \zeta\rangle_{\mathcal{X}} + \langle (p_0(s), p_1(s)), \mathcal{B}^*\zeta\rangle_{\mathbb{C}^2})\,\mathrm{d}s$$

and using definition of $\mathcal{B}^*$ provided in Theorem 3.5.2, we can see that the components of z comprise the unique weak solution of (3.1.1) with boundary conditions (3.5.1).

Theorem 3.5.7. *If $z_0 \in \mathcal{X}_0^\perp$ and $p_0, p_1 \in L^2_{\text{loc}}([0,\infty), \mathbb{C})$ then* (3.1.1), (3.1.3), (3.5.1) *has a unique weak solution $z \in C([0,\infty), \mathcal{X}_0^\perp) \cap H^1_{\text{loc}}((0,\infty), \mathcal{D}(\mathcal{A}_p^*)')$ and for every $T > 0$ there exists $C = C(T) > 0$ such that*

$$\|z\|_{H^1((0,T), \mathcal{D}(\mathcal{A}_p^*)')} + \|z\|_{C([0,T], \mathcal{X}_0^\perp)} \leq C(\|z^0\|_{\mathcal{X}} + \|(p_0, p_1)\|_{L^2((0,T);\mathbb{C}^2)}) \qquad (3.5.11)$$

for all $z^0 \in \mathcal{X}_0^\perp$ and $(p_0, p_1) \in L^2((0,T); \mathbb{C}^2)$. Moreover, if $z^0 \in \mathcal{Z}$ and $p_0, p_1 \in H^1((0,T), \mathbb{C})$ satisfy the compatibility condition $\mathcal{G}z^0 = (p_0(0), p_1(0))$ then the solution z is in $C([0,T], \mathcal{Z}) \cap C^1([0,T], \mathcal{X}_0^\perp)$.

Proof. The first statement was already explained above and the estimate (3.5.11) can be shown from (3.5.10) and Theorem B.3.1, while the second statement is a direct application of [**77, Proposition 10.1.8**]. □

Remark 3.5.8. As in the proof of Theorem 3.5.6, it can be shown that $\mathcal{B}^*$ is an admissible observation operator for the semigroup generated by $\mathcal{A}_p$. This implies the following: For any $z_0 \in \mathcal{D}(\mathcal{A}_p)$ the solution of the (unforced) system satisfies

$$\|u(\cdot, 0)\|_{L^2(0,T)} + \|u(\cdot, \ell)\|_{L^2(0,T)} \leq C_T \|z_0\|_{\mathcal{X}}.$$

By a standard density argument, one can use this to define the traces $u(\cdot, 0), u(\cdot, \ell) \in L^2(0,T)$ for the solution corresponding to the initial state $z_0 \in \mathcal{X}_0^\perp$. Note that these traces do not make sense by the usual trace theorem for Sobolev spaces because in general $u \in C([0,T], L^2(0,\ell))$. This is sometimes referred as a *hidden regularity property* of solutions, see [**44, 46, 50**]. The hidden regularity property will be revisited in Chapter 4 using different tools and methods.

The controllability result Theorem 3.5.6 still holds even if there is only one forcing function that is applied to either of the tanks. In this case, the control operator would be either the first or second component of $\mathcal{B}$ according to where the control pressure is applied. The results can be also extended for two tanks with different horizontal cross sections. Now let us consider the case where the forcing is applied only at the left tank. In this case, the boundary operator $\mathcal{G}_0 : \mathcal{Z}_0 \to \mathbb{C}$ is defined by $\mathcal{G}_0(A, u, h_0, h_\ell) = A(0) - \gamma h_0$, where $\mathcal{Z}_0 := \{(A, u, h_0, h) \in H^1(0,\ell)^2 \times \mathbb{C}^2 : A(\ell) = \gamma h_\ell\} \cap \mathcal{X}_0^\perp$ is the corresponding solution space.

Theorem 3.5.9. *In the situation of Theorem 3.5.6, where $\mathcal{G}$ is replaced by $\mathcal{G}_0$, the pair $(\mathcal{A}_p, \mathcal{B}_0)$ is not approximately controllable for any time $0 < \tau < \tau^*$, where $\mathcal{B}_0$ is the control operator associated with the boundary control system $(\mathcal{F}, \mathcal{G}_0)$. In particular, $(\mathcal{A}_p, \mathcal{B}_0)$ is not exactly controllable for any time $0 < \tau < \tau^*$.*

Proof. From (3.4.3), there exists a positive integer M such that $|\frac{\tau^* \Im\lambda_{n+1}}{2\pi} - n| < \frac{1}{4}$ whenever $n > M$. By symmetry of the eigenvalues we have $|\frac{\tau^* \Im\lambda_{n-1}}{2\pi} - n| < \frac{1}{4}$ for all $n < -M$. Now according to the Generalized Kadec $\frac{1}{4}$-Theorem [**81, Corollary 2, p. 196**], the system of exponentials $(e^{i2n\pi t/\tau^*})_{0\le|n|\le M} \cup (e^{\lambda_{n+1}t})_{|n|>M}$ forms a Riesz basis for $L^2(0,\tau^*)$. Let $(g_n)_{n\in\mathbb{Z}}$ be the sequence biorthogonal to this Riesz basis. Given $0 < \tau < \tau^*$, let us take a nonzero element $F_1 \in L^2(0, \tau^* - \tau)$ such that

$$\int_0^{\tau^*-\tau} F_1(t)\overline{g_n(\tau + t)}\,\mathrm{d}t = 0, \qquad 0 \le |n| \le M,$$

that is, F_1 is in the orthogonal complement of the subspace of $L^2(0, \tau^* - \tau)$ spanned by the functions $(g_n(\tau + \cdot))_{0\le|n|\le M}$. Define the nonzero element $F \in L^2(0, \tau^*)$ by $F(t) = 0$ if $0 \le t \le \tau$ and $F(t) = F_1(t - \tau)$ if $\tau < t \le \tau^* = 2\ell/\sqrt{\alpha A_e}$.

Define

$$z = \sum_{|n|>M} \left(\langle F, g_n\rangle_{L^2(0,\tau^*)} \frac{\|z^*_{n+1}\|_{\mathcal{X}}}{\psi_{n+1}(0)} \right) \frac{z^*_{n+1}}{\|z^*_{n+1}\|_{\mathcal{X}}}.$$

This is a nonzero element of $\mathcal{X}_0^\perp$ because $(\langle F, g_n\rangle_{L^2(0,\tau^*)} \|z^*_{n+1}\|_{\mathcal{X}} \psi_{n+1}(0)^{-1})_{|n|>M}$ is nonzero element in ℓ^2. Note that by the uniqueness of the coefficients in a series of the elements of the Riesz basis, we must have $\langle F, g_n\rangle_{L^2(0,\tau^*)} \|z^*_{n+1}\|_{\mathcal{X}} \psi_{n+1}(0)^{-1} = \langle z, \tilde{z}_{n+1}\rangle_{\mathcal{X}}$ for all $|n| > M$ and so

$$\mathcal{B}_0^* e^{\mathcal{A}_p^* t} z = \sum_{|n|>M} e^{\lambda_{n+1}t} \langle z, \tilde{z}_{n+1}\rangle_{\mathcal{X}} \frac{\psi_{n+1}(0)}{\|z^*_{n+1}\|_{\mathcal{X}}} = \sum_{|n|>M} \langle F, g_n\rangle_{L^2(0,\tau^*)} e^{\lambda_{n+1}t} = F(t).$$

The terms with indices $0 \le |n| \le M$ vanish by construction of F. Hence there exists $z \in \mathcal{X}_0^\perp \setminus \{0\}$ such that $\mathcal{B}_0^* e^{\mathcal{A}_p^*(\cdot)} z = 0$ in $L^2(0,\tau)$. Therefore $\mathcal{N}(\mathcal{B}_0^* e^{\mathcal{A}_p^*(\cdot)}) \ne \{0\}$ so that the adjoint system $(\mathcal{A}_p^*, \mathcal{B}_0^*)$ is not approximately observable in time τ for any $0 < \tau < \tau^*$. The theorem follows from the duality of approximate observability and approximate controllability. □

3.6 SYSTEM WITH DISTRIBUTED CONTROL

One could also consider external control pressure applied to a part of the elastic tube, e.g. [**63, 78, 13**]. In this case, the linearized momentum equation becomes

$$\frac{\partial u}{\partial t} = -\alpha \frac{\partial A}{\partial x} - \beta u + P_c \chi_{[a,b]}, \tag{3.6.1}$$

where $P_c \in L^2_{\mathrm{loc}}([0,\infty), L^2(0,\ell))$, $0 < a < b < \ell$. In the literature, the control has to vanish at the endpoints of the subinterval $[a,b]$ where it is applied, however, we consider the general case where this vanishing condition is not assumed.

In the present situation, the control operator $\mathcal{B}_1 : L^2(0,\ell) \to \mathcal{X}_0^\perp$ is bounded and given by $\mathcal{B}_1 P_c = (0, P_c\chi_{[a,b]}, 0, 0)$. For each $z = (A, u, h_0, h) \in \mathcal{X}_0^\perp$ and $P_c \in L^2(0,\ell)$ we have $\langle \mathcal{B}_1 P_c, z\rangle_{\mathcal{X}} = \frac{1}{\alpha}\langle P_c, u\chi_{[a,b]}\rangle_{L^2(0,\ell)}$. Thus, the operator $\mathcal{B}_1^* : \mathcal{X}_0^\perp \to L^2(0,\ell)$ is given by $\mathcal{B}_1^*(A, u, h_0, h) = \frac{1}{\alpha}u\chi_{[a,b]}$. We have the following result, whose proof is similar as in the previous section, and hence it is omitted.

Theorem 3.6.1. *The pair $(\mathcal{A}_p, \mathcal{B}_1)$ is exactly controllable in time τ if $\tau > \frac{2\ell}{\sqrt{\alpha A_e}}$.*

3.7 CHARACTERIZATION OF CONTROLS

In this section we present a theorem which characterizes the control (p_0, p_1) described in Theorem 3.5.6. This problem has been considered for wave equations with either Dirichlet or Neumman boundary control using variational techniques, see [**82**] for instance. Instead of working with the specific case stated in Theorem 3.5.6, we consider a more general framework that includes the particular set-up of the said theorem. We prepare with a lemma.

Lemma 3.7.1. *Let $\mathcal{X}$ and $\mathcal{U}$ be complex Hilbert spaces. Suppose that $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ generates a $\mathcal{C}_0$-group on $\mathcal{X}$ and $\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{D}(\mathcal{A}^*)')$ is an admissible control operator for $(e^{\mathcal{A}t})_{t\in\mathbb{R}}$. Given $T > 0$, $z^0, z^1 \in \mathcal{X}$ and $u \in L^2((0,T);\mathcal{U})$, the solution $z \in C([0,T],\mathcal{X})$ in $\mathcal{D}(\mathcal{A}^*)'$ of the initial-value problem*

$$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}u(t), \qquad z(0) = z^0, \tag{3.7.1}$$

satisfies $z(T; z^0, u) = z^1$ if and only if

$$\int_0^T \langle u(t), \Psi_t^e\zeta\rangle_{\mathcal{U}}\,\mathrm{d}t + \langle z^0 - e^{-\mathcal{A}T}z^1, \zeta\rangle_{\mathcal{X}} = 0, \qquad \forall \zeta \in \mathcal{X} \tag{3.7.2}$$

where $\Psi_t^e \in \mathcal{L}(\mathcal{X}; L^2([0,\infty);\mathcal{U}))$ is the unique extension of the output map $\Psi_t \in \mathcal{L}(\mathcal{D}(\mathcal{A}^*); L^2([0,\infty);\mathcal{U}))$ defined by $(\Psi_t\zeta)(\tau) = \mathbf{1}_{\{0\le\tau\le t\}}\mathcal{B}^*e^{-\mathcal{A}^*\tau}\zeta$ for $\zeta \in \mathcal{D}(\mathcal{A}^*)$.

Proof. Recall from Proposition B.3.3 that the linear map Ψ_t^e exists and it is bounded due to the admissibility of the observation operator $\mathcal{B}^*$ under the semigroup generated by $\mathcal{A}^*$, and hence also for $(-\mathcal{A})^* = -\mathcal{A}^*$. Multiplying (3.7.1) by $e^{-\mathcal{A}^*t}\zeta$ for $\zeta \in \mathcal{D}(\mathcal{A}^{*2})$ we get

$$\langle \dot{z}(t), e^{-\mathcal{A}^*t}\zeta\rangle_{\mathcal{D}(\mathcal{A}^*)'\times\mathcal{D}(\mathcal{A}^*)} = \langle z(t), \mathcal{A}^*e^{-\mathcal{A}^*t}\zeta\rangle_{\mathcal{X}} + \langle u(t), \mathcal{B}^*e^{-\mathcal{A}^*t}\zeta\rangle_{\mathcal{U}}.$$

Integrating from 0 to T, the above equality implies that

$$\begin{aligned}
\langle z(T), e^{-\mathcal{A}^*T}\zeta\rangle_{\mathcal{X}} - \langle z^0, \zeta\rangle_{\mathcal{X}} &= \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\langle z(t), e^{-\mathcal{A}^*t}\zeta\rangle_{\mathcal{D}(\mathcal{A}^*)'\times\mathcal{D}(\mathcal{A}^*)}\,\mathrm{d}t \\
&= \int_0^T \langle \dot{z}(t), e^{-\mathcal{A}^*t}\zeta\rangle_{\mathcal{D}(\mathcal{A}^*)'\times\mathcal{D}(\mathcal{A}^*)} + \left\langle z(t), \frac{\mathrm{d}}{\mathrm{d}t}e^{-\mathcal{A}^*t}\zeta\right\rangle_{\mathcal{X}}\mathrm{d}t \\
&= \int_0^T \left\langle z(t), \frac{\mathrm{d}}{\mathrm{d}t}e^{-\mathcal{A}^*t}\zeta + \mathcal{A}^*e^{-\mathcal{A}^*t}\zeta\right\rangle_{\mathcal{X}} + \langle u(t), \mathcal{B}^*e^{-\mathcal{A}^*t}\zeta\rangle_{\mathcal{U}}\,\mathrm{d}t \\
&= \int_0^T \langle u(t), \mathcal{B}^*e^{-\mathcal{A}^*t}\zeta\rangle_{\mathcal{U}}\,\mathrm{d}t
\end{aligned}$$

Thus (3.7.2) holds for all $\zeta \in \mathcal{D}(\mathcal{A}^{*2})$ and by density this holds for all $\zeta \in \mathcal{X}$. Suppose that the solution of (3.7.1) satisfies $z(T; z^0, u) = z^1$ then according to what we have shown (3.7.2) holds. Conversely, suppose that (3.7.2) holds. Then the above calculations imply that $\langle z(T) - z^1, e^{-\mathcal{A}^*T}\zeta\rangle_{\mathcal{X}} = 0$ for all $\zeta \in \mathcal{X}$ and since $e^{-\mathcal{A}^*T}$ is bijective we have $z(T) = z^1$. □

The set-up of the following theorem takes place in complexified Hilbert spaces. Given a real Hilbert space $\mathcal{X}_{\mathbb{R}}$ we let $\mathcal{X} = \{x_1 + ix_2 : x_1, x_2 \in \mathcal{X}_{\mathbb{R}}\}$. The space $\mathcal{X}$ can be equipped with the inner product

$$\langle x_1 + ix_2, y_1 + iy_2\rangle_{\mathcal{X}} := \langle x_1, y_1\rangle_{\mathcal{X}_{\mathbb{R}}} + i\langle x_2, y_1\rangle_{\mathcal{X}_{\mathbb{R}}} - i\langle x_1, y_2\rangle_{\mathcal{X}_{\mathbb{R}}} + \langle x_2, y_2\rangle_{\mathcal{X}_{\mathbb{R}}}$$

with corresponding norm $\|x_1 + ix_2\|^2_{\mathcal{X}} = \|x_1\|^2_{\mathcal{X}_{\mathbb{R}}} + \|x_2\|^2_{\mathcal{X}_{\mathbb{R}}}$. This makes $\mathcal{X}$ a Hilbert space called a *complexified Hilbert space.*

Theorem 3.7.2. *Let $\mathcal{X}$ and $\mathcal{U}$ be complexified Hilbert spaces. Suppose that $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ generates a $\mathcal{C}_0$-group, $\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{D}(\mathcal{A}^*)')$ is an admissible control operator for $(e^{\mathcal{A}t})_{t\in\mathbb{R}}$ and $z^0, z^1 \in \mathcal{X}$. Assume that the following three conditions hold.*

(i) The pair $(\mathcal{A}, \mathcal{B})$ is exactly controllable in time $T > 0$.

(ii) The real Hilbert space $\mathcal{X}_{\mathbb{R}}$ is invariant under $(e^{\mathcal{A}t})_{t\in\mathbb{R}}$.

(iii) It holds that $\Psi^e_t\zeta \in L^2((0,T); \mathcal{U}_{\mathbb{R}})$ for every $\zeta \in \mathcal{X}_{\mathbb{R}}$.

For each fix $w^0, w^1 \in \mathcal{X}_{\mathbb{R}}$, define the cost functional $\mathcal{J}(\,\cdot\,, w^0, w^1) : \mathcal{X}_{\mathbb{R}} \to \mathbb{R}$ by

$$\mathcal{J}(\zeta) := \mathcal{J}(\zeta, w^0, w^1) = \frac{1}{2}\int_0^T \|\Psi^e_t\zeta\|^2_{\mathcal{U}}\,\mathrm{d}t + \langle w^0 - e^{-\mathcal{A}T}w^1, \zeta\rangle_{\mathcal{X}_{\mathbb{R}}}.$$

Let $\zeta^\star$ and $\vartheta^\star$ be the unique minimizers of the $\mathcal{J}(\,\cdot\,, \Re z^0, \Re z^1)$ and $\mathcal{J}(\,\cdot\,, \Im z^0, \Im z^1)$, respectively. Then $u^\star(t) = \Psi^e_t(\zeta^\star + i\vartheta^\star)$ satisfies $z(T; z^0, u^\star) = z^1$. Moreover, $u^\star$ is optimal in the L^2-sense, i.e.,

$$\|u^\star\|_{L^2((0,T);\mathcal{U})} = \min\{\|v\|_{L^2((0,T);\mathcal{U})} : v \in L^2((0,T);\mathcal{U}) \text{ and } z(T; z^0, v) = z^1\}.$$

Proof. We begin by noting from the antilinearity of (3.7.2) in ζ that (3.7.2) is equivalent to the same statement but with $\mathcal{X}$ replaced by $\mathcal{X}_{\mathbb{R}}$. First we consider the case where $z^0, z^1 \in \mathcal{X}_{\mathbb{R}}$. By this assumption together with (ii) we have $\langle z^0 - e^{-\mathcal{A}T}z^1, \zeta\rangle_{\mathcal{X}} \in \mathbb{R}$ for all $\zeta \in \mathcal{X}_{\mathbb{R}}$. If $\zeta^\star$ is the unique minimizer of $\mathcal{J}$, then

$$\begin{aligned} 0 \;&=\; \lim_{\epsilon\to 0}\frac{\mathcal{J}(\zeta^\star + \epsilon\zeta) - \mathcal{J}(\zeta^\star)}{\epsilon} \;=\; \int_0^T \Re\langle \Psi^e_t\zeta^\star, \Psi^e_t\zeta\rangle_{\mathcal{U}}\,\mathrm{d}t + \langle z^0 - e^{-\mathcal{A}T}z^1, \zeta\rangle_{\mathcal{X}} \\ &=\; \int_0^T \langle u^\star(t), \Psi^e_t\zeta\rangle_{\mathcal{U}}\,\mathrm{d}t + \langle z^0 - e^{-\mathcal{A}T}z^1, \zeta\rangle_{\mathcal{X}} \end{aligned}$$

for all $\zeta \in \mathcal{X}_{\mathbb{R}}$ and the last equality is due to assumption (iii). According to Lemma 3.7.1 and the previous remark, we conclude that $u^\star(t) = \Psi^e_t\zeta^\star$ satisfies $z(T; z^0, u) = z^1$.

Let us prove that $\mathcal{J}$ has a unique minimizer. The proof is standard but we include it here for the sake of completeness. Because $(\mathcal{A}, \mathcal{B})$ is exactly controllable in time

$T>0$ and $(e^{\mathcal{A}t})_{t\in\mathbb{R}}$ is a group then the pair $(-\mathcal{A},\mathcal{B})$ is also exactly controllable in time $T>0$ and thus there exists $c_T>0$ such that

$$|\mathcal{J}(\zeta)| \geq \Big(\frac{c_T}{2}\|\zeta\|_{\mathcal{X}} - \|z^0 - e^{-\mathcal{A}T}z^1\|_{\mathcal{X}}\Big)\|\zeta\|_{\mathcal{X}}, \qquad \forall \zeta \in \mathcal{X}_{\mathbb{R}}$$

and so $\mathcal{J}$ is coercive, that is, $\lim_{\|\zeta\|_{\mathcal{X}}\to\infty}\mathcal{J}(\zeta)=\infty$. For $\lambda\in(0,1)$ and distinct $\zeta_1,\zeta_2\in\mathcal{X}_{\mathbb{R}}$ let $\zeta=\zeta_1-\zeta_2\neq 0$. Then

$$\begin{aligned}\mathcal{J}(\lambda\zeta_1+(1-\lambda)\zeta_2) &= -\frac{\lambda(1-\lambda)}{2}\int_0^T \|\Psi_t^e\zeta\|_{\mathcal{U}}^2\,\mathrm{d}t + \lambda\mathcal{J}(\zeta_1)+(1-\lambda)\mathcal{J}(\zeta_2)\\ &\leq \lambda\mathcal{J}(\zeta_1)+(1-\lambda)\mathcal{J}(\zeta_2) - \frac{\lambda(1-\lambda)c_T}{2}\|\zeta\|_{\mathcal{X}}^2\end{aligned}$$

for which strict convexity of $\mathcal{J}$ follows. Moreover,

$$\begin{aligned}&|\mathcal{J}(\zeta_1)-\mathcal{J}(\zeta_2)|\\ &\leq \frac{1}{2}\big|\|\Psi_t^e\zeta_1\|^2_{L^2((0,T);\mathcal{U})} - \|\Psi_t^e\zeta_2\|^2_{L^2((0,T);\mathcal{U})}\big| + \|z^0-e^{-\mathcal{A}T}z^1\|_{\mathcal{X}}\|\zeta_1-\zeta_2\|_{\mathcal{X}}\\ &\leq \Big(\frac{1}{2}\|\Psi_t^e\|^2_{\mathcal{L}(\mathcal{X},L^2((0,\infty);\mathcal{U}))}(\|\zeta_1\|_{\mathcal{X}}+\|\zeta_2\|_{\mathcal{X}}) + \|z^0-e^{-\mathcal{A}T}z^1\|_{\mathcal{X}}\Big)\|\zeta_1-\zeta_2\|_{\mathcal{X}}\end{aligned}$$

for every $\zeta_1,\zeta_2\in\mathcal{X}_{\mathbb{R}}$. Thus $\mathcal{J}$ is a continuous, coercive, strictly convex functional and therefore it has a unique minimizer.

If $v\in\mathcal{U}_T(z^1)$ then taking $\zeta=\zeta^\star$ in Lemma 3.7.1 we get

$$\begin{aligned}\|u^\star\|^2_{L^2((0,T);\mathcal{U})} &= \int_0^T \|\Psi^e\zeta^\star\|^2_{\mathcal{U}}\,\mathrm{d}t \;=\; -\langle z^0-e^{-\mathcal{A}T}z^1,\zeta^\star\rangle_{\mathcal{X}}\\ &= \int_0^T\langle v(t),\Psi^e\zeta^\star\rangle_{\mathcal{U}}\,\mathrm{d}t \;\leq\; \|v\|_{L^2((0,T);\mathcal{U})}\|u^\star\|_{L^2((0,T);\mathcal{U})}\end{aligned}$$

and so $\|u^\star\|_{L^2((0,T);\mathcal{U})}\leq\|v\|_{L^2((0,T);\mathcal{U})}$. Now suppose that $z^0,z^1\in\mathcal{X}$ and $\zeta^\star$ and $\vartheta^\star$ are the unique minimizers of the cost functionals $\mathcal{J}(\,\cdot\,,\Re z^0,\Re z^1)$ and $\mathcal{J}(\,\cdot\,,\Im z^0,\Im z^1)$, respectively. Since $u^\star(t)=\Psi_t^e\zeta^\star\in L^2((0,T);\mathcal{U}_{\mathbb{R}})$ and $v^\star(t)=\Psi_t^e\vartheta^\star\in L^2((0,T);\mathcal{U}_{\mathbb{R}})$ are the optimal controls steering $\Re z^0$ to $\Re z^1$ and $\Im z^0$ to $\Im z^1$, respectively, then $u^\star+iv^\star$ is a control steering z^0 to z^1 according to Lemma 3.7.1. Let us prove that $u^\star+iv^\star$ is optimal. Suppose that $w\in L^2((0,T);\mathcal{U})$ is a control steering z^0 to z^1. Since $\Re w,\Im w\in L^2((0,T);\mathcal{U}_{\mathbb{R}})$, using conditions (i), (iii) and Lemma 3.7.1 it follows that $\Re w$ and $\Im w$ are controls steering $\Re z^0$ to $\Re z^1$ and $\Im z^0$ to $\Im z^1$, respectively. By the optimality of controls $u^\star$ and $v^\star$ we must have $\|u^\star\|_{L^2((0,T);\mathcal{U}_{\mathbb{R}})}\leq\|\Re w\|_{L^2((0,T);\mathcal{U}_{\mathbb{R}})}$ and $\|v^\star\|_{L^2((0,T);\mathcal{U}_{\mathbb{R}})}\leq\|\Im w\|_{L^2((0,T);\mathcal{U}_{\mathbb{R}})}$ and so

$$\|u^\star+iv^\star\|^2_{L^2((0,T);\mathcal{U})} = \|u^\star\|^2_{L^2((0,T);\mathcal{U}_{\mathbb{R}})}+\|v^\star\|^2_{L^2((0,T);\mathcal{U}_{\mathbb{R}})} \leq \|w\|^2_{L^2((0,T);\mathcal{U})}$$

proving the optimality of $u^\star+iv^\star$. □

Now, we apply the abstract result Theorem 3.7.2 to our problem. It can be shown that all assumptions of the previous theorem hold for our particular problem for any $T>\frac{2\ell}{\sqrt{\alpha A_e}}$. For our model with boundary control, the cost functional $\mathcal{J}:(\mathcal{X}_0^\perp)_{\mathbb{R}}\to\mathbb{R}$ is given by

$$\mathcal{J}(\zeta^0)=\frac{1}{2}\int_0^T(|v(t,0)|^2+|v(t,\ell)|^2)\,\mathrm{d}t + \langle w^0,\zeta^0\rangle_{\mathcal{X}} - \langle w^1,\zeta(T)\rangle_{\mathcal{X}}$$

where $w^0, w^1 \in (\mathcal{X}_0^\perp)_\mathbb{R}$, $\zeta = (B, v, g_0, g_\ell)$ is the solution of the adjoint problem

$$\begin{cases} \dfrac{\partial}{\partial t}B(t,x) = -A_e\dfrac{\partial}{\partial x}v(t,x), \quad 0 < t < T,\ 0 < x < \ell, \\ \dfrac{\partial}{\partial t}v(t,x) = -\alpha\dfrac{\partial}{\partial x}B(t,x) + \beta v(t,x), \quad 0 < t < T,\ 0 < x < \ell, \\ g_0'(t) = -\dfrac{A_e}{A_T}v(t,0), \quad 0 < t < T, \\ g_\ell'(t) = \dfrac{A_e}{A_T}v(t,\ell), \quad 0 < t < T, \\ B(t,0) = \gamma g_0(t), \quad B(t,\ell) = \gamma g_\ell(t), \qquad 0 < t < T, \\ B(0,x) = B^0(x), \quad v(0,x) = v^0(x), \quad 0 < x < \ell, \\ g_0(0) = g_0^0, \quad g_\ell(0) = g_\ell^0 \end{cases} \tag{3.7.3}$$

and $(B^0, v^0, g_0^0, g_\ell^0) \in (\mathcal{X}_0^\perp)_\mathbb{R}$. If $\zeta^\star \in (\mathcal{X}_0^\perp)_\mathbb{R}$ and $\vartheta^\star \in (\mathcal{X}_0^\perp)_\mathbb{R}$ are the unique minimizers in the conclusion of Theorem 3.7.2 and $(B^\star, v^\star, g_0^\star, g_\ell^\star)$ is the corresponding solution of the adjoint problem (3.7.3) with initial data $\zeta^\star + i\vartheta^\star$ then $p_0(t) = v^\star(t,0) \in L^2((0,T);\mathbb{C})$ and $p_1(t) = -v^\star(t,\ell) \in L^2((0,T);\mathbb{C})$ is a pair of control satisfying $z(T; z^0, (p_0, p_1)) = z^1$ and the pair is optimal in the L^2-sense.

For the model with interior control, $\mathcal{J} : (\mathcal{X}_0^\perp)_\mathbb{R} \to \mathbb{R}$ takes the form

$$\mathcal{J}(\zeta^0) = \frac{1}{2}\int_0^T\int_a^b |v(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t + \langle z^0, \zeta^0\rangle_\mathcal{X} - \langle z^1, \zeta(T)\rangle_\mathcal{X}$$

where ζ is the solution of (3.7.3). Using similar notations as in the previous paragraph, $P(t) = v^\star(t,\cdot) \in L^2((0,T); L^2((a,b);\mathbb{C}))$ is an L^2-optimal control for which $z(T; z^0, P) = z^1$.

3.8 LEGENDRE TAU APPROXIMATION OF THE LINEARIZED SYSTEM

For the rest of this chapter, we are interested in computing numerically the solution of the linearized system

$$\begin{cases} \dfrac{\partial A}{\partial t} = -A_e\dfrac{\partial u}{\partial x}, & t > 0,\ 0 < x < \ell, \\ \dfrac{\partial u}{\partial t} = -\alpha\dfrac{\partial A}{\partial x} - \beta u, & t > 0,\ 0 < x < \ell, \\ \dfrac{dh_0}{dt} = -\dfrac{A_e}{A_T}u(t,0), & t > 0, \\ \dfrac{dh_\ell}{dt} = \dfrac{A_e}{A_T}u(t,\ell), & t > 0, \\ A(t,0) = \gamma h_0(t), \qquad A(t,\ell) = \gamma h_\ell(t), & t > 0, \\ A(0,x) = A^0(x), \qquad u(0,x) = u^0(x), & 0 < x < \ell, \\ h_0(0) = h_0^0, \qquad h_\ell(0) = h_\ell^0. \end{cases} \tag{3.8.1}$$

The scheme in [**39**] using Legendre tau approximations will be used. To do this, the system is diagonalized by using an appropriate transformation decomposing the wave components that propagate to the left and to the right. The solution of the original problem can be obtained by using the fact that the semigroups generated by the two systems are similar up to a suitable time scaling.

3.8.1 Diagonalization

The state space that we consider is the real Hilbert space $\mathcal{X} = L^2((0,\ell);\mathbb{R})^2 \times \mathbb{R}^2$. Define the map $S : \mathcal{X} \to \mathcal{X}$

$$S(\phi^-, \phi^+, \eta_0, \eta_\ell) = \frac{1}{\sqrt{2}} \left(\sqrt{A_e}(\phi^+ - \phi^-), \sqrt{\alpha}(\phi^+ + \phi^-), \frac{\sqrt{A_e}}{A_T}\eta_0, -\frac{\sqrt{A_e}}{A_T}\eta_\ell \right).$$

The map S is clearly invertible and its inverse $S^{-1} : \mathcal{X} \to \mathcal{X}$ is given by

$$S^{-1}(A, u, h_0, h_\ell) = \frac{1}{\sqrt{2}} \left(\frac{u}{\sqrt{\alpha}} - \frac{A}{\sqrt{A_e}}, \frac{u}{\sqrt{\alpha}} + \frac{A}{\sqrt{A_e}}, \frac{2A_T}{\sqrt{A_e}}h_0, -\frac{2A_T}{\sqrt{A_e}}h_\ell \right).$$

Recall the generator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ for the semigroup associated with the system (3.8.1) defined in Section 3.2. Define $\mathcal{B} : \mathcal{D}(\mathcal{B}) \subset \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{B}(\phi^-, \phi^+, \eta_0, \eta_\ell) = \begin{pmatrix} \phi^-_x - \frac{\beta}{2c}(\phi^+ + \phi^-) \\ -\phi^+_x - \frac{\beta}{2c}(\phi^+ + \phi^-) \\ -(\phi^+(0) + \phi^-(0)) \\ -(\phi^+(\ell) + \phi^-(\ell)) \end{pmatrix}$$

where $c = \sqrt{\alpha A_e}$ and $\mathcal{D}(\mathcal{B}) = \{z \in \mathcal{X} : Sz \in \mathcal{D}(\mathcal{A})\} = \{(\phi^-, \phi^+, \eta_0, \eta_\ell) \in \mathcal{X} : \phi^\pm \in H^1(0,\ell), \eta_0 = \frac{A_T}{\gamma}(\phi^+(0) - \phi^-(0)), \eta_\ell = -\frac{A_T}{\gamma}(\phi^+(\ell) - \phi^-(\ell))\}$. A direct calculation shows that

$$c\mathcal{B} = S^{-1}\mathcal{A}S.$$

From [**25, pp. 60-61**] the operator $\mathcal{B}$ generates a strongly continuous group and is similar to the group generated by rescaling $\mathcal{A}$, more precisely,

$$e^{\mathcal{B}t} = S^{-1}e^{c^{-1}\mathcal{A}t}S, \qquad \forall\, t \in \mathbb{R}.$$

Moreover, their spectra are related by $\sigma(\mathcal{A}) = c\sigma(\mathcal{B})$. Thus, the original system can be solved using the new generator $\mathcal{B}$ and the transformation S via

$$e^{\mathcal{A}t}z_0 = Se^{c\mathcal{B}t}S^{-1}z_0 \tag{3.8.2}$$

for any $z_0 \in \mathcal{X}$ and $t \in \mathbb{R}$.

3.8.2 The Numerical Scheme

Let $p_k : [-1,1] \to \mathbb{R}$, $k = 0,1,2,\ldots$, be the Legendre polynomial of degree k and $N \geq 2$ be fixed. The goal is to derive a finite-dimensional system associated to the approximate functions

$$\phi^-(t,x) = \sum_{k=0}^{N} \phi^-_k(t)p_k(2x/\ell - 1) \tag{3.8.3}$$

$$\phi^+(t,x) = \sum_{k=0}^{N-1} \phi^+_k(t)p_k(2x/\ell - 1) \tag{3.8.4}$$

such that the system

$$\frac{d}{dt}(\phi^-,\phi^+,\eta_0,\eta_\ell)^\top = \mathcal{B}(\phi^-,\phi^+,\eta_0,\eta_\ell)^\top \tag{3.8.5}$$

is satisfied as well as the compatibility conditions for $\phi^\pm$, η_0 and η_ℓ in the definition of $\mathcal{D}(\mathcal{B})$. Take note that the degrees of freedom for ϕ^- and ϕ^+ are different. This choice will be justified later.

Define the rescaled kth degree Legendre polynomial $\tilde{p}_k : [0,\ell] \to \mathbb{R}$ by $\tilde{p}_k(x) = p_k(2x/\ell - 1)$. Using the orthogonality property of the Legendre polynomials

$$(p_k, p_j)_{L^2(-1,1)} = 2\delta_{kj}/(2k+1),$$

one can deduce the following orthogonality property of the basis functions $\tilde{p}_k$

$$\int_0^\ell \tilde{p}_k(x)\tilde{p}_j(x)\,\mathrm{d}x = \frac{\ell}{2}\int_{-1}^1 p_k(\xi)p_j(\xi)\,\mathrm{d}\xi = \frac{\ell}{2k+1}\delta_{kj}.$$

The derivatives of the Legendre polynomials can be written as linear combinations of the lower order Legendre polynomials. More precisely,

$$p_k' = \begin{cases} \displaystyle\sum_{j=0}^{k/2-1} (4j+3)p_{2j+1}, & \text{if } k \text{ is even,} \\ \displaystyle\sum_{j=0}^{(k-1)/2} (4j+1)p_{2j}, & \text{if } k \text{ is odd.} \end{cases}$$

Taking the inner product in $L^2(0,\ell)$ of the first equation of (3.8.5) with $\tilde{p}_l$ for $l = 0,\dots,N-1$ and the second equation with $\tilde{p}_l$ for $l = 0,\dots,N-2$, and using the fact that $\tilde{p}_k' = \frac{2}{\ell}p_k'$, we obtain the finite-dimensional system

$$\begin{aligned} \frac{d}{dt}(\phi_0^-,\dots,\phi_{N-1}^-)^\top(t) &= D_N(\phi_0^-,\dots,\phi_N^-)^\top(t) - M_N(\phi_0^-,\dots,\phi_{N-1}^-)^\top(t) \\ &\quad - M_N(\phi_0^+,\dots,\phi_{N-1}^+)^\top(t) \qquad (3.8.6) \\ \frac{d}{dt}(\phi_0^+,\dots,\phi_{N-2}^+)^\top(t) &= -D_{N-1}(\phi_0^+,\dots,\phi_{N-1}^+)^\top(t) - M_{N-1}(\phi_0^+,\dots,\phi_{N-2}^+)^\top(t) \\ &\quad - M_{N-1}(\phi_0^-,\dots,\phi_{N-2}^-)^\top(t) \qquad (3.8.7) \end{aligned}$$

where D_N is the $N \times (N+1)$ matrix given by

$$D_N = \frac{2}{\ell}\begin{pmatrix} 0 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & \cdots & 3 & 0 & 3 \\ 0 & 0 & 0 & 5 & \cdots & 0 & 5 & 0 \\ \vdots & & & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & & & & \ddots & 2N-5 & 0 & 2N-5 \\ \vdots & & & & & 0 & 2N-3 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 2N-1 \end{pmatrix}, \qquad \text{if } N \text{ is even}$$

or

$$D_N = \frac{2}{\ell}\begin{pmatrix} 0 & 1 & 0 & 1 & \cdots & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 & \cdots & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 & \cdots & 5 & 0 & 5 \\ \vdots & & & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & & & & \ddots & 2N-5 & 0 & 2N-5 \\ \vdots & & & & & 0 & 2N-3 & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 & 0 & 2N-1 \end{pmatrix}, \qquad \text{if } N \text{ is odd}$$

and M_N is the $N \times N$ diagonal matrix

$$M_N = \frac{\beta}{2c}\operatorname{diag}(1, 1, \dots, 1).$$

To treat the boundary conditions, we impose that $(\phi^-(t), \phi^+(t), \eta_0(t), \eta_\ell(t)) \in \mathcal{D}(\mathcal{B})$ for every $t > 0$. Using $p_k(\pm 1) = (\pm 1)^k$ this means that

$$\eta_0(t) = \frac{A_T}{\gamma}\left(-\sum_{k=0}^{N}(-1)^k\phi_k^-(t) + \sum_{k=0}^{N-1}(-1)^k\phi_k^+(t)\right) \tag{3.8.8}$$

$$\eta_\ell(t) = \frac{A_T}{\gamma}\left(\sum_{k=0}^{N}\phi_k^-(t) - \sum_{k=0}^{N-1}\phi_k^+(t)\right). \tag{3.8.9}$$

Solving for ϕ_N^- and ϕ_{N-1}^+ in (3.8.8) and (3.8.9) gives us the linear system

$$\begin{pmatrix} \phi_N^- \\ \phi_{N-1}^+ \end{pmatrix}(t) = T_N z_N(t) \tag{3.8.10}$$

where $z_N = (\phi_0^-, \dots, \phi_{N-1}^-, \phi_0^+, \dots, \phi_{N-2}^+, \eta_0, \eta_\ell)^\top$ and T_N is the $2 \times (2N+1)$ matrix

$$T_N = \left(\begin{array}{ccccc|ccccc|cc} -1 & 0 & \dots & -1 & 0 & 1 & 0 & \dots & 0 & 1 & -\frac{\gamma}{2A_T} & \frac{\gamma}{2A_T} \\ 0 & 1 & \dots & 0 & 1 & 0 & -1 & \dots & -1 & 0 & -\frac{\gamma}{2A_T} & -\frac{\gamma}{2A_T} \end{array}\right)$$

if N is even while

$$T_N = \left(\begin{array}{ccccc|ccccc|cc} 0 & -1 & \dots & -1 & 0 & 0 & 1 & \dots & 0 & 1 & \frac{\gamma}{2A_T} & \frac{\gamma}{2A_T} \\ 1 & 0 & \dots & 0 & 1 & -1 & 0 & \dots & -1 & 0 & \frac{\gamma}{2A_T} & -\frac{\gamma}{2A_T} \end{array}\right)$$

if N is odd. This is the place where we need the degrees of freedom of ϕ^- and ϕ^+ to be distinct. Not only that, we need that the sum of their degrees should be odd. Otherwise, if either the degrees are both even or both odd, then (3.8.8) and (3.8.9) is not uniquely solvable in ϕ_N^- and ϕ_{N-1}^+.

When ϕ^- and ϕ^+ are given by (3.8.3) and (3.8.4), respectively, then the differential equations for η_0 and η_ℓ in (3.8.5) are

$$\frac{d\eta_0}{dt}(t) = -\sum_{k=0}^{N}(-1)^k\phi_k^-(t) - \sum_{k=0}^{N-1}(-1)^k\phi_k^+(t) \tag{3.8.11}$$

$$\frac{d\eta_\ell}{dt}(t) = -\sum_{k=0}^{N}\phi_k^-(t) - \sum_{k=0}^{N-1}\phi_k^+(t). \tag{3.8.12}$$

Introducing the variable $\bar{z}_N = (z_N^\top, \phi_N^-, \phi_{N-1}^+)^\top$ the differential equations (3.8.11) and (3.8.12) can be combined into a single system given by

$$\frac{d}{dt}(\eta_0, \eta_\ell)^\top(t) = K_N \bar{z}_N(t) \tag{3.8.13}$$

where $K_N = [R_N \; R_{N-1} \; O_{2\times 2} \; r_N]$ is a $2 \times (2N+3)$ matrix with

$$R_N \;=\; \begin{pmatrix} -1 & 1 & -1 & \dots & (-1)^N \\ -1 & -1 & -1 & \dots & -1 \end{pmatrix} \in \mathbb{R}^{2\times N}$$

and r_N is the 2×2 matrix consisting of the last two columns of R_N. Combining (3.8.6), (3.8.7), (3.8.10) and (3.8.13) we obtain the system of differential equations

$$\dot{z}_N(t) = A_N \bar{z}_N(t), \qquad \begin{pmatrix} \phi_N^- \\ \phi_{N-1}^+ \end{pmatrix}(t) = T_N z_N(t) \tag{3.8.14}$$

where the $(2N+1) \times (2N+3)$ matrix A_N is given by

$$\begin{aligned} A_N &= E_N - F_N \\ E_N &= \begin{pmatrix} \tilde{D}_N & O_{N\times(N-1)} & O_{N\times 2} & d_N & O_{N\times 1} \\ O_{(N-1)\times N} & -\tilde{D}_{N-1} & O_{(N-1)\times 2} & O_{(N-1)\times 1} & -d_{N-1} \\ & & K_N & & \end{pmatrix} \\ F_N &= \begin{pmatrix} M_N & \tilde{M}_N & O_{N\times 3} & m_N \\ L_{N-1} & M_{N-1} & O_{(N-1)\times 3} & O_{(N-1)\times 1} \\ & O_{2\times(2N+3)} & & \end{pmatrix} \end{aligned}$$

where $D_N = (\tilde{D}_N \; d_N)$, $M_N = (\tilde{M}_N \; m_N)$, d_N, m_N are the last columns of D_N and M_N, respectively, and $L_{N-1} = (M_{N-1} \; O_{(n-1)\times 1})$.

To solve the system (3.8.14) numerically, it is convenient to rewrite it as a system in terms of z_N . This can be done by partitioning the matrix A_N. Partitioning A_N as $A_N = (A_N^1 \; A_N^2)$, where A_N^1 is $(2N+1) \times (2N+1)$ and A_N^2 is $(2N+1) \times 2$, and using the second equation in (3.8.14), the right hand side of the differential equation in (3.8.14) is given by

$$A_N \bar{z}_N = A_N^1 z_N + A_N^2 \begin{pmatrix} \phi_N^- \\ \phi_{N-1}^+ \end{pmatrix} = (A_N^1 + A_N^2 T_N) z_N.$$

Therefore the system (3.8.14) can be expressed as an ODE in z_N by

$$\dot{z}_N(t) \;=\; (A_N^1 + A_N^2 T_N) z_N(t), \qquad t > 0. \tag{3.8.15}$$

Algorithm
Input: Initial Data $z^0 = (A^0, u^0, h_0^0, h_\ell^0)$ and $N \geq 2$.
Output: Approximate solution of the system $\dot{z}(t) = \mathcal{A}z(t)$, $z(0) = z^0$.

(1) Compute $S^{-1}z^0 = (\phi^-, \phi^+, \eta_0, \eta_\ell)$.

(2) Determine $\phi_k^- = (\phi^-, \tilde{p}_k)_{L^2(0,\ell)}$ for $k = 0, \dots, N-1$ and $\phi_k^+ = (\phi^+, \tilde{p}_k)_{L^2(0,\ell)}$ for $k = 0, \dots, N-2$.

(3) Set $z_N^0 = (\phi_0^-, \dots, \phi_{N-1}^-, \phi_0^+, \dots, \phi_{N-2}^+, \eta_0, \eta_\ell)$.

(4) Assemble the matrices D_N, M_N, T_N and K_N.

(5) Set $B_N = A_N^1 + A_N^2 T_N$.

(6) Compute the solution of the IVP $\dot{z}_N(t) = B_N z_N(t)$, $z_N(0) = z_N^0$.

(7) Compute $z(t/c) = S z_N(t)$.

3.8.3 **Convergence Analysis**

In this subsection we review the variational form of the Trotter-Kato Theorem in Ito, Kappel and Salamon [**38**] and apply it to prove the convergence of the numerical scheme presented in Section 3.8.2. However, the following discussion is a simplified version of the more general setting given in [**38**].

Let V be a real Hilbert space equipped with an inner product $(\cdot,\cdot)$. Suppose that A generates a $\mathcal{C}_0$-semigroup such that $\|e^{At}\| \leq e^{\omega t}$ for $t \geq 0$. Denote by U the space $D(A)$ equipped with the graph norm $\|u\|_U = (\|u\|^2 + \|Au\|^2)^{1/2}$. Let $U^N \subset U$ and $V^N \subset V$ be subspaces for all $N \in \mathbb{N}$ and $\Xi^N \in \mathcal{L}(U^N, V^N)$ be an isomorphism. In applications, U^N and V^N are finite-dimensional and hence the existence of Ξ^N implies that $\dim U^N = \dim V^N$. Let $\tilde{A}^N \in \mathcal{L}(U^N, V^N)$ and define $A^N \in \mathcal{L}(V^N)$ by

$$A^N = \tilde{A}^N (\Xi^N)^{-1}.$$

Then A^N generates a strongly continuous semigroup on V^N for all $N \in \mathbb{N}$.

Let $\pi^N : V \to V^N$ be the canonical orthogonal projection of V onto V^N satisfying $\pi^N v = v$ for all $v \in V^N$. Thus, $\|\pi^N\| \leq 1$ for all $N \in \mathbb{N}$ so that the projections $(\pi^N)_N$ are uniformly bounded. To approximate the solution of the Cauchy problem $\dot{z}(t) = Az(t)$, $t > 0$, $z(0) = v$, first we project v onto V^N via the projection π^N and then solve the finite-dimensional system $\dot{z}_N(t) = A^N z(t)$, $t > 0$, $z_N(0) = \pi^N v$. For large N, we hope that z_N is close to z in some sense. Sufficient conditions for this to happen is given by the following stability and consistency conditions.

(S) *Stability.* There exists $\omega \in \mathbb{R}$ such that $(A^N v^N, v^N) \leq \omega \|v^N\|^2$ for all $v^N \in V^N$.

(C) *Consistency*

(a) For any $v \in V$ there exists $v^N \in V^N$ for all $N \in \mathbb{N}$ such that

$$\lim_{N\to\infty} \|v^N - v\| = 0.$$

(b) There exists $D \subset U$ such that $\overline{(\lambda_0 I - A)D} = V$ for some $\lambda_0 > \omega$ where ω is the constant in (S). For any $u \in D$ there exists $u^N \in U^N$ for $N \in \mathbb{N}$ such that

$$\lim_{N\to\infty} \|\Xi^N u^N - u\| = 0$$

and

$$\lim_{N\to\infty} \|A^N \Xi^N u^N - Au\| = 0.$$

For the proof of the following theorem, we refer to [**38**].

Theorem 3.8.1. *If the stability* (S) *and the consistency* (C) *conditions are satisfied then* $\lim_{N\to\infty} e^{A^N t}\pi^N v = e^{At} v$ *for all* $v \in V$ *uniformly on* $[0,T]$ *for all* $T > 0$.

Let P_N be the space of polynomials on $(0,\ell)$ of degree at most N and $V^N = P_{N-1} \times P_{N-2} \times \mathbb{R}^2 \subset \mathcal{X}$. Denote by Π^N the orthogonal projection of $L^2(0,\ell)$ onto P_N and by $\pi^N : \mathcal{X} \to V^N$ the orthogonal projection of $\mathcal{X}$ onto V^N given by

$$\pi^N(\phi^-, \phi^+, \eta_0, \eta_\ell) = (\Pi^{N-1}\phi^-, \Pi^{N-2}\phi^+, \eta_0, \eta_\ell).$$

Let U^N be the subspace of all $(\tilde{\phi}^-, \tilde{\phi}^+, \eta_0, \eta_\ell) \in V^{N+1}$ such that there exist unique real numbers α and β satisfying

$$\begin{aligned}
\tilde{\phi}^- &= \Pi^{N-1}\tilde{\phi}^- + \alpha\tilde{p}_N \\
\tilde{\phi}^+ &= \Pi^{N-2}\tilde{\phi}^+ + \beta\tilde{p}_{N-1} \\
\eta_0 &= \frac{A_T}{\gamma}(-\tilde{\phi}^-(0) + \tilde{\phi}^+(0)) \\
\eta_\ell &= \frac{A_T}{\gamma}(\tilde{\phi}^-(\ell) - \tilde{\phi}^+(\ell)).
\end{aligned}$$

It is clear that U^N is a subspace of $D(\mathcal{A})$. We claim that $\Xi^N : U^N \to V^N$ defined by

$$\Xi^N(\tilde{\phi}^-, \tilde{\phi}^+, \eta_0, \eta_\ell) = (\Pi^{N-1}\tilde{\phi}^-, \Pi^{N-2}\tilde{\phi}^+, \eta_0, \eta_\ell)$$

is an isomorphism. First let us show that Ξ^N is injective. For this purpose, suppose that $\Xi^N(\tilde{\phi}^-, \tilde{\phi}^+, \eta_0, \eta_\ell) = (0,0,0,0)$. This implies that $\tilde{\phi}^- = \alpha\tilde{p}_N$, $\tilde{\phi}^+ = \beta\tilde{p}_{N-1}$, $-\tilde{\phi}^-(0) + \tilde{\phi}^+(0) = 0$ and $\tilde{\phi}^-(\ell) - \tilde{\phi}^+(\ell) = 0$. The first two equations imply that $\tilde{\phi}^-(0) = (-1)^N\alpha$, $\tilde{\phi}^+(0) = (-1)^{N-1}\beta$, $\tilde{\phi}^-(\ell) = \alpha$ and $\tilde{\phi}^+(\ell) = \beta$. Plugging these into the remaining two equations we obtain $\alpha + \beta = 0$ and $\alpha - \beta = 0$, which imply $\alpha = \beta = 0$. Therefore $(\tilde{\phi}^-, \tilde{\phi}^+, \eta_0, \eta_\ell) = (0,0,0,0)$ and hence Ξ^N is injective. Next let us show that Ξ^N is surjective. Let $(\tilde{\phi}^-, \tilde{\phi}^+, \eta_0, \eta_\ell) \in V^N$ and let (α, β) solve the system

$$\begin{cases}
\alpha + \beta = (-1)^{N-1}\left(\dfrac{\gamma}{A_T}\eta_0 + \tilde{\phi}^-(0) - \tilde{\phi}^+(0)\right), \\
\alpha - \beta = \dfrac{\gamma}{A_T}\eta_\ell - \tilde{\phi}^-(\ell) + \tilde{\phi}^+(\ell).
\end{cases}$$

If we take $\phi^- = \tilde{\phi}^- + \alpha\tilde{p}_N$ and $\phi^+ = \tilde{\phi}^+ + \beta\tilde{p}_{N-1}$ then we have $(\phi^-, \phi^+, \eta_0, \eta_\ell) \in U^N$ and $\Xi^N(\phi^-, \phi^+, \eta_0, \eta_\ell) = (\tilde{\phi}^-, \tilde{\phi}^+, \eta_0, \eta_\ell)$, and so Ξ^N is surjective. Hence Ξ^N is an isomorphism.

Define $\tilde{B}^N : U^N \to V^N$ by

$$\tilde{B}^N(\tilde{\phi}_N^-, \tilde{\phi}_N^+, \eta_0, \eta_\ell) = \begin{pmatrix}
(\tilde{\phi}_N^-)_x - \frac{\beta}{2c}(\tilde{\phi}_N^+ + \Pi^{N-1}\tilde{\phi}_N^-) \\
-(\tilde{\phi}_N^+)_x - \frac{\beta}{2c}(\Pi^{N-2}\tilde{\phi}_N^+ + \Pi^{N-2}\tilde{\phi}_N^-) \\
-(\tilde{\phi}_N^+(0) + \tilde{\phi}_N^-(0)) \\
-(\tilde{\phi}_N^+(\ell) + \tilde{\phi}_N^-(\ell))
\end{pmatrix}$$

and $B^N : V^N \to V^N$ by $B^N = \tilde{B}^N(\Xi^N)^{-1}$. The approximate system (3.8.15) can be rewritten in terms of the operator B^N as

$$\dot{z}_N(t) = B^N z_N(t), \qquad z_N(0) = \pi^N S^{-1} z^0$$

where $z^0 = (A^0, u^0, h_0^0, h_\ell^0) \in \mathcal{X}$.

Theorem 3.8.2. *For every $z \in \mathcal{X}$ and $T > 0$ we have*

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \|e^{\mathcal{A}t}z - Se^{cB^N t}\pi^N S^{-1}z\|_{\mathcal{X}} = 0. \tag{3.8.16}$$

Proof. Let $y_N = (\tilde\phi_N^-, \tilde\phi_N^+, \eta_0, \eta_\ell) \in U^N$. Define a new inner product on $\mathcal{X}$ by

$$\begin{aligned}((\tilde\phi^-, \tilde\phi^+, \eta_0, \eta_\ell), (\tilde\psi^-, \tilde\psi^+, \theta_0, \theta_\ell))_1 &= (\tilde\phi^-, \tilde\psi^-)_{L^2(0,\ell)} + (\tilde\phi^+, \tilde\psi^+)_{L^2(0,\ell)} \\ &+ \frac{\gamma}{2A_T}(\eta_0\theta_0 + \eta_\ell\theta_\ell).\end{aligned}$$

Similar to the proof of dissipativity of $\mathcal{A}$ in Theorem 3.2.2, it can be checked that $\mathcal{B}$ is dissipative with respect to the inner product $(\cdot,\cdot)_1$. Similarly, $(B^N z^N, z^N)_1 \le 0 \cdot \|z^N\|_1^2$ for all $z^N \in V^N$ and this proves stability.

It remains to show consistency. In this case we choose $D = D(\mathcal{B})$ and $\lambda_0 = 1$. Condition (C,a) is guaranteed since $\|\pi^N z - z\|_1 \to 0$ for all $z \in \mathcal{X}$. Let $y = (\phi^-, \phi^+, \eta_0, \eta_\ell) \in D(\mathcal{B})$ and define $y^N = (\tilde\phi_N^-, \tilde\phi_N^+, \eta_{0N}, \eta_{\ell N}) \in U^N$ by

$$\begin{aligned}\tilde\phi_N^-(x) &= \phi^-(0) + \int_0^x \Pi^{N-1}\phi_x^-(\xi)\,\mathrm{d}\xi \\ \tilde\phi_N^+(x) &= \phi^+(0) + \int_0^x \Pi^{N-2}\phi_x^+(\xi)\,\mathrm{d}\xi \\ \eta_{0N} &= \frac{A_T}{\gamma}(-\tilde\phi_N^-(0) + \tilde\phi_N^+(0)) \\ \eta_{\ell N} &= \frac{A_T}{\gamma}(\tilde\phi_N^-(\ell) - \tilde\phi_N^+(\ell))\end{aligned}$$

Let $z^N = \Xi^N y^N = (\Pi^{N-1}\tilde\phi_N^-, \Pi^{N-2}\tilde\phi_N^+, \eta_{0N}, \eta_{\ell N}) \in V^N$. For each $x \in [0,\ell]$ we have by Cauchy-Schwarz inequality

$$|\tilde\phi_N^-(x) - \phi^-(x)| \le \int_0^\ell |\Pi^{N-1}\phi_x^-(\xi) - \phi_x^-(\xi)|\,\mathrm{d}\xi \le \ell^{1/2}\|\Pi^{N-1}\phi_x^- - \phi_x^-\|_{L^2(0,\ell)}.$$

Therefore $\|\tilde\phi_N^- - \phi^-\|_\infty \to 0$. Similarly, $\|\tilde\phi_N^+ - \phi^+\|_\infty \to 0$. In particular we have $\eta_{\ell N} \to \frac{A_T}{\gamma}(\tilde\phi^-(\ell) - \tilde\phi^+(\ell)) = \eta_\ell$. By definition, $\eta_{0N} = \eta_0$ for all N. Moreover, since $(\Pi^N)_N$ is uniformly bounded

$$\|\Pi^{N-1}\tilde\phi_N^- - \phi^-\|_{L^2(0,\ell)} \le \|\Pi^{N-1}(\tilde\phi_N^- - \phi^-)\|_{L^2(0,\ell)} + \|\Pi^{N-1}\phi^- - \phi^-\|_{L^2(0,\ell)} \to 0.$$

Similarly, $\|\Pi^{N-2}\tilde\phi_N^+ - \phi^+\|_{L^2(0,\ell)} \to 0$. Consequently,

$$\|\Xi^N y^N - y\|_1 = \|z^N - y\|_1 \to 0.$$

Because $(\tilde\phi_N^-)_x = \Pi^{N-1}\phi_x^-$ and $(\tilde\phi_N^+)_x = \Pi^{N-2}\phi_x^+$ it follows that $(\tilde\phi_N^-)_x \to \phi_x^-$ and $(\tilde\phi_N^+)_x \to \phi_x^+$ both in $L^2(0,\ell)$. These imply that

$$\|B^N\Xi^N y^N - By\|_1 = \|\tilde B^N y^N - By\|_1 \to 0$$

Thus, the consistency condition (C) holds. Invoking Theorem 3.8.2 we have

$$\lim_{N\to\infty}\sup_{t\in[0,\tau]}\|e^{\mathcal{B}t}y - e^{B^N t}\pi^N y\|_1 = 0 \tag{3.8.17}$$

for every $y \in \mathcal{X}$ and $\tau > 0$. From (3.8.2) and the equivalence of the norm $\|\cdot\|_1$ and the norm in $\mathcal{X}$, the following estimate

$$\begin{aligned}\|e^{\mathcal{A}t}z - Se^{cB^N t}\pi^N S^{-1}z\|_{\mathcal{X}} &= \|Se^{c\mathcal{B}t}S^{-1}z - Se^{cB^N t}\pi^N S^{-1}z\|_{\mathcal{X}} \\ &\le C\|S\|_{\mathcal{L}(\mathcal{X})}\|e^{c\mathcal{B}t}S^{-1}z - e^{cB^N t}\pi^N S^{-1}z\|_1.\end{aligned}$$

holds for some constant $C > 0$. Taking the supremum over all $t \in [0,T]$ and using (3.8.17) with $\tau = cT$ we conclude that (3.8.16) is satisfied. This completes the proof of the theorem. □

3.8.4 Numerical Results

In our simulations, the parameters listed in the following table were used. Typical values were taken from [**60**].

name	meaning	typical size	unit in cgs
s	thickness of the tube material	0.1	cm
r_0	inner rest radius of the tube	1	cm
A_0	rest cross section of the tube	π	cm^2
E	Young's modulus of the material	4.1×10^4	g/s^2/cm
p_0	ambient pressure	10^6	g/s^2/cm
g	gravitational constant	0.981	cm/s^2
ρ	constant density of the fluid	0.998	g/cm^2
μ_0	viscosity of the fluid	0.009	g/cm/s

The length of the tube and the overall volume of water are given by $\ell = 180$ cm and $V = 10^4$ cm^3, respectively. Each tank has a cross sectional area $A_T = 50A_0$ and for simplicity we suppose that there are no forcing pressures applied on the top of each tank, i.e., $p_{f0} = p_{f\ell} = 0$. With these parameters we have $\alpha = 6.5380\times10^4$, $\beta = 0.0721$ and $\gamma = 1.5005 \times 10^{-5}$. Consequently, the equilibrium cross section and equilibrium level heights are approximately given by $A_e \approx 3.1420$ and $h_{0e} = h_{\ell e} \approx 30.0288$, respectively.

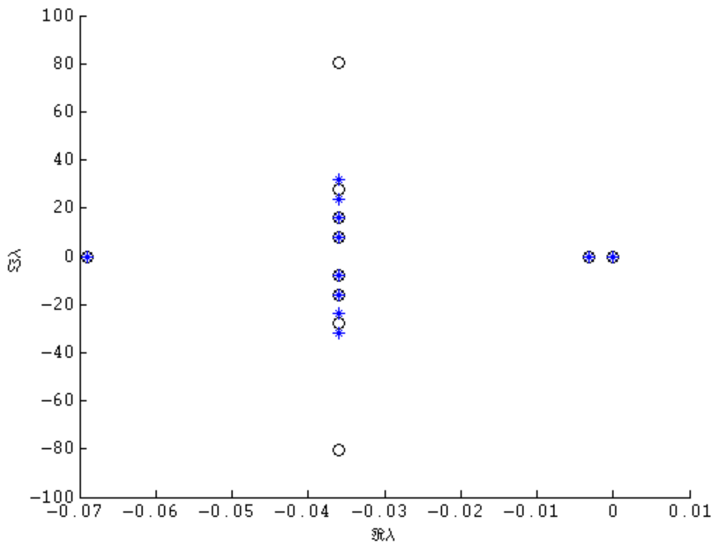

Figure 3.1.: Eigenvalues of the matrix cB_5 ($\circ$) and the generator $\mathcal{A}$ ($*$).

It can be seen in Figure 3.1 that the eigenvalues of the approximating matrix cB_N lie on a single line except for those three that lie on the real axis. For comparison we plotted the first 11 eigenvalues of the generator $\mathcal{A}$ using the one we have computed theoretically, see (3.4.2). We can see the first 7 out of 11 eigenvalues of cB_5 and $\mathcal{A}$ are close to each other.

In the implementation, we use the trapezoidal rule to compute the inner products in Step 2 of the algorithm. The differential equation in Step 6 is solved using the

ODE solver `ode45` in MATLAB. It can be observed that by increasing the number of Legendre polynomials we get more oscillations in the numerical solution. This is due to the fact that the matrix cB_N will have more eigenvalues with large imaginary part. It can be seen that the deviations of the level heights converge to zero. Also,

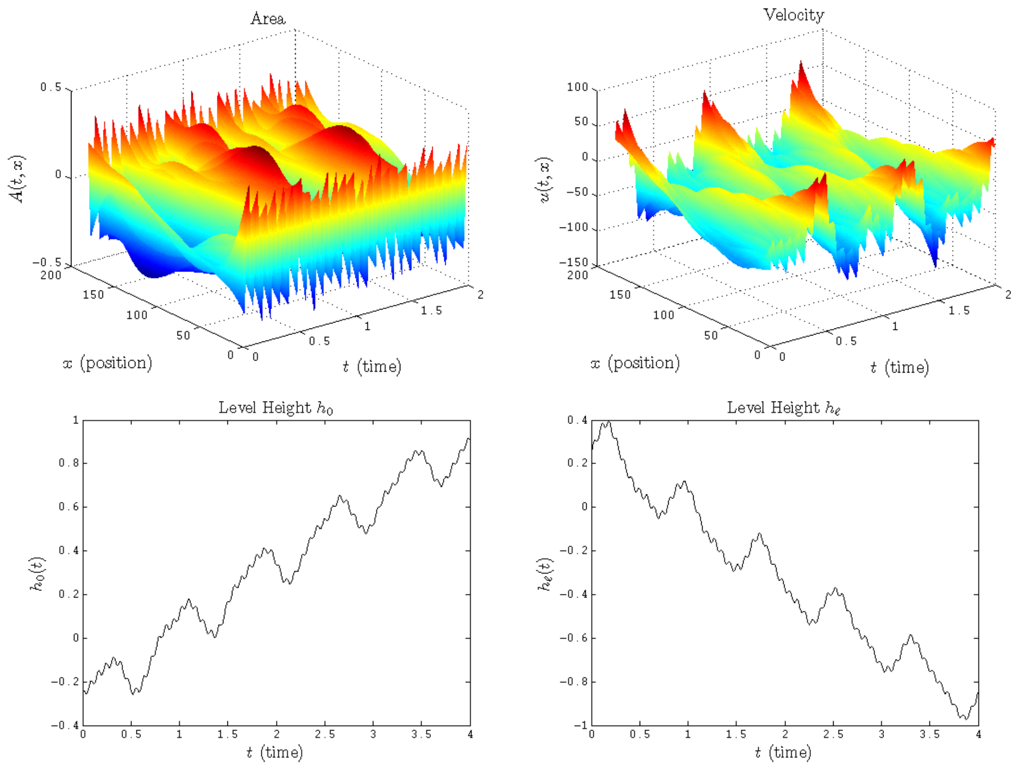

Figure 3.2.: Approximate solution (deviations from the equilibrium) of the linearized 2-tank model with the initial conditions $A^0(x) = 0.25(2x/\ell - 1)$, $u^0(x) = -50\sin(\pi x/\ell + 1)$, $h_0^0 = -0.25$ and $h_\ell^0 = 0.25$ using 5 Legendre polynomials: area (upper left), velocity (upper right), level height in the left tank (lower left) and level height in the right tank (upper right).

we have $\|u(T,\cdot)\|_{L^\infty(0,\ell)} = 0.0021$ and $\|A(T,\cdot)\|_{L^\infty(0,\ell)} = 1.3568 \times 10^{-6}$ for $T = 2000$. These confirm the theoretical result Theorem 3.4.4.

4

LINEAR SYSTEMS WITH VARIABLE COEFFICIENTS

To establish the well-posedness of the nonlinear system (2.6.5) using an iteration scheme, as a first step one has to linearize it by freezing some of the coefficients. Therefore linear systems with variable coefficients will be studied. This chapter is devoted to the existence, uniqueness and regularity of solutions of a hyperbolic system of first order linear differential equations on a bounded interval as well as a hyperbolic system with dynamic boundary conditions.

First, we deal with a hyperbolic system with variable coefficients. Two usual notions of solutions are considered, *weak* and *strong*. The weak solutions are defined in Lebesgue space L^2 or a weighted version of it. The strong solutions are also defined in L^2 but are the limit of smoother functions that satisfy a system that is a regularization of the original system. The existence and uniqueness of weak solutions will be established based on a priori estimates that hold for smoother functions and the Hahn-Banach and Riesz Representation Theorems. This procedure can be dated back to Friedrichs. The strong solutions satisfy the energy estimates and as a consequence will have more regularity. In particular, it will be shown that they have L^2-trace at the boundary. By an approximation or mollification argument, the weak solutions are actually strong solutions. This is sometimes called the *weak equals strong argument.*

After dealing with existence and uniqueness of weak and strong solutions, the regularity of the solutions will be considered. In particular, the weak solutions will be in the Sobolev space H^k for some positive integer k as long as the data are sufficiently regular and satisfy appropriate compatibility conditions. This will be done using a priori estimates in Sobolev spaces.

Finally, we will consider a linear hyperbolic system with variable coefficients coupled with an ordinary differential equation at the boundary and prove well-posedness in L^2. In the case where the coefficients are constant, it will be shown that the weak solution obtained from the variational method coincides with the one given by the theory of $\mathcal{C}_0$-semigroups.

4.1 A VARIATIONAL EQUATION

In this section we prove the existence and uniqueness of solutions of a variational problem. The framework introduced here will include the problems we consider in this chapter, i.e., boundary value problems, initial boundary value problems and coupled PDE-ODE systems with variable coefficients.

Let X and Z be real Hilbert spaces and Y be a subspace of X. Suppose that $\Lambda : Y \to X$, $\Psi : Y \to Z$ and $\Phi : Y \to Z$ are linear operators. Let $W = \ker \Phi$ and we assume that W and $\Lambda(W)$ are both nontrivial. Given $F \in X$ and $G \in Z$ we consider the variational problem: Find $u \in X$ such that

$$(u, \Lambda w)_X = (F, w)_X + (G, \Psi w)_Z, \qquad \forall\, w \in W. \tag{4.1.1}$$

For the differential equations we consider, Ψ is a trace operator while Λ and Φ are the differential and trace operators associated with the adjoint problem. We note that the space of test functions W need not be dense with respect to the topology of the space X. For the examples in the succeeding sections, X will be the dual of the solution space.

Theorem 4.1.1. *Suppose that there exist $\gamma > 0$ and $C > 0$ such that*

$$\gamma\|w\|_X^2 + \|\Psi w\|_Z^2 \le C\left(\frac{1}{\gamma}\|\Lambda w\|_X^2 + \|\Phi w\|_Z^2\right), \qquad \forall\, w \in Y. \tag{4.1.2}$$

Then the variational equation (4.1.1) has a solution $u \in X$ satisfying

$$\gamma\|u\|_X^2 \le C\left(\frac{1}{\gamma}\|F\|_X^2 + \|G\|_Z^2\right). \tag{4.1.3}$$

In addition, the solution is unique if $\Lambda(W)$ is dense in X.

Proof. By assumption, the restriction $\Lambda : W \to X$ of Λ to W is injective, and therefore it has a left inverse $\Lambda^{-1} : \Lambda(W) \subset X \to W$. According to (4.1.2)

$$\gamma\|\Lambda^{-1}\varphi\|_X^2 + \|\Psi\Lambda^{-1}\varphi\|_Z^2 \le \frac{C}{\gamma}\|\varphi\|_X^2, \qquad \forall\, \varphi \in \Lambda(W). \tag{4.1.4}$$

Define the linear map $\ell : \Lambda(W) \to \mathbb{R}$ by

$$\ell\varphi = (F, \Lambda^{-1}\varphi)_X + (G, \Psi\Lambda^{-1}\varphi)_Z,$$

for $\varphi \in \Lambda(W)$. We equipped $\Lambda(W)$ with the norm $\|\cdot\|_X$. The Cauchy-Schwarz inequality and (4.1.4) imply that

$$\begin{aligned}
|\ell\varphi|^2 &\le \|F\|_X^2\|\Lambda^{-1}\varphi\|_X^2 + \|G\|_Z^2\|\Psi\Lambda^{-1}\varphi\|_Z^2 \\
&\le \left(\frac{1}{\gamma}\|F\|_X^2 + \|G\|_Z^2\right)(\gamma\|\Lambda^{-1}\varphi\|_X^2 + \|\Psi\Lambda^{-1}\varphi\|_Z^2) \\
&\le \frac{C}{\gamma}\left(\frac{1}{\gamma}\|F\|_X^2 + \|G\|_Z^2\right)\|\varphi\|_X^2
\end{aligned}$$

for all $\varphi \in \Lambda(W)$. Thus $\ell \in [\Lambda(W)]'$ and

$$\gamma\|\ell\|_{[\Lambda(W)]'}^2 \le C\left(\frac{1}{\gamma}\|F\|_X^2 + \|G\|_Z^2\right).$$

According to the Hahn-Banach Theorem, ℓ admits an extension $\tilde{\ell} \in X'$ such that $\|\tilde{\ell}\|_{X'} = \|\ell\|_{[\Lambda(W)]'}$. From the Riesz Representation Theorem there is a unique $u \in X$ such that $\|u\|_X = \|\tilde{\ell}\|_{X'}$ and $(u, v)_X = \tilde{\ell}v$ for all $v \in X$. In particular, for every $w \in W$

$$(u, \Lambda w)_X = \tilde{\ell}\Lambda w = \ell\Lambda w = (F, w)_X + (G, \Psi w)_Z.$$

Thus u is a solution of the variational equation (4.1.1) and it satisfies the estimate (4.1.3). Suppose that u_1 and u_2 solve (4.1.1). Then $(u_1 - u_2, \Lambda w) = 0$ for every $w \in W$. If $\Lambda(W)$ is dense in X then $u_1 - u_2 = 0$ and thus the solution of (4.1.1) is unique. □

The idea of the proof of Theorem 4.1.1 can be traced back to the work of Friedrichs [**28**]. The same idea has been used in [**9, 15, 41**]. The constant γ is introduced because the a priori estimates will be derived in weighted Lebesque spaces. This parameter is useful as well for the nonlinear analysis.

In the context of differential equations, the variational equation (4.1.1) can be derived by multiplying the differential equation by appropriate test functions and formally integrate by parts. To prove the existence of solutions of the variational equation (4.1.1), one has to prove the *abstract a priori estimate* (4.1.2). For hyperbolic partial differential equations, the a priori estimates can be obtained with the help of symmetrizers. This will be the topic of Sections 4.5, 4.6 and 4.7. Before dealing with partial differential equations, we will first illustrate how Theorem 4.1.1 can be used to prove well-posedness of a system of ordinary differential equations. This will be done in the succeeding section.

To prove uniqueness, a sufficient condition is to show that for each $v \in X$ there exists $w \in Y$ with $\Lambda w = v$ and $\Phi w = 0$. This corresponds to a homogeneous dual problem. In most cases, the well-posedness of the dual problem follows from the primal problem after time reversal. However, the criterion that the solution lies in the space Y is not known a priori. In the context of PDEs a different approach in proving uniqueness will be provided.

4.2 LINEAR ORDINARY DIFFERENTIAL EQUATIONS

Consider the ordinary differential equation

$$\begin{cases} h'(t) = H(t)h(t) + f(t), \quad t \in (0,T), \\ h(0) = h_0 \end{cases} \tag{4.2.1}$$

where $T > 0$, $h : (0,T) \to \mathbb{R}^m$, $h_0 \in \mathbb{R}^m$, $H \in L^\infty((0,T);\mathbb{R}^{m\times m})$ and $f \in L^2((0,T);\mathbb{R}^m)$. The goal is to determine the existence and uniqueness of a function $u \in H^1(0,T)$ satisfying (4.2.1). This can be done in several ways. We only discuss the fixed-point method and the variational method based on Theorem 4.1.1. The first method is classical. The ordinary differential equation (4.2.1) is rewritten as an integral equation. The existence and uniqueness of a solution of the integral equation can be obtained using the so-called Banach Fixed-Point Theorem. In the energy method, the ordinary differential equation is rewritten in variational form. One acquires the existence and uniqueness of a solution of the variational equation by proving an a priori estimate and using Theorem 4.1.1.

4.2.1 The Fixed-Point Method

Suppose that $h \in H^1(0,T)$ satisfies (4.2.1). Integration yields

$$h(t) = h_0 + \int_0^t H(s)h(s) + f(s)\,\mathrm{d}s, \qquad t \in (0,T). \tag{4.2.2}$$

On the other hand, a function $h \in L^2(0,T)$ satisfying the integral equation (4.2.2) is absolutely continuous and its weak derivative is given by the integrand in (4.2.2). Thus $h \in H^1(0,T)$ satisfies (4.2.1) in $L^2(0,T)$. The integral equation enables us to reduce the problem of determining the existence of a solution of (4.2.1) to the problem of determining a fixed point of the map $\mathcal{M} : L^2(0,T) \to L^2(0,T)$ defined by

$$(\mathcal{M}h)(t) = h_0 + \int_0^t H(s)h(s) + f(s)\,\mathrm{d}s, \qquad t \in (0,T)$$

It can be easily checked that $\mathcal{M}$ is well-defined, i.e., it maps $L^2(0,T)$ into itself. Given $t \in [0,T]$, the Cauchy Schwarz inequality implies

$$\begin{aligned}\int_0^t |(\mathcal{M}h_1)(s) - (\mathcal{M}h_2)(s)|^2\,\mathrm{d}s &= \int_0^t \left|\int_0^s H(\tau)(h_1(\tau) - h_2(\tau))\,\mathrm{d}\tau\right|^2 \mathrm{d}s \\ &\le Tt\|H\|^2_{L^\infty(0,T)}\|h_1 - h_2\|^2_{L^2(0,T)}.\end{aligned}$$

for $h_1, h_2 \in L^2(0,T)$. By induction, we show that for every $t \in [0,T]$ and positive integer N

$$\|\mathcal{M}^N h_1 - \mathcal{M}^N h_2\|^2_{L^2(0,t)} \le \frac{(Tt)^N}{N!}\|H\|^{2N}_{L^\infty(0,T)}\|h_1 - h_2\|^2_{L^2(0,T)}. \tag{4.2.3}$$

The case where $N = 1$ has been already shown. Suppose that (4.2.3) holds for some N. Applying the Cauchy Schwarz inequality and the induction hypothesis we have

$$\begin{aligned}\|\mathcal{M}^{N+1}h_1 - \mathcal{M}^{N+1}h_2\|^2_{L^2(0,t)} &= \int_0^t \left|\int_0^s H(\tau)(\mathcal{M}^N h_1(\tau) - \mathcal{M}^N h_2(\tau))\,\mathrm{d}\tau\right|^2 \mathrm{d}s \\ &\le \int_0^t T\frac{(Ts)^N}{N!}\|H\|^{2(N+1)}_{L^\infty(0,T)}\|h_1 - h_2\|^2_{L^2(0,T)}\,\mathrm{d}s \\ &\le \frac{(Tt)^{N+1}}{(N+1)!}\|H\|^{2(N+1)}_{L^\infty(0,T)}\|h_1 - h_2\|^2_{L^2(0,T)}.\end{aligned}$$

Taking $t = T$ in (4.2.3) we get

$$\|\mathcal{M}^N h_1 - \mathcal{M}^N h_2\|_{L^2(0,T)} \le \frac{T^N}{\sqrt{N!}}\|H\|^N_{L^\infty(0,T)}\|h_1 - h_2\|_{L^2(0,T)}.$$

As a consequence, the map $\mathcal{M}$ is continuous and $\mathcal{M}^N$ is a strict contraction for N large enough. Now the existence of a unique fixed point of $\mathcal{M}$ follows immediately from the following theorem.

Theorem 4.2.1 (Banach Fixed-Point Theorem)**.** *Let $T : X \to X$ be a map on a complete metric space X with metric d. If there exists a positive integer N such that T^N is a strict contraction, i.e., there exists a constant $c \in (0,1)$ such that $d(T^N x, T^N y) \le c d(x,y)$ for all $x, y \in X$, then T has a unique fixed point.*

The proof can be found in [**58, Theorem 1.1.3**].

4.2.2 The Energy Method

A function $h \in L^2(0,T)$ is called a *weak solution* of (4.2.1) if the variational equation

$$(h, \eta' + H^\top \eta)_{L^2(0,T)} = -h_0 \cdot \eta(0) - (f, \eta)_{L^2(0,T)} \tag{4.2.4}$$

holds for every $\eta \in H^1(0,T)$ such that $\eta(T) = 0$. If h is a weak solution of (4.2.1) then necessarily $h \in H^1(0,T)$ and $h' = Hh + f$ in the weak sense. This can be seen immediately from (4.2.4) by taking $\eta \in \mathscr{D}(0,T)$. In addition, integrating by parts we obtain $h(0) = h_0$. As a result, the variational equation (4.2.4) is equivalent to the ordinary differential equation (4.2.1).

The existence and uniqueness of a weak solution of (4.2.1) relies on an a priori estimate that will be derived using the following proposition. For the proof we refer to [**9, p. 283**].

Proposition 4.2.2. *For each $\eta \in e^{\gamma t} H^1(-\infty, T)$ and $\gamma \geq 1$ we have*

$$\int_{-\infty}^{T} e^{-2\gamma t} |\eta(t)|^2 \, \mathrm{d}t \leq \frac{1}{\gamma^2} \int_{-\infty}^{T} e^{-2\gamma t} |\eta'(t)|^2 \, \mathrm{d}t.$$

As a consequence we have the following estimate.

Corollary 4.2.3. *For each $\gamma \geq 1$ and $\eta \in H^1(0,T)$ such that $\eta(T) = 0$ we have*

$$\int_{0}^{T} e^{2\gamma t} |\eta(t)|^2 \, \mathrm{d}t \leq \frac{1}{\gamma^2} \int_{0}^{T} e^{2\gamma t} |\eta'(t)|^2 \, \mathrm{d}t. \tag{4.2.5}$$

Proof. Extending η by zero for $t > T$ we have $\eta \in H^1(0, \infty)$. Define $\zeta \in e^{\gamma t} H^1(-\infty, T)$ by $\zeta(t) = \eta(T - t)$. Proposition 4.2.2 and the change of variable $s = T - t$ imply

$$\begin{aligned} \int_{0}^{T} e^{2\gamma t} |\eta(t)|^2 \, \mathrm{d}t &= \int_{-\infty}^{T} e^{-2\gamma(s-T)} |\zeta(s)|^2 \, \mathrm{d}s \\ &\leq \frac{1}{\gamma^2} \int_{-\infty}^{T} e^{-2\gamma(s-T)} |\zeta'(s)|^2 \, \mathrm{d}s. \end{aligned} \tag{4.2.6}$$

Using $\zeta'(s) = -\eta'(T - s)$ and the change of variable $t = T - s$ we have

$$\begin{aligned} \int_{-\infty}^{T} e^{-2\gamma(s-T)} |\zeta'(s)|^2 \, \mathrm{d}s &= \int_{-\infty}^{T} e^{-2\gamma(s-T)} |\eta'(T-s)|^2 \, \mathrm{d}s \\ &= \int_{0}^{T} e^{2\gamma t} |\eta'(t)|^2 \, \mathrm{d}t. \end{aligned} \tag{4.2.7}$$

The estimate (4.2.5) now follows from (4.2.6) and (4.2.7). □

With the estimate (4.2.5), it is now possible to derive an a priori estimate needed in the well-posedness of (4.2.4). This a priori estimate will be also used in the PDE-ODE systems of Section 4.20.

Theorem 4.2.4. *Let $A \in L^\infty((0,T); \mathbb{R}^{m \times m})$. There exist constants $C > 0$ and $\gamma_0 \geq 1$ depending only on $\|A\|_{L^\infty(0,T)}$ such that for all $\eta \in H^1(0,T)$ and for all $\gamma \geq \gamma_0$ we have*

$$|\eta(0)|^2 + \gamma \|e^{\gamma t} \eta\|^2_{L^2(0,T)} \leq \frac{C}{\gamma} \|e^{\gamma t}(\eta' + A\eta)\|^2_{L^2(0,T)} + C e^{2\gamma T} |\eta(T)|^2. \tag{4.2.8}$$

Proof. First, suppose that $\eta \in H^1(0,T)$ satisfies $\eta(T) = 0$. According to Corollary 4.2.3 and the triangle inequality we have

$$\gamma \|e^{\gamma t} \eta\|^2_{L^2(0,T)} \leq \frac{2}{\gamma} \|e^{\gamma t}(\eta' + A\eta)\|^2_{L^2(0,T)} + \frac{2}{\gamma} \|A\|^2_{L^\infty(0,T)} \|e^{\gamma t} \eta\|^2_{L^2(0,T)}. \tag{4.2.9}$$

For sufficiently large γ, the second term on the right hand side of (4.2.9) can be absorbed by the term on the left hand side. Thus there are constants $C > 0$ and $\gamma_0 \geq 1$ both depending only on the L^∞-norm of A such that for all $\gamma \geq \gamma_0$

$$\gamma \|e^{\gamma t} \eta\|^2_{L^2(0,T)} \leq \frac{C}{\gamma} \|e^{\gamma t}(\eta' + A\eta)\|^2_{L^2(0,T)}. \tag{4.2.10}$$

Define $\eta(t) = 0$ for $t > T$ and $w(t) = e^{\gamma(T-t)}\eta(T-t)$ for $-\infty < t < T$. Then $w \in H^1(-\infty, T)$ and therefore it satisfies the weighted Sobolev estimate

$$\|w\|^2_{L^\infty(-\infty,T)} \leq \gamma\|w\|^2_{L^2(-\infty,T)} + \frac{1}{\gamma}\|w'\|^2_{L^2(-\infty,T)} \tag{4.2.11}$$

for all $\gamma > 0$, see the proof of Proposition 4.16.1. Since $w'(t) = -\gamma e^{\gamma(T-t)}\eta(T-t) - e^{\gamma(T-t)}\eta'(T-t)$ the above estimate implies that for some $C > 0$ there holds

$$e^{2\gamma(T-t)}|\eta(T-t)|^2 \leq C\left(\gamma\|e^{\gamma t}\eta\|^2_{L^2(0,T)} + \frac{1}{\gamma}\|e^{\gamma t}\eta'\|^2_{L^2(0,T)}\right) \tag{4.2.12}$$

for all $t \in [0, T]$. Choosing $t = T$ in (4.2.12), writing $\eta' = (\eta' + A\eta) - A\eta$ and using the same argument as before we obtain, by increasing γ_0 if necessary, that for all $\gamma \geq \gamma_0$

$$|\eta(0)|^2 \leq C\left(\gamma\|e^{\gamma t}\eta\|^2_{L^2(0,T)} + \frac{1}{\gamma}\|e^{\gamma t}(\eta' + A\eta)\|^2_{L^2(0,T)}\right) \tag{4.2.13}$$

for some $C > 0$. The estimate

$$|\eta(0)|^2 + \gamma\|e^{\gamma t}\eta\|^2_{L^2(0,T)} \leq \frac{C}{\gamma}\|e^{\gamma t}(\eta' + A\eta)\|^2_{L^2(0,T)} \tag{4.2.14}$$

follows from (4.2.10) and (4.2.13).

Now suppose that $\eta \in H^1(0,T)$. Define $\zeta \in H^1(0,T)$ by $\zeta(t) = \eta(t) - \eta(T)$ for $0 < t < T$. Applying (4.2.14) to ζ, using the triangle inequality and the fact that $2\gamma\|e^{\gamma t}\|^2_{L^2(0,T)} = e^{2\gamma T} - 1$ we obtain (4.2.8). $\square$

We are now in position to use Theorem 4.1.1 in proving that (4.2.4) is well-posed. We take $X = e^{-\gamma t}L^2(0,T)$, $Y = H^1(0,T)$ and $Z = \mathbb{R}^m$. The operators Λ, Ψ and Φ are given by $\Lambda\eta = (\eta' + H^\top\eta)$, $\Psi\eta = \eta(0)$ and $\Phi\eta = \eta(T)$ for all $\eta \in Y$, respectively. Thus the variational equation (4.2.4) can be written in the form

$$(e^{-2\gamma t}h, \Lambda\eta)_X = (-e^{-2\gamma t}f, \eta)_X + (-h_0, \Psi\eta)_Z, \qquad \forall\, \eta \in W \tag{4.2.15}$$

where $W = \{\eta \in Y : \eta(T) = 0\}$. Note that the set X coincides with $L^2(0,T)$.

Theorem 4.2.5. *Let $h_0 \in \mathbb{R}^m$, $H \in L^\infty(0,T)$ and $f \in L^2(0,T)$. Then (4.2.1) has a unique weak solution $h \in L^2(0,T)$. Furthermore, $h \in H^1(0,T)$ and it satisfies the energy estimates*

$$\gamma\|e^{-\gamma t}h\|^2_{L^2(0,T)} \leq C\left(\frac{1}{\gamma}\|e^{-\gamma t}f\|^2_{L^2(0,T)} + |h_0|^2\right) \tag{4.2.16}$$

and

$$\|e^{-\gamma t}h'\|^2_{L^2(0,T)} \leq C(\|e^{-\gamma t}f\|^2_{L^2(0,T)} + |h_0|^2) \tag{4.2.17}$$

for all $\gamma \geq \gamma_0$ for some $C > 0$ and $\gamma_0 \geq 1$ both depending only on $\|H\|_{L^\infty(0,T)}$.

Proof. Using the notations of the paragraph preceding the theorem, the a priori estimate (4.1.3) follows directly from Theorem 4.2.4. Hence Theorem 4.1.1 implies the existence of $g \in X$ such that

$$(g, \Lambda\eta)_X = (-e^{-2\gamma t}f, \eta)_X + (-h_0, \Psi\eta)_Z, \qquad \forall\, \eta \in W,$$

and it satisfies

$$\gamma\|g\|_X^2 \leq C\left(\frac{1}{\gamma}\|e^{-2\gamma t}f\|_X^2 + |h_0|^2\right). \tag{4.2.18}$$

Then $h = e^{2\gamma t}g \in L^2(0,T)$ is a weak solution of (4.2.1) and it satisfies (4.2.16) due to (4.2.18). From the discussion at the beginning of this section, we already know that the weak solution h lies in $H^1(0,T)$ and it satisfies $h' = Hh + f$ in $L^2(0,T)$. The estimate (4.2.17) follows from the differential equation $h' = Hh + f$ and (4.2.16). Given $f \in X$, the dual problem $\eta' + H^\top \eta = f$, $\eta(T) = 0$ admits a solution $\eta \in H^1(0,T)$, which was just shown for the forward problem. Hence $\Lambda(W) = X$ and therefore the weak solution is unique by Theorem 4.1.1. □

The fixed point method takes less effort than the energy method. However, the advantage of the latter is that it shows directly from the a priori estimate the continuous dependence of the solution with respect to the data. We would like to extend the energy method presented above to a hyperbolic system of partial differential equations on a bounded interval. Of course the derivation of the a priori estimates will now be more technical. This will be the goal of the succeeding sections.

4.3 LINEAR HYPERBOLIC SYSTEM OF PDES

Consider the hyperbolic system of first order linear partial differential equations with variable coefficients

$$\partial_t u(t,x) + A(t,x)\partial_x u(t,x) + R(t,x)u(t,x) = f(t,x), \quad (t,x) \in (0,T)\times(0,1), \tag{4.3.1}$$

where u takes values in $\mathbb{R}^n$. The system (4.3.1) is supplied with the boundary conditions

$$B_0 u(t,0) = g_0(t), \qquad t \in (0,T), \tag{4.3.2}$$

$$B_1 u(t,1) = g_1(t), \qquad t \in (0,T), \tag{4.3.3}$$

and initial condition

$$u(0,x) = u_0(x), \qquad x \in (0,1). \tag{4.3.4}$$

The data f, g_0, g_1, u_0 and the coefficients A, R, B_0, B_1 are contained in appropriate function spaces that will be specified precisely in the succeeding sections.

The aim is to prove the well-posedness of the system (4.3.1)−(4.3.4) in L^2 and the regularity of the solutions under additional smoothness and compatibility conditions on the initial data, boundary data and the coefficients. Following the framework in [**9**] the first step is to provide well-posedness for the pure boundary value problem

$$\begin{cases} \partial_t u(t,x) + A(t,x)\partial_x u(t,x) + R(t,x)u(t,x) = f(t,x), & (t,x) \in \mathbb{R}\times(0,1), \\ B_0 u(t,0) = g_0(t), \quad t \in \mathbb{R}, \\ B_1 u(t,1) = g_1(t), \quad t \in \mathbb{R}, \end{cases}$$

where A and R are infinitely differentiable. Afterwards, the initial value problem (4.3.1)−(4.3.4) with homogeneous initial data $u_0 = 0$ will be considered. This is done by extending the data f, g_0, and g_1 by zero outside the interval $(0,T)$ and analyzing the associated boundary value problem. Thanks to a causality principle the restriction of the solution of the boundary value problem is the solution of the homogeneous initial boundary value problem. Finally, the well-posedness of the general initial boundary value problem will be established from the homogeneous case using lifting and approximation arguments.

In the previous section, we have seen that a weak solution of the ordinary differential equation (4.2.1) automatically satisfies the differential equation in the sense of L^2 and the initial condition is satisfied. However, for the initial value problem (4.3.1)−(4.3.4) this is not immediate. In order to show that the weak solution, which is in L^2 initially, satisfies the partial differential equation in some sense we need to consider the space of L^2 functions u for which the left hand side of (4.3.1) also lies in L^2 in the sense of distributions. Furthermore, it will be shown that such functions admit traces on the boundary in certain spaces and this information will help us explain how the weak solution satisfies the boundary and the initial conditions in some sense.

For simplicity, we rewrite the boundary conditions in a single equation. Setting $\Omega = (0,1)$, wefine $u_{|\partial\Omega}(t) = (u(t,0), u(t,1))$, $g = (g_0, g_1)$,

$$B = \begin{pmatrix} B_0 & O_{p\times n} \\ O_{(n-p)\times n} & B_1 \end{pmatrix} \in \mathbb{R}^{n\times 2n}. \tag{4.3.5}$$

Here $O_{k\times j}$ denotes the $k \times j$ zero matrix. The boundary conditions (4.3.2) and (4.3.3) can be written as $Bu_{|\partial\Omega} = g$. Whenever there are matrices defined at the boundaries, we combine them into a single matrix using the same form as (4.3.5).

4.4 GRAPH SPACES AND THEIR TRACES

Let $\mathcal{O}$ be a non-empty open subset of $\mathbb{R}^2$, $A \in W^{1,\infty}(\mathcal{O})$ and $R \in L^\infty(\mathcal{O})$. Consider the linear operator $L : H^1(\mathcal{O}) \to L^2(\mathcal{O})$ defined by

$$Lu = \partial_t u + A\partial_x u + Ru.$$

By duality, we can extend the definition of L for $u \in L^1_{\text{loc}}(\mathcal{O})$ in the sense of distributions. Define $L : L^1_{\text{loc}}(\mathcal{O}) \to \mathscr{D}(\mathcal{O})'$ by

$$Lu(\varphi) = (Lu, \varphi)_{\mathscr{D}(\mathcal{O})'\times\mathscr{D}(\mathcal{O})} = \int_{\mathcal{O}} u \cdot L^*\varphi \,\mathrm{d}x\,\mathrm{d}t, \quad \forall\, \varphi \in \mathscr{D}(\mathcal{O})$$

where L^* denotes the formal adjoint of L given by

$$L^*\varphi = -\,\partial_t\varphi - A^\top\partial_x\varphi - (\partial_x A)^\top\varphi + R^\top\varphi. \tag{4.4.1}$$

By the definition of distributional derivatives, it can be seen that

$$Lu = \partial_t u + \partial_x(Au) - (\partial_x A)u + Ru$$

for all $u \in L^1_{\text{loc}}(\mathcal{O})$ in the sense of distributions.

Given $u \in L^2(\mathcal{O})$, we have

$$|Lu(\varphi)| \le \|u\|_{L^2(\mathcal{O})}\|L^*\varphi\|_{L^2(\mathcal{O})} \le C\|u\|_{L^2(\mathcal{O})}\|\varphi\|_{H^1(\mathcal{O})}$$

for all $\varphi \in \mathscr{D}(\mathcal{O})$ and for some constant $C > 0$ independent of u and φ. Therefore Lu has a unique extension, denoted by the same notation, to a bounded linear functional from $H^1_0(\mathcal{O})$ to $\mathbb{R}$. Furthermore, $\|Lu\|_{H^{-1}(\mathcal{O})} \le C\|u\|_{L^2(\mathcal{O})}$ for all $u \in L^2(\mathcal{O})$, showing that $L \in \mathcal{L}(L^2(\mathcal{O}); H^{-1}(\mathcal{O}))$.

Given $u \in L^2(\mathcal{O})$, suppose that there exists $C > 0$ such that

$$|Lu(\varphi)| \leq C\|\varphi\|_{L^2(\mathcal{O})}, \qquad \forall\, \varphi \in \mathscr{D}(\mathcal{O}). \tag{4.4.2}$$

The density of $\mathscr{D}(\mathcal{O})$ in $L^2(\mathcal{O})$ implies that Lu can be extended to a linear functional in $L^2(\mathcal{O})'$ and denote this extension by $\widetilde{Lu}$ at the moment. It can be seen that $\widetilde{Lu}_{|H_0^1(\mathcal{O})} = Lu$ where L is regarded as a bounded linear operator from $L^2(\mathcal{O})$ to $H^{-1}(\mathcal{O})$. This equality follows immediately from the fact that it holds for all $u \in \mathscr{D}(\mathcal{O})$ and that both operators $\widetilde{Lu}$ and Lu are continuous. For this reason, we simply write Lu for $\widetilde{Lu}$.

From the Riesz Representation Theorem, there exists a unique $f \in L^2(\mathcal{O})$ such that $Lu(\varphi) = (f, \varphi)_{L^2(\mathcal{O})}$ for all $\varphi \in L^2(\mathcal{O})$ whenever (4.4.2) holds. Identifying $L^2(\mathcal{O})$ with its dual, we write $Lu = f$. Thus, $Lu = f$ for some $f \in L^2(\mathcal{O})$, with $u \in L^2(\mathcal{O})$, is equivalent to

$$(u, L^*\varphi)_{L^2(\mathcal{O})} = (f, \varphi)_{L^2(\mathcal{O})}, \qquad \forall\, \varphi \in \mathscr{D}(\mathcal{O}).$$

If $u \in H^1(\mathcal{O})$ then from the definition of weak derivatives it follows that

$$(u, L^*\varphi)_{L^2(\mathcal{O})} = (\partial_t u + A\partial_x u + Ru, \varphi)_{L^2(\mathcal{O})}, \qquad \forall\, \varphi \in \mathscr{D}(\mathcal{O}).$$

Therefore $Lu = \partial_t u + A\partial_x u + Ru$ in the weak sense. Thus, the operator L defined in the sense of distributions and the differential operator $\partial_t + A\partial_x + R$ coincide in $H^1(\mathcal{O})$.

For $\theta \in \mathscr{C}^\infty(\overline{\mathcal{O}}; \mathbb{R})$ the distribution $\theta Lu \in \mathscr{D}(\mathcal{O})'$ is defined by

$$\theta Lu(\varphi) = Lu(\theta\varphi) = (u, L^*(\theta\varphi))_{L^2(\mathcal{O})}, \quad \forall \varphi \in \mathscr{D}(\mathcal{O}).$$

The product rule for smooth functions implies

$$\begin{aligned} (u, L^*(\theta\varphi))_{L^2(\mathcal{O})} &= (u, \theta L^*\varphi - (\partial_t\theta + A^\top \partial_x\theta)\varphi)_{L^2(\mathcal{O})} \\ &= (\theta u, L^*\varphi)_{L^2(\mathcal{O})} - ((\partial_t\theta + (\partial_x\theta)A)u, \varphi)_{L^2(\mathcal{O})}. \end{aligned}$$

Therefore $\theta Lu = L(\theta u) - (\partial_t\theta + (\partial_x\theta)A)u$ in the sense of distributions.

Consider the following subspace of $L^2(\mathcal{O})$

$$E(\mathcal{O}) = \{u \in L^2(\mathcal{O}) : Lu \in L^2(\mathcal{O})\}.$$

Induced by the graph norm

$$\|u\|_{E(\mathcal{O})} = (\|u\|^2_{L^2(\mathcal{O})} + \|Lu\|^2_{L^2(\mathcal{O})})^{\frac{1}{2}}$$

$E(\mathcal{O})$ becomes a Hilbert space, called a *graph space*. Furthermore, the zero order terms of L is immaterial in the definition of $E(\mathcal{O})$, that is,

$$E(\mathcal{O}) = \{u \in L^2(\mathcal{O}) : \partial_t u + \partial_x(Au) \in L^2(\mathcal{O})\}.$$

The space $E(\mathcal{O})$ is closed under multiplication of functions in $\mathscr{C}_b^\infty(\overline{\mathcal{O}}; \mathbb{R})$ and if $u_j \to u$ in $E(\mathcal{O})$ then $\theta u_j \to \theta u$ in $E(\mathcal{O})$ for every $\theta \in \mathscr{C}_b^\infty(\overline{\mathcal{O}}; \mathbb{R})$.

Theorem 4.4.1. *Let $\mathcal{O}_1$ be a nonempty open subset of $\mathcal{O}$. If $u \in E(\mathcal{O})$ then $u_{|\mathcal{O}_1} \in E(\mathcal{O}_1)$ and*

$$L(u_{|\mathcal{O}_1}) = (Lu)_{|\mathcal{O}_1}. \tag{4.4.3}$$

If $(u_j)_j \subset E(\mathcal{O})$ satisfies $u_j \to u$ in $E(\mathcal{O})$ then $u_{j|\mathcal{O}_1} \to u_{|\mathcal{O}_1}$ in $E(\mathcal{O}_1)$.

Proof. It is clear that $u_{|\mathcal{O}_1} \in L^2(\mathcal{O}_1)$. Every $\varphi \in \mathscr{D}(\mathcal{O}_1)$ can be considered as an element of $\mathscr{D}(\mathcal{O})$ by defining φ to be zero outside $\mathcal{O}_1$. With this, we have

$$\begin{aligned}(u_{|\mathcal{O}_1}, L^*\varphi)_{L^2(\mathcal{O}_1)} &= \int_{\mathcal{O}_1} u_{|\mathcal{O}_1} \cdot L^*\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_{\mathcal{O}} u \cdot L^*\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\mathcal{O}} Lu \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t = \int_{\mathcal{O}_1} (Lu)_{|\mathcal{O}_1} \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Thus $L(u_{|\mathcal{O}_1}) = (Lu)_{|\mathcal{O}_1}$ and therefore $u_{|\mathcal{O}_1} \in E(\mathcal{O}_1)$. If $u_j \to u$ in $E(\mathcal{O})$ then $u_{j|\mathcal{O}_1} \to u_{|\mathcal{O}_1}$ in $L^2(\mathcal{O}_1)$ and from (4.4.3)

$$L(u_{j|\mathcal{O}_1}) = (Lu_j)_{|\mathcal{O}_1} \to (Lu)_{|\mathcal{O}_1} = L(u_{|\mathcal{O}_1})$$

in $L^2(\mathcal{O}_1)$. Therefore $u_{j|\mathcal{O}_1} \to u_{|\mathcal{O}_1}$ in $E(\mathcal{O}_1)$. □

The trace operator $\Gamma : H^1(\mathbb{R}\times(0,1)) \to H^{\frac{1}{2}}(\mathbb{R}\times\{0,1\})$ can be extended to $E(\mathbb{R}\times(0,1))$ thanks to Theorem 4.4.2 below. Identifying the elements of $H^{\frac{1}{2}}(\mathbb{R}\times\{0,1\})$ and $H^{\frac{1}{2}}(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$ we sometimes write $\Gamma u = (u_{|x=0}, u_{|x=1})$ for $u \in H^1(\mathbb{R}\times(0,1))$. Before proving the following trace theorem, we need to construct a continuous right inverse of the trace operator. Since the trace operator Γ is onto, it follows that $\Gamma\Gamma^* > 0$ and hence $\Gamma\Gamma^*$ is invertible, see [**77, Proposition 12.1.3**]. Here, $\Gamma^* \in \mathcal{L}(H^{\frac{1}{2}}(\mathbb{R}\times\{0,1\}); H^1(\mathbb{R}\times(0,1)))$ denotes the adjoint of Γ. Define

$$\Gamma_R = \Gamma^*(\Gamma\Gamma^*)^{-1} \in \mathcal{L}(H^{\frac{1}{2}}(\mathbb{R}\times\{0,1\}); H^1(\mathbb{R}\times(0,1))). \tag{4.4.4}$$

By definition, $\Gamma\Gamma_R = \mathrm{id}$, where id is the identity map of $H^{\frac{1}{2}}(\mathbb{R}\times\{0,1\})$. The operator Γ_R is sometimes called a *lifting operator* because functions defined on the boundary are lifted in the domain in a continuous way. An alternative way of proving the existence of lifting operators is presented in Adams [**1**].

Given two pair of functions (f_1, f_2) and (g_1, g_2) we define the componentwise product

$$(f_1, f_2) \otimes (g_1, g_2) = (f_1 g_1, f_2, g_2) \tag{4.4.5}$$

whenever the products $f_1 g_1$ and $f_2 g_2$ are meaningful. This definition will be applied to pairs of traces at $x = 0$ and $x = 1$.

Theorem 4.4.2. *Suppose that $A \in W^{1,\infty}(\mathbb{R}\times(0,1))$ is invertible and constant outside a compact subset of $\mathbb{R}\times(0,1)$ and $R \in L^\infty(\mathbb{R}\times(0,1))$.*

1. *The set $\mathscr{D}(\mathbb{R}\times[0,1])$ is dense in $E(\mathbb{R}\times(0,1))$.*

2. *For each $u \in E(\mathbb{R}\times(0,1))$ define $\Gamma_g u : H^{\frac{1}{2}}(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R}) \to \mathbb{R}$ by*

$$\Gamma_g u(\varphi_0, \varphi_1) = \lim_{j\to\infty} (\Gamma u_j, (\varphi_0, \varphi_1))_{L^2(\mathbb{R})\times L^2(\mathbb{R})} \tag{4.4.6}$$

where $(u_j)_j \subset H^1(\mathbb{R}\times(0,1))$ satisfies $u_j \to u$ in $E(\mathbb{R}\times(0,1))$. Then $\Gamma_g u \in [H^{\frac{1}{2}}(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})]' \simeq H^{-\frac{1}{2}}(\mathbb{R}) \times H^{-\frac{1}{2}}(\mathbb{R})$ and $\Gamma_g \in \mathcal{L}(E(\mathbb{R}\times(0,1)); H^{-\frac{1}{2}}(\mathbb{R}) \times H^{-\frac{1}{2}}(\mathbb{R}))$. Furthermore,

$$\Gamma_g u = \Gamma u, \qquad \forall\, u \in H^1(\mathbb{R}\times(0,1)). \tag{4.4.7}$$

3. If $\theta \in \mathscr{C}_b^\infty(\mathbb{R} \times [0,1];\mathbb{R})$ and $u \in E(\mathbb{R} \times (0,1))$ then $\Gamma_g(\theta u) = \Gamma\theta \otimes \Gamma_g u$ where

$$(\Gamma\theta \otimes \Gamma_g u)(\varphi_0, \varphi_1) := \Gamma_g u(\Gamma\theta \otimes (\varphi_0, \varphi_1)) \tag{4.4.8}$$

for $(\varphi_0, \varphi_1) \in H^{\frac{1}{2}}(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$.

Proof. (1) follows using mollifiers, see [**9, p. 258**]. Let $v \in H^1(\mathbb{R} \times (0,1))$ and $\varphi = (\varphi_0, \varphi_1) \in H^{\frac{1}{2}}(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$. By Green's Formula

$$\begin{aligned}(\Gamma v, \varphi)_{L^2(\mathbb{R})\times L^2(\mathbb{R})} &= \int_{\mathbb{R}} \Gamma v(t,1) \cdot \varphi_1(t)\,\mathrm{d}t - \int_{\mathbb{R}} \Gamma v(t,0) \cdot (-\varphi_0(t))\,\mathrm{d}t \\ = \int_{\mathbb{R}}\int_0^1 A^{-1}Lv \cdot \Gamma_R(-\varphi_0, \varphi_1)\,\mathrm{d}x\,\mathrm{d}t &- \int_{\mathbb{R}}\int_0^1 v \cdot (A^{-1}L)^*\Gamma_R(-\varphi_0, \varphi_1)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

where

$$(A^{-1}L)^*\varphi = -\partial_t(A^{-\top}\varphi) - \partial_x\varphi + R^\top A^{-\top}\varphi. \tag{4.4.9}$$

Thus there exists a constant $C > 0$ independent of v and φ such that

$$|(\Gamma v, \varphi)_{L^2(\mathbb{R})\times L^2(\mathbb{R})}| \leq C\|v\|_{E(\mathbb{R}\times(0,1))}\|\varphi\|_{H^{\frac{1}{2}}(\mathbb{R})\times H^{\frac{1}{2}}(\mathbb{R})}. \tag{4.4.10}$$

Therefore the limit in (4.4.6) exists and from (1) the limit is independent of the approximating sequence $(u_j)_j \subset H^1(\mathbb{R} \times (0,1))$ as long as $u_j \to u \in E(\mathbb{R} \times (0,1))$. From (4.4.6) and (4.4.10) we have $\Gamma_g u \in H^{-\frac{1}{2}}(\mathbb{R}) \times H^{-\frac{1}{2}}(\mathbb{R})$ and $\Gamma_g \in \mathcal{L}(E(\mathbb{R} \times (0,1)); H^{-\frac{1}{2}}(\mathbb{R}) \times H^{-\frac{1}{2}}(\mathbb{R}))$. The equality (4.4.7) follows immediately from the definition of Γ_g and the inclusion $L^2(\mathbb{R}) \times L^2(\mathbb{R}) \subset H^{-\frac{1}{2}}(\mathbb{R}) \times H^{-\frac{1}{2}}(\mathbb{R})$.

If $u \in H^1(\mathbb{R} \times (0,1))$ then (4.4.7) implies

$$\Gamma_g(\theta u) = \Gamma(\theta u) = \Gamma\theta \otimes \Gamma u = \Gamma\theta \otimes \Gamma_g u. \tag{4.4.11}$$

Now (4.4.8) follows from property (1), (4.4.11) and the continuity of Γ_g. □

If $a \in W^{1,\infty}(\mathcal{O})$ and $u \in H^{\frac{1}{2}}(\mathcal{O})$ then $au \in H^{\frac{1}{2}}(\mathcal{O})$. We extend this definition of product for $u \in H^{-\frac{1}{2}}(\mathcal{O})$ by duality.

Definition 4.4.3. Let $u \in H^{-\frac{1}{2}}(\mathcal{O})$ and $a \in W^{1,\infty}(\mathcal{O};\mathbb{R}^{n\times n})$. The product $au \in H^{-\frac{1}{2}}(\mathcal{O})$ is defined by

$$\langle au, \varphi\rangle_{H^{-\frac{1}{2}}(\mathcal{O})\times H^{\frac{1}{2}}(\mathcal{O})} = \langle u, a^\top\varphi\rangle_{H^{-\frac{1}{2}}(\mathcal{O})\times H^{\frac{1}{2}}(\mathcal{O})}, \qquad \varphi \in H^{\frac{1}{2}}(\mathcal{O}).$$

For each $u \in E(\mathbb{R}\times(0,1))$, define the trace operators $\Gamma_g^i u : H^{\frac{1}{2}}(\mathbb{R}) \to \mathbb{R}$ for $i = 0,1$ by

$$\Gamma_g^0 u(\varphi) = \Gamma_g u(\varphi, 0), \qquad \Gamma_g^1 u(\varphi) = \Gamma_g u(0, \varphi), \qquad \varphi \in H^{\frac{1}{2}}(\mathbb{R}).$$

Then $\Gamma_g^i \in \mathcal{L}(E(\mathbb{R} \times (0,1)); H^{-\frac{1}{2}}(\mathbb{R}))$ for $i = 0,1$. If $u \in H^1(\mathbb{R} \times (0,1))$ then $\Gamma_g^i u = \Gamma u_{|x=i}$ for $i = 0,1$. By a standard density argument, we have the generalized Green's identity

$$\begin{aligned}&\int_{\mathbb{R}}\int_0^1 Lu \cdot v\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}}\int_0^1 u \cdot L^*v\,\mathrm{d}x\,\mathrm{d}t \\ &= \langle A_{|x=1}\Gamma_g^1 u, \Gamma v_{|x=1}\rangle_{H^{-\frac{1}{2}}(\mathbb{R})\times H^{\frac{1}{2}}(\mathbb{R})} - \langle A_{|x=0}\Gamma_g^0 u, \Gamma v_{|x=0}\rangle_{H^{-\frac{1}{2}}(\mathbb{R})\times H^{\frac{1}{2}}(\mathbb{R})}\end{aligned} \tag{4.4.12}$$

for all $u \in E(\mathbb{R} \times (0,1))$ and $v \in H^1(\mathbb{R} \times (0,1))$.

We also need traces of functions in $E(Q_T)$ where $Q_T = (0,T) \times (0,1)$ which will be used for initial boundary value problems. This has been done in [**3**] for general Lipschitz domains in [**41**] for general graph spaces. It is shown in [**3**] that $\mathscr{D}(\overline{Q}_T)$ is dense in $E(Q_T)$. This information allows us to extend the trace operator $\Gamma : H^1(Q_T) \to H^{\frac{1}{2}}(\partial Q_T)$ to functions in $E(Q_T)$ as we have done in the case of the graph space $E(\mathbb{R} \times (0,1))$. Given $u \in E(Q_T)$ define $\Gamma_g u : H^{\frac{1}{2}}(\partial Q_T) \to \mathbb{R}$ by

$$\Gamma_g u(\varphi) = \lim_{j\to\infty} (\Gamma u_j, A_\partial \varphi)_{L^2(\partial Q_T)}, \qquad \varphi \in H^{\frac{1}{2}}(Q_T),$$

where

$$A_\partial = -\mathbf{1}_{\{x=0\}} + \mathbf{1}_{\{x=1\}} - A^{-\top}\mathbf{1}_{\{t=0\}} + A^{-\top}\mathbf{1}_{\{t=T\}}, \qquad \text{in } \partial Q_T$$

and $(u_j)_j \subset H^1(Q_T)$ and $u_j \to u$ in $E(Q_T)$. Here, $\mathbf{1}_S$ denotes the indicator function of a set S. As in the previous theorem we have $\Gamma_g u \in H^{-\frac{1}{2}}(\partial Q_T)$ and $\Gamma_g \in \mathcal{L}(E(Q_T); H^{-\frac{1}{2}}(\partial Q_T))$. Moreover, if $u \in H^1(Q)$ then $\Gamma_g u = A_\partial^\top \Gamma u$ and $\Gamma_g(\theta u) = \theta_{|\partial Q_T} \Gamma_g u$ for every $\theta \in \mathscr{C}^\infty(\overline{Q}_T; \mathbb{R})$ and $u \in E(Q_T)$.

The next step is to localize the trace defined in the previous discussion. Given a nonempty $\Sigma \subset \partial Q_T$ we define

$$\mathcal{V}(\Sigma) = \{\varphi \in H^{\frac{1}{2}}(\partial Q_T) : \operatorname{supp} \varphi \subset \Sigma\}. \tag{4.4.13}$$

It is known that $\mathcal{V}(\Sigma)$ is dense in $L^2(\Sigma)$, see [**77, Theorem 13.6.10**]. Denote by $V(\Sigma)$ the completion of $\mathcal{V}(\Sigma)$ with respect to the norm of $H^{\frac{1}{2}}(\partial Q_T)$. Thus we have the Gelfand triple

$$V(\Sigma) \subset L^2(\Sigma) \subset V(\Sigma)'. \tag{4.4.14}$$

If $\varphi \in V(\Sigma)$ then there exists a sequence $(\varphi_j)_j \subset \mathcal{V}(\Sigma)$ such that $\|\varphi_j - \varphi\|_{H^{\frac{1}{2}}(\partial Q_T)} \to 0$. If $a \in W^{1,\infty}(\Sigma)$ then we have $a^\top \varphi_j \in \mathcal{V}(\Sigma)$ and $\|a^\top\varphi_j - a^\top\varphi\|_{H^{\frac{1}{2}}(\partial Q_T)} \to 0$. Hence $a^\top \varphi \in V(\Sigma)$. As a result, we can define the product $au \in V(\Sigma)'$ where $u \in V(\Sigma)'$ and $a \in W^{1,\infty}(\Sigma)$ by

$$\langle au, \varphi\rangle_{V(\Sigma)'\times V(\Sigma)} = \langle u, a^\top\varphi\rangle_{V(\Sigma)'\times V(\Sigma)}, \qquad \varphi \in V(\Sigma). \tag{4.4.15}$$

Let us denote $\Sigma_0 = \{0\} \times (0,1)$, $\Sigma_1 = (0,T) \times \{0\}$, $\Sigma_2 = (0,T) \times \{1\}$ and $\Sigma_3 = \{T\} \times (0,1)$. Given $u \in E(Q_T)$ we define the generalized trace $u_{|\Sigma_1} : V(\Sigma_1) \to \mathbb{R}$ of u on Σ_1 by

$$u_{|\Sigma_1}(\varphi) = -\lim_{j\to\infty} \langle \Gamma_g u, \varphi_j\rangle_{H^{-\frac{1}{2}}(\partial Q_T)\times H^{\frac{1}{2}}(\partial Q_T)}, \qquad \varphi \in V(\Sigma_1), \tag{4.4.16}$$

where $(\varphi_j)_j \subset \mathcal{V}(\Sigma_1)$ and $\|\varphi_j - \varphi\|_{H^{\frac{1}{2}}(\partial Q_T)} \to 0$. By definition, we have

$$|u_{|\Sigma_1}(\varphi)| \le \|\Gamma_g u\|_{H^{-\frac{1}{2}}(\partial Q_T)} \|\varphi\|_{H^{\frac{1}{2}}(\partial Q_T)}.$$

Thus $u_{|\Sigma_1} \in V(\Sigma_1)'$ and $\|u_{|\Sigma_1}\|_{V(\Sigma_1)'} \le \|\Gamma_g u\|_{H^{-\frac{1}{2}}(\partial Q_T)}$. In particular, $u \mapsto u_{|\Sigma_1} \in \mathcal{L}(E(Q_T); V(\Sigma_1)')$ because Γ_g is bounded. It follows from the definition that

$$\langle u_{|\Sigma_1}, \varphi\rangle_{V(\Sigma_1)'\times V(\Sigma_1)} = -\langle \Gamma_g u, \varphi\rangle_{H^{-\frac{1}{2}}(\partial Q_T)\times H^{\frac{1}{2}}(\partial Q_T)} \tag{4.4.17}$$

for all $u \in E(Q_T)$ and $\varphi \in \mathcal{V}(\Sigma_1)$. Also,

$$u_{|\Sigma_1} = (\Gamma u)_{|\Sigma_1}, \qquad \forall\, u \in H^1(Q_T). \tag{4.4.18}$$

The other trace operators are defined as follows

$$\begin{aligned}
\langle u_{|\Sigma_2}, \varphi_2 \rangle_{V(\Sigma_2)' \times V(\Sigma_2)} &= \lim_{j\to\infty} \langle \Gamma_g u, \varphi_{2j} \rangle_{H^{-\frac{1}{2}}(\partial Q_T) \times H^{\frac{1}{2}}(\partial Q_T)} \\
\langle u_{|\Sigma_0}, \varphi_0 \rangle_{V(\Sigma_0)' \times V(\Sigma_0)} &= -\lim_{j\to\infty} \langle \Gamma_g u, A(0,\cdot)^\top \varphi_{0j} \rangle_{H^{-\frac{1}{2}}(\partial Q_T) \times H^{\frac{1}{2}}(\partial Q_T)} \\
\langle u_{|\Sigma_3}, \varphi_3 \rangle_{V(\Sigma_3)' \times V(\Sigma_3)} &= \lim_{j\to\infty} \langle \Gamma_g u, A(T,\cdot)^\top \varphi_{3j} \rangle_{H^{-\frac{1}{2}}(\partial Q_T) \times H^{\frac{1}{2}}(\partial Q_T)}
\end{aligned}$$

where $\varphi_i \in V(\Sigma_i)$, $\varphi_{ij} \in \mathcal{V}(\Sigma_i)$ and $\|\varphi_{ij} - \varphi_i\|_{H^{\frac{1}{2}}(\partial Q_T)} \to 0$ for $i = 0, 2, 3$. The properties of the trace $u_{|\Sigma_1}$ are carried by these traces as well.

Let us simplify the notation for the traces we have introduced in this section. For functions u in $E(\mathbb{R} \times (0,1))$ we shall also use the notations $u_{|\partial\Omega}$, $u_{|x=0}$ and $u_{|x=1}$ for $\Gamma_g u$, $\Gamma_g^0 u$ and $\Gamma_g^1 u$, respectively. If $u \in E(Q_T)$ then similarly we also denote $u_{|x=0}$, $u_{|x=1}$, $u_{|t=0}$ and $u_{|t=T}$ for $u_{|\Sigma_1}$, $u_{|\Sigma_2}$, $u_{|\Sigma_0}$, and $u_{|\Sigma_3}$, respectively. Moreover, setting $\Omega = (0,1)$ we let $u_{|\partial\Omega} = (u_{|x=0}, u_{|x=1})$ for $u \in E(Q_T)$.

4.5 A PRIORI ESTIMATES IN $e^{\gamma t}L^2$ WITH SMOOTH COEFFICIENTS

4.5.1 Functional Boundary Symmetrizers

Consider the first order differential operator

$$L = \partial_t + A\partial_x + R.$$

The goal of this subection is to prove an a priori estimate necessary for well-posedness under the following assumptions on the coefficients A and R.

(H1) $A \in \mathscr{C}_b^\infty(\mathbb{R} \times [0,1]; \mathbb{R}^{n\times n})$ has a bounded inverse and is constant outside a compact set of $\mathbb{R} \times [0,1]$

(H2) $R \in \mathscr{C}_b^\infty(\mathbb{R} \times [0,1]; \mathbb{R}^{n\times n})$

We also assume that the boundary matrices B_0 and B_1 satisfy

(H3) $B_0 \in \mathscr{C}_b^\infty(\mathbb{R}; \mathbb{R}^{p\times n})$ and $B_1 \in \mathscr{C}_b^\infty(\mathbb{R}; \mathbb{R}^{(n-p)\times n})$ are constant outside a compact set of $\mathbb{R}$ and have full ranks

The a priori estimates are derived in the weighted space $L^2(\mathbb{R} \times (0,1); e^{-\gamma t}\, \mathrm{d}t\, \mathrm{d}x)$ where $\gamma \geq 1$ is sufficiently large. For this reason, we also introduce the differential operator $L_\gamma = L + \gamma I_n$, where $\gamma \geq 1$. Let $P^\gamma(x) = -A(t,x)^{-1}\partial_t - \gamma A(t,x)^{-1}$. Then P^γ is a first order partial differential operator in the variable t with parameters $x \in [0,1]$ and $\gamma \geq 1$. From (H1) it can be shown that for all $x \in [0,1]$, $\{P^\gamma(x)\}_{\gamma\geq 1}$ is a family of pseudo-differential operators of order 1 in the variable t and their symbols are $p(t,\delta,\gamma;x) = -(i\delta + \gamma)A(t,x)^{-1}$. Here, δ is the frequency associated with the Fourier variable t.

Definition 4.5.1. A *functional boundary symmetrizer* for (A, B) is a family $\{R^\gamma : \gamma \geq \gamma_0\} \subset \mathscr{C}^1([0,1]; \mathcal{L}(L^2(\mathbb{R})))$, where $\gamma_0 \geq 1$, such that

1. there exists $M > 0$ such that
$$\sup_{\gamma \geq \gamma_0} \|R^\gamma\|_{\mathscr{C}^1([0,1];\mathcal{L}(L^2(\mathbb{R})))} \leq M, \tag{4.5.1}$$
2. $R^\gamma(x)$ is self-adjoint for all $x \in [0,1]$ and $\gamma \geq \gamma_0$,
3. $\Re(R^\gamma(x)P^\gamma(x)) \in \mathcal{L}(L^2(\mathbb{R}))$ and there exists $C > 0$ such
$$\Re(R^\gamma(x)P^\gamma(x)) \geq C\gamma \tag{4.5.2}$$
holds for all $x \in [0,1]$ and $\gamma \geq \gamma_0$, and
4. there exist $\alpha, \beta > 0$ such that
$$-\nu(x)(R^\gamma(x)u, u)_{L^2(\mathbb{R})} \geq \alpha\|u\|^2_{L^2(\mathbb{R})} - \beta\|B_x u\|^2_{L^2(\mathbb{R})} \tag{4.5.3}$$
for all $x = 0, 1$, $t \in \mathbb{R}$ and $u \in L^2(\mathbb{R})$, where $\nu(0) = -1$ and $\nu(1) = 1$.

The condition (4.5.3) allows us to control the trace $u_{|\partial\Omega}$.

Theorem 4.5.2. *If (A, B) has a functional boundary symmetrizer then there exist $\gamma_0 \geq 1$ and $C > 0$ such that for all $\gamma \geq \gamma_0$ and $u \in e^{\gamma t}H^1(\mathbb{R} \times (0,1))$ we have*
$$\begin{aligned} &\gamma \int_{\mathbb{R}} \int_0^1 e^{-2\gamma t}|u(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} e^{-2\gamma t}|u(t)_{|\partial\Omega}|^2 \,\mathrm{d}t \\ &\leq C\left(\frac{1}{\gamma}\int_{\mathbb{R}}\int_0^1 e^{-2\gamma t}|(Lu)(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} e^{-2\gamma t}|Bu(t)_{|\partial\Omega}|^2 \,\mathrm{d}t\right). \end{aligned} \tag{4.5.4}$$

Proof. It is enough to prove the estimate in the case where $R = 0$. Indeed, if (4.5.4) holds for $R = 0$ then by the triangle inequality
$$\begin{aligned} &\gamma \int_{\mathbb{R}} \int_0^1 e^{-2\gamma t}|u(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} e^{-2\gamma t}|u(t)_{|\partial\Omega}|^2 \,\mathrm{d}t \\ &\leq C\bigg(\frac{1}{\gamma}\int_{\mathbb{R}}\int_0^1 e^{-2\gamma t}|(Lu)(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\|R\|^2_{L^\infty}}{\gamma}\int_{\mathbb{R}}\int_0^1 e^{-2\gamma t}|u(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_{\mathbb{R}} e^{-2\gamma t}|Bu(t)_{|\partial\Omega}|^2 \,\mathrm{d}t\bigg). \end{aligned}$$
for every $\gamma \geq \gamma_0$ and $u \in e^{\gamma t}H^1(\mathbb{R} \times (0,1))$. The second term on the right hand side can be absorbed by the first term on the left hand side for sufficiently large γ. For if $\gamma_1 = \sqrt{2C}\|R\|_{L^\infty}$ then for every $\gamma \geq \max(\gamma_0, \gamma_1)$ there exists a $C > 0$ such that (4.5.4) holds for $R \neq 0$.

By a standard density argument, it is enough to prove the estimate (4.5.4) for all $u \in \mathscr{D}(\mathbb{R} \times [0,1])$. Since $R^\gamma(x)$ is self-adjoint
$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathbb{R}} R^\gamma(x)u(t,x) \cdot u(t,x)\,\mathrm{d}t \\ &= \int_{\mathbb{R}} \frac{\mathrm{d}R^\gamma}{\mathrm{d}x}(x)u(t,x) \cdot u(t,x)\,\mathrm{d}t + 2\int_{\mathbb{R}} \Re R^\gamma(x)\partial_x u(t,x) \cdot u(t,x)\,\mathrm{d}t \\ &= \int_{\mathbb{R}} \frac{\mathrm{d}R^\gamma}{\mathrm{d}x}(x)u(t,x) \cdot u(t,x)\,\mathrm{d}t + 2\int_{\mathbb{R}} \Re R^\gamma(x)P^\gamma(x)u(t,x) \cdot u(t,x)\,\mathrm{d}t \\ &\quad + 2\Re\int_{\mathbb{R}} R^\gamma(x)A(t,x)^{-1}L_\gamma u(t,x) \cdot u(t,x)\,\mathrm{d}t \\ &=: I_1(x) + I_2(x) + I_3(x). \end{aligned} \tag{4.5.5}$$

According to (4.5.1) we have

$$|I_1(x)| \le M \int_{\mathbb{R}} |u(t,x)|^2 \,\mathrm{d}t, \qquad \forall\, x \in [0,1]. \tag{4.5.6}$$

From (4.5.2), the term I_2 can be estimated from below

$$I_2(x) \ge 2C\gamma \int_{\mathbb{R}} |u(t,x)|^2 \,\mathrm{d}t, \qquad \forall\, x \in [0,1] \tag{4.5.7}$$

By Young's inequality and (4.5.1) we obtain

$$|I_3(x)| \le C_1\Big(\frac{1}{\epsilon\gamma}\int_{\mathbb{R}} |L_\gamma u(t,x)|^2 \,\mathrm{d}t + \epsilon\gamma \int_{\mathbb{R}} |u(t,x)|^2 \,\mathrm{d}t\Big), \qquad \forall\, x \in [0,1]. \tag{4.5.8}$$

for some $C_1 > 0$ independent of x and u and $\epsilon > 0$.

Therefore from (4.5.5)−(4.5.8)

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathbb{R}} R^\gamma(x)u(t,x)\cdot u(t,x)\,\mathrm{d}t\\ &\ge ((2C - C_1\epsilon)\gamma - M)\int_{\mathbb{R}} |u(t,x)|^2 \,\mathrm{d}t - \frac{C_1}{\epsilon\gamma}\int_{\mathbb{R}} |L_\gamma u(t,x)|^2 \,\mathrm{d}t\end{aligned}$$

Chossing $\epsilon = C/C_1$, integrating over $[0,1]$ and rearranging the terms

$$\begin{aligned}&(C\gamma - M)\int_{\mathbb{R}}\int_0^1 |u(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}} R^\gamma(1)u(t,1)\cdot u(t,1)\,\mathrm{d}t\\ &+\int_{\mathbb{R}} R^\gamma(0)u(t,0)\cdot u(t,0)\,\mathrm{d}t \le \frac{C_1^2}{C\gamma}\int_{\mathbb{R}}\int_0^1 |L_\gamma u(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Using (4.5.3) and choosing $\gamma \ge \max(\gamma_0, 2M/C)$ we can see that

$$\begin{aligned}&\frac{C\gamma}{2}\int_{\mathbb{R}}\int_0^1 |u(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t + \alpha\int_{\mathbb{R}} |u(t)_{|\partial\Omega}|^2 \,\mathrm{d}t\\ &\le \frac{C_1^2}{C\gamma}\int_{\mathbb{R}}\int_0^1 |L_\gamma u(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t + \beta\int_{\mathbb{R}} |Bu(t)_{|\partial\Omega}|^2 \,\mathrm{d}t.\end{aligned} \tag{4.5.9}$$

Replacing u by $e^{-\gamma t}u$, which is also an element of $\mathscr{D}(\mathbb{R}\times[0,1])$ provided that u is, and using $L_\gamma(e^{-\gamma t}u) = e^{-\gamma t}Lu$, the a priori estimate (4.5.4) follows from (4.5.9). □

4.5.2 **Kreiss Symmetrizers**

For boundary value problems with Friedrichs symmetrizer and dissipative boundary conditions, there is a natural functional boundary symmetrizer induced by the Friedrichs symmetrizer. However, there are boundary value problems that do not have dissipative boundary conditions but still admit a functional boundary symmetrizer, for example the system that we are considering here.

In 1970, Kreiss [**45**] introduced a class of symmetrizers for which energy estimates can be also obtained. The author considered the case of constant coefficients and proposed that it also can be done for the variable coefficient case. Later on, it has been verified that this holds [**9, 15, 55**]. In this section, we define the global and local Kreiss symmetrizers and see how global Kreiss symmetrizers induce a functional boundary symmetrizer. Our approach follows from Benzoni-Gavage and Serre [**9**].

Define $\mathbb{C}^+ = \{z \in \mathbb{C} : \Re z \ge 0\}$, $\mathbb{C}^+_* = \mathbb{C}^+ \setminus \{0\}$, $\mathbb{X} = \mathbb{R}\times[0,1]\times\mathbb{C}^+_*$ and $\mathbb{X}_0 = \mathbb{R}\times\{0,1\}\times\mathbb{C}^+_*$. For $X = (t,x,\tau) \in \mathbb{X}$ we let $\mathcal{A}(X) = -\tau A(t,x)^{-1}$.

Definition 4.5.3. A matrix-valued map $\mathcal{R} : \mathscr{C}^\infty(\mathbb{X}; \mathbb{C}^{n\times n})$ is called a *global Kreiss symmetrizer* for (A, B) if $\mathcal{R}(X)$ is Hermitian for all $X \in \mathbb{X}$, $t \mapsto \mathcal{R}(t, x, \tau)$ is constant outside a compact subset of $\mathbb{R}$, $(t, \delta) \mapsto \mathcal{R}(t, x, \gamma + i\delta) \in S^0(\mathbb{R}_t \times \mathbb{R}_\delta)$, there exist constants $\alpha, \beta, C > 0$ such that

$$\Re(\mathcal{R}(X)\mathcal{A}(X)) \geq (C\Re\tau)I_n, \qquad \forall\ X = (t, x, \tau) \in \mathbb{X}, \tag{4.5.10}$$

and

$$-\nu(x)\mathcal{R}(X) + \beta B_x(t)^\top B_x(t) \geq \alpha I_n, \qquad \forall\ X = (t, x, \tau) \in \mathbb{X}_0. \tag{4.5.11}$$

Theorem 4.5.4. *If (A, B) has a global Kreiss symmetrizer then it has a functional boundary symmetrizer.*

Proof. For the sake of completeness, we include a proof of this theorem which basically follows from the one given in [**9**]. By assumption $\mathcal{R}(x) := \mathcal{R}(\cdot, x, \gamma + i\,\cdot) \in S^0(\mathbb{R}_t \times \mathbb{R}_\delta)$ for $\gamma \geq 1$. Therefore $\{\operatorname{Op}^\gamma(\mathcal{R}(x))\}_{\gamma\geq 1}$ is a family of pseudo-differential operators of order 0.

There is no reason for $\operatorname{Op}^\gamma(R(x))$ to be symmetric. For this reason we *symmetrize* it. We claim that

$$x \mapsto R^\gamma(x) := \Re\operatorname{Op}^\gamma(R(x)) = \frac{1}{2}(\operatorname{Op}^\gamma(\mathcal{R}(x)) + \operatorname{Op}^\gamma(\mathcal{R}(x))^*) \tag{4.5.12}$$

defines a functional boundary symmetrizer. The operator $R^\gamma(x)$ is clearly symmetric for every $x \in [0, 1]$ and $\gamma \geq 1$. According to [**2, Exercise 5.3**], there exists $C > 0$ independent of x and γ such that

$$\|\operatorname{Op}^\gamma(\mathcal{R}(x))\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq C \sum_{i,j\in\{0,1\}} \|\partial_t^i\partial_\delta^j\mathcal{R}(x)\|_{L^\infty(\mathbb{R}\times\mathbb{R})} \tag{4.5.13}$$

Therefore, there exists a constant $M_1 > 0$ such that $\|\operatorname{Op}^\gamma(\mathcal{R}(x))\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq M_1$ for every $x \in [0, 1]$ and $\gamma \geq 1$. In particular, $\|R^\gamma(x)\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq M_1$ from (4.5.12).

It can be seen from the dominated convergence theorem that $\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{Op}^\gamma(\mathcal{R}(x)) = \operatorname{Op}^\gamma\left(\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{R}(x)\right)$ and as in (4.5.13) there exists $M_2 > 0$ such that

$$\left\|\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{Op}^\gamma(\mathcal{R}(x))\right\|_{\mathcal{L}(L^2(\mathbb{R}))} = \left\|\operatorname{Op}^\gamma\left(\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{R}(x)\right)\right\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq M_2 \tag{4.5.14}$$

for every $x \in [0, 1]$ and $\gamma \geq 1$. Thus $R^\gamma \in \mathscr{C}^1([0, 1]; \mathcal{L}(L^2(\mathbb{R})))$ satisfies (4.5.1) with $\gamma_0 = 1$ and $M = \max(M_1, M_2)$.

It remains to verify (4.5.2) and (4.5.3), which is possible if we take γ_0 large enough. From Theorem C.2.1 and the fact that $\mathcal{R}(x)$ is Hermitian, there exists a family $\{q(x)\}_{\gamma\geq 1}$ of order -1 symbols such that

$$\operatorname{Op}^\gamma(q(x)) = \operatorname{Op}^\gamma(\mathcal{R}(x)) - \operatorname{Op}^\gamma(\mathcal{R}(x))^* = -2(R^\gamma(x) - \operatorname{Op}^\gamma(\mathcal{R}(x))).$$

Hence $R^\gamma(x) = \operatorname{Op}^\gamma(\mathcal{R}(x) - \frac{1}{2}q(x))$.

Because B is independent of the frequency δ it follows

$$-\nu(x)R^\gamma(x) - \frac{\nu(x)}{2}\operatorname{Op}^\gamma(q(x)) + \beta B_x^\top B_x = \operatorname{Op}^\gamma(-\nu(x)\mathcal{R}(x) + \beta B_x^\top B_x)$$

for $x \in \{0,1\}$. Applying (4.5.11) and Garding's inequality Theorem C.2.2 yield

$$\Re\left(-\nu(x)R^\gamma(x) + \beta B_x^\top B_x - \frac{\nu(x)}{2}\operatorname{Op}^\gamma(q(x))u, u\right)_{L^2(\mathbb{R})} \geq \frac{\alpha}{4}\|u\|^2_{L^2(\mathbb{R})} \tag{4.5.15}$$

for all $u \in L^2(\mathbb{R})$. Since $\{q(x)\}_{\gamma\geq 1}$ is a family of order -1, (C.2.1) implies that

$$|(\operatorname{Op}^\gamma(q(x))u, u)_{L^2(\mathbb{R})}| \leq \|\operatorname{Op}^\gamma(q(x))u\|_{L^2(\mathbb{R})}\|u\|_{L^2(\mathbb{R})} \leq \frac{C}{\gamma}\|u\|^2_{L^2(\mathbb{R})} \tag{4.5.16}$$

for some $C > 0$ independent of γ. Therefore choosing γ_0 large enough so that $\frac{\alpha}{4} - \frac{C}{2\gamma} \geq \frac{\alpha}{8}$, we have from (4.5.15) and (4.5.16) that

$$-\nu(x)(R^\gamma(x)u, u)_{L^2(\mathbb{R})} + \beta\|B_x u\|^2_{L^2(\mathbb{R})} \geq \frac{\alpha}{8}\|u\|^2_{L^2(\mathbb{R})}$$

for all $x \in \{0,1\}$, $\gamma \geq \gamma_0$ and $u \in L^2(\mathbb{R})$, which verifies (4.5.3) in Definition 4.5.1.

It remains to verify that (4.5.2) is satisfied. From Theorem C.2.1 there exists a family $\{s(x)\}_{\gamma\geq 1}$ of order 0 such that

$$R^\gamma(x)P^\gamma(x) = \operatorname{Op}^\gamma(\mathcal{R}(x)\mathcal{A}(x) + s(x)) \tag{4.5.17}$$

Using (4.5.10), (4.5.17) and sharp Garding's inequality Theorem C.2.3, it holds that

$$\Re\langle[R^\gamma(x)P^\gamma(x) - C\gamma I_n - \operatorname{Op}^\gamma(s(x))]u, u\rangle_{H_\gamma^{-\frac{1}{2}}(\mathbb{R})\times H_\gamma^{\frac{1}{2}}(\mathbb{R})} \geq -C_1\|u\|^2_{L^2(\mathbb{R})}. \tag{4.5.18}$$

for some $C_1 > 0$ and for all $u \in H^{\frac{1}{2}}(\mathbb{R})$. Because $s(x)$ is of order 0 it holds that

$$\langle\operatorname{Op}^\gamma(s(x))u, u\rangle_{H_\gamma^{-\frac{1}{2}}(\mathbb{R})\times H_\gamma^{\frac{1}{2}}(\mathbb{R})} = (\operatorname{Op}^\gamma(s(x))u, u)_{L^2(\mathbb{R})} \leq C\|u\|^2_{L^2(\mathbb{R})} \tag{4.5.19}$$

for all $u \in H^{\frac{1}{2}}(\mathbb{R})$ for some constant $C > 0$ independent of x, γ and u. Similarly, since $\Re R^\gamma(x)P^\gamma(x)$ is of order 0, for γ sufficiently large we have from (4.5.18) and (4.5.19) that

$$(\Re R^\gamma(x)P^\gamma(x)u, u)_{L^2(\mathbb{R})} \geq C\gamma\|u\|_{L^2(\mathbb{R})} \tag{4.5.20}$$

for all $u \in H^{\frac{1}{2}}(\mathbb{R})$ and for some $C > 0$ independent of u. Since $H^{\frac{1}{2}}(\mathbb{R})$ is densely embedded in $L^2(\mathbb{R})$ and the operator $\Re R^\gamma(x)P^\gamma(x)$ is bounded in $L^2(\mathbb{R})$ it follows that (4.5.20) also holds for all $u \in L^2(\mathbb{R})$. □

The symmetrizers in Definition 4.5.3 are defined on the whole time-space-frequency set $\mathbb{X}$. In the following we introduce a local version of this symmetrizer. These local symmetrizers can be used as building blocks in obtaining global symmmetrizers, cf. Lemma 4.5.6 below.

Definition 4.5.5. A *local Kreiss symmterizer* for (A, B) at $X \in \mathbb{X}$ is a Hermitian matrix-valued map $r \in \mathscr{C}^\infty(\mathscr{V}(X); \mathbb{C}^{n\times n})$, where $\mathscr{V}(X)$ is some neighborhood of X in $\mathbb{X}$, such that there exists a map $T \in \mathscr{C}^\infty(\mathscr{V}(X), \mathbb{C}^{n\times n})$ satisfying the following conditions

1. $T(Y) \in GL_n(\mathbb{C})$ for all $Y \in \mathscr{V}(X)$
2. there exists $C > 0$ such that for all $Y = (t, x, \tau) \in \mathscr{V}(X)$

$$\Re(r(Y)T(Y)^{-1}\mathcal{A}(Y)T(Y)) \geq C(\Re\tau)I_n \tag{4.5.21}$$

3. if in addition, $Y \in \mathscr{V}(X) \cap \mathbb{X}_0$, there exist $\alpha, \beta > 0$ independent of Y such that

$$-\nu(x) r(Y) + \beta (B_x(t) T(Y))^* B_x(t) T(Y) \geq \alpha I_n. \tag{4.5.22}$$

The inequalities (4.5.21) and (4.5.22) can be viewed as local versions of (4.5.10) and (4.5.11), respectively.

Lemma 4.5.6. *Suppose that A and B satisfy (H1) and (H3), respectively. If (A, B) has a local Kreiss symmetrizer at every point in $\mathbb{X}_1 := \{X = (t, x, \tau) \in \mathbb{X} : |\tau| = 1\}$ then (A, B) has a global Kreiss symmetrizer.*

Proof. Suppose that A is constant in $B_M := \{(t, x) \in \mathbb{R} \times [0, 1] : |t| > M\}$ and B is constant in $\{t \in \mathbb{R} : |t| > M\}$. By homogeneity it is enough to construct the global symmetrizer $\mathcal{R}$ on the compact set $\mathbb{K} := \{(t, x, \tau) \in \mathbb{X} : |t| \leq M, |\tau| = 1\} \subset \mathbb{X}_1$. Indeed, we can define

$$\mathcal{R}(t, x, \tau) = \begin{cases} \mathcal{R}(-M, x, \tau/|\tau|), & \text{if } t < -M, \\ \mathcal{R}(t, x, \tau/|\tau|), & \text{if } |t| \leq M, \\ \mathcal{R}(M, x, \tau/|\tau|), & \text{if } t > M, \end{cases}$$

for $x \in [0, 1]$ and $\tau \in \mathbb{C}^+_*$.

By assumption, for each $X \in \mathbb{K}$ there exists a pair $(r_X, \mathscr{V}(X))$ such that $\mathscr{V}(X)$ is a neighborhood of X in $\mathbb{X}$ and $r_X \in \mathscr{C}^\infty(\mathscr{V}(X); \mathbb{C}^{n\times n})$ is a local Kreiss symmetrizer for (A, B) at X. The collection $\{\mathscr{V}(X) \in \mathbb{K} : X \in \mathbb{K}\}$ forms a covering of $\mathbb{K}$ consisting of open sets in $\mathbb{X}$. By compactness of $\mathbb{K}$, there exists a finite sequence $X_1, \ldots, X_I \in \{X \in \mathbb{K} : x \in (0, 1)\}$ and $X_{I+1}, \ldots, X_{I+J} \in \{X \in \mathbb{K} : x = 0, 1\}$ such that $\{\mathscr{V}(X_i) : 1 \leq i \leq I + J\}$ still covers $\mathbb{K}$. Let $\{\varphi_i : 1 \leq i \leq I + J\}$ denote a partition of unity subordinate to this subcover, i.e., $\varphi_i \in \mathscr{D}(\mathscr{V}(X_i))$, $0 \leq \varphi_i \leq 1$ and $\sum_{i=1}^{I+J} \varphi_i \equiv 1$ on $\mathbb{K}$.

Let T_{X_i} be the invertible matrix-valued map associated with r_{X_i}. Then the map

$$\mathcal{R}(X) = \sum_{i=1}^{I+J} \varphi_i(X) (T_{X_i}(X)^*)^{-1} r_{X_i}(X) T_{X_i}(X)^{-1}, \qquad X \in \mathbb{K}, \tag{4.5.23}$$

after extending it to the whole of $\mathbb{X}$ by homogeneity, is the required global Kreiss symmetrizer for (A, B). See [**9, pp. 231–232**] for details. □

The remaining task is to derive a local Kreiss symmetrizer at every point in $\mathbb{X}_1$. For this, we need the following additional hypothesis on the coefficient matrix A.

(H4) A is smoothly diagonalizable with p positive eigenvalues and $n - p$ negative eigenvalues.

For each $X \in \mathbb{X}$ such that $\Re\tau > 0$ the matrix-valued map $\mathcal{A}(X) = -\tau A(t, x)^{-1}$ is hyperbolic, i.e., its eigenvalues have nonzero real parts. This follows immediately from (H4) and $\sigma(\mathcal{A}(X)) = \{-\tau\lambda^{-1} : \lambda \in \sigma(A(t, x))\}$. Given $X = (t, x, \tau) \in \mathbb{X}$ such that $\Re\tau > 0$, consider the Dunford-Taylor integral

$$P_-(X) = \frac{1}{2\pi i} \int_{\mathcal{C}} (z I_n - \mathcal{A}(X))^{-1} \, \mathrm{d}z$$

where $\mathcal{C}$ is a positively oriented Jordan curve in the left-half of the complex plane enclosing all the eigenvalues of $\mathcal{A}(X)$ with negative real parts. Then $E_-(X) := E^s(\mathcal{A}(X)) = \text{ran } P_-(X)$ and

$$E_+(X) := E^u(\mathcal{A}(X)) = \ker P_-(X) = \text{ran } P_+(X)$$

where $P_+ = I_n - P_-$. The spectral projectors $P_\pm$ are $\mathscr{C}^\infty$ in (t,x) and analytic in τ.

Now, we extend E_- and E_+ up to points in $\mathbb{X}$ where $\Re\tau = 0$. For each $X = (t,x,i\delta) \in \mathbb{X}$ we define

$$P_\pm(t,x,i\delta) = P_\pm(t,x,1+i\delta)$$

By definition, we have the following continuity of $P_\pm$ up to the boundary of $\mathbb{X}$

$$P_\pm(X) = \lim_{\mathbb{X}\ni Y\to X} P_\pm(Y), \qquad \forall\ X \in \mathbb{X}. \tag{4.5.24}$$

Define $E_\pm(t,x,i\delta) = \text{ran } P_\pm(t,x,i\delta)$. Thus $E_\pm(t,x,i\delta) = E_\pm(t,x,\sigma+i\delta)$ for every $\sigma > 0$ and $(t,x,\delta) \in \mathbb{R}\times[0,1]\times(\mathbb{R}\setminus\{0\})$.

4.5.3 UKL and Local Kreiss Symmetrizers

In order to derive local Kreiss Symmetrizers we need an additional assumption on the boundary matrices. The following condition is called the *Uniform Kreiss-Lopatinskiĭ condition*, abbreviated as UKL.

(H5) There exists $C > 0$ such that for all $t \in \mathbb{R}$ we have

$$|V| \le C|B_0(t)V|, \qquad \forall\ V \in E^u(A(t,0)), \tag{4.5.25}$$

and

$$|V| \le C|B_1(t)V|, \qquad \forall\ V \in E^s(A(t,1)). \tag{4.5.26}$$

Let $X \in \mathbb{X}_1$ with $\Re\tau > 0$ and $\mathscr{V}(X)$ be a neighborhood of X in $\mathbb{X}$ such that the spectral projections $P_-(X)$ and $P_+(X)$ of $\mathbb{C}^n$ onto $E_-(X)$ and $E_+(X)$, respectively, are well defined. Denote by $\lambda_1(t,x),\dots,\lambda_p(t,x)$ the positive eigenvalues of $A(t,x)^{-1}$ and by $\lambda_{p+1}(t,x),\dots,\lambda_n(t,x)$ the negative eigenvalues. Let $z_i(t,x)$ be an eigenvector of $A(t,x)^{-1}$ associated with the eigenvalue $\lambda_i(t,x)$.

Writing each z_i as column vectors we denote the change of basis matrix by

$$T_0 = (z_1 \ \cdots \ z_n).$$

Define $T : \mathscr{V}(X) \to \mathbb{C}^{n\times n}$ by $T(Y) = T_0(t,x)$ for all $Y = (t,x,\tau) \in \mathscr{V}(X)$. Then $T \in \mathscr{C}^\infty(\mathscr{V}(X);\mathbb{C}^{n\times n})$ and we have

$$T(Y)^{-1}\mathcal{A}(Y)T(Y) = \begin{pmatrix} -\tau\Sigma^+(t,x) & O_{p\times(n-p)} \\ O_{(n-p)\times p} & -\tau\Sigma^-(t,x) \end{pmatrix} \tag{4.5.27}$$

where $\Sigma^+ = \text{diag}(\lambda_1,\dots,\lambda_p)$ and $\Sigma^- = \text{diag}(\lambda_{p+1},\dots,\lambda_n)$.

Consider the Hermitian matrix-vaued map $r \in \mathscr{C}^\infty(\mathscr{V}(X);\mathbb{C}^{n\times n})$ defined by

$$r(Y) = \begin{pmatrix} -I_p & O_{p\times(n-p)} \\ O_{(n-p)\times p} & \mu I_{n-p} \end{pmatrix}, \qquad Y \in \mathscr{V}(X), \tag{4.5.28}$$

where $\mu \geq 1$. From (4.5.27) and (4.5.28)

$$r(Y)T(Y)^{-1}\mathcal{A}(Y)T(Y) = \begin{pmatrix} \tau\Sigma^+(t,x) & O_{p\times(n-p)} \\ O_{(n-p)\times p} & -\mu\tau\Sigma^-(t,x) \end{pmatrix}, \tag{4.5.29}$$

for every $Y \in \mathscr{V}(X)$. Therefore, there exists $C = C(\mu) > 0$ such that

$$\Re(r(Y)T(Y)^{-1}\mathcal{A}(Y)T(Y)) \geq C(\Re\tau)I_n$$

for all $Y = (t,x,\tau) \in \mathscr{V}(X)$.

Now consider the case where $x = 0$ and $\Re\tau > 0$. Each vector $v \in \mathbb{C}^n$ is decomposed into $v = \binom{v^-}{v^+}$ where $v^- \in \mathbb{C}^p$ consists of the first p entries of v and $v^+ \in \mathbb{C}^{n-p}$ consists of the rest. Since $E_-(Y) = \operatorname{span}\{z_i(t,x) : 1 \leq j \leq p\}$ we have

$$P_-(Y)T(Y)v = \sum_{j=1}^{n} P_-(Y)v_j z_j(t,x) = \sum_{j=1}^{p} v_j z_j(t,x) = T(Y)\begin{pmatrix} v^- \\ 0 \end{pmatrix}.$$

Therefore

$$|T(Y)|^{-1}|P_-(Y)T(Y)v| \leq |v^-| \leq |T(Y)^{-1}||P_-(Y)T(Y)v| \tag{4.5.30}$$

Similarly, using the fact that $E_+(Y) = \operatorname{span}\{z_i(t,x) : p+1 \leq j \leq n\}$ we have

$$|T(Y)|^{-1}|P_+(Y)T(Y)v| \leq |v^+| \leq |T(Y)^{-1}||P_+(Y)T(Y)v|. \tag{4.5.31}$$

By the UKL condition (H5) we have for each $v \in \mathbb{C}^n$ and $Y \in \mathscr{V}(X)$

$$\begin{aligned} |P_-(Y)T(Y)v| &\leq C|B_0(t)P_-(Y)T(Y)v| \\ &\leq C|B_0(t)(I_n - P_+(Y))T(Y)v| \\ &\leq C(|B_0(t)T(Y)v| + |B_0(t)||P_+(Y)T(Y)v|). \end{aligned} \tag{4.5.32}$$

Using (4.5.30)–(4.5.32) we obtain

$$\begin{aligned} r(Y)v\cdot v &= -2|v^-|^2 + \mu|v^+|^2 + |v^-|^2 \\ &\geq -2|T(Y)^{-1}|^2|P_-(Y)T(Y)v|^2 + \mu|T(Y)|^{-2}|P_+(Y)T(Y)v|^2 \\ &\quad + |T(Y)^{-1}|^2|P_-(Y)T(Y)v|^2 \\ &\geq -4C|T(Y)^{-1}|^2(|B_0(t)T(Y)v|^2 + |B_0(t)|^2|P_+(Y)T(Y)v|^2) \\ &\quad + \mu|T(Y)|^{-2}|P_+(Y)T(Y)v|^2 + |T(Y)^{-1}|^2|P_-(Y)T(Y)v|^2 \\ &\geq -C_1|B_0(t)T(Y)v|^2 + C_2|P_-(Y)T(Y)v|^2 \\ &\quad + (C_3\mu - C_4)|P_+(Y)T(Y)v|^2. \end{aligned}$$

Choosing $\mu \geq 1$ large enough, applying the Pythagorean identity and the fact that T is invertible yield

$$r(Y)v\cdot v + \beta|B_0(t)T(Y)v|^2 \geq \alpha|v|^2, \qquad \forall v \in \mathbb{C}^n,\ Y \in \mathscr{V}(X)$$

for some $\alpha, \beta > 0$ independent of v and Y.

In the case where $x = 1$, then the local Kreiss symmetrizer can be chosen to be

$$r(Y) = \begin{pmatrix} -\mu I_p & O_{p\times(n-p)} \\ O_{(n-p)\times p} & I_{n-p} \end{pmatrix}, \qquad Y \in \mathscr{V}(X). \tag{4.5.33}$$

The details are the same as in the case where $x = 0$ and therefore we omit them here. Thus, (4.5.22) holds.

Suppose that $X \in \mathbb{X}_1$ and $\Re\tau = 0$. If $0 < x < 1$ then r can be taken to be the local Kreiss symmetrizer at X. If $x = 0$ then passing to the limit of the projections, see (4.5.24), we still have

$$|P_-(Y)T(Y)v| \le C(|B_0(t)T(Y)v| + |B_0(t)||P_+(Y)T(Y)v|)$$

for all $Y \in \mathscr{V}(X)$. The procedure of constructing local symmetrizers are now the same with the help of the latter inequality. Therefore we have shown the following theorem.

Theorem 4.5.7. *If (H1), (H2), (H4) and (H5) hold, then* (A, B) *has a local Kreiss symmetrizer at every point in* $\mathbb{X}_1$.

Combining Theorem 4.5.2, Theorem 4.5.4, Lemma 4.5.6 and Theorem 4.5.7 we have the following theorem.

Theorem 4.5.8. *Assume that (H1)–(H5) hold. Then the a priori estimate* (4.5.4) *holds for all* $u \in e^{\gamma t}H^1(\mathbb{R} \times (0, 1))$ *and all* $\gamma \ge \gamma_0$ *for some* $\gamma_0 \ge 1$.

4.6 A PRIORI ESTIMATES IN $e^{\gamma t}L^2$ WITH LIPSCHITZ COEFFICIENTS

The a priori estimate (4.5.4) applies to problems with smooth coefficients. In this section, we would like to prove this a priori estimate in the case where A and B are only Lipschitz. More precisely, we suppose that the coefficients are compositions of $\mathscr{C}^\infty$-matrix fields and a function in $W^{1,\infty}$. All throughout this section, we assume the following hypotheses.

(FS) *Friedrichs Symmetrizability.* Let $\mathcal{U} \subset \mathbb{R}^n$ open and convex. The differential operator

$$L_w = \partial_t + A(w)\partial_x$$

is Friedrichs symmetrizable for all $w \in \mathcal{U}$, i.e., there exists a symmetric positive-definite matrix-valued function $S \in \mathscr{C}^\infty(\mathcal{U}; \mathbb{R}^{n\times n})$, called the *Friedrichs symmetrizer*, that is bounded as well as its derivatives, $S(w)A(w)$ is symmetric for all $w \in \mathcal{U}$, and there exists $\alpha > 0$ such that $S(w) \ge \alpha I_n$ for all $w \in \mathcal{U}$.

(D) *Diagonalizability.* It holds that $A \in \mathscr{C}^\infty(\mathcal{U}; \mathbb{R}^{n\times n})$ and for each $w \in \mathcal{U}$, $A(w)$ is diagonalizable with p positive eigenvalues and $n - p$ negative eigenvalues. In particular, $A(w)$ is invertible and has n independent eigenvectors.

(UKL) *Uniform Kreiss-Lopatinskiĭ Condition.* The boundary matrices satisfy $B_0 \in \mathscr{C}^\infty(\mathcal{U}; \mathbb{R}^{p\times n})$, $B_1 \in \mathscr{C}^\infty(\mathcal{U}; \mathbb{R}^{(n-p)\times n})$ are of full rank and there exists $C > 0$ such that for all $w \in \mathcal{U}$

$$|V| \le C|B_0(w)V|, \qquad \text{for all } V \in E^u(A(w))$$

and

$$|V| \le C|B_1(w)V|, \qquad \text{for all } V \in E^s(A(w))$$

where $E^u(A)$ and $E^s(A)$ denote the unstable and stable subspaces of a matrix A, respectively.

Friedrichs symmetrizability is used in deriving pointwise in time estimates. The diagonalizability assumption implies that we are in the non-characteristic case. Finally, the Uniform Kreiss-Lopatinskiĭ Condition tells us what forms of the boundary conditions are appropriate.

Let $\mathbb{X} = \mathcal{U} \times \mathbb{C}_*^+$ and $\mathbb{X}_1 = \{(w,\tau) \in \mathbb{X} : |\tau| = 1\}$. In nonlinear analysis we also need to consider the range of the frozen coefficient and how it is involved in the a priori estimate. For this reason we introduce the following set. For each compact subset $\mathcal{K}$ of $\mathcal{U}$ and for each $K > 0$ let

$$\mathbb{W}(K,\mathcal{K}) := \{v \in W^{1,\infty}(\mathbb{R} \times (0,1)) : \operatorname{ran} v \subset \mathcal{K},\ \|v\|_{W^{1,\infty}} \leq K\}.$$

By replacing the (pseudo)-differential operator $P^\gamma(x) = \mathrm{Op}^\gamma(\mathcal{A}(X))$ in Definition 4.5.1 by its paradifferential version, we can similarly define a functional boundary symmetrizer for coefficients with limited regularity. Let $A_v = A(v)$ and $B_v = B(v)$.

Definition 4.6.1. Let $v \in \mathbb{W}(K,\mathcal{K})$. A *functional boundary symmetrizer* for (A_v, B_v) is a two-parameter family of self-adjoint operators $\{R_v^\gamma(x) : \gamma \geq \gamma_0,\ x \in [0,1]\}$, where $\gamma_0 \geq 1$, such that

1. $R_v^\gamma \in W^{1,\infty}([0,1]; \mathcal{L}(L^2(\mathbb{R})))$ is uniformly bounded in $\gamma \geq \gamma_0$,
2. there exists $C > 0$ such that for all $x \in [0,1]$ and $\gamma \geq \gamma_0$,
$$\Re(R_v^\gamma(x) T^{\chi,\gamma}_{\mathcal{A}_v(x)}) \geq C\gamma \tag{4.6.1}$$
where $\mathcal{A}_v(x) = -(\gamma + i\delta)A(v(\cdot,x))^{-1}$, $\delta \in \mathbb{R}$, and $T^{\chi,\gamma}_{\mathcal{A}_v(x)}$ is the paradifferential operator with parameters $x \in [0,1]$ and γ associated to the symbol $\mathcal{A}_v(x) \in \Gamma^1_1$ and an admissible frequency cut-off function χ,
3. and there exist $\alpha, \beta > 0$ such that
$$-\nu(x)\langle R_v^\gamma(x)u, u\rangle_{L^2(\mathbb{R})} + \beta\|T^{\chi,\gamma}_{B_v(x)}u\|^2_{L^2(\mathbb{R})} \geq \alpha\|u\|^2_{L^2(\mathbb{R})} \tag{4.6.2}$$
for $x \in \{0,1\}$ and $u \in L^2(\mathbb{R})^n$, where $\nu(0) = -1$ and $\nu(1) = 1$.

We note that the constants α, β and C appearing in Definition 4.6.1 may depend only on K and $\mathcal{K}$ but are independent of $v \in \mathbb{W}(K,\mathcal{K})$. As in the smooth case, a functional boundary symmetrizer induces an a priori estimate in a weighted Lebesgue space.

Theorem 4.6.2. *Suppose that (A_v, B_v) has a functional boundary symmetrizer. Let $v \in \mathbb{W}(K,\mathcal{K})$. There exist $C = C(K,\mathcal{K}) > 0$ and $\gamma_0 = \gamma_0(K,\mathcal{K}) \geq 1$ such that for every $u \in \mathscr{D}(\mathbb{R} \times [0,1])$ and $\gamma \geq \gamma_0$ we have*

$$\begin{aligned}&\gamma\|u\|^2_{L^2(\mathbb{R}\times(0,1))} + \|u_{|\partial\Omega}\|^2_{L^2(\mathbb{R})}\\ &\leq C\left(\frac{1}{\gamma}\|\partial_x u - T^{\chi,\gamma}_{A_v}u\|^2_{L^2(\mathbb{R}\times(0,1))} + \|T^{\chi,\gamma}_{B_v}u_{|\partial\Omega}\|^2_{L^2(\mathbb{R})}\right).\end{aligned} \tag{4.6.3}$$

Proof. With the aid of the equality

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathbb{R}} R_v^\gamma(x)u(x)\cdot u(x)\,\mathrm{d}x\\ &= \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}x}R_v^\gamma(x)u(x)\cdot u(x)\,\mathrm{d}x + 2\int_{\mathbb{R}} \Re R_v^\gamma(x) T^{\chi,\gamma}_{\mathcal{A}_v(x)}u(x)\cdot u(x)\,\mathrm{d}x\\ &\quad + 2\int_{\mathbb{R}} \Re R_v^\gamma(x)(\partial_x u(x) - T^{\chi,\gamma}_{\mathcal{A}_v(x)}u(x))\cdot u(x)\,\mathrm{d}x\end{aligned}$$

the proof uses the same method as in Theorem 4.5.2 but using Definition 4.6.1 instead of Definition 4.5.1. □

The following tells us that in order to prove (4.6.4), it is enough to replace $P_v^\gamma = -A(v)^{-1}(\gamma I_n + \partial_t)$ and B_v by their paradifferential version.

Corollary 4.6.3. *In the situation of Theorem 4.6.2, suppose in addition that $R \in L^\infty(\mathbb{R}\times(0,1);\mathbb{R}^{n\times n})$ satisfies $\|R\|_{L^\infty} \leq \varrho$. Then there are constants $C = C(\varrho, K, \mathcal{K}) > 0$ and $\gamma_0 = \gamma_0(\varrho, K, \mathcal{K}) \geq 1$ such that the a priori estimate*

$$\begin{aligned}
&\gamma\|e^{-\gamma t}u\|^2_{L^2(\mathbb{R}\times(0,1))} + \|e^{-\gamma t}u_{|\partial\Omega}\|^2_{L^2(\mathbb{R})}\\
&\leq C\left(\frac{1}{\gamma}\|e^{-\gamma t}L_v u\|^2_{L^2(\mathbb{R}\times(0,1))} + \|e^{-\gamma t}B_v u_{|\partial\Omega}\|^2_{L^2(\mathbb{R})}\right)
\end{aligned}\tag{4.6.4}$$

holds for every $u \in e^{\gamma t}H^1(\mathbb{R}\times(0,1))$ and $\gamma \geq \gamma_0$.

Proof. Using a usual absorption argument, we can assume without loss of generality that $R = 0$, see the proof of Theorem 4.6.2. Note that from (C.3.4) we have

$$T^{\chi,\gamma}_{\mathcal{A}_v(x)} = T^{\chi,\gamma}_{-(\gamma+i\delta)A_v(x)^{-1}} = -\gamma T^{\chi,\gamma}_{A_v(x)^{-1}} - T^{\chi,\gamma}_{A_v(x)^{-1}}\partial_t.$$

Thus, for each $x \in (0,1)$ we have according to [Theorem C.20, GS]

$$\begin{aligned}
\|P^\gamma_{v(x)}u(x) - T^{\chi,\gamma}_{\mathcal{A}_v(x)}u(x)\|_{L^2(\mathbb{R})} &\leq \gamma\|A_v(x)^{-1}u(x) - T^{\chi,\gamma}_{A_v(x)^{-1}}u(x)\|_{L^2(\mathbb{R})}\\
&\quad + \|A_v(x)^{-1}\partial_t u(x) - T^{\chi,\gamma}_{A_v(x)^{-1}}\partial_t u(x)\|_{L^2(\mathbb{R})}\\
&\leq C\|A_v\|_{L^\infty}\|u(x)\|_{L^2(\mathbb{R})}
\end{aligned}\tag{4.6.5}$$

for all $u \in \mathscr{D}(\mathbb{R}\times(0,1))$. Upon squaring both sides of (4.6.5) and integrating over $x \in (0,1)$ we see that

$$\|P^\gamma_v u - T^{\chi,\gamma}_{\mathcal{A}_v}u\|^2_{L^2(\mathbb{R}\times(0,1))} \leq C\|u\|^2_{L^2(\mathbb{R}\times(0,1))}. \tag{4.6.6}$$

for some $C = C(K,\mathcal{K}) > 0$. Similarly, from Theorem C.3.3 there exists $C = C(K,\mathcal{K}) > 0$ with

$$\|B_v u_{|\partial\Omega} - T^{\chi,\gamma}_{B_v}u_{|\partial\Omega}\|^2_{L^2(\mathbb{R})} \leq \frac{C}{\gamma}\|u_{|\partial\Omega}\|^2_{L^2(\mathbb{R})}. \tag{4.6.7}$$

By the triangle inequality, (4.6.6) and (4.6.7) we have

$$\begin{aligned}
&\frac{1}{\gamma}\|\partial_x u - T^{\chi,\gamma}_{\mathcal{A}_v}u\|^2_{L^2(\mathbb{R}\times(0,1))} + \|T^{\chi,\gamma}_{B_v}u_{|\partial\Omega}\|^2_{L^2(\mathbb{R})}\\
&\leq C\Bigg(\frac{1}{\gamma}\|\partial_x u - P^\gamma_v u\|^2_{L^2(\mathbb{R}\times(0,1))} + \frac{1}{\gamma}\|u\|^2_{L^2(\mathbb{R}\times(0,1))}\\
&\qquad + \|B_v u_{|\partial\Omega}\|^2_{L^2(\mathbb{R})} + \frac{1}{\gamma}\|u_{|\partial\Omega}\|^2_{L^2(\mathbb{R})}\Bigg)
\end{aligned}\tag{4.6.8}$$

From (4.6.3), (4.6.8) and $\partial_x - P^\gamma_v = A_v^{-1}L^\gamma_v$, there exist constants $\gamma_0 = \gamma_0(K,\mathcal{K}) \geq 1$ and $C = C(K,\mathcal{K}) > 0$ such that if $\gamma \geq \gamma_0$ then

$$\gamma\|u\|^2_{L^2(\mathbb{R}\times(0,1))} + \|u_{|\partial\Omega}\|^2_{L^2(\mathbb{R})} \leq C\left(\frac{1}{\gamma}\|L^\gamma_v u\|^2_{L^2(\mathbb{R}\times(0,1))} + \|B_v u_{|\partial\Omega}\|^2_{L^2(\mathbb{R})}\right)$$

for all $u \in \mathscr{D}(\mathbb{R}\times(0,1))$. Replacing u by $e^{-\gamma t}u$ and using the density of $\mathscr{D}(\mathbb{R}\times(0,1))$ in $e^{\gamma t}H^1(\mathbb{R}\times(0,1))$ we obtain (4.6.4). □

For the existence of functional boundary symmetrizers for (A_v, B_v), sufficient conditions are the smooth diagonalizability of A and the uniform Kreiss-Lopatinskiĭ condition. As in the case of smooth coefficients, the functional boundary symmetrizers can be constructed from Kreiss symmetrizers, and these can be obtained first locally and then globally after homogeneity and compactness arguments. As before we introduce the following local symmetrizers.

Definition 4.6.4. Let $A \in \mathscr{C}^\infty(\mathcal{U}; \mathbb{R}^{n\times n})$, $B_0 \in \mathscr{C}^\infty(\mathcal{U}; \mathbb{R}^{p\times n})$, $B_1 \in \mathscr{C}^\infty(\mathcal{U}; \mathbb{R}^{(n-p)\times n})$ and $v \in \mathbb{W}(K, \mathcal{K})$. A *local Kreiss symmetrizer* for (A_v, B_v) at $\underline{X} = (\underline{t}, \underline{x}, \underline{\tau}) \in \mathbb{R} \times [0,1] \times \mathbb{C}^+_*$ is a Hermitian matrix-valued function $r \in \mathscr{C}^\infty(\tilde{\mathcal{U}} \times \mathcal{O}; \mathbb{C}^{n\times n})$, where $\tilde{\mathcal{U}} \times \mathcal{O}$ is open in $\mathcal{U} \times \mathbb{C}^+_*$ and $v(\mathcal{V}(\underline{t}, \underline{x})) \subset \tilde{\mathcal{U}}$ for some neighbourhood $\mathcal{V}(\underline{t}, \underline{x})$ of $(\underline{t}, \underline{x})$ in $\mathbb{R} \times [0,1]$, such that there exists an invertible matrix-valued function $T \in \mathscr{C}^\infty(\tilde{\mathcal{U}} \times \mathcal{O}; GL(n, \mathbb{C}))$ with the following properties

(a) there exists $C > 0$ such that

$$\Re(r(X)T(X)^{-1}\mathcal{A}(X)T(X)) \geq (C\Re\tau)I_n, \tag{4.6.9}$$

where $\mathcal{A}(X) = -\tau A(v(t,x))^{-1}$, for all $X = (v(t,x), \tau)$ with $(t,x,\tau) \in \mathcal{V}(\underline{t}, \underline{x}) \times \mathcal{O}$

(b) and if in addition, $\underline{X} \in \mathbb{R} \times \{0,1\} \times \mathbb{C}^+_*$, then there exist $\alpha, \beta > 0$ such that for all $(t,x,\tau) \in \mathcal{V}(\underline{t}, \underline{x}) \times \mathcal{O}$ we have

$$-\nu(x)r(X) + \beta T(X)^* B_v(x)^\top B_v(x) T(X) \geq \alpha I_n \tag{4.6.10}$$

where $X = (v(t,x), \tau)$.

Theorem 4.6.5. *Suppose that (D) and (UKL) hold and let $v \in \mathbb{W}(K, \mathcal{K})$. Then (A_v, B_v) has a local Kreiss symmetrizer at every point in $\mathbb{X}_M := [-M, M] \times [0,1] \times \{\tau \in \mathbb{C}^+ : |\tau| = 1\}$ for every $M > 0$.*

Proof. The construction is the same as in Subsections 4.5.2 and 4.5.3. For the sake of completeness we provide the main ideas. We start with the case where $\Re\tau > 0$. The matrix $\mathcal{A}(w,\tau) = -\tau A(w)^{-1}$ is hyperbolic for all $w \in \mathcal{U}$. Indeed, we have

$$E_-(w,\tau) := E^s(\mathcal{A}(w,\tau)) = E^u(A(w)), \quad E_+(w,\tau) := E^u(\mathcal{A}(w,\tau)) = E^s(A(w)).$$

These show that $E_-(w,\tau)$ and $E_+(w,\tau)$ are independent of τ as long as $\Re\tau > 0$.

Let $\underline{X} = (\underline{t}, \underline{x}, \underline{\tau}) \in \mathbb{X}_M$ be such that $\Re\underline{\tau} > 0$ and $\tilde{\mathcal{U}} \times \mathcal{O}$ be an open set in $\mathcal{U} \times \mathbb{C}^+_*$ containing $(v(\underline{t}, \underline{x}), \underline{\tau})$, where $\tilde{\mathcal{U}}$ and $\mathcal{O}$ are open sets in $\mathcal{U}$ and $\mathbb{C}^+_* \cap \{\Re\tau > 0\}$, respectively. By continuity of v, there exists an open set $\mathcal{V}(\underline{t}, \underline{x})$ in $\mathbb{R} \times [0,1]$ such that $v(\mathcal{V}(\underline{t}, \underline{x})) \subset \tilde{\mathcal{U}}$. For each $w \in \tilde{\mathcal{U}}$ we let $T_0(w) \in \mathscr{C}^\infty(\mathcal{U}; GL(n, \mathbb{C}))$ be the matrix consisting of the eigenvectors of $A(t,x)^{-1}$, arranged in such a way that the first p columns correspond to the p positive eigenvalues and the rest correspond to the $n-p$ negative eigenvalues. Then $A(w)^{-1}$ can be diagonalized as

$$T_0(w)^{-1}A(w)^{-1}T_0(w) = \begin{pmatrix} \Sigma^+(w) & O_{p\times(n-p)} \\ O_{(n-p)\times p} & \Sigma^-(w) \end{pmatrix} \tag{4.6.11}$$

where $\Sigma^+(w) = \operatorname{diag}(\lambda_1(w), \ldots, \lambda_p(w))$ and $\Sigma^-(w) = \operatorname{diag}(\lambda_{p+1}(w), \ldots, \lambda_n(w))$ are the diagonal matrices with the positive eigenvalues and negative eigenvalues of

$A(w)^{-1}$ as entries, respectively. Define $T(w,\tau) = T_0(w)$ for all $(w,\tau) \in \tilde{\mathcal{U}} \times \mathcal{O}$. Then we have

$$T(w,\tau)^{-1}\mathcal{A}(w,\tau)T(w,\tau) = \begin{pmatrix} -\tau\Sigma^+(w) & O_{p\times(n-p)} \\ O_{(n-p)\times p} & -\tau\Sigma^-(w) \end{pmatrix} \tag{4.6.12}$$

Suppose $0 < \underline{x} < 1$. Then the Hermitian matrix

$$r(w,\tau) = \begin{pmatrix} -I_p & O_{p\times(n-p)} \\ O_{(n-p)\times p} & \mu I_{n-p} \end{pmatrix} \tag{4.6.13}$$

can be chosen to be a local Kreiss symmetrizer at $\underline{X}$ for any $\mu \geq 1$ and T defined above is the associated invertible-matrix valued function.

If $\underline{x} = 0$ the same form of $r(w,\tau)$ given by (4.6.13) is possible for sufficiently large μ. This is the place where one requires the Kreiss-Lopantiskiĭ condition. Reducing $\tilde{\mathcal{U}}$ if necessary, we can assume without loss of generality that the spectral projections $P_-(w,\tau)$ and $P_+(w,\tau)$ onto $E_-(w,\tau)$ and $E_+(\tau,w)$, respectively, are well-defined. These projections can be written as Dunford-Taylor integrals and by a classical argument in Kato they can be chosen so that they are $\mathscr{C}^\infty$ in w and analytic in τ. Since $E_-(w,\tau)$ and $E_+(w,\tau)$ are independent of τ then $P_-(w,\tau)$ and $P_+(w,\tau)$ are also independent of τ. By (UKL), for all $V \in \mathbb{C}^n$ and $(w,\tau) \in \tilde{\mathcal{U}} \times \mathcal{O}$ we have

$$\begin{aligned} \|P_-(w,\tau)V\| &\leq C\|B_0P_-(w,\tau)V\| = C\|B_0(V - P_+(w,\tau)V)\| \\ &\leq C_1(\|B_0V\| + \|P_+(w,\tau)V\|). \end{aligned} \tag{4.6.14}$$

With this estimate it can be shown, as in Subection 4.5.3, that for sufficiently large μ, r given by (4.6.13) is a local Kreiss symmetrizer at $\underline{X}$. If $\underline{x} = 1$ then analogously, one can choose

$$r(w,\tau) = \begin{pmatrix} -\mu I_p & O_{p\times(n-p)} \\ O_{(n-p)\times p} & I_{n-p} \end{pmatrix} \tag{4.6.15}$$

where μ is again sufficiently large.

The next step is to construct symmetrizers at points with $\Re\tau = 0$ of the frequency set $\mathbb{C}^+_* \cap \{|\tau| = 1\} = \{\pm i\}$. However, for nonzero real number δ, $E_-(w, i\delta)$ is not the stable subspace of $\mathcal{A}(w,i\delta)$ anymore. Note that $E_-(w,i\delta)$ is the zero subspace. Instead, we extend the definition of $E_-(w,\tau)$ by continuity, or equivalently, the definition of the spectral projections $P_-(w,\tau)$. For each $(w,\delta) \in \mathcal{U} \times (\mathbb{R} \setminus \{0\})$ we define

$$P_\pm(w,i\delta) = P_\pm(w,\sigma + i\delta)$$

where $\sigma > 0$. This definition of $P_\pm$ is independent on σ as long as it is a positive real number. Moreover, one immediately have the continuity of the projections up to the boundary of the frequency set

$$\lim_{\mathbb{X}\ni(z,\tau)\to(w,i\delta)} P_\pm(z,\tau) = P_\pm(w,i\delta).$$

We define $E_\pm(w,\tau) := \operatorname{ran} P_\pm(w,\tau)$, for $\Re\tau = 0$.

Suppose that $\underline{X} = (\underline{t}, \underline{x}, \underline{\tau}) \in \mathbb{X}_M$ where $\Re\underline{\tau} = 0$. The neighborhoods $\tilde{\mathcal{U}}, \mathcal{O}$, and $\mathcal{V}$ along with matrices r and T are the same as in the construction above. If $0 < \underline{x} < 1$ then we choose r as in (4.6.13). If $\underline{x} = 0$, by passing to the limit of projections in (4.6.13) we still have the estimate

$$\|P_-(w,\tau)V\| \leq C\|B_0V\| + \|P_+(w,\tau)V\|$$

for all $V \in \mathbb{C}^n$ and $(w,\tau) \in \tilde{\mathcal{U}} \times \mathcal{O}$. Once we have this estimate we can proceed in exactly the same manner as before. The case $\underline{x} = 1$ is analogous. $\square$

We are now in position to state and prove the main theorem of this section.

Theorem 4.6.6. *Assume that (D) and (UKL) hold. Let* $v \in \mathbb{W}(K,\mathcal{K})$ *and* $R \in L^\infty(\mathbb{R} \times (0,1); \mathbb{R}^{n\times n})$ *be such that* $\|R\|_{L^\infty} \leq \varrho$. *Then* (A_v, B_v) *has a functional boundary symmetrizer and hence the a priori estimate* (4.6.4) *holds for every* $u \in e^{\gamma t}H^1(\mathbb{R} \times (0,1))$.

Proof. Fix $M > 0$ sufficiently large. Given $(t,x,\tau) \in \mathbb{X}_M$ let r_v be the local Kreiss symmetrizer at (t,x,τ) and as in Lemma 4.5.6 we construct a global symmetrizer $\mathcal{R}_v$ which is homogeneous degree 0 in τ. With the construction provided by the partition of unity, see (4.5.23), we have

$$\mathcal{R}_v(t,x,\tau) = \sum_j P_j(X)^* r_j(X) P_j(X) \tag{4.6.16}$$

with $X = (v(t,x),\tau)$ and $P_j(X) = \varphi_j(X)^{\frac{1}{2}} T_j(X)^{-1}$. From the construction this sum is finite. It can be shown that $\mathcal{S} = \sum_j P_j^* P_j$ is uniformly bounded from below.

The matrix-valued function

$$\mathcal{R}_v(x) : (t,\tau) = (t, \gamma + i\delta) \mapsto \mathcal{R}_v(t,x,\gamma+i\delta)$$

for all $x \in (0,1)$ satisfies $\mathcal{R}_v(x) \in \Gamma_1^0(\mathbb{R}_t \times \mathbb{R}_\delta)$ with parameter $\gamma \geq 1$ since $v \in W^{1,\infty}(\mathbb{R}\times\Omega)$ and $\mathcal{R}_v$ is homogeneous degree 0 in τ. As in [**9, pp. 231–232**], the local estimates (4.5.21) and (4.5.22) can be extended to a global estimate in the sense that there are some constants $\alpha, \beta, C > 0$ depending only on $(\varrho, K, \mathcal{K})$ such that

$$-\nu(x)\mathcal{R}_v(t,x,\tau) + \beta B_x(v(t,x))^\top B_x(v(t,x)) \geq \alpha I_n, \tag{4.6.17}$$

for every $(t,x) = \mathbb{R} \times \{0,1\}$ and

$$\Re(\mathcal{R}_v(t,x,\tau)\mathcal{A}_v(t,x,\tau)) \geq (C\Re\tau) I_n, \tag{4.6.18}$$

for every for $(t,x) = \mathbb{R}\times(0,1)$. It follows that for each $x \in (0,1)$, $\{T^{\chi,\gamma}_{\mathcal{R}_v(x)}\}_{\gamma\geq 1}$ is a family of paradifferential operators of order 0, and their operator norm in $\mathcal{L}(L^2(\mathbb{R}))$ is uniform in $\gamma \geq \gamma_0$ and as well in $x \in [0,1]$ since their symbols are Lipschitz in the parameter x, see [**17, Theorem 4.4**] and [**54, Chapter 5**].

Let us construct the functional boundary symmetrizer. The symmetrizer is the paradifferential version of the one constructed in Theorem 4.5.4, cf. (4.5.12). Consider the operator

$$R_v^\gamma(x) := \frac{1}{2}(T^{\chi,\gamma}_{\mathcal{R}_v(x)} + (T^{\chi,\gamma}_{\mathcal{R}_v(x)})^*).$$

It follows that $R_v^\gamma(x)$ is a self-adjoint bounded operator in $\mathcal{L}(L^2(\mathbb{R}))$. As in the proof of Theorem 4.5.4, there exists $M_1 = M_1(K,\mathcal{K}) > 0$ such that

$$\sup_{\gamma\geq 1} \|T^{\chi,\gamma}_{\mathcal{R}_v}\|_{W^{1,\infty}((0,1);\mathcal{L}(L^2(\mathbb{R})))} \leq M_1, \tag{4.6.19}$$

see [**17, Theorem 4.4**] and [**54, Chapter 5**].

From Theorem C.3.4, $\{R_v^\gamma(x) - T^{\chi,\gamma}_{\mathcal{R}_v(x)}\}_{\gamma\geq 1}$ is a family of paradifferential operators of order -1. According to (C.2.1), for every $u \in L^2(\mathbb{R})$ we have

$$\|R_v^\gamma(x)u - T^{\chi,\gamma}_{\mathcal{R}_v(x)}u\|_{L^2(\mathbb{R})} \leq \frac{C}{\gamma}\|u\|_{L^2(\mathbb{R})} \tag{4.6.20}$$

for some $C > 0$ independent of u, x and γ. If

$$Q(x) = -\nu(x)\mathcal{R}_v(t,x,\tau) + \beta B_x(v(t,x))^\top B_x(v(t,x))$$

then from (4.6.17) we have $Q(x) + Q(x)^* \geq 2\alpha I_n$ for $x \in \{0,1\}$, where we used the fact that $\mathcal{R}_v^* = \mathcal{R}_v$. Also, we have

$$T^{\chi,\gamma}_{Q(x)} = -\nu(x)T^{\chi,\gamma}_{\mathcal{R}_v(x)} + \beta(T^{\chi,\gamma}_{B_v(x)})^* T^{\chi,\gamma}_{B_v(x)} + \mathcal{Q}^\gamma(x), \quad x \in \{0,1\}, \tag{4.6.21}$$

where $\{\mathcal{Q}^\gamma(x)\}_{\gamma\geq 1}$ is a family of operators of order -1. By Garding's inequality Theorem C.3.5 and a standard absorption argument

$$\Re(-\nu(x)T^{\chi,\gamma}_{\mathcal{R}_v(x)}u,u)_{L^2(\mathbb{R})} + \beta\|T^{\chi,\gamma}_{B_v(x)}u\|^2_{L^2(\mathbb{R})} \geq \frac{\alpha}{4}\|u\|^2_{L^2(\mathbb{R})} \tag{4.6.22}$$

for γ large enough. Using (4.6.20), (4.6.22) and the fact that $R_v^\gamma(x)$ is self-adjoint we obtain

$$\begin{aligned}
&(-\nu(x)R_v^\gamma(x)u,u)_{L^2(\mathbb{R})} + \beta\|T^{\chi,\gamma}_{B_v(x)}u\|^2_{L^2(\mathbb{R})}\\
&= -\nu(x)\{\Re((R_v^\gamma(x) - T^{\chi,\gamma}_{\mathcal{R}_v(x)})u,u)_{L^2(\mathbb{R})} + \Re(T^{\chi,\gamma}_{\mathcal{R}_v(x)}u,u)_{L^2(\mathbb{R})}\} + \beta\|T^{\chi,\gamma}_{B_v(x)}u\|^2_{L^2(\mathbb{R})}\\
&\geq \left(\frac{\alpha}{4} - \frac{C}{\gamma}\right)\|u\|^2_{L^2(\mathbb{R})} \;\geq\; \frac{\alpha}{8}\|u\|^2_{L^2(\mathbb{R})}
\end{aligned}$$

for $x \in \{0,1\}$ and γ large enough. Therefore for $x \in \{0,1\}$ and $\gamma \geq \gamma_0$, where γ_0 is large enough

$$(-\nu(x)R_v^\gamma(x)u,u)_{L^2(\mathbb{R})} + \beta\|T^{\chi,\gamma}_{B_v(x)}u\|^2_{L^2(\mathbb{R})} \geq \frac{\alpha}{8}\|u\|^2_{L^2(\mathbb{R})}.$$

This proves (4.6.2) after renaming the constant α.

It remains to prove (4.6.1). From (4.6.16) and the form of the local symmetrizers (4.6.13) and (4.6.15) we have

$$\begin{aligned}
&\mathcal{R}_v(x,t,\gamma)\mathcal{A}_v(t,x,\tau)\\
&= \sum_j \varphi_j(X)^{\frac{1}{2}}(T_j(X)^{-1})^* r_j(X)\varphi_j(X)^{\frac{1}{2}}T_j(X)^{-1}\mathcal{A}(X)T_j(X)T_j(X)^{-1}\\
&= \sum_j P_j(X)^*(r_j(X)T_j(X)^{-1}\mathcal{A}(X)T_j(X))P_j(X)\\
&= \sum_j P_j(X)^*(\tau\Delta_{0j}(X))P_j(X)
\end{aligned} \tag{4.6.23}$$

where Δ_{0j} are diagonal matrices independent of τ and $\Re\Delta_{0j}(X) \geq C_j I_n$ for each j. Hence Δ_{0j} is homogeneous degree 0 in $\tau = \gamma + i\delta$, and we have $\Delta_{0j}(X) = \Delta_{0j}(w,\tau) \in \Gamma_1^0(\mathbb{R}_t \times \mathbb{R}_\delta)$. From $\Delta_{0j}(X) + \Delta_{0j}(X)^* \geq 2C_j I_n$ and Garding's inequality Theorem C.3.5 we have

$$\Re(T^{\chi,\gamma}_{\Delta_{0j}}u,u)_{L^2(\mathbb{R})} \geq \frac{C_j}{2}\|u\|^2_{L^2(\mathbb{R})} \tag{4.6.24}$$

for every $u \in L^2(\mathbb{R})$. Now the symbol of $R_v^\gamma T_{\mathcal{A}_v}^{\chi,\gamma}$ differs from the symbol $\mathcal{R}_v\mathcal{A}_v$ by a symbol of order 0 as in (4.5.17), so by (4.6.23), (4.6.24), Theorem C.3.4 and a standard error estimate

$$\begin{aligned}
\Re(R_v^\gamma(x)T_{\mathcal{A}_v}^{\chi,\gamma}u,u)_{L^2(\mathbb{R})} &\geq \sum_j \Re(T_{\mathcal{R}_v\mathcal{A}_v}^{\chi,\gamma}u,u)_{L^2(\mathbb{R})} - C\|u\|^2_{L^2(\mathbb{R})} \\
&\geq \sum_j \gamma\Re((T_{P_j}^{\chi,\gamma})^* T_{\Delta_{0j}}^{\chi,\gamma} T_{P_j}^{\chi,\gamma}u,u)_{L^2(\mathbb{R})} - C\|u\|^2_{L^2(\mathbb{R})} \\
&\geq \gamma\sum_j \Re(T_{\Delta_{0j}}^{\chi,\gamma} T_{P_j}^{\chi,\gamma}u, T_{P_j}^{\chi,\gamma}u)_{L^2(\mathbb{R})} - C\|u\|^2_{L^2(\mathbb{R})} \\
&\geq \frac{C\gamma}{2}\sum_j \|T_{P_j}^{\chi,\gamma}u\|^2_{L^2(\mathbb{R})} - C\|u\|^2_{L^2(\mathbb{R})}. \qquad (4.6.25)
\end{aligned}$$

However we have, since $\sum_j P_j^* P_j \geq \sigma I_N$ for some $\sigma > 0$, by Garding's inequality C.3.5

$$\sum_j \Re(T_{P_j^*P_j}^{\chi,\gamma}u,u)_{L^2(\mathbb{R})} = \Re(T_{\sum_j P_j^*P_j}^{\chi,\gamma}u,u)_{L^2(\mathbb{R})} \geq \frac{\sigma}{2}\|u\|^2_{L^2(\mathbb{R})}. \qquad (4.6.26)$$

Because the symbol of $(T_{P_j}^{\chi,\gamma})^* T_{P_j}^{\chi,\gamma}$ differs from the symbol of $T_{P_j^*P_j}^{\chi,\gamma}$ by a symbol of order -1 we have

$$\|T_{P_j}^{\chi,\gamma}u\|^2_{L^2(\mathbb{R})} = ((T_{P_j}^{\chi,\gamma})^* T_{P_j}^{\chi,\gamma}u,u)_{L^2(\mathbb{R})} \geq \Re(T_{P_j^*P_j}^{\chi,\gamma}u,u)_{L^2(\mathbb{R})} - \frac{C}{\gamma}\|u\|^2_{L^2(\mathbb{R})}. \quad (4.6.27)$$

Choosing γ_0 sufficiently large, we obtain from (4.6.25)−(4.6.27) that

$$\Re(R_v^\gamma(x)T_{\mathcal{A}_v}^{\chi,\gamma}u,u)_{L^2(\mathbb{R})} \geq C\gamma\|u\|^2_{L^2(\mathbb{R})}$$

for all $\gamma \geq \gamma_0$. Thus (4.6.1) is satisfied. This completes the proof that R_v^γ is a functional boundary symmetrizer for (A_v, B_v). Consequently, the a priori estimate (4.6.4) follows from Corollary 4.6.3. $\square$

4.7 A PRIORI ESTIMATES IN $e^{-\gamma t}L^2$ FOR THE ADJOINT OPERATOR

The weak solutions of the partial differential equations we consider satisfy a variational equality where the test functions lie in a space associated with the dual problem. For this, we need to prove the a priori estimates on a subspace of the dual of the solution space. The goal of this section is to derive such a priori estimates using the same assumptions in the previous sections.

We begin with the case where the coefficients are smooth.

Lemma 4.7.1. *Let (H1) and (H3) be satisfied. Then there exist matrix-valued maps $N_0, C_0, M_1 \in \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^{(n-p)\times n})$ and $N_1, C_1, M_0 \in \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^{p\times n})$, which are constant outside a compact subset of $\mathbb{R}$, such that*

$$A(t,x) = M_x(t)^\top B_x(t) + C_x(t)^\top N_x(t), \qquad \forall\ (t,x) \in \mathbb{R}\times\{0,1\}. \qquad (4.7.1)$$

Proof. We only prove the case where $x = 0$. Since B_0 is of full rank, there exists another full rank matrix $N_0 \in \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^{(n-p)\times n})$ such that

$$\begin{pmatrix} B_0 \\ N_0 \end{pmatrix} \in \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^{n\times n}) \qquad (4.7.2)$$

is invertible. Let us decompose its inverse into two blocks $(Y_0\ D_0)$ where $Y_0 \in \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^{n\times p})$ and $D_0 \in \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^{n\times(n-p)})$. Thus

$$Y_0 B_0 + D_0 N_0 = (Y_0\ D_0)\begin{pmatrix} B_0 \\ N_0 \end{pmatrix} = I_n. \tag{4.7.3}$$

Multiplying both sides by $A(t,0)$, it can be seen that (4.7.1) with $x = 0$ holds where $M_0(t) = (A(t,0)Y_0(t))^\top$ and $C_0(t) = (A(t,0)D_0(t))^\top$. Because the matrices B_0 and $A(\cdot,0)$ are constant outside a compact subset of $\mathbb{R}$, the matrices N_0, Y_0, D_0, M_0 and C_0 can also be chosen to be constant outside a compact subset of $\mathbb{R}$. □

In the following discussions, we will show that if (A,B) satisfies the UKL condition (H5) then $(-A^\top, C)$ also satisfies the UKL condition, i.e., there exists $C > 0$ such that for all $t \in \mathbb{R}$ we have

$$|U| \le C|C_0(t)U|, \qquad \forall\, U \in E^u(-A(t,0)^\top), \tag{4.7.4}$$

and

$$|U| \le C|C_1(t)U|, \qquad \forall\, U \in E^s(-A(t,1)^\top). \tag{4.7.5}$$

Suppose that $t \in \mathbb{R}$. By (H5) there exists a constant $C > 0$ such that

$$|V| \le C|B_0(t)V|, \qquad \forall\, V \in E^u(A(t,0)) = E^u(A(t,0)^{-1}). \tag{4.7.6}$$

Let $U \in E^u(-A(t,0)^\top) = E^s(A(t,0)^{-\top})$ and $V \in E^u(A(t,0)^{-1})$. Define $v(s) = e^{-sA(t,0)^{-1}}V$ and $u(s) = e^{sA(t,0)^{-\top}}U$. By assumption, we have $v(s) \to 0$ and $u(s) \to 0$ as $s \to \infty$. Note that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}(u(s)\cdot A(t,0)v(s)) &= \dot u(s)\cdot A(t,0)v(s)) + A(t,0)^\top u(s)\cdot \dot v(s) \\ &= A(t,0)^{-\top}u(s)\cdot A(t,0)v(s) - A(t,0)^\top u(s)\cdot A(t,0)^{-1}v(s) \\ &= 0. \end{aligned}$$

Thus $u(s)\cdot A(t,0)v(s) = u(0)\cdot A(t,0)v(0) = U\cdot A(t,0)V$ for all $s \ge 0$. Letting $s \to \infty$ it follows that $U\cdot A(t,0)V = 0$ whenever $U \in E^u(-A(t,0)^\top)$ and $V \in E^u(A(t,0))$. From Lemma 4.7.1 there exists $C > 0$ independent of U and t such that

$$|U| \le C(|M_0(t)U| + |C_0(t)U|). \tag{4.7.7}$$

Since $B_0(t) : E^u(A(t,0)) \to \mathbb{C}^p$ is an isomorphism we have

$$\begin{aligned} |M_0(t)U| &= \sup_{W\in\mathbb{C}^p}\frac{|M_0(t)U\cdot W|}{|W|} = \sup_{V\in E^u(A(t,0))}\frac{|M_0(t)U\cdot B_0(t)V|}{|B_0(t)V|} \\ &\le C \sup_{V\in E^u(A(t,0))}\frac{|M_0(t)U\cdot B_0(t)V|}{|V|} \end{aligned} \tag{4.7.8}$$

according to (4.7.6). However, if $V \in E^u(A(t,0))$ and $U \in E^s(A(t,0)^\top)$ then by (4.7.1)

$$\begin{aligned} M_0(t)U\cdot B_0(t)V &= (A(t,0)^\top U - N_0(t)^\top C_0(t)U)\cdot V \\ &= -C_0(t)U\cdot N_0(t)V. \end{aligned}$$

Thus by the Cauchy-Schwarz inequality

$$\frac{|M_0(t)U \cdot B_0(t)V|}{|V|} = \frac{|C_0(t)U \cdot N_0(t)V|}{|V|} \leq |C_0(t)U| \|N_0(t)\|. \tag{4.7.9}$$

Now, (4.7.4) follows from (4.7.7)−(4.7.9). The proof of (4.7.5) is analogous.

Using Theorem 4.5.8 and changing the time variable t by $-t$ one obtains the following a priori estimate in terms of the formal adjoint L^* of L given by (4.4.1) and a boundary matrix C in Lemma 4.7.1. Recall that

$$C = \begin{pmatrix} C_0 & O_{(n-p)\times n} \\ O_{p\times n} & C_1 \end{pmatrix}.$$

Theorem 4.7.2. *Assume that (H1)−(H5) hold. Then there exist $C^* > 0$ and $\gamma_0^* \geq 1$ such that the a priori estimate*

$$\begin{aligned} &\gamma \|e^{\gamma t}\varphi\|^2_{L^2(\mathbb{R}\times(0,1))} + \|e^{\gamma t}\varphi_{|\partial\Omega}\|^2_{L^2(\mathbb{R})} \\ &\leq C^* \left(\frac{1}{\gamma} \|e^{\gamma t} L^* \varphi\|^2_{L^2(\mathbb{R}\times(0,1))} + \|e^{\gamma t} C \varphi_{|\partial\Omega}\|^2_{L^2(\mathbb{R})} \right) \end{aligned} \tag{4.7.10}$$

holds for all $\varphi \in e^{-\gamma t} H^1(\mathbb{R}\times(0,1))$ and $\gamma \geq \gamma_0^*$.

The previous theorem gives us an a priori estimate of the adjoint operator L^* in the case where the coefficients are smooth. In the case where the coefficients have limited regularity we have the following analogous results.

Lemma 4.7.3. *Assume that (D) holds and suppose that the boundary matrices $B_0 \in \mathscr{C}^\infty(\mathcal{U}; \mathbb{R}^{p\times n})$ and $B_1 \in \mathscr{C}^\infty(\mathcal{U}; \mathbb{R}^{(n-p)\times n})$ have full ranks at each point of $\mathcal{U}$. Then there exist matrix-valued maps $N_0, C_0, M_1 \in \mathscr{C}^\infty(\mathcal{U}; \mathbb{R}^{(n-p)\times n})$ and $N_1, C_1, M_0 \in \mathscr{C}^\infty(\mathcal{U}; \mathbb{R}^{p\times n})$ such that*

$$A(w) = M_x(w)^\top B_x(w) + C_x(w)^\top N_x(w), \qquad \forall\ (w,x) \in \mathcal{U} \times \{0,1\}. \tag{4.7.11}$$

Theorem 4.7.4. *In the framework of Theorem 4.6.6, there exist constants $C^* = C^*(\varrho, K, \mathcal{K}) > 0$ and $\gamma_0^* = \gamma_0^*(\varrho, K, \mathcal{K}) \geq 1$ such that for every $\varphi \in e^{-\gamma t} H^1(\mathbb{R}\times(0,1))$ and $\gamma \geq \gamma_0^*$ we have*

$$\begin{aligned} &\gamma \|e^{\gamma t}\varphi\|^2_{L^2(\mathbb{R}\times(0,1))} + \|e^{\gamma t}\varphi_{|\partial\Omega}\|^2_{L^2(\mathbb{R})} \\ &\leq C^* \left(\frac{1}{\gamma} \|e^{\gamma t} L_v^* \varphi\|^2_{L^2(\mathbb{R}\times(0,1))} + \|e^{\gamma t} C_v \varphi_{|\partial\Omega}\|^2_{L^2(\mathbb{R})} \right). \end{aligned} \tag{4.7.12}$$

4.8 WEAK AND STRONG SOLUTIONS FOR THE BVP

Two types of solutions of the pure boundary value problem

$$\begin{cases} Lu = \partial_t u + A\partial_x u + Ru = f, & -\infty < t < \infty,\ 0 < x < 1, \\ Bu_{|\partial\Omega} = g, & -\infty < t < \infty, \end{cases} \tag{4.8.1}$$

in the weighted Lebesgue space $e^{\gamma t} L^2(\mathbb{R}\times(0,1))$ will be defined in this section. This definition applies to systems where the coefficients A and B are at least Lipschitz and the coefficient R is bounded.

Definition 4.8.1. Let $f \in e^{\gamma t}L^2(\mathbb{R} \times (0,1))$ and $g \in e^{\gamma t}L^2(\mathbb{R})$ where $\gamma \in \mathbb{R}$. A function $u \in e^{\gamma t}L^2(\mathbb{R} \times (0,1))$ is called a *weak solution* of the BVP if for every $\varphi \in e^{-\gamma t}H^1(\mathbb{R} \times (0,1))$ such that $C\varphi_{|\partial\Omega} = 0$ we have

$$\begin{aligned}\int_{\mathbb{R}}\int_0^1 u \cdot L^*\varphi \,\mathrm{d}x\,\mathrm{d}t &= \int_{\mathbb{R}}\int_0^1 f \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}} g_1 \cdot M_1\varphi_{|x=1}\,\mathrm{d}t \\ &\quad + \int_{\mathbb{R}} g_0 \cdot M_0\varphi_{|x=0}\,\mathrm{d}t, \end{aligned} \tag{4.8.2}$$

where C_0, C_1, M_0 and M_1 are the matrices in Lemma 4.7.1.

Since $\mathscr{D}(\mathbb{R} \times (0,1))$ is contained in the space of test functions $\{\varphi \in e^{-\gamma t}H^1(\mathbb{R} \times (0,1)) : C\varphi_{|\partial\Omega} = 0\}$, the space of test functions in Definition 4.8.1 is dense in the solution space $e^{\gamma t}L^2(\mathbb{R}\times(0,1))$. The following theorem tells us how the weak solution satisfies the BVP (4.8.1) in some sense.

Theorem 4.8.2. *If $u \in e^{\gamma t}L^2(\mathbb{R} \times (0,1))$ is a weak solution of (4.8.1) then $u \in e^{\gamma t}E(\mathbb{R}\times(0,1))$, and in particular, $u_{|\partial\Omega} \in e^{\gamma t}H^{-\frac{1}{2}}(\mathbb{R})$. The equation $Lu = f$ holds in $e^{\gamma t}L^2(\mathbb{R}\times(0,1))$ in the sense of distributions and the boundary conditions $B_0u_{|x=0} = g_0$ and $B_1u_{|x=1} = g_1$ hold in $e^{\gamma t}H^{-\frac{1}{2}}(\mathbb{R})$.*

Proof. The fact that $Lu = f$ in the sense of distributions follows immediately from (4.8.2) by taking $\varphi \in \mathscr{D}(\mathbb{R} \times (0,1))$. Furthermore,

$$L(e^{-\gamma t}u) = -\gamma e^{-\gamma t}u + e^{-\gamma t}f \in L^2(\mathbb{R} \times (0,1)).$$

Thus $e^{-\gamma t}u \in E(\mathbb{R} \times (0,1))$. By Green's identity (4.4.12), Lemma 4.7.1 and (4.8.2) we have

$$\begin{aligned}&\langle B_1u_{|x=1}, M_1\varphi_{|x=1}\rangle_{e^{\gamma t}H^{-\frac{1}{2}}(\mathbb{R})\times e^{-\gamma t}H^{\frac{1}{2}}(\mathbb{R})} - \langle B_0u_{|x=0}, M_0\varphi_{|x=0}\rangle_{e^{\gamma t}H^{-\frac{1}{2}}(\mathbb{R})\times e^{-\gamma t}H^{\frac{1}{2}}(\mathbb{R})} \\ &= \int_{\mathbb{R}} g_1 \cdot M_1\varphi_{|x=1}\,\mathrm{d}t - \int_{\mathbb{R}} g_0 \cdot M_0\varphi_{|x=0}\,\mathrm{d}t\end{aligned} \tag{4.8.3}$$

for every $\varphi \in e^{-\gamma t}H^1(\mathbb{R} \times (0,1))$ be such that $C\varphi_{|\partial\Omega} = 0$.

Let $\psi \in e^{-\gamma t}H^{\frac{1}{2}}(\mathbb{R})$ and $\phi \in e^{-\gamma t}H^1(\mathbb{R}\times(0,1))$ such that $\phi_{|x=0} = \psi$ and $\phi_{|x=1} = 0$. Define

$$\varphi(t,x) = A(t,x)^{-\top}\begin{pmatrix} Y_0(t)^\top \\ D_0(t)^\top \end{pmatrix}^{-1}\begin{pmatrix} \phi(t,x) \\ O_{(n-p)\times 1} \end{pmatrix}$$

where Y_0 and D_0 are the matrices in the proof of Lemma 4.7.1. Then $\varphi \in e^{-\gamma t}H^1(\mathbb{R}\times(0,1))$ satisfies $M_0\varphi_{|x=0} = Y_0^\top A(t,0)^\top\varphi_{|x=0} = \psi$, $C_0\varphi_{|x=0} = D_0^\top A(t,0)^\top\varphi_{|x=0} = 0$ and $\varphi_{|x=1} = 0$. With this φ in (4.8.3) we have

$$\langle B_0u_{|x=0}, \psi\rangle_{e^{\gamma t}H^{-\frac{1}{2}}(\mathbb{R})\times e^{-\gamma t}H^{\frac{1}{2}}(\mathbb{R})} = \int_{\mathbb{R}} g_0 \cdot \psi\,\mathrm{d}t$$

for all $\psi \in e^{-\gamma t}H^{\frac{1}{2}}(\mathbb{R})$. This means that $B_0u_{|x=0} = g_0$ holds in $e^{\gamma t}H^{-\frac{1}{2}}(\mathbb{R})$. The other boundary condition is similar. □

A stronger type of solutions for the boundary value problem (4.8.1) is given in the following definition.

Definition 4.8.3. A function $u \in e^{\gamma t}L^2(\mathbb{R} \times (0,1))$ is called a *strong solution* of (4.8.1) if there exist sequences $(f_j)_j \subset e^{\gamma t}L^2(\mathbb{R} \times (0,1))$, $(g_j)_j \subset e^{\gamma t}H^{\frac{1}{2}}(\mathbb{R})$ and $(u_j)_j \subset e^{\gamma t}H^1(\mathbb{R} \times (0,1))$ satisfying

$$\begin{cases} Lu_j = f_j, & -\infty < t < \infty,\ 0 < x < 1, \\ Bu_{j|\partial\Omega} = g_j, & -\infty < t < \infty, \end{cases} \tag{4.8.4}$$

where $f_j \to f$ in $e^{\gamma t}L^2(\mathbb{R} \times (0,1))$, $g_j \to g$ in $e^{\gamma t}L^2(\mathbb{R})$ and $u_j \to u$ in $e^{\gamma t}L^2(\mathbb{R} \times (0,1))$.

The reason why the above definition is stronger than the one given in Definition 4.8.1 is because every strong solution is a weak solution. Indeed, if u is a strong solution of (4.8.1) and $(u_j)_j$, $(f_j)_j$, and $(g_j)_j$ are the corresponding sequences then Green's identity implies that

$$\begin{aligned} \int_{\mathbb{R}}\int_0^1 u_j \cdot L^*\varphi \,\mathrm{d}x\,\mathrm{d}t &= \int_{\mathbb{R}}\int_0^1 f_j \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}} g_{1j} \cdot M_1\varphi_{|x=1}\,\mathrm{d}t \\ &\quad + \int_{\mathbb{R}} g_{0j} \cdot M_0\varphi_{|x=0}\,\mathrm{d}t \end{aligned} \tag{4.8.5}$$

for every $\varphi \in e^{-\gamma t}H^1(\mathbb{R} \times (0,1))$ such that $C\varphi_{|\partial\Omega} = 0$. Passing to the limit $j \to \infty$ in (4.8.5) shows that u is a weak solution. It will be shown later that for sufficiently large γ, weak and strong solutions coincide.

Theorem 4.8.4. *Let u be a strong solution of the boundary value problem* (4.8.1) *and $(u_j)_j \subset e^{\gamma t}H^1(\mathbb{R}\times(0,1))$ be the corresponding sequence given in Definition 4.8.3. Then $u_j \to u$ in $e^{\gamma t}E(\mathbb{R} \times (0,1))$, and in particular $u_{j|\partial\Omega} \to u_{|\partial\Omega}$ in $e^{\gamma t}H^{-\frac{1}{2}}(\mathbb{R})$.*

Proof. The limit $u_j \to u$ in $e^{\gamma t}E(\mathbb{R} \times (0,1))$ follows immediately from the fact that

$$L(e^{-\gamma t}u_j) = -\gamma e^{-\gamma t}u_j + e^{-\gamma t}f_j \to -\gamma e^{-\gamma t}u + e^{-\gamma t}f = L(e^{-\gamma t}u)$$

in $L^2(\mathbb{R} \times (0,1))$. The convergence of the traces follows from the continuity of the generalized trace operator. □

Note that in this section we exhibit basic properties of weak and strong solutions without proving any existence nor uniqueness. This will be done however in Section 4.10 for smooth coefficients and in Section 4.12 for coefficients that are at least Lipschitz.

4.9 WEAK AND STRONG SOLUTIONS FOR THE IBVP

In this section we define the weak and strong solutions in $L^2(Q_T)$ of the initial boundary value problem

$$\begin{cases} Lu = \partial_t u + A\partial_x u + Ru = f, & 0 < t < T,\ 0 < x < 1 \\ Bu_{|\partial\Omega} = g, & 0 < t < T, \\ u_{|t=0} = u_0, & 0 < x < 1. \end{cases} \tag{4.9.1}$$

Definition 4.9.1. Let $f \in L^2(Q_T)$, $g \in L^2(0,T)$ and $u_0 \in L^2(0,1)$. A function $u \in L^2(Q_T)$ is called a *weak solution* of the initial-boundary value problem (4.9.1) if

$$\begin{aligned}\int_0^T \int_0^1 u \cdot L^*\varphi \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_0^1 f \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T g_1 \cdot M_1 \varphi_{|x=1} \,\mathrm{d}t \\ &\quad + \int_0^T g_0 \cdot M_0 \varphi_{|x=0} \,\mathrm{d}t + \int_0^1 u_0 \cdot \varphi_{|t=0} \,\mathrm{d}x \qquad (4.9.2)\end{aligned}$$

holds for all $\varphi \in H^1(Q_T)$ such that $C\varphi_{|\partial\Omega} = 0$ and $\varphi_{|t=T} = 0$.

Since $\mathscr{D}(Q_T) \subset \{\varphi \in H^1(Q_T) : C\varphi_{|\partial\Omega} = 0, \varphi_{|t=T} = 0\}$, it follows that the space of test functions in Definition 4.9.1 is dense in the solutions space $L^2(Q_T)$. Recall from (4.4.1) and (4.4.9) that the formal adjoint of L and $A^{-1}L$ are given by

$$L^* v = -\partial_t v - \partial_x (A^\top v) + R^\top v$$

and

$$(A^{-1}L)^* v = -\partial_t (A^{-\top} v) - \partial_x v + R^\top A^{-\top} v.$$

Thus for each $v \in H^1(Q_T)$ we have $L^*(A^{-\top}v) = (A^{-1}L)^* v$ and the Green's identity

$$\begin{aligned}&\int_0^T \int_0^1 Lu \cdot A^{-\top} v \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_0^1 u \cdot L^*(A^{-\top} v) \,\mathrm{d}x\,\mathrm{d}t + \langle \Gamma_g u, \Gamma v \rangle_{H^{-\frac{1}{2}}(\partial Q_T) \times H^{\frac{1}{2}}(\partial Q_T)} \qquad (4.9.3)\end{aligned}$$

for all $u \in E(Q_T)$ and $v \in H^1(Q_T)$. With this version of the generalized Green's identity we are able to prove the following theorem stating how the weak solution satisfies the IBVP (4.9.1) in some sense.

Theorem 4.9.2. *If $u \in L^2(Q_T)$ is a weak solution of* (4.9.1) *then $u \in E(Q_T)$. The equation $Lu = f$ holds in $L^2(Q_T)$ in the sense of distributions and the boundary and initial conditions are satisfied in the following sense*

$$B_0 u_{|x=0} = g_0 \quad \text{in } V(\Sigma_1)', \qquad (4.9.4)$$

$$B_1 u_{|x=1} = g_1 \quad \text{in } V(\Sigma_2)', \qquad (4.9.5)$$

$$u_{|t=0} = u_0 \quad \text{in } V(\Sigma_0)'. \qquad (4.9.6)$$

Proof. By taking $\varphi \in \mathscr{D}(Q_T)$ in the definition, the equation $Lu = f$ holds in the sense of distributions and hence $u \in E(Q_T)$. Given $\psi \in \mathcal{V}(\Sigma_1)$, let $\phi \in H^1(Q_T)$ be such that $\Gamma\phi = \psi$ and

$$\varphi(t,x) = \begin{pmatrix} Y_0(t)^\top \\ D_0(t)^\top \end{pmatrix}^{-1} \begin{pmatrix} \phi(t,x) \\ O_{(n-p)\times 1} \end{pmatrix}.$$

Then $\varphi \in H^1(Q_T)$ and $C_0(t)A(t,0)^{-\top}\varphi(t,0) = D_0(t)^\top A(t,0)^\top A(t,0)^{-\top}\varphi(t,0) = 0$ for a.e. $t \in (0,T)$. Furthermore $C_1(t)A(t,1)^{-\top}\varphi(t,1) = 0$ for a.e. $t \in (0,1)$ and $\varphi(0,x) = 0$ and $\varphi(T,x) = 0$ for a.e. $x \in (0,1)$ since the support of ψ lies in Σ_1. From (4.4.16), (4.9.2) and the generalized Green's identity (4.9.3) we have

$$\begin{aligned}\langle B_0 u_{|\Sigma_1}, \psi \rangle_{V(\Sigma_1)' \times V(\Sigma_1)} &= -\langle \Gamma_g u, B_0^\top \psi \rangle_{H^{-\frac{1}{2}}(\partial Q_T) \times H^{\frac{1}{2}}(\partial Q_T)} \\ &= \int_0^T g_0(t) \cdot M_0(t) A^{-\top}(t,0) B_0(t)^\top \psi(t,0) \,\mathrm{d}t \\ &= \int_0^T g_0(t) \cdot \psi(t,0) \,\mathrm{d}t.\end{aligned}$$

for each $\psi \in \mathcal{V}(\Sigma_1)$ since $B_0(t)A(t,0)^{-1}M_0(t)^\top = B_0(t)Y_0(t) = I_p$. Therefore (4.9.4) holds. A similar argument shows that (4.9.5) holds as well.

Let us prove (4.9.6). For $\varphi \in \mathcal{V}(\Sigma_0)$ we let $\phi \in H^1(Q_T)$ such that $\phi_{|\partial\Omega} = \varphi$. Then $C\phi_{|\partial\Omega} = 0$, $\phi_{|t=T} = 0$ and so

$$\begin{aligned} \langle u_{|\Sigma_0}, \varphi\rangle_{V(\Sigma_0)'\times V(\Sigma_0)} &= -\left\langle \Gamma_g u, A(0,\cdot)^\top \varphi\right\rangle_{H^{-\frac{1}{2}}(\partial Q_T)\times H^{\frac{1}{2}}(\partial Q_T)} \\ &= \int_0^1 u_0(x)\cdot\varphi(0,x)\,\mathrm{d}x \end{aligned}$$

from (4.9.2) and (4.9.3). Thus $u_{|\Sigma_0} = u_0$ in $V(\Sigma_0)'$. □

We can also introduce a stronger notion of solution for the IBVP (4.9.1).

Definition 4.9.3. A function $u \in L^2(Q_T)$ is called a *strong solution* of (4.9.1) if there exist sequences $(u_j)_j \in H^1(Q_T)$, $(f_j)_j \in L^2(Q_T)$, $(g_j)_j \in H^{\frac{1}{2}}(0,T)$ and $(u_{0j})_j \in H^{\frac{1}{2}}(0,1)$ such that

$$\begin{cases} Lu_j = f_j, & 0<t<T,\ 0<x<1, \\ Bu_{j|\partial\Omega} = g_j, & 0<t<T, \\ u_{j|t=0} = u_{0j}, & 0<x<1, \end{cases}$$

with $u_j \to u$ and $f_j \to f$ in $L^2(Q_T)$, $g_j \to g$ in $L^2(0,T)$ and $u_{0j} \to u_0$ in $L^2(0,1)$.

It can be easily seen that every strong solution of (4.9.1) is also a weak solution. The convergence of the sequence approximating a strong solution can be improved to $E(Q_T)$. The proof of the following theorem is similar to the proof of Theorem 4.8.4 and therefore we omit the details.

Theorem 4.9.4. *If u is a strong solution of (4.9.1) and $(u_j)_j \subset H^1(Q_T)$ is a corresponding approximating sequence of u then $u_j \to u$ in $E(Q_T)$. In particular, $u_{j|\Sigma_i} \to u_{|\Sigma_i}$ in $V(\Sigma_i)'$ for $i = 1,2,3,4$.*

4.10 BVP WITH SMOOTH COEFFICIENTS

In order to apply Theorem 4.1.1, we take $X = e^{-\gamma t}L^2(\mathbb{R}\times(0,1))$, $Y = e^{-\gamma t}H^1(\mathbb{R}\times(0,1))$ and $Z = e^{-\gamma t}L^2(\mathbb{R})$. Define $\Lambda : Y \to X$, $\Phi : Y \to Z$ and $\Psi : Y \to Z$ by

$$\Lambda\varphi = L^*\varphi, \qquad \Phi\varphi = C\varphi_{|\partial\Omega}, \qquad \Psi\varphi = (M_0\varphi_{|x=0}, -M_1\varphi_{|x=1}),$$

for all $\varphi \in Y$. The variational equation (4.8.2) can now be written in the form

$$(e^{-2\gamma t}u, \Lambda\varphi)_X = (e^{-2\gamma t}f, \varphi)_X + (e^{-2\gamma t}(g_0,g_1), \Psi\varphi)_Z, \qquad \forall\, \varphi \in \ker\Phi. \tag{4.10.1}$$

Theorem 4.10.1. *Assume that (H1)–(H5) hold. Then there exists $\gamma_0 \geq 1$ such that for all $\gamma \geq \gamma_0$, $f \in e^{\gamma t}L^2(\mathbb{R}\times(0,1))$ and $g \in e^{\gamma t}L^2(\mathbb{R})$ the boundary value problem (4.2.7) has a weak solution $u \in e^{\gamma t}L^2(\mathbb{R}\times(0,1))$ satisfying the energy estimate*

$$\gamma\|e^{-\gamma t}u\|^2_{L^2(\mathbb{R}\times(0,1))} \leq C\left(\frac{1}{\gamma}\|e^{-\gamma t}f\|^2_{L^2(\mathbb{R}\times(0,1))} + \|e^{-\gamma t}g\|^2_{L^2(\mathbb{R})}\right) \tag{4.10.2}$$

for some $C > 0$.

Proof. With the notations in the paragraph preceding the theorem, the estimate (4.1.2) holds for all $\varphi \in Y$ according to Theorem 4.7.2. Thus, according to Theorem 4.1.1, taking supremum norms of M_0 and M_1, there exists $v \in X$ such that

$$(v, \Lambda\varphi)_X = (e^{-2\gamma t} f, \varphi)_X + (e^{-2\gamma t}(g_0, g_1), \Psi\varphi)_Z, \qquad \forall\, \varphi \in \ker \Phi.$$

Then $u = e^{2\gamma t} v \in e^{2\gamma t} X = e^{\gamma t} L^2(\mathbb{R} \times (0,1))$ satisfies (4.10.1), and so u is a weak solution of (4.8.1). The energy estimate (4.10.2) is a consequence of (4.2.8). $\square$

We define $\mathcal{E}(\mathbb{R} \times (0,1))$ to be the set of all functions $\varphi \in E(\mathbb{R} \times (0,1))$ such that $\varphi_{|\partial\Omega} \in L^2(\mathbb{R})$ and there exists a sequence $(\varphi_j)_j \subset H^1(\mathbb{R} \times (0,1))$ satisfying

$$\lim_{j\to\infty} \|u_j - u\|_{E(\mathbb{R}\times(0,1))} + \|u_{j|\partial\Omega} - u_{|\partial\Omega}\|_{L^2(\mathbb{R})} = 0.$$

It is clear that $H^1(\mathbb{R} \times (0,1)) \subset \mathcal{E}(\mathbb{R} \times (0,1))$. It can be shown that $\mathcal{E}(\mathbb{R} \times (0,1))$ is the completion of $H^1(\mathbb{R} \times (0,1))$ with respect to the norm

$$\|u\|_{\mathcal{E}(\mathbb{R}\times(0,1))} := (\|u\|^2_{E(\mathbb{R}\times(0,1))} + \|u_{|\partial\Omega}\|^2_{L^2(\mathbb{R})})^{\frac{1}{2}}.$$

The proof is similar to Theorem 4.13.4 below. The space $\mathcal{E}^*(\mathbb{R}\times(0,1))$ can be defined similarly by replacing L by L^* in the definition.

Remark 4.10.2. The a priori estimate (4.7.12) in Theorem 4.7.4 is valid for all functions $\varphi \in e^{-\gamma t}\mathcal{E}^*(\mathbb{R} \times (0,1))$. Indeed, (4.7.12) holds for φ_j where φ_j is the approximating sequence for φ, and hence for φ by passing to the limit $j \to \infty$.

Theorem 4.10.3. *For all $u \in e^{\gamma t}\mathcal{E}(\mathbb{R} \times (0,1))$ and $w \in e^{-\gamma t}\mathcal{E}^*(\mathbb{R} \times (0,1))$ we have*

$$\begin{aligned} &\int_{\mathbb{R}} \int_0^1 u(t,x) \cdot L^* w(t,x)\,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}} \int_0^1 Lu(t,x) \cdot w(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\mathbb{R}} A(t,0)u(t,0) \cdot w(t,0)\,\mathrm{d}t - \int_{\mathbb{R}} A(t,1)u(t,1) \cdot w(t,1)\,\mathrm{d}t. \end{aligned} \tag{4.10.3}$$

Proof. Using integration by parts, (4.10.3) holds for all $u, w \in \mathscr{D}(\mathbb{R} \times (0,1))$. By a density argument, (4.10.3) holds for all $u \in e^{\gamma t} H^1(\mathbb{R} \times (0,1))$ and $w \in e^{-\gamma t} H^1(\mathbb{R} \times (0,1))$. The conclusion now follows from the definition of the spaces $e^{\gamma t}\mathcal{E}(\mathbb{R} \times (0,1))$ and $e^{-\gamma t}\mathcal{E}^*(\mathbb{R} \times (0,1))$. $\square$

The following theorem implies that strong solutions have L^2-traces at the boundary and the convergence of the traces given in Theorem 4.8.4 can be improved.

Theorem 4.10.4. *Assume that (H1)–(H5) hold. There exists $\gamma_0 \geq 1$ such that if $u \in e^{\gamma t} L^2(\mathbb{R} \times (0,1))$ is a strong solution of (4.8.1) then $u_{|\partial\Omega} \in e^{\gamma t} L^2(\mathbb{R})$ and u satisfies the energy estimate*

$$\begin{aligned} &\gamma \|e^{-\gamma t} u\|^2_{L^2(\mathbb{R}\times(0,1))} + \|e^{-\gamma t} u_{|\partial\Omega}\|^2_{L^2(\mathbb{R})} \\ &\leq C\left(\frac{1}{\gamma}\|e^{-\gamma t} f\|^2_{L^2(\mathbb{R}\times(0,1))} + \|e^{-\gamma t} g\|^2_{L^2(\mathbb{R})}\right) \end{aligned} \tag{4.10.4}$$

for some $C > 0$ and for all $\gamma \geq \gamma_0$. If $(u_j)_j \subset e^{\gamma t} H^1(\mathbb{R} \times (0,1))$ is the sequence associated with u then $u_{j|\partial\Omega} \to u_{|\partial\Omega}$ in $e^{\gamma t} L^2(\mathbb{R})$. In particular the strong solution is unique and $u \in e^{\gamma t}\mathcal{E}(\mathbb{R} \times (0,1))$.

Proof. Let u be a strong solution of (4.8.1) and $(u_j)_j$, $(f_j)_j$ and $(g_j)_j$ be the corresponding sequence stated in Definition 4.8.3. Applying the a priori estimate (4.5.4) to $u_j - u_k$ and the fact that $Lu_j = f_j$ and $Bu_{j|\partial\Omega} = g_j$ for all n we have

$$\begin{aligned}&\gamma\|e^{-\gamma t}(u_j-u_k)\|^2_{L^2(\mathbb{R}\times(0,1))} + \|e^{-\gamma t}(u_{j|\partial\Omega}-u_{k|\partial\Omega})\|^2_{L^2(\mathbb{R})}\\ &\le C\left(\frac{1}{\gamma}\|e^{-\gamma t}(f_j-f_k)\|^2_{L^2(\mathbb{R}\times(0,1))} + \|e^{-\gamma t}(g_j-g_k)\|^2_{L^2(\mathbb{R})}\right)\end{aligned}$$

for some $C>0$ and for all $\gamma \ge \gamma_0$ where γ_0 is the constant in the statement of Theorem 4.5.2. Thus $(u_j)_j$ and $(u_{j|\partial\Omega})_j$ are Cauchy sequences in $e^{\gamma t}L^2(\mathbb{R}\times(0,1))$ and $e^{\gamma t}L^2(\mathbb{R})$, respectively. By definition we already have $u_j \to u$ in $e^{\gamma t}L^2(\mathbb{R}\times(0,1))$. Let $v \in e^{\gamma t}L^2(\mathbb{R})$ such that $u_{j|\partial\Omega} \to v$ in $e^{\gamma t}L^2(\mathbb{R})$. From Theorem 4.8.4 we have $u_{j|\partial\Omega} \to u_{|\partial\Omega}$ in $e^{\gamma t}H^{-\frac{1}{2}}(\mathbb{R})$. Since the embedding $e^{\gamma t}L^2(\mathbb{R}) \subset e^{\gamma t}H^{-\frac{1}{2}}(\mathbb{R})$ is continuous we must have $u_{|\partial\Omega} = v$. Applying the a priori estimate (4.5.4) to u_j and passing to the limit, we can see that the energy estimate (4.10.4) is satisfied. The uniqueness of the strong solution follows from (4.10.4). □

Theorem 4.10.5. *Suppose that that (H1)–(H5) hold. There exists $\tilde{\gamma}_0 \ge 1$ such that for all $\gamma \ge \tilde{\gamma}_0$, a weak solution $u \in e^{\gamma t}L^2(\mathbb{R}\times(0,1))$ of* (4.8.1) *is a strong solution. In particular, this weak solution is unique, has a trace $u_{|\partial\Omega} \in e^{\gamma t}L^2(\mathbb{R})$, and the energy estimate* (4.10.4) *is satisfied by the weak solution u. The boundary condition $Bu_{|\partial\Omega} = g$ holds in $e^{\gamma t}L^2(\mathbb{R})$.*

Proof. The first statement will be proved even in the case where the coefficients are only Lipschitz, cf. Theorem 4.12.2. An alternative proof is to use the regularity result Theorem 4.10.6 below and apply a standard approximation argument, see [**9, pp. 260–262**] for details. The rest of the theorem follows from Theorem 4.10.4. □

The following regularity theorem can be shown as in [**9, 15**].

Theorem 4.10.6. *In the situation of Theorem 4.10.5, for all $k \in \mathbb{N}_0$ there exists $\gamma_k \ge 1$ such that for all $\gamma \ge \gamma_k$, if $f \in e^{\gamma t}H^k(\mathbb{R}\times(0,1))$ and $g \in e^{\gamma t}H^k(\mathbb{R})$ then the weak solution u of the BVP* (4.8.1) *lies in $e^{\gamma t}H^k(\mathbb{R}\times(0,1))$ and satisfies $u_{|\partial\Omega} \in e^{\gamma t}H^k(\mathbb{R})$. There exists $C_k > 0$ such that*

$$\begin{aligned}&\gamma\|u\|^2_{e^{\gamma t}H^k_\gamma(\mathbb{R}\times(0,1))} + \|u_{|\partial\Omega}\|^2_{e^{\gamma t}H^k_\gamma(\mathbb{R})}\\ &\le C_k\left(\frac{1}{\gamma}\|f\|^2_{e^{\gamma t}H^k_\gamma(\mathbb{R}\times(0,1))} + \|g\|^2_{e^{\gamma t}H^k_\gamma(\mathbb{R})}\right).\end{aligned} \tag{4.10.5}$$

Furthermore, there exists a sequence $(u_j)_j \subset e^{\gamma t}H^{k+1}_\gamma(\mathbb{R}\times(0,1))$ such that $u_j \to u$ in $e^{\gamma t}H^k_\gamma(\mathbb{R}\times(0,1))$, $Lu_j \to Lu$ in $e^{\gamma t}H^k_\gamma(\mathbb{R}\times(0,1))$ and $u_{j|\partial\Omega} \to u_{|\partial\Omega}$ in $e^{\gamma t}H^k_\gamma(\mathbb{R})$.

4.11 IBVP WITH CONSTANT COEFFICIENTS

In this section we study the well-posedness of the IBVP (4.9.1) and we restrict ourselves to the case where the coefficient are constants. We refer the readers to the paper of Rauch and Massey [**64**] for the case of smooth coefficients. The results of this section will be used in a PDE-ODE system that we consider in Section 4.21.

All throughout this section, we suppose that the $A \in \mathbb{R}^{n\times n}$ is invertible with p positive eigenvalues and $n-p$ negative eigenvalues, $R \in \mathbb{R}^{n\times n}$, $B_0 \in \mathbb{R}^{p\times n}$, $B_1 \in \mathbb{R}^{(n-p)\times n}$, B_0 and B_1 have full ranks and the UKL condition (H5) is satisfied. We begin with L^2-well-posedness. This theorem will be shown even in the case where the coefficient is Lipschitz, cf. Theorem 4.13.10.

Theorem 4.11.1. *For each $f \in L^2(Q_T)$, $g \in L^2(0,T)$ and $u_0 \in L^2(0,1)$ the initial-boundary value problem (4.9.1) has a unique weak solution $u \in L^2(Q_T)$. This weak solution is a strong solution, $u_{|\partial\Omega} \in L^2(0,T)$ and u satisfies the energy estimate*

$$\begin{aligned} &e^{-2\gamma T}\|u\|^2_{CL^2(Q_T)} + \gamma\|e^{-\gamma t}u\|^2_{L^2(Q_T)} + \|e^{-\gamma t}u_{|\partial\Omega}\|^2_{L^2(0,T)} \\ &\le C\left(\|u_0\|^2_{L^2(0,1)} + \frac{1}{\gamma}\|e^{-\gamma t}f\|^2_{L^2(Q_T)} + \|e^{-\gamma t}g\|^2_{L^2(0,T)}\right) \end{aligned}$$

for all $\gamma \ge \gamma_0$ for some $C > 0$ and some $\gamma_0 \ge 1$. Furthermore, there exists $(u_j)_j \subset H^1(Q_T)$ such that $u_j \to u$ in $CL^2(Q_T) \cap E(Q_T)$ and $u_{j|\partial\Omega} \to u_{|\partial\Omega}$ in $L^2(0,T)$.

Now we prove additional regularity of the solution of the IBVP (4.9.1) under the assumption that the data are also regular and satisfy compatibility conditions. The argument relies on the following a priori estimate.

Theorem 4.11.2. *Let $k \in \mathbb{N}_0$. There exists $\gamma_k \ge 1$ and $C_k > 0$ such that the a priori estimate*

$$\begin{aligned} &e^{-\gamma T}\sum_{|\alpha|\le k}\gamma^{2(k-|\alpha|)}\sup_{\tau\in[0,T]}\|\partial^\alpha u(\tau)\|^2_{L^2(0,1)} + \gamma\|e^{-\gamma t}u\|^2_{H^k_\gamma(Q_T)} + \|e^{-\gamma t}u_{|\partial\Omega}\|^2_{H^k_\gamma(0,T)} \\ &\le C_k\left(\sum_{j=0}^{k}\|\partial_t^j u_{|t=0}\|^2_{H^{k-j}(0,1)} + \frac{1}{\gamma}\|e^{-\gamma t}Lu\|^2_{H^k_\gamma(Q_T)} + \|e^{-\gamma t}Bu_{|\partial\Omega}\|^2_{H^k_\gamma(0,T)}\right). \end{aligned}$$

holds for all $u \in H^{k+1}(Q_T)$ and $\gamma \ge \gamma_k$.

The proof of this theorem can be done as in the case of variable coefficients, cf. Section 4.19. The proof is therefore omitted. We begin in the homogeneous case.

Theorem 4.11.3. *Suppose that $f \in H^k(Q_T)$ and $g \in H^k(0,T)$ satisfy $\partial_t^j f_{|t=0} = 0$ and $g^{(j)}(0) = 0$ for all $0 \le j \le k-1$. The weak solution of*

$$Lu = f, \qquad Bu_{|\partial\Omega} = g, \qquad u_{|t=0} = 0 \tag{4.11.1}$$

lies in $CH^k(Q_T)$ and has trace $u_{|\partial\Omega} \in H^k(0,T)$. Furthermore, there exists a sequence $(u_j)_j \subset H^{k+1}(Q_T)$ such that $u_j \to u$ in $CH^k(Q_T)$, $Lu_j \to Lu$ in $H^k(Q_T)$ and $u_{m|\partial\Omega} \to u_{|\partial\Omega}$ in $H^k(0,T)$.

Proof. Let $\tilde f \in e^{\gamma t}H^k(\mathbb{R}\times(0,1))$ and $\tilde g \in e^{\gamma t}H^k(\mathbb{R})$ be extensions of f and g such that $\tilde f_{|t<0} = 0$ and $\tilde g_{|t<0} = 0$. Such extensions exist due to the assumptions on f and g at $t = 0$. From Theorem 4.10.6, the solution of the BVP $L\tilde u = \tilde f$, $B\tilde u_{|\partial\Omega} = \tilde g$ satisfies $\tilde u \in e^{\gamma t}H^k(\mathbb{R}\times(0,1))$ and $\tilde u_{|\partial\Omega} \in e^{\gamma t}H^k(\mathbb{R})$. Moreover, there exists a sequence $(\tilde u_j)_j \subset e^{\gamma t}H^{k+1}_\gamma(\mathbb{R}\times(0,1))$ such that $\tilde u_j \to \tilde u$ in $e^{\gamma t}H^k_\gamma(\mathbb{R}\times(0,1))$, $L\tilde u_j \to L\tilde u$ in $e^{\gamma t}H^k_\gamma(\mathbb{R}\times(0,1))$ and $\tilde u_{j|\partial\Omega} \to \tilde u_{|\partial\Omega}$ in $e^{\gamma t}H^k_\gamma(\mathbb{R})$. Let $u_j = \tilde u_{j|Q_T}$. Applying the a priori estimate to $u_j - u_k$ in Theorem 4.11.2 shows that $(u_j)_j$ and $(u_{j|\partial\Omega})_j$ are Cauchy sequences in $CH^k(Q_T) \cap H^k(Q_T)$ and $H^k(0,T)$, respectively. If $u = \tilde u_{|Q_T}$ then u is the weak solution of the IBVP and satisfies the conclusion of the theorem while the sequence $(u_j)_j$ is the required sequence in the statement of the theorem. □

We say that the data $(u_0, f, g) \in H^k(0,1) \times H^k(Q_T) \times H^k(0,T)$ satisfies the compatibility condition up to order $k-1$ if

$$B_y u_i(y) = D^i g_y(0), \qquad i = 0, \dots, k-1, \; y = 0, 1, \tag{4.11.2}$$

where

$$u_i = -A\partial_x u_{i-1} - Ru_{i-1} - \partial_t^{i-1} f_{|t=0}, \qquad i = 1, \dots, k. \tag{4.11.3}$$

Theorem 4.11.4. *Let k be a positive integer. If $f \in H^k(Q_T)$, $g \in H^k(0,T)$ and $u_0 \in H^k(0,1)$ satisfy the compatibility condition up to order $k-1$ then the weak solution of the IBVP*

$$Lu = f, \qquad Bu_{|\partial\Omega} = g, \qquad u_{|t=0} = u_0 \tag{4.11.4}$$

satisfies $u \in CH^k(Q_T)$ and $u_{|\partial\Omega} \in H^k(0,T)$. There is a sequence $(u_j)_j \subset H^{k+1}(Q_T)$ with the properties $u_j \to u$ in $CH^k(Q_T)$, $Lu_j \to Lu$ in $H^k(Q_T)$ and $u_{j|\partial\Omega} \to u_{|\partial\Omega}$ in $H^k(0,T)$. Moreoever, u satisfies the energy estimate

$$\begin{aligned} &e^{-\gamma T} \sum_{|\alpha| \le k} \gamma^{2(k-|\alpha|)} \sup_{\tau \in [0,T]} \|\partial^\alpha u(\tau)\|^2_{L^2(0,1)} + \gamma \|e^{-\gamma t} u\|^2_{H^k_\gamma(Q_T)} + \|e^{-\gamma t} u_{|\partial\Omega}\|^2_{H^k_\gamma(0,T)} \\ &\le C_k \left(\sum_{j=0}^{k} \|u_j\|^2_{H^{k-j}(0,1)} + \frac{1}{\gamma} \|e^{-\gamma t} f\|^2_{H^k_\gamma(Q_T)} + \|e^{-\gamma t} g\|^2_{H^k_\gamma(0,T)} \right). \end{aligned} \tag{4.11.5}$$

for all $\gamma \ge \gamma_k$ for some $C_k > 0$ and $\gamma_k \ge 1$.

Proof. First suppose that $u_0 \in H^{k+\frac{1}{2}}(0,1)$. From [**1, pp. 216-217**], there exists a function $u_a \in H^{k+1}(\mathbb{R} \times (0,1))$ such that $\partial_t^i (u_a)_{|t=0} = u_i$ for every $i = 0, \dots, k-1$. Let $f_a = f - Lu_a \in H^k(Q_T)$ and $g_a = g - Bu_{a|\partial\Omega} \in H^k(0,T)$. From (4.11.3) we have $\partial_t^i f_{a|t=0} = 0$ for $i = 0, \dots, k-1$ and from the the compatibility conditions (4.11.2) it holds that $D^i g_a(0) = 0$ for $i = 0, \dots, k-1$. According to Theorem 4.11.3, the weak solution of the homogeneous IBVP

$$Lu_h = f_a, \qquad Bu_{h|\partial\Omega} = g_a, \qquad u_{h|t=0} = 0$$

satisfies $u_h \in CH^k(Q_T)$ and $u_{h|\partial\Omega} \in H^k(Q_T)$. Then the solution of the IBVP (4.11.4) is given by $u = u_h + u_a$ and therefore $u \in CH^k(Q_T)$ and $u_{|\partial\Omega} \in H^k(0,T)$. The sequence $(u_{jh} + u_a)_j \subset H^{k+1}(Q_T)$, where $(u_{jh})_j$ is the sequence in Theorem 4.11.3 corresponding to u_h, has the desired properties.

For the case where $u_0 \in H^k(0,1)$, one can find a sequence $(u_{j0})_j \subset H^{k+\frac{1}{2}}(0,1)$ such that (u_{j0}, f, g) is still compatible up to order $k-1$, see [**64**] or the proof of Theorem 4.21.2 below. Thanks to the a priori estimate in Theorem 4.11.2 the desired results can be shown, see the proof of Theorem 4.19.5 and Remark 4.19.7. □

4.12 BVP WITH LIPSCHITZ COEFFICIENTS

We turn to the boundary value problem where the coefficients are Lipschitz. As in Theorem 4.10.1, one can prove the following theorem using the a priori estimate in Theorem 4.7.4 instead of Theorem 4.7.2.

Theorem 4.12.1. *In the framework of Theorem 4.6.6, there exists* $\gamma_0 = \gamma_0(\varrho, K, \mathcal{K}) \geq 1$ *such that for all* $\gamma \geq \gamma_0$, $R \in L^\infty(\mathbb{R} \times (0,1); \mathbb{R}^{n\times n})$ *with* $\|R\|_{L^\infty(\mathbb{R}\times(0,1))} \leq \varrho$, $v \in \mathbb{W}(K, \mathcal{K})$, $f \in e^{\gamma t} L^2(\mathbb{R} \times (0,1))$ *and* $g \in e^{\gamma t} L^2(\mathbb{R})$, *the boundary value problem* (4.8.1) *has a weak solution* $u \in e^{\gamma t} L^2(\mathbb{R} \times (0,1))$ *satisfying the energy estimate*

$$\gamma \|e^{-\gamma t} u\|^2_{L^2(\mathbb{R}\times(0,1))} \leq C \left(\frac{1}{\gamma} \|e^{-\gamma t} f\|^2_{L^2(\mathbb{R}\times(0,1))} + \|e^{-\gamma t} g\|^2_{L^2(\mathbb{R})} \right)$$

for some $C = C(\varrho, K, \mathcal{K}) > 0$.

We show that the weak solution of (4.8.1) is actually a strong solution provided that γ is large enough.

Theorem 4.12.2. *Suppose that the hypotheses of Theorem 4.6.6 hold. Then every weak solution* $u \in e^{\gamma t} L^2(\mathbb{R} \times (0,1))$ *of* (4.8.1) *is a strong solution and* $u \in e^{\gamma t} \mathcal{E}(\mathbb{R} \times (0,1))$. *In particular,* (4.8.1) *has a unique weak solution satisfying the energy estimate*

$$\begin{aligned} &\gamma \|e^{-\gamma t} u\|^2_{L^2(\mathbb{R}\times(0,1))} + \|e^{-\gamma t} u_{|\partial\Omega}\|^2_{L^2(\mathbb{R})} \\ &\leq C \left(\frac{1}{\gamma} \|e^{-\gamma t} f\|^2_{L^2(\mathbb{R}\times(0,1))} + \|e^{-\gamma t} g\|^2_{L^2(\mathbb{R})} \right) \end{aligned} \tag{4.12.1}$$

for every $\gamma \geq \gamma_0$, *for some* $\gamma_0 = \gamma_0(\varrho, K, \mathcal{K}) \geq 1$ *and* $C = C(\varrho, K, \mathcal{K}) > 0$.

To prove this we need a few lemmas. Let $\rho \in \mathscr{D}(\mathbb{R})$ be a mollifier with support in $(-1,1)$ and $\int_\mathbb{R} \rho(t)\,\mathrm{d}t = 1$. Define $\rho_\epsilon(t) = \epsilon^{-1}\rho(t/\epsilon)$. Denote by R_ϵ the convolution operator corresponding to ρ_ϵ, that is,

$$R_\epsilon u := \rho_\epsilon \star u = \mathrm{Op}(\mathscr{F}\rho_\epsilon) u.$$

Then $R_\epsilon \in \mathcal{L}(H^r(\mathbb{R}), H^s(\mathbb{R}))$ for all $r, s \in \mathbb{R}$ and $\epsilon \in (0,1)$. However $(R_\epsilon)_{0<\epsilon<1}$ is uniformly bounded only as operators of order $m \geq 0$.

The first lemma tells us that the trace operator and the convolution operator R_ϵ commute when applied to elements of the graph space $e^{\gamma t} E(\mathbb{R} \times (0,1))$.

Lemma 4.12.3. *Let* $u \in e^{\gamma t} E(\mathbb{R} \times (0,1))$. *Then* $(R_\epsilon(e^{-\gamma t} u))_{|\partial\Omega} = R_\epsilon(e^{-\gamma t} u_{|\partial\Omega}) \in H^{+\infty}(\mathbb{R})$ *for every* $\epsilon \in (0,1)$.

Proof. Fix $\epsilon \in (0,1)$. The fact that $R_\epsilon(e^{-\gamma t} u_{|\partial\Omega}) \in H^{+\infty}(\mathbb{R})$ for every $\epsilon \in (0,1)$ follows from $R_\epsilon \in \mathcal{L}(H^r(\mathbb{R}), H^s(\mathbb{R}))$ for all $r, s \in \mathbb{R}$ and $\epsilon \in (0,1)$. Since $e^{-\gamma t} u \in E(\mathbb{R} \times (0,1))$, there exists a sequence $(u_j^\gamma)_j \in \mathscr{D}(\mathbb{R} \times [0,1])$ with the property $u_j^\gamma \to e^{-\gamma t} u$ in $E(\mathbb{R} \times (0,1))$. Because u_j^γ is smooth one has $(R_\epsilon u_j^\gamma)_{|\partial\Omega} = R_\epsilon((u_j^\gamma)_{|\partial\Omega})$ for all j. By continuity of the generalized trace operator we have $(u_j^\gamma)_{|\partial\Omega} \to e^{-\gamma t} u_{|\partial\Omega}$ in $H^{-\frac{1}{2}}(\mathbb{R})$ and since $R_\epsilon \in \mathcal{L}(H^{-\frac{1}{2}}(\mathbb{R}))$ it follows that $R_\epsilon((u_j^\gamma)_{|\partial\Omega}) \to R_\epsilon(e^{-\gamma t} u_{|\partial\Omega})$ in $H^{-\frac{1}{2}}(\mathbb{R})$. Since $R_\epsilon \in \mathcal{L}(L^2(\mathbb{R}), H^1(\mathbb{R}))$ we have $R_\epsilon u_j^\gamma \to R_\epsilon(e^{-\gamma t} u)$ in $H^1(\mathbb{R})$ and by the continuity of the trace operator it follows wthat $(R_\epsilon u_j^\gamma)_{|\partial\Omega} \to (R_\epsilon(e^{-\gamma t} u))_{|\partial\Omega}$ in $H^{\frac{1}{2}}(\mathbb{R})$. Finally the continuity of the embedding $H^{\frac{1}{2}}(\mathbb{R}) \subset H^{-\frac{1}{2}}(\mathbb{R})$ implies that $(R_\epsilon(e^{-\gamma t} u))_{|\partial\Omega} = R_\epsilon(e^{-\gamma t} u_{|\partial\Omega})$. $\square$

The second lemma shows that differentiation with respect to space and convolution with respect to time associated with the mollifier ρ_ϵ commute for elements of the graph space $E(\mathbb{R} \times (0,1))$.

Lemma 4.12.4. *For each $u \in E(\mathbb{R} \times (0,1))$ it holds that $\partial_x(\rho_\epsilon \star u) = \rho_\epsilon \star (\partial_x u)$ in $H^{-1}(\mathbb{R} \times (0,1))$.*

Proof. For simplicity we let $\mathcal{O} = \mathbb{R} \times (0,1)$. Take $\varphi \in \mathscr{D}(\mathcal{O})$. By Fubini's Theorem and the change of variable $\sigma = t - s$ we have

$$\begin{aligned}
\langle \partial_x(\rho_\epsilon \star u), \varphi \rangle_{H^{-1}(\mathcal{O}) \times H_0^1(\mathcal{O})} &= -\int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}} \rho_\epsilon(t-s) u(s,x) \partial_x \varphi(t,x) \, \mathrm{d}s \, \mathrm{d}t \, \mathrm{d}x \\
&= -\int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}} \rho_\epsilon(\sigma) \partial_x \varphi(\sigma + s, x) u(s,x) \, \mathrm{d}\sigma \, \mathrm{d}s \, \mathrm{d}x \\
&= \int_{\mathbb{R}} \rho_\epsilon(\sigma) \langle \partial_x u, \varphi(\sigma + \cdot, \cdot) \rangle_{H^{-1}(\mathcal{O}) \times H_0^1(\mathcal{O})} \, \mathrm{d}\sigma
\end{aligned}$$

Recall that $\partial_x u = A_v^{-1} L_v u - \partial_t(A_v^{-1} u) + (\partial_t A_v^{-1}) u$. Computations similar as above show that

$$\int_{\mathbb{R}} \rho_\epsilon(\sigma) \langle w, \varphi(\sigma + \cdot, \cdot) \rangle_{H^{-1}(\mathcal{O}) \times H_0^1(\mathcal{O})} \, \mathrm{d}\sigma = \langle \rho_\epsilon \star w, \varphi \rangle_{H^{-1}(\mathcal{O}) \times H_0^1(\mathcal{O})}$$

for all $w \in L^2(\mathcal{O})$. Thus

$$\begin{aligned}
&\int_{\mathbb{R}} \rho_\epsilon(\sigma) \langle \partial_x u, \varphi(\sigma + \cdot, \cdot) \rangle_{H^{-1}(\mathcal{O}) \times H_0^1(\mathcal{O})} \, \mathrm{d}\sigma \\
&= \langle \rho_\epsilon \star w, \varphi \rangle_{H^{-1}(\mathcal{O}) \times H_0^1(\mathcal{O})} - \int_{\mathbb{R}} \rho_\epsilon(\sigma) \langle \partial_t(A_v^{-1} u), \varphi(\sigma + \cdot, \cdot) \rangle_{H^{-1}(\mathcal{O}) \times H_0^1(\mathcal{O})} \, \mathrm{d}\sigma
\end{aligned} \tag{4.12.2}$$

where $w = A_v^{-1} L_v u + (\partial_t A_v^{-1}) u \in L^2(\mathcal{O})$. Let us consider the integral on the right hand side of (4.12.2). Integrating by parts and using the fact that R_ϵ is a convolution operator with respect to t, so that R_ϵ and ∂_t commute, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \rho_\epsilon(\sigma) \langle \partial_t(A_v^{-1} u), \varphi(\sigma + \cdot, \cdot) \rangle_{H^{-1}(\mathcal{O}) \times H_0^1(\mathcal{O})} \, \mathrm{d}\sigma \\
&= -\int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}} \rho_\epsilon(\sigma) A_v^{-1} u(s,x) \partial_t \varphi(s + \sigma, x) \, \mathrm{d}\sigma \, \mathrm{d}s \, \mathrm{d}x \\
&= -\int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}} \rho_\epsilon(\sigma) A_v^{-1} u(t - \sigma, x) \partial_t \varphi(t,x) \, \mathrm{d}t \, \mathrm{d}\sigma \, \mathrm{d}x \\
&= -\int_0^1 \int_{\mathbb{R}} (\rho_\epsilon \star (A_v^{-1} u))(t,x) \partial_t \varphi(t,x) \, \mathrm{d}t \, \mathrm{d}x \\
&= \int_0^1 \int_{\mathbb{R}} \partial_t(\rho_\epsilon \star (A_v^{-1} u))(t,x) \varphi(t,x) \, \mathrm{d}t \, \mathrm{d}x \\
&= \int_0^1 \int_{\mathbb{R}} (\rho_\epsilon \star \partial_t(A_v^{-1} u))(t,x) \varphi(t,x) \, \mathrm{d}t \, \mathrm{d}x \\
&= \langle \rho_\epsilon \star \partial_t(A_v^{-1} u), \varphi \rangle_{H^{-1}(\mathcal{O}) \times H_0^1(\mathcal{O})}
\end{aligned}$$

Therefore using $\partial_x u = w - \partial_t(A_v^{-1} u)$ we have

$$\int_{\mathbb{R}} \langle \rho_\epsilon(\sigma) \partial_x u, \varphi(\sigma + \cdot, \cdot) \rangle_{H^{-1}(\mathcal{O}) \times H_0^1(\mathcal{O})} \, \mathrm{d}\sigma = \langle \rho_\epsilon \star (\partial_x u), \varphi \rangle_{H^{-1}(\mathcal{O}) \times H_0^1(\mathcal{O})}$$

Since $\mathscr{D}(\mathcal{O})$ is dense in $H_0^1(\mathcal{O})$ it follows that $\partial_x(\rho_\epsilon \star u) = \rho_\epsilon \star (\partial_x u)$ in $H^{-1}(\mathcal{O})$. □

The following lemma is a generalization of Friedrichs Lemma, see Theorem C.1.1 and (C.1.4).

Lemma 4.12.5. *For each* $u \in e^{\gamma t}E(\mathbb{R}\times(0,1))$ *and* $\epsilon \in (0,1)$ *we have* $[P_v^\gamma, R_\epsilon]e^{-\gamma t}u \in L^2(\mathbb{R}\times(0,1))$, $[B_v, R_\epsilon]e^{-\gamma t}u_{|\partial\Omega} \in H^{\frac{1}{2}}(\mathbb{R})$,

$$\lim_{\epsilon\to 0} \|[P_v^\gamma, R_\epsilon]e^{-\gamma t}u\|_{L^2(\mathbb{R}\times(0,1))} = 0$$

and

$$\lim_{\epsilon\to 0} \|[B_v, R_\epsilon]e^{-\gamma t}u_{|\partial\Omega}\|_{L^2(\mathbb{R})} = 0.$$

To prove Theorem 4.12.2, we regularized the weak solution using the smoothing operator R_ϵ. This will give us more regularity in time. Using the PDE and Lemma 4.12.5 we can obtain additional regularity in space. The sequence of regularizations satisfy a boundary value problem that is an approximation of the original boundary value problem, and hence, the weak solution is a strong solution. This is the main idea of the proof below.

Proof of Theorem 4.12.2. Let us define the following regularized functions

$$u_\epsilon^\gamma = R_\epsilon(e^{-\gamma t}u), \qquad F_\epsilon^\gamma = R_\epsilon(A_v^{-1}e^{-\gamma t}f), \qquad g_\epsilon^\gamma = R_\epsilon(e^{-\gamma t}g),$$

where u is the weak solution of the BVP (4.8.1). For each $\epsilon > 0$, we have $u_\epsilon^\gamma \in L^2((0,1);H^{+\infty}(\mathbb{R}))$, $F_\epsilon^\gamma \in L^2((0,1);H^{+\infty}(\mathbb{R}))$, $g_\epsilon^\gamma \in H^{+\infty}(\mathbb{R})$ and as $\epsilon \to 0$ we have $u_\epsilon^\gamma \to e^{-\gamma t}u$ in $L^2(\mathbb{R}\times(0,1))$, $F_\epsilon^\gamma \to A_v^{-1}e^{-\gamma t}f$ in $L^2(\mathbb{R}\times(0,1))$ and $g_\epsilon^\gamma \to e^{-\gamma t}g$ in $L^2(\mathbb{R})$. According to Theorem 4.8.2, the weak solution u lies in $e^{\gamma t}E(\mathbb{R}\times(0,1))$.

We claim that $u_\epsilon^\gamma \in H^1(\mathbb{R}\times(0,1))$. Recall that

$$\partial_x e^{-\gamma t}u = P_v^\gamma e^{-\gamma t}u + A_v^{-1}e^{-\gamma t}f. \tag{4.12.3}$$

From Lemma 4.12.4 and (4.12.3)

$$\partial_x u_\epsilon^\gamma = R_\epsilon P_v^\gamma e^{-\gamma t}u + F_\epsilon^\gamma = F_\epsilon^\gamma + P_v^\gamma u_\epsilon^\gamma - [R_\epsilon, P_v^\gamma]e^{-\gamma t}u. \tag{4.12.4}$$

By construction $F_\epsilon^\gamma, P_v^\gamma u_\epsilon^\gamma \in L^2(\mathbb{R}\times(0,1))$. According to the previous lemma $[P_v^\gamma, R_\epsilon]e^{-\gamma t}u \in L^2(\mathbb{R}\times(0,1))$. Therefore $\partial_x u_\epsilon^\gamma \in L^2(\mathbb{R}\times(0,1))$ and as a result $u_\epsilon^\gamma \in H^1(\mathbb{R}\times(0,1))$. Applying Theorem 4.6.6 to $e^{\gamma t}(u_\epsilon^\gamma - u_{\epsilon'}^\gamma) \in e^{\gamma t}H^1(\mathbb{R}\times(0,1))$ and using Lemma 4.12.3 and (4.12.4) we have

$$\begin{aligned}
&\gamma\|(u_\epsilon^\gamma - u_{\epsilon'}^\gamma)\|^2_{L^2(\mathbb{R}\times(0,1))} + \|(u_\epsilon^\gamma - u_{\epsilon'}^\gamma)_{|\partial\Omega}\|^2_{L^2(\mathbb{R})} \\
&\le C\left(\frac{1}{\gamma}\|(\partial_x - P_v^\gamma)(u_\epsilon^\gamma - u_{\epsilon'}^\gamma)\|^2_{L^2(\mathbb{R}\times(0,1))} + \|B_v(u_\epsilon^\gamma - u_{\epsilon'}^\gamma)_{|\partial\Omega}\|^2_{L^2(\mathbb{R})}\right) \\
&\le C\left(\frac{1}{\gamma}\|F_\epsilon^\gamma - F_{\epsilon'}^\gamma\|^2_{L^2(\mathbb{R}\times(0,1))} + \frac{1}{\gamma}\|[P_v^\gamma, R_\epsilon - R_{\epsilon'}]e^{-\gamma t}u\|^2_{L^2(\mathbb{R}\times(0,1))}\right. \\
&\qquad\left. + \|[B_v, R_\epsilon - R_{\epsilon'}]e^{-\gamma t}u_{|\partial\Omega}\|^2_{L^2(\mathbb{R})} + \|g_\epsilon^\gamma - g_{\epsilon'}^\gamma\|^2_{L^2(\mathbb{R})}\right).
\end{aligned}$$

Using Lemma 4.12.5, we conclude that $(u_\epsilon^\gamma)_{\epsilon>0}$ and $((u_\epsilon^\gamma)_{|\partial\Omega})_{\epsilon>0}$ are Cauchy sequences in $L^2(\mathbb{R}\times(0,1))$ and $L^2(\mathbb{R})$, respectively. We already know that $u_\epsilon^\gamma \to e^{-\gamma t}u$ in $L^2(\mathbb{R}\times(0,1))$. From (4.12.4)

$$L_v u_\epsilon^\gamma = A_v F_\epsilon^\gamma - \gamma u_\epsilon^\gamma - A_v[R_\epsilon, P_v^\gamma]e^{-\gamma t}u. \tag{4.12.5}$$

Passing to the limit in (4.12.5) we have $L_v u_\epsilon^\gamma \to e^{-\gamma t}f - \gamma e^{-\gamma t}u = L_v(e^{-\gamma t}u)$ in $L^2(\mathbb{R}\times(0,1))$. Thus $u_\epsilon^\gamma \to e^{-\gamma t}u$ in $E(\mathbb{R}\times(0,1))$. The continuity of the generalized trace operator implies that $(u_\epsilon^\gamma)_{|\partial\Omega} \to e^{-\gamma t}u_{|\partial\Omega}$ in $H^{-\frac{1}{2}}(\mathbb{R})$, and hence in $L^2(\mathbb{R})$.

We see from (4.12.4) and Lemma 4.12.4 that $w_\epsilon := e^{\gamma t}u_\epsilon^\gamma \in e^{\gamma t}H^1(\mathbb{R}\times(0,1))$ satisfies the system

$$L_v w_\epsilon = e^{\gamma t}A_v F_\epsilon^\gamma - e^{\gamma t}A_v[R_\epsilon, P_v^\gamma]e^{-\gamma t}u =: f_\epsilon$$
$$B_v(w_\epsilon)_{|\partial\Omega} = e^{\gamma t}[B_v, R_\epsilon]e^{-\gamma t}u_{|\partial\Omega} + e^{\gamma t}g_\epsilon^\gamma =: h_\epsilon.$$

Since $f_\epsilon \in e^{\gamma t}L^2(\mathbb{R}\times(0,1))$, $h_\epsilon \in e^{\gamma t}H^{\frac{1}{2}}(\mathbb{R})$, $f_\epsilon \to f$ in $e^{\gamma t}L^2(\mathbb{R}\times(0,1))$ and $h_\epsilon \to g$ in $e^{\gamma t}L^2(\mathbb{R})$, it follows that u is a strong solution of (4.8.1). The energy estimate (4.12.1) follows from the a priori estimate (4.6.4) applied first to u_ϵ and then passing to the limit $\epsilon \to 0$. The uniqueness of weak solution of (4.8.1) is a consequence of the energy estimate (4.12.1).

The above arguments show that there exists a sequence $(u_j)_j \subset e^{\gamma t}H^1(\mathbb{R}\times(0,1))$ such that $u_j \to u$ in $e^{\gamma t}E(\mathbb{R}\times(0,1))$ and $u_{j|\partial\Omega} \to u_{|\partial\Omega}$ in $e^{\gamma t}L^2(\mathbb{R})$. Therefore $u \in e^{\gamma t}\mathcal{E}(\mathbb{R}\times(0,1))$. □

In studying initial-boundary value problems, the following causality principle will be used. For the proof, we refer to [**9, Theorem 9.13**].

Theorem 4.12.6 (Principle of Causality). *Let $\tau \in \mathbb{R}$. If $f \in L^2(\mathbb{R}\times(0,1))$ and $g \in L^2(\mathbb{R})$ satisfy $f_{|t<\tau} = 0$ and $g_{|t<\tau} = 0$ then the weak solution of* (4.8.1) *also satisfies $u_{|t<\tau} = 0$.*

4.13 IBVP WITH LIPSCHITZ COEFFICIENTS

The proof of existence and uniqueness of weak solutions for the IBVP (4.9.1) is slightly different from the one we have already done for the BVP (4.8.1). Theorem 4.1.1 is not applicable at the moment since a suitable a priori estimate is not available at this point. If the initial data in (4.9.1) is zero, then (4.9.2) is similar to (4.8.2). With this observation, one can prove well-posedness of the homogeneous IBVP by using results for the BVP and the Causality Principle Theorem 4.12.6. Thanks to this procedure we obtain an a priori estimate for the IBVP with homogeneous initial data. By a duality argument, an a priori estimate for the IBVP will be proved, and with this estimate, Theorem 4.1.1 can now be applied to prove the well-posedness of the general IBVP (4.9.1).

The passage from initial-boundary value problems to pure boundary value problems requires a technical step of extending a function in $W^{1,\infty}(Q_T)$ to a function in $W^{1,\infty}(\mathbb{R}\times(0,1))$. This is possible thanks to a standard reflection argument, see Adams [**1, p. 84**].

Theorem 4.13.1. *For each $v \in W^{1,\infty}(Q_T)$ there exists $V \in W^{1,\infty}(\mathbb{R}^2)$ such that $\|v\|_{W^{1,\infty}(Q_T)} = \|V\|_{W^{1,\infty}(\mathbb{R}^2)}$ and v and V have the same range.*

With abuse of notation, we denote by the same notation v the extension V of v stated in Theorem 4.13.1. In this section, we let $\mathbb{W}(K,\mathcal{K})$ denote the set of all functions $v \in W^{1,\infty}(Q_T)$ such that $\operatorname{ran} v \subset \mathcal{K}$ and $\|v\|_{W^{1,\infty}(Q_T)} \le K$.

Theorem 4.13.2. *Suppose that (D) and (UKL) hold. Let $f \in L^2(Q_T)$, $g \in L^2(0,T)$, $v \in \mathbb{W}(K,\mathcal{K})$ and $R \in L^\infty(Q_T)$ with $\|R\|_{L^\infty} \le \varrho$. The homogeneous initial-boundary value problem*

$$L_v u = f, \qquad B_v u_{|\partial\Omega} = g, \qquad u_{|t=0} = 0 \tag{4.13.1}$$

has a unique weak solution. Furthermore, the weak solution is a strong solution and it satisfies $u_{|\partial\Omega} \in L^2(0,T)$ *and the energy estimate*

$$\begin{aligned}
&\gamma\|e^{-\gamma t}u\|^2_{L^2(Q_T)} + \|e^{-\gamma t}u_{|\partial\Omega}\|^2_{L^2(0,T)} \\
&\leq C\left(\frac{1}{\gamma}\|e^{-\gamma t}f\|^2_{L^2(Q_T)} + \|e^{-\gamma t}g\|^2_{L^2(0,T)}\right)
\end{aligned} \tag{4.13.2}$$

for all $\gamma \geq \gamma_0$ *for some* $\gamma_0 = \gamma_0(\varrho, K, \mathcal{K}) \geq 1$ *and* $C = C(\varrho, K, \mathcal{K}) > 0$*. In particular, the boundary condition* $B_v u_{|\partial\Omega} = g$ *holds in* $L^2(0,T)$*.*

Proof. Let $\tilde{f}$ and $\tilde{g}$ be the extensions of f and g by zero outside $(0,T)$ and let $\tilde{u} \in e^{\gamma t}\mathcal{E}(\mathbb{R} \times (0,1))$ be the unique weak solution of the BVP $L_v\tilde{u} = \tilde{f}$, $B_v\tilde{u}_{|\partial\Omega} = \tilde{g}$. We know that this weak solution is strong and by Theorem 4.12.6 $u_{|t<0} = 0$. Let $(\tilde{u}_j)_j \in e^{\gamma t}H^1(\mathbb{R} \times (0,1))$ be the sequence of functions approximating $\tilde{u}$ in the proof of Theorem 4.12.2. In particular, $\tilde{u}_j$ satisfies a BVP

$$L_v\tilde{u}_j = \tilde{f}_j, \qquad B_v\tilde{u}_j = \tilde{g}_j \tag{4.13.3}$$

where $\tilde{f}_j \to \tilde{f}$ in $e^{\gamma t}L^2(\mathbb{R}\times(0,1))$ and $\tilde{g}_j \to \tilde{g}$ in $e^{\gamma t}L^2(\mathbb{R})$. By replacing the mollifiers ρ_ϵ by $\epsilon^{-1}\rho((x-a)/\epsilon)$ for some $a > 0$ small enough in the proof of Theorem 4.12.2, so that they are supported in $\{t > 0\}$, we have $\tilde{u}_{j|t<0} = 0$ for each j.

From (4.13.3) and integration by parts we have

$$\begin{aligned}
\int_0^T\int_0^1 \tilde{u}_j \cdot L_v^*\varphi \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_0^1 \tilde{f}_j \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \tilde{g}_{1j} \cdot M_1(v)\varphi_{|x=1}\,\mathrm{d}t \\
&\quad + \int_0^T \tilde{g}_{0j} \cdot M_0(v)\varphi_{|x=0}\,\mathrm{d}t
\end{aligned} \tag{4.13.4}$$

for all $\varphi \in H^1(Q_T)$ such that $C_v\varphi_{|\partial\Omega} = 0$ and $\varphi_{|t=T} = 0$. Passing to the limit in (4.13.4) yields

$$\begin{aligned}
\int_0^T\int_0^1 u \cdot L_v^*\varphi \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_0^1 f \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T g_1 \cdot M_1(v)\varphi_{|x=1}\,\mathrm{d}t \\
&\quad + \int_0^T g_0 \cdot M_0(v)\varphi_{|x=0}\,\mathrm{d}t
\end{aligned}$$

where $u = \tilde{u}_{|Q_T}$. Thus u is a weak solution of the initial boundary value problem (4.13.1).

Because $\tilde{u}_j \to \tilde{u}$ in $e^{\gamma t}E(\mathbb{R} \times (0,1))$ we also have $u_j := \tilde{u}_{j|Q_T} \to u$ in $E(Q_T)$ from Theorem 4.4.1 and in particular $u_{j|x=0} \to u_{|x=0}$ in $V(\Sigma_1)'$ and $u_{j|x=1} \to u_{|x=1}$ in $V(\Sigma_2)'$. However we already have $u_{j|\partial\Omega} \to u_{|\partial\Omega}$ in $L^2(0,T)$. Thus $u_{|\partial\Omega} \in L^2(0,T)$ since the second inclusion in (4.4.14) is continuous. Likewise, $u_{j|t=0} = 0$ for all j so that $u_{|t=0} = 0$ in $L^2(0,1)$. Because $(u_j)_j \subset H^1(Q_T)$, $(f_j)_j \subset L^2(Q_T)$ and $(g_j)_j \subset H^{\frac{1}{2}}(0,T)$ satisfy $L_v u_j = f_j$, $Bu_{j|\partial\Omega} = g_j$, $u_{j|t=0} = 0$, the weak solution constructed above is a strong solution.

As the function $\tilde{u}_j$ satisfies the boundary value problem (4.13.3), it also satisfies the energy estimate

$$\gamma\|e^{-\gamma t}\tilde{u}_j\|^2_{L^2(Q_T)} + \|e^{-\gamma t}\tilde{u}_{j|\partial\Omega}\|^2_{L^2(0,T)} \leq C\left(\frac{1}{\gamma}\|e^{-\gamma t}\tilde{f}_j\|^2_{L^2(\mathbb{R}\times(0,1))} + \|\tilde{g}_j\|^2_{L^2(\mathbb{R})}\right).$$

according to Theorem 4.12.2. Letting $j \to \infty$ and recalling that $\tilde{f}$ and $\tilde{g}$ vanish for $t \in (-\infty, 0) \cup (T, \infty)$, it follows that the energy estimate (4.13.2) is satisfied by the weak solution that we have constructed.

It remains to prove that the weak solution of the IBVP is unique. For this, we suppose that u_1 and u_2 are any weak solutions and let $w = u_1 - u_2 \in L^2(Q_T)$. Then w is a weak solution of the homogeneous IBVP

$$L_v w = 0, \qquad B_v w_{|\partial\Omega} = 0, \qquad w_{|t=0} = 0.$$

This means that

$$\int_0^T \int_0^1 w \cdot L_v^* \varphi \,\mathrm{d}x \,\mathrm{d}t = 0 \tag{4.13.5}$$

for all $\varphi \in H^1(Q_T)$ such that $C_v \varphi_{|\partial\Omega} = 0$ and $\varphi_{|t=T} = 0$. Fix $\tau \in (0, T)$. Let $\theta_\tau \in \mathscr{D}(\mathbb{R})$ be a cut-off function such that $\theta_\tau(t) = 1$ for $t \leq \tau$ and $\theta_\tau(t) = 0$ for $t \geq T$. Let $\tilde{w}$ be the extension of w by zero outside $(0, T)$. Take $\psi \in e^{-\gamma t} H^1(\mathbb{R} \times (0, 1))$ with $C_v \psi_{|\partial\Omega} = 0$. From the equality $L_v^*(\theta_\tau \psi) = \theta_\tau L_v^* \psi - \theta_\tau' \psi$ we have

$$\begin{aligned} \int_{\mathbb{R}} \int_0^1 \theta_\tau \tilde{w} \cdot L_v^* \psi \,\mathrm{d}x \,\mathrm{d}t &= \int_0^T \int_0^1 w \cdot L_v^*(\theta_\tau \psi) \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_0^1 \theta_\tau' w \cdot \psi \,\mathrm{d}x \,\mathrm{d}t \\ &= \int_{\mathbb{R}} \int_0^1 \theta_\tau' \tilde{w} \cdot \psi \,\mathrm{d}x \,\mathrm{d}t \end{aligned} \tag{4.13.6}$$

where the second equality is based on (4.13.5) with φ replaced by $\theta_\tau \psi$. This is possible since $\theta_\tau \psi \in H^1(Q_T)$, $C(\theta_\tau \psi)_{|\partial\Omega} = 0$ and $(\theta_\tau \psi)_{|t=T} = 0$. Therefore, from (4.13.6) we can see that $z := \theta_\tau \tilde{w}$ satisfies the boundary value problem $L_v z = \theta_\tau' \tilde{w}$, $B_v z_{|\partial\Omega} = 0$. By construction, $\theta_\tau' \tilde{w} = 0$ for $t < \tau$ and therefore $z = 0$ for $t < \tau$ by Theorem 4.12.6. Consequently, $w = 0$ a.e. in Q_τ. Since $\tau \in (0, T)$ is arbitrary, we have $w = 0$ a.e. in Q_T and thus the uniqueness of weak solutions for (4.13.1). $\square$

Using Friedrichs symmetrizability we can prove an a priori estimate which includes terms that are pointwise-in-time.

Theorem 4.13.3. *In the framework of Theorem 4.13.2, suppose in addition that (FS) holds. For all $u \in H^1(Q_T)$ satisfying $u_{|t=0} = 0$ there exist constants $C = C(\varrho, K, \mathcal{K}) > 0$ and $\gamma_0 = \gamma_0(\varrho, K, \mathcal{K}) \geq 1$ such that*

$$\begin{aligned} & e^{-2\gamma T} \|u\|_{CL^2(Q_T)}^2 + \gamma \|e^{-\gamma t} u\|_{L^2(Q_T)}^2 + \|e^{-\gamma t} u_{|\partial\Omega}\|_{L^2(0,T)}^2 \\ & \leq C \left(\frac{1}{\gamma} \|e^{-\gamma t} L_v u\|_{L^2(Q_T)}^2 + \|e^{-\gamma t} B_v u_{|\partial\Omega}\|_{L^2(0,T)}^2 \right) \end{aligned} \tag{4.13.7}$$

for all $\gamma \geq \gamma_0$.

Proof. We use the same notation as in the proof of Theorem 4.13.2. Take $u \in H^1(Q_T)$ satisfying $u_{|t=0} = 0$. Thanks to Theorem 4.13.2 we already have

$$\begin{aligned} & \gamma \|e^{-\gamma t} u\|_{L^2(Q_T)}^2 + \|e^{-\gamma t} u_{|\partial\Omega}\|_{L^2(0,T)}^2 \\ & \leq C \left(\frac{1}{\gamma} \|e^{-\gamma t} L_v u\|_{L^2(Q_T)}^2 + \|e^{-\gamma t} B_v u_{|\partial\Omega}\|_{L^2(0,T)}^2 \right) \end{aligned} \tag{4.13.8}$$

by taking $f = L_v u$ and $g = B_v u_{|\partial\Omega}$.

Denote by S_v the Friedrichs symmetrizer of A_v and let $\sigma = \sigma(K, \mathcal{K}) > 0$ be a constant independent of v such that $\sigma I_n \leq S_v \leq \sigma^{-1} I_n$. Define $u_\gamma = e^{-\gamma t} u$ so that $L_v^\gamma u_\gamma = e^{-\gamma t} L_v u$. Since S_v is symmetric we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_0^\tau \int_0^1 S_v u_\gamma \cdot u_\gamma \,\mathrm{d}x\,\mathrm{d}t \qquad (4.13.9)\\
&= \int_0^\tau \int_0^1 (\partial_t S_v) u_\gamma \cdot u_\gamma + S_v \partial_t u_\gamma \cdot u_\gamma + S_v u_\gamma \cdot \partial_t u_\gamma \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\tau \int_0^1 (\partial_t S_v) u_\gamma \cdot u_\gamma + 2\Re S_v \partial_t u_\gamma \cdot u_\gamma \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\tau \int_0^1 (\partial_t S_v) u_\gamma \cdot u_\gamma + 2\Re[S_v (L_v^\gamma - \gamma) u_\gamma \cdot u_\gamma] - 2\Re[S_v A_v \partial_x u_\gamma \cdot u_\gamma] \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Integrating by parts gives us

$$\begin{aligned}
&\int_0^\tau \int_0^1 S_v A_v \partial_x u_\gamma \cdot u_\gamma \,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau S_v(t,1) A_v(t,1) u_\gamma(t,1) \cdot u_\gamma(t,1) \,\mathrm{d}t \\
&- \int_0^\tau S_v(t,0) A_v(t,0) u_\gamma(t,0) \cdot u_\gamma(t,0) \,\mathrm{d}t - \int_0^\tau \int_0^1 u_\gamma \cdot \partial_x (A_v^\top S_v u_\gamma) \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

However we have $\partial_x (A_v^\top S_v u_\gamma) = (\partial_x (A_v^\top S_v)) u_\gamma + A_v^\top S_v \partial_x u_\gamma$ and thus

$$\begin{aligned}
&2 \int_0^\tau \int_0^1 \Re[S_v A_v \partial_x u_\gamma \cdot u_\gamma] \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\tau S_v(t,1) A_v(t,1) u_\gamma(t,1) \cdot u_\gamma(t,1) \,\mathrm{d}t - \int_0^\tau S_v(t,0) A_v(t,0) u_\gamma(t,0) \cdot u_\gamma(t,0) \,\mathrm{d}t \\
&- \int_0^\tau \int_0^1 (\partial_x (A_v^\top S_v)) u_\gamma \cdot u_\gamma \,\mathrm{d}x\,\mathrm{d}t \qquad (4.13.10)
\end{aligned}$$

Therefore from (4.13.9) and (4.13.10) we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_0^\tau \int_0^1 S_v u_\gamma \cdot u_\gamma \,\mathrm{d}x\,\mathrm{d}t \; &= \; - \int_0^\tau S_v(t,1) A_v(t,1) u_\gamma(t,1) \cdot u_\gamma(t,1) \,\mathrm{d}t \\
&+ \int_0^\tau S_v(t,0) A_v(t,0) u_\gamma(t,0) \cdot u_\gamma(t,0) \,\mathrm{d}t \\
&+ \int_0^\tau \int_0^1 (\partial_t S_v + \partial_x (A_v^\top S_v)) u_\gamma \cdot u_\gamma + 2\Re[S_v (L_v^\gamma - \gamma) u_\gamma \cdot u_\gamma] \,\mathrm{d}x\,\mathrm{d}t. \qquad (4.13.11)
\end{aligned}$$

By Cauchy-Schwarz inequality and Young's inequality and $\sigma I_n \leq S_v$ we have

$$\begin{aligned}
&\|u_\gamma(\tau)\|^2_{L^2(0,1)} \qquad (4.13.12)\\
&\leq C \left((1+\gamma) \|u_\gamma\|^2_{L^2(Q_\tau)} + \frac{1}{\gamma} \|L_v^\gamma u_\gamma\|^2_{L^2(Q_\tau)} + \|(u_\gamma)_{|\partial\Omega}\|^2_{L^2(0,\tau)} \right)
\end{aligned}$$

for every $\tau \in [0, T]$. Therefore (4.13.7) follows from (4.13.8) and (4.13.12). □

With Friedrichs symmetrizability, additional regularity in time is possible for the weak solution of (4.13.1). Furthermore, the solution lies on a subspace of the graph space $E(Q_T)$.

We let $\mathcal{E}(Q_T)$ be the space of all functions $\varphi \in E(Q_T)$ such that $\varphi_{|\partial Q_T} \in L^2(\partial Q_T)$ and there exists a sequence $(\varphi_j)_j \subset H^1(Q_T)$ with the property that

$$\lim_{j\to\infty} \|u_j - u\|_{E(Q_T)} + \|u_{j|\partial Q_T} - u_{|\partial Q_T}\|_{L^2(\partial Q_T)} = 0. \tag{4.13.13}$$

Obviously, we have $H^1(Q_T) \subset \mathcal{E}(Q_T)$. The space $\mathcal{E}^*(Q_T)$ is also defined in a similar manner where L is replaced by L^*.

Theorem 4.13.4. *The space $\mathcal{E}(Q_T)$ is the completion of $H^1(Q_T)$ with respect to the norm*

$$\|u\|_{\mathcal{E}(Q_T)} := (\|u\|^2_{E(Q_T)} + \|u_{|\partial Q_T}\|^2_{L^2(\partial Q_T)})^{\frac{1}{2}}. \tag{4.13.14}$$

Proof. Denote by $\tilde{\mathcal{E}}(Q_T)$ the completion of $H^1(Q_T)$ under the norm $\|\cdot\|_{\mathcal{E}(Q_T)}$. Let $u \in \mathcal{E}(Q_T)$ and $(u_j)_j \subset H^1(Q_T)$ be its corresponding sequence. From (4.13.13) it follows that $(u_j)_j$ is a Cauchy sequence in $\tilde{\mathcal{E}}(Q_T)$. Thus, $u_j \to w$ in $\tilde{\mathcal{E}}(Q_T)$ for some $w \in \tilde{\mathcal{E}}(Q_T)$. In particular, $u_j \to w$ in $E(Q_T)$. By uniqueness of limits in $E(Q_T)$ it follows that $u = w$ and so $u \in \tilde{\mathcal{E}}(Q_T)$.

Conversely, suppose that $u \in \tilde{\mathcal{E}}(Q_T)$ so that there exists $(u_j)_j \subset H^1(Q_T)$ such that $u_j \to u$ in $\tilde{\mathcal{E}}(Q_T)$. Thus (4.13.13) holds and $u \in E(Q_T)$. It remains to show that $u_{|\partial Q_T} \in L^2(\partial Q_T)$. It follows from (4.13.13) that there exists $v \in L^2(\partial Q_T)$ such that $u_{j|\partial Q_T} \to v$ in $L^2(\partial Q_T)$. Since $u_j \to u$ in $E(Q_T)$ we have $u_{j|x=0} \to u_{|x=0}$ in $V(\Sigma_1)'$. Because the inclusion $L^2(0,T) \subset V(\Sigma_1)'$ is continuous we have $u_{|x=0} = v_{|\Sigma_1}$. Similarly, we have $u_{|\Sigma_i} = v_{|\Sigma_i}$ for $i = 0, 2, 3$. Therefore $u_{|\partial Q_T} = v$ on ∂Q_T and so $u_{|\partial Q_T} \in L^2(\partial Q_T)$. Hence $u \in \mathcal{E}(Q_T)$ and this proves the other inclusion. $\square$

As in the proof of Theorem 4.10.3, we have the following generalized Green's identity.

Theorem 4.13.5. *For every $u \in \mathcal{E}(Q_T)$ and $\varphi \in \mathcal{E}^*(Q_T)$ we have*

$$\begin{aligned}\int_0^T\int_0^1 u \cdot L^*\varphi \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_0^1 Lu \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T A(t,1)u(t,1)\cdot\varphi(t,1)\,\mathrm{d}t \\ &\quad + \int_0^T A(t,0)u(t,0)\cdot\varphi(t,0)\,\mathrm{d}t - \int_0^1 u(T,x)\cdot\varphi(T,x)\,\mathrm{d}x \\ &\quad + \int_0^1 u(0,x)\cdot\varphi(0,x)\,\mathrm{d}x. \end{aligned} \tag{4.13.15}$$

Corollary 4.13.6. *In the situation of Theorem 4.13.3, the solution u of* (4.13.1) *lies in $CL^2(Q_T) \cap \mathcal{E}(Q_T)$ and satisfies the energy estimate*

$$\begin{aligned} &e^{-2\gamma T}\|u\|^2_{CL^2(Q_T)} + \gamma\|e^{-\gamma t}u\|^2_{L^2(Q_T)} + \|e^{-\gamma t}u_{|\partial\Omega}\|^2_{L^2(0,T)} \\ &\le C\left(\frac{1}{\gamma}\|e^{-\gamma t}f\|^2_{L^2(Q_T)} + \|e^{-\gamma t}g\|^2_{L^2(0,T)}\right). \end{aligned} \tag{4.13.16}$$

Moreover, there exists a sequence $(u_j)_j \in H^1(Q_T)$ such that $u_j \to u$ in $CL^2(Q_T) \cap E(Q_T)$ and $u_{j|\partial\Omega} \to u_{|\partial\Omega}$ in $L^2(0,T)$.

Proof. We know from Theorem 4.13.2 that the weak solution u of (4.13.1) is a strong one. Let $(u_j)_j \subset H^1(Q_T)$ be a sequence corresponding to the strong solution u. Applying the a priori estimate (4.13.7) to $u_j - u_k$, we can see that $(u_j)_j$ and $(u_{j|\partial\Omega})_j$ are Cauchy sequences in $CL^2(Q_T)$ and $L^2(0,T)$, respectively. Since we already know that $u_j \to u$ in $L^2(Q_T)$ we must have $u_j \to u$ in $CL^2(Q_T)$ as well since the inclusion $CL^2(Q_T) \subset L^2(Q_T)$ is continuous. From Theorem 4.9.4, $u_j \to u$ in $E(Q_T)$ and $u_{j|\Sigma_i} \to u_{|\Sigma_i}$ in $V(\Sigma_i)'$ for every $i = 0, 1, 2, 3$. According (4.4.14) we have $u_{j|\partial\Omega} \to u_{|\partial\Omega}$ in $L^2(0, T)$. Finally, since $u_j \to u$ in $CL^2(Q_T)$ we have $u_{j|t=\tau} \to u_{|t=\tau}$ in $L^2(0, 1)$ for every $\tau \in [0, T]$ and so $u \in \mathcal{E}(Q_T)$. □

With a duality argument, the a priori estimate in Theorem 4.13.3 can be improved to all functions $u \in H^1(Q_T)$.

Theorem 4.13.7. *In the situation of Theorem 4.13.3, there are constants $C = C(\varrho, K, \mathcal{K}) > 0$ and $\gamma_0 = \gamma_0(\varrho, K, \mathcal{K}) \geq 1$ such that the a priori estimate*

$$\begin{aligned} &e^{-2\gamma T}\|u\|^2_{CL^2(Q_T)} + \gamma\|e^{-\gamma t}u\|^2_{L^2(Q_T)} + \|e^{-\gamma t}u_{|\partial\Omega}\|^2_{L^2(0,T)} \\ &\leq C\left(\|u_{|t=0}\|^2_{L^2(0,1)} + \frac{1}{\gamma}\|e^{-\gamma t}L_v u\|^2_{L^2(Q_T)} + \|e^{-\gamma t}B_v u_{|\partial\Omega}\|^2_{L^2(0,T)}\right) \end{aligned} \tag{4.13.17}$$

holds for all $u \in H^1(Q_T)$ and $\gamma \geq \gamma_0$.

Proof. Suppose that $F \in e^{-\gamma t}L^2(Q_T)$, $G \in e^{-\gamma t}L^2(0, T)$ and $u \in H^1(Q_T)$. Let z be the solution of the IBVP

$$L_v^* z = F, \qquad C_v z_{|\partial\Omega} = G, \qquad z_{|t=T} = 0.$$

The dual version of Corollary 4.13.6 implies that z satisfies the energy estimate

$$\begin{aligned} &\|z_{|t=0}\|^2_{L^2(0,1)} + \gamma\|e^{\gamma t}z\|^2_{L^2(Q_T)} + \|e^{\gamma t}z_{|\partial\Omega}\|^2_{L^2(0,T)} \\ &\leq C\left(\frac{1}{\gamma}\|e^{\gamma t}F\|^2_{L^2(Q_T)} + \|e^{\gamma t}G\|^2_{L^2(0,T)}\right) \end{aligned} \tag{4.13.18}$$

and $z \in \mathcal{E}^*(Q_T)$. Using the generalized Green's identity (4.13.15) for u and z

$$\begin{aligned} &\int_{Q_T} u \cdot F \,\mathrm{d}x\,\mathrm{d}t + \int_0^T N_{1v}u_{|x=1} \cdot G_1 \,\mathrm{d}t - \int_0^T N_{0v}u_{|x=0} \cdot G_0 \,\mathrm{d}t \\ &= \int_{Q_T} L_v u \cdot z \,\mathrm{d}x\,\mathrm{d}t - \int_0^T B_{1v}u_{|x=1} \cdot M_{1v}z_{|x=1} \,\mathrm{d}t + \int_0^T B_{0v}u_{|x=0} \cdot M_{0v}z_{|x=0} \,\mathrm{d}t \\ &\quad + \int_0^1 u_{|t=0} \cdot z_{|t=0} \,\mathrm{d}x. \end{aligned} \tag{4.13.19}$$

Taking $G_0 = G_1 = 0$ in (4.13.19), using the Cauchy-Schwarz inequality and the estimate (4.13.18) we have

$$\begin{aligned} \int_{Q_T} u \cdot F \,\mathrm{d}x\,\mathrm{d}t &\leq C(\|e^{\gamma t}z\|_{L^2(Q_T)}\|e^{-\gamma t}L_v u\|_{L^2(Q_T)} \\ &\quad + \|e^{\gamma t}z_{|\partial\Omega}\|_{L^2(0,T)}\|e^{-\gamma t}B_v u_{|\partial\Omega}\|_{L^2(0,T)} + \|z_{|t=0}\|_{L^2(0,1)}\|u_{|t=0}\|_{L^2(0,1)}) \\ &\leq \frac{C}{\sqrt{\gamma}}\|e^{\gamma t}F\|_{L^2(Q_T)}Q(u) \end{aligned} \tag{4.13.20}$$

where

$$Q(u) := \|u_{|t=0}\|_{L^2(0,1)} + \frac{1}{\sqrt{\gamma}}\|e^{-\gamma t}L_v u\|_{L^2(Q_T)} + \|e^{-\gamma t}B_v u_{|\partial\Omega}\|_{L^2(0,T)}$$

Dividing by the norm involving F and taking the supremum over all $F \in e^{\gamma t} L^2(Q_T)$ in (4.13.20) yields

$$\sqrt{\gamma} \| e^{-\gamma t} u \|_{L^2(Q_T)} \leq C Q(u). \tag{4.13.21}$$

Similarly, letting $F = 0$ and G arbitrary in (4.13.19) we have

$$\| e^{-\gamma t} N_v u_{|\partial\Omega} \|_{L^2(0,T)} \leq C Q(u). \tag{4.13.22}$$

Define the $2n \times 2n$ matrix-valued function

$$P = \begin{pmatrix} B_{0v} & O_{p\times n} \\ N_{0v} & O_{(n-p)\times n} \\ O_{(n-p)\times n} & B_{1v} \\ O_{p\times n} & N_{1v} \end{pmatrix}$$

where N_{1v} are the matrices in Lemma 4.7.1. Note that P is invertible and hence

$$\begin{aligned} \| e^{-\gamma t} u_{|\partial\Omega} \|_{L^2(0,T)} &= C \| e^{-\gamma t} P^{-1} P u_{|\partial\Omega} \|_{L^2(0,T)} \\ &\leq C (\| e^{-\gamma t} B_v u_{|\partial\Omega} \|_{L^2(0,T)} + \| e^{-\gamma t} N_v u_{|\partial\Omega} \|_{L^2(0,T)}). \end{aligned} \tag{4.13.23}$$

Revisiting the proof of Theorem 4.13.3, we have

$$\| e^{-\gamma\tau} u(\tau) \|^2_{L^2(0,1)} \leq C (Q(u)^2 + \gamma \| e^{-\gamma t} u \|^2_{L^2(Q_T)}) \tag{4.13.24}$$

for all $\tau \in [0, T]$. The main difference here is the occurrence of the tern $u_{|t=0}$, which does not appear in Theorem 4.13.3 due to the assumption on u there. The conclusion now follows form (4.13.21)–(4.13.24). □

There is also a corresponding a priori estimate for the dual problem. We leave the details of this estimate to the reader. The proof of the following corollary follows from the dual version of (4.13.17) and the definition of $\mathcal{E}^*(Q_T)$.

Corollary 4.13.8. *In the situation of Theorem 4.13.3, there exist $C = C(\varrho, K, \mathcal{K}) > 0$ and $\gamma_0 = \gamma_0(\varrho, K, \mathcal{K}) \geq 1$ such that a priori estimate*

$$\begin{aligned} &\| u_{|t=0} \|^2_{L^2(0,1)} + \gamma \| e^{\gamma t} u \|^2_{L^2(Q_T)} + \| e^{\gamma t} u_{|\partial\Omega} \|^2_{L^2(0,T)} \\ &\leq C \left(e^{2\gamma T} \| u_{|t=T} \|^2_{L^2(0,1)} + \frac{1}{\gamma} \| e^{\gamma t} L_v^* u \|^2_{L^2(Q_T)} + \| e^{\gamma t} C_v u_{|\partial\Omega} \|^2_{L^2(0,T)} \right) \end{aligned} \tag{4.13.25}$$

holds for all $u \in \mathcal{E}^*(Q_T)$ and $\gamma \geq \gamma_0$.

For the coupled PDE-ODE system that will be discussed in Section 4.20, the a priori estimate (4.13.25) will be used.

Theorem 4.13.9. *Suppose that the hypotheses of Theorem 4.13.3 hold. Then the inhomogeneous IBVP* (4.9.1) *has a unique weak solution and there exist $C = C(\varrho, K, \mathcal{K}) > 0$ and $\gamma_0 = \gamma_0(\varrho, K, \mathcal{K}) \geq 1$ such that*

$$\gamma \| e^{-\gamma t} u \|^2_{L^2(Q_T)} \leq C \left(\| u_0 \|^2_{L^2(0,1)} + \frac{1}{\gamma} \| e^{-\gamma t} f \|^2_{L^2(Q_T)} + \| e^{-\gamma t} g \|^2_{L^2(0,T)} \right). \tag{4.13.26}$$

holds for every $\gamma \geq \gamma_0$.

Proof. We apply Theorem 4.1.1. Let $X = e^{-\gamma t}L^2(Q_T)$, $Y = H^1(Q_T)$ and $Z = e^{-\gamma t}L^2(0,T) \times e^{-\gamma t}L^2(0,T) \times L^2(0,1)$. Define $\Lambda : Y \to X$, $\Psi : Y \to Z$, $\Phi : Y \to Z$ by

$$\Lambda\varphi = L_v^*\varphi, \quad \Psi\varphi = (M_{0v}\varphi_{|x=0}, -M_{1v}\varphi_{|x=1}, \varphi_{|t=0}), \quad \Phi\varphi = (C\varphi_{|\partial\Omega}, \varphi_{|t=T}).$$

for $\varphi \in Y$. The variational equation (4.9.2) can be written as

$$(e^{-2\gamma t}u, \Lambda w)_X = (e^{-2\gamma t}f, w)_X + ((e^{-2\gamma t}g_0, e^{-2\gamma t}g_1, u_0), \Psi w)_Z \tag{4.13.27}$$

for all $w \in W = \ker\Phi$. The existence of a solution for (4.13.27) satisfying (4.13.26) follows the same lines of argument as in the proof of Theorem 4.10.1 thanks to the dual version of the a priori estimate (4.13.17). The uniqueness of weak solutions follows from the uniqueness of weak solutions for homogeneous problems stated in Theorem 4.13.3. □

To close this section, we show that the weak solution of (4.9.1) given in Theorem 4.13.9 is a strong solution.

Theorem 4.13.10. *In the situation of Theorem 4.13.3, the weak solution u is a strong solution, $u \in CL^2(Q_T) \cap \mathcal{E}(Q_T)$. There exists a sequence $(u_j)_j \subset H^1(Q_T)$ such that $u_j \to u$ in $CL^2(Q_T) \cap E(Q_T)$ and $u_{j|\partial\Omega} \to u_{|\partial\Omega}$ in $L^2(0,T)$. Furthermore, there exist $\gamma_0 = \gamma_0(\varrho, K, \mathcal{K}) \geq 1$ and $C = C(\varrho, K, \mathcal{K}) > 0$ such that u satisfies the energy estimate*

$$\begin{aligned} &e^{-2\gamma T}\|u\|^2_{CL^2(Q_T)} + \gamma\|e^{-\gamma t}u\|^2_{L^2(Q_T)} + \|e^{-\gamma t}u_{|\partial\Omega}\|^2_{L^2(0,T)} \\ &\leq C\left(\|u_0\|^2_{L^2(0,1)} + \frac{1}{\gamma}\|e^{-\gamma t}f\|^2_{L^2(Q_T)} + \|e^{-\gamma t}g\|^2_{L^2(0,T)}\right) \end{aligned} \tag{4.13.28}$$

for every $\gamma \geq \gamma_0$.

Proof. Suppose that $u_0 \in L^2(0,1)$. Let $(u_{0j})_j \subset H^1(0,1)$ be such that $u_{0j} \to u_0$ in $L^2(0,1)$. Let u_{jc} be the weak solution of the Cauchy problem

$$Lu_{jc} = 0, \qquad u_{jc|t=0} = u_{0j},$$

where u_{0j} is extended to the whole of $\mathbb{R}$. From Theorem [**9, Theorem 2.9**], $u_{jc} \in CH^1(Q_T) \subset H^1(Q_T)$, and so $Bu_{jc|\partial\Omega} \in H^{\frac{1}{2}}(0,T)$. Using Green's identity we have

$$\begin{aligned} \int_{Q_T} u_{jc} \cdot L_v^*\varphi \,\mathrm{d}x\,\mathrm{d}t &= -\int_0^T B_{1v}u_{jc|x=1} \cdot M_{1v}\varphi\,\mathrm{d}t + \int_0^T B_{0v}u_{jc|x=0} \cdot M_{0v}\varphi\,\mathrm{d}t \\ &\quad + \int_0^1 u_{0j} \cdot \varphi_{|t=0}\,\mathrm{d}x \end{aligned} \tag{4.13.29}$$

for all $\varphi \in H^1(Q_T)$ such that $C\varphi_{|\partial\Omega} = 0$ and $\varphi_{|t=T} = 0$.

Consider the homogeneous initial-boundary value problem

$$L_v u_{jh} = f, \qquad B_v u_{jh|\partial\Omega} = g - Bu_{jc|\partial\Omega}, \qquad u_{jh|t=0} = 0.$$

From Corollary 4.13.6 this problem has a strong solution and hence for each positive integer j there exists $w_{jh} \in H^1(Q_T)$, $F_j \in L^2(Q_T)$ and $G_j \in H^{\frac{1}{2}}(0,T)$ such that $L_v w_{jh} = F_j$, $B_v w_{jh|\partial\Omega} = G_j$, $w_{jh|t=0} = 0$,

$$\|w_{jh} - u_{jh}\|_{E(Q_T)\cap CL^2(Q_T)} + \|w_{jh|\partial\Omega} - u_{jh|\partial\Omega}\|_{L^2(0,T)} < \frac{1}{j}$$

and

$$\|F_j - f\|_{L^2(Q_T)} + \|G_j - (g - Bu_{jc|\partial\Omega})\|_{L^2(0,T)} < \frac{1}{j}.$$

Thus w_{jh} satisfies the variational equality

$$\begin{aligned}\int_{Q_T} w_{jh} \cdot L_v^* \varphi \,\mathrm{d}x\,\mathrm{d}t &= \int_{Q_T} F_j \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T G_{1j} \cdot M_{1v}\varphi \,\mathrm{d}t \\ &\quad + \int_0^T G_{0j} \cdot M_{0v}\varphi \,\mathrm{d}t \qquad (4.13.30)\end{aligned}$$

for all $\varphi \in H^1(Q_T)$ such that $C\varphi_{|\partial\Omega} = 0$ and $\varphi_{|t=T} = 0$.

Define $w_j = w_{jh} + u_{jc}$. From (4.13.29) and (4.13.30), it can be seen that $w_j \in H^1(Q_T)$ solves the initial-boundary value problem

$$L_v w_j = F_j, \qquad B_v w_{j|\partial\Omega} = G_j + Bu_{jc|\partial\Omega}, \qquad w_{j|t=0} = u_{0j}.$$

Applying the a priori estimate (4.13.17) for $w_j - w_k$ and using $F_j \to f$ in $L^2(Q_T)$ and $G_j + Bu_{jc|\partial\Omega} \to g$ in $L^2(0,T)$ show that $(w_j)_j$ is a Cauchy sequence in $CL^2(Q_T)$. Let w be the limit of $(w_j)_j$ in $CL^2(Q_T)$. Thus w is a strong solution of the inhomogeneous IBVP (4.9.1). Because strong solutions are weak and weak solutions are unique, we must have $u = w$ where u is the weak solution of (4.9.1). It can be checked that $(w_j)_j \subset H^1(Q_T)$ is an approximating sequence for u satisfying all the desired properties stated in the theorem. Applying the a priori estimate (4.13.17) to w_j and then passing to the limit proves (4.13.28). □

We end this section with a simple remark that will be used in Section 4.20.

Remark 4.13.11. According to Green's identity (4.10.2) and Theorem 4.13.10, the weak solution u of the IBVP (4.9.1) satisfies

$$\begin{aligned}\int_0^T\int_0^1 u \cdot L_v^*\varphi \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_0^1 f \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T A_v(t,1)u(t,1) \cdot \varphi(t,1)\,\mathrm{d}t \\ &\quad + \int_0^T A_v(t,0)u(t,0) \cdot \varphi(t,0)\,\mathrm{d}t - \int_0^1 u(T,x) \cdot \varphi(T,x)\,\mathrm{d}x \\ &\quad + \int_0^1 u_0(x) \cdot \varphi(0,x)\,\mathrm{d}x.\end{aligned}$$

for every $\varphi \in \mathcal{E}^*(Q_T)$. In particular, (4.9.2) holds for every $\varphi \in \mathcal{E}^*(Q_T)$ with the properties $C\varphi_{|\partial\Omega} = 0$ and $\varphi_{|t=T} = 0$. On the other hand, if u satisfies (4.9.2) for every $\varphi \in \mathcal{E}^*(Q_T)$ such that $C\varphi_{|\partial\Omega} = 0$ and $\varphi_{|t=T} = 0$ then u must be the unique solution of (4.9.2).

4.14 SOME CLASSICAL SOBOLEV ESTIMATES

Our next goal is to prove the regularity of weak solutions for boundary value problems and initial-boundary value problems where the coefficients are smooth. Again the results rely on a priori estimates, but now in the setting of Sobolev spaces. In preparation we state the following various results on Sobolev spaces.

Proposition 4.14.1. *Let Ω be an open cube or a strip in $\mathbb{R}^d$. For all real numbers $s, t \geq 0$ such that $s + t > 0$, if $u \in H^s(\Omega)$ and $v \in H^t(\Omega)$ then $uv \in H^r(\Omega)$*

for all $0 \le r \le \min(s,t)$ *such that* $r + d/2 < s + t$. *Furthermore, there exists* $C = C(r,s,t,\Omega) > 0$ *such that*

$$\|uv\|_{H^r(\Omega)} \le C\|u\|_{H^s(\Omega)}\|v\|_{H^t(\Omega)}$$

In particular, $H^s(\Omega)$ *is a Banach algebra for all* $s > d/2$.

Proof. The proof follows from a well-known result in the case $\Omega = \mathbb{R}^d$, e.g. [**9, Theorem C.10**]. Indeed, we recall that given a real $q \ge 0$ there exists a continuous operator $E_q : H^q(\Omega) \to H^q(\mathbb{R}^d)$ such that $(E_q u)_{|\Omega} = u$ and

$$\|E_q u\|_{H^q(\mathbb{R}^d)} \le C_q\|u\|_{H^q(\Omega)}$$

for some constant $C_q = C_q(\Omega) > 0$ independent of $u \in H^q(\Omega)$, see e.g. [**1, p. 207–208**]. Then $uv = (E_s u\, E_t v)_{|\Omega} \in H^r(\Omega)$ and

$$\|uv\|_{H^r(\Omega)} \le \|E_s u\, E_t u\|_{H^r(\mathbb{R}^d)} \le C\|E_s u\|_{H^s(\mathbb{R}^d)}\|E_t v\|_{H^t(\mathbb{R}^d)} \le C\|u\|_{H^s(\Omega)}\|v\|_{H^t(\Omega)}.$$

This proves the proposition. □

By induction, if $s_1, \ldots, s_N \ge 0$ are real numbers such that $s_1 + \cdots + s_N > 0$ and if $u_i \in H^{s_i}(\Omega)$ for all $1 \le i \le N$ then $u_1 \cdots u_N \in H^r(\Omega)$ whenever $0 \le r \le \min_{1\le i\le N} s_i$ and $r + d/2 < s_1 + \cdots + s_N$, and moreover, we have the estimate

$$\|u_1 \cdots u_N\|_{H^r(\Omega)} \le C\|u_1\|_{H^{s_1}(\Omega)} \cdots \|u_N\|_{H^{s_N}(\Omega)} \tag{4.14.1}$$

for some $C > 0$ independent of u_i for $1 \le i \le N$.

In a similar way the following commutator estimate can be shown.

Proposition 4.14.2. *Let* Ω *be an open cube or a strip in* $\mathbb{R}^d$, $s \ge [d/2]+2$, $a \in H^s(\Omega)$ *and* $u \in H^{s-1}(\Omega)$. *Then for all* $1 \le |\alpha| \le s$ *we have*

$$\|[\partial^\alpha, a]u\|_{L^2(\Omega)} \le C\|a\|_{H^s(\Omega)}\|u\|_{H^{|\alpha|-1}(\Omega)}.$$

Proposition 4.14.3. *Let* Ω *be an open cube or a strip in* $\mathbb{R}^d$, $s > d/2$ *and* $F \in \mathscr{C}^\infty(\mathbb{R})$ *such that* $F(0) = 0$. *If* $u \in H^s(\Omega)$ *then* $F(u) \in H^s(\Omega)$ *and there exists a continuous function* $C : [0,\infty) \to [0,\infty)$ *such that*

$$\|F(u)\|_{H^s(\Omega)} \le C(\|u\|_{L^\infty(\Omega)})\|u\|_{H^s(\Omega)}.$$

Proof. The proof uses the same ideas as in the proof of the Proposition 4.14.1. We note that the extension operator $E_q : H^q(\Omega) \to H^q(\mathbb{R}^d)$ can be chosen, e.g. successive application of Seeley's reflection argument [**1, p. 84**], in such a way that $\|u\|_{L^\infty(\mathbb{R}^d)} \le C(q,\Omega)\|u\|_{L^\infty(\Omega)}$. Using the same extension argument as above and [**9, Theorem C.12**] one can prove the proposition. □

Similarly, using [**9, Corollary C.3**] one can prove the following.

Proposition 4.14.4. *Let* Ω *be an open cube or a strip in* $\mathbb{R}^d$, $s > d/2$ *and* $F \in \mathscr{C}^\infty(\mathbb{R})$. *Then there exists a continuous function* $C : [0,\infty) \to (0,\infty)$ *such that for all* $u, v \in H^s(\Omega)$ *we have*

$$\|F(u) - F(v)\|_{H^s(\Omega)} \le C(\max(\|u\|_{H^s(\Omega)}, \|v\|_{H^s(\Omega)}))\|u - v\|_{H^s(\Omega)}.$$

4.15 A PRIORI ESTIMATES IN SOBOLEV SPACES WITH TIME INTERVAL $\mathbb{R}$

The proof of the regularity of solutions also relies on an a priori estimate, but now in weighted Sobolev spaces. All throughout this section we let $\Omega = (0, 1)$. Let $v \in H^m(\mathbb{R}\times\Omega)$ taking values on a compact set $\mathcal{K} \subset \mathcal{U}$, $\|v\|_{W^{1,\infty}(\mathbb{R}\times\Omega)} \le K$, $\|v\|_{H^m(\mathbb{R}\times\Omega)} \le R$ and $u \in \mathscr{D}(\mathbb{R} \times \overline{\Omega})$. First we estimate in terms of the norm $\|\cdot\|_{H^m_\gamma}$, where $m \ge 3$ is an integer. We divide the derivation of the estimates into pure time derivatives and mixed derivatives.

4.15.1 **Estimates on Time Derivatives**

Applying the a priori estimate (4.6.4) to $w = \partial_t^\alpha u$ for $\alpha = 0, 1, \ldots, m$ one obtains

$$\begin{aligned}
&\sqrt{\gamma}\|\partial_t^\alpha u\|_{L^2(\Omega;L^2(\mathbb{R}))} + \|(\partial_t^\alpha u)_{|\partial\Omega}\|_{L^2(\mathbb{R})} \\
&\quad\le c\left(\frac{1}{\sqrt{\gamma}}\|L_v^\gamma \partial_t^\alpha u\|_{L^2(\Omega;L^2(\mathbb{R}))} + \|B(\partial_t^\alpha u)_{|\partial\Omega}\|_{L^2(\mathbb{R})}\right).
\end{aligned} \tag{4.15.1}$$

Since B is a constant matrix, the boundary terms on the right hand side of (4.15.1) are given by

$$\begin{aligned}
\sum_{\alpha=0}^{m} \gamma^{m-\alpha}\|B(\partial_t^\alpha u)_{|\partial\Omega}\|_{L^2(\mathbb{R})} &= \sum_{\alpha=0}^{m} \gamma^{m-\alpha}\|\partial_t^\alpha (Bu_{|\partial\Omega})\|_{L^2(\mathbb{R})} \\
&= \|Bu_{|\partial\Omega}\|_{H^m_\gamma(\mathbb{R})}.
\end{aligned} \tag{4.15.2}$$

Here the trace and the derivative commute since u is smooth. The term $L_v^\gamma \partial_t^\alpha u$ is more involved. We rewrite it as

$$L_v^\gamma \partial_t^\alpha u = A(v)\partial_t^\alpha (A(v)^{-1} f) + A(v)[A(v)^{-1} L_v^\gamma, \partial_t^\alpha] u \tag{4.15.3}$$

where $f = L_v^\gamma u$.

For the first term on the right hand side of (4.15.3) we write

$$A(v)\partial_t^\alpha (A(v)^{-1} f) = A(v)\partial_t^\alpha (\mathcal{A}(v) f) + A(v)A(0)^{-1}\partial_t^\alpha f \tag{4.15.4}$$

where $\mathcal{A}(v) = A(v)^{-1} - A(0)^{-1}$ satisfies $\mathcal{A}(0) = 0$. Taking the L^2-norm in (4.15.4) and applying the triangle inequality

$$\|A(v)\partial_t^\alpha (A(v)^{-1} f)\|_{L^2(\mathbb{R}\times\Omega)} \le C\|\partial_t^\alpha (\mathcal{A}(v) f)\|_{L^2(\mathbb{R}\times\Omega)} + C\|f\|_{H^\alpha(\mathbb{R}\times\Omega)}. \tag{4.15.5}$$

Here and below, C is a generic positive constant which depends only on m, $\mathcal{K}$ and K. Let us estimate the first term on the right hand side of (4.15.5). Since the case $\alpha = 0$ is nothing but the L^2-estimate (4.6.4) we only need to consider the case where $\alpha \ge 1$. If $\alpha = 1$ then $\partial_t(\mathcal{A}(v)f) = (\partial_t \mathcal{A}(v))f + \mathcal{A}(v)\partial_t f$ for which can be estimated immediately

$$\gamma^{m-1}\|\partial_t(\mathcal{A}(v)f)\|_{L^2(\mathbb{R}\times\Omega)} \le C\gamma^{m-1}\|f\|_{H^1(\mathbb{R}\times\Omega)} \le C\|f\|_{H^m_\gamma(\mathbb{R}\times\Omega)}.$$

Suppose that $\alpha \ge 2$. Then using Proposition 4.14.1 and (1.1.18)

$$\begin{aligned}
\gamma^{m-\alpha}\|\partial_t^\alpha(\mathcal{A}(v)f)\|_{L^2(\mathbb{R}\times\Omega)} &\le C\gamma^{m-\alpha}\|v\|_{H^\alpha(\mathbb{R}\times\Omega)}\|f\|_{H^\alpha(\mathbb{R}\times\Omega)} \\
&\le C\|v\|_{H^\alpha(\mathbb{R}\times\Omega)}\|f\|_{H^\alpha_\gamma(\mathbb{R}\times\Omega)}
\end{aligned}$$

Therefore it holds that for all $\alpha = 0, 1, \dots, m$

$$\gamma^{m-\alpha}\|A(v)\partial_t^\alpha(A(v)^{-1}f)\|_{L^2(\mathbb{R}\times\Omega)} \le C(1 + \|v\|_{H^m(\mathbb{R}\times\Omega)})\|f\|_{H^m_\gamma(\mathbb{R}\times\Omega)} \tag{4.15.6}$$

We can rewrite the commutator in (4.15.3) in terms of derivatives with respect to t only. Indeed, a straightforward computation gives us

$$A(v)[A(v)^{-1}L_v^\gamma, \partial_t^\alpha]u = A(v)[\partial_t^\alpha, A(v)^{-1}]\partial_t u + \gamma A(v)[\partial_t^\alpha, A(v)^{-1}]u. \tag{4.15.7}$$

Writing $A(v)^{-1} = (A(v)^{-1} - A(0)^{-1}) + A(0)^{-1}$, applying commutator estimate Proposition 4.14.2 (and this is the place where we need the assumption $m \ge 3$) in each term of (4.15.7) together with (1.1.18) and Propsition 4.14.3 we have

$$\gamma^{m-\alpha}\|A(v)[A(v)^{-1}L_v^\gamma, \partial_t^\alpha]u\|_{L^2(\mathbb{R}\times\Omega)} \le C\|v\|_{H^m(\mathbb{R}\times\Omega)}\|u\|_{H^m_\gamma(\mathbb{R}\times\Omega)}. \tag{4.15.8}$$

Applying (4.15.6) and (4.15.8) in (4.15.3) and then taking the sum yields

$$\begin{aligned}&\sum_{\alpha=0}^m \gamma^{m-\alpha}\|L_v^\gamma\partial_t^\alpha u\|_{L^2(\Omega;L^2(\mathbb{R}))} \\ &\le C(1 + \|v\|_{H^m(\mathbb{R}\times\Omega)})(\|L_v^\gamma u\|_{H^m_\gamma(\mathbb{R}\times\Omega)} + \|u\|_{H^m_\gamma(\mathbb{R}\times\Omega)}).\end{aligned} \tag{4.15.9}$$

Thus according to (4.15.1), (4.15.2) and (4.15.9) we have the following estimates on the time derivatives

$$\begin{aligned}&\sqrt{\gamma}\|u\|_{L^2(\Omega;H^m_\gamma(\mathbb{R}))} + \|u_{|\partial\Omega}\|_{H^m_\gamma(\mathbb{R})} \\ &\le \frac{C}{\sqrt{\gamma}}(1 + \|v\|_{H^m(\mathbb{R}\times\Omega)})\|L_v^\gamma u\|_{H^m_\gamma(\mathbb{R}\times\Omega)} + C\|Bu_{|\partial\Omega}\|_{H^m_\gamma(\mathbb{R})} \\ &\quad + \frac{C}{\sqrt{\gamma}}(1 + \|v\|_{H^m(\mathbb{R}\times\Omega)})\|u\|_{H^m_\gamma(\mathbb{R}\times\Omega)} =: CN(u, v).\end{aligned}$$

It is important to note that on the right hand side, the norms of v are independent of γ.

4.15.2 Estimates on Spatial and Mixed Derivatives

To obtain estimates involving derivatives with respect to x we use the operator L_v^γ. We show by strong induction that

$$\gamma^{m-k-\alpha+1/2}\|\partial_x^k\partial_t^\alpha u\|_{L^2(\mathbb{R}\times\Omega)} \le CN(u, v)$$

holds for all k and α such that $k + \alpha \le m$. The case $k = 0$ only involves time-derivatives and hence the basis step was already established. Suppose we have shown that for all j and α such that $j = 0, \dots, k$ and $j + \alpha \le m$ we have

$$\gamma^{m-(j-1)-\alpha-1/2}\|\partial_x^j\partial_t^\alpha u\|_{L^2(\mathbb{R}\times\Omega)} \le CN(u, v) \tag{4.15.10}$$

We show that this also holds for $k + 1$ and α such that $k + 1 + \alpha \le m$. First, by applying $\partial_x^k\partial_t^\alpha$ to the equality

$$\partial_x u = \mathcal{A}(v)(f - \partial_t u - \gamma u) + A(0)^{-1}(f - \partial_t u - \gamma u), \tag{4.15.11}$$

one obtains

$$\begin{aligned}\partial_x^{k+1}\partial_t^\alpha u &= \partial_x^k\partial_t^\alpha[\mathcal{A}(v)(f-\partial_t u-\gamma u)]\\ &\quad + A(0)^{-1}(\partial_x^k\partial_t^\alpha f-\partial_x^k\partial_t^{\alpha+1}u-\gamma\partial_x^k\partial_t^\alpha u).\end{aligned} \tag{4.15.12}$$

The first term in (4.15.12) may be expanded using the Leibniz's rule as

$$\partial_x^k\partial_t^\alpha[\mathcal{A}(v)(f-\partial_t u-\gamma u)] = \sum_{j=0}^{k}\sum_{l=0}^{\alpha} c_{jl}\partial_x^{k-j}\partial_t^{\alpha-l}\mathcal{A}(v)\,\partial_x^j\partial_t^l(f-\partial_t u-\gamma u) \tag{4.15.13}$$

for some nonnegative constants c_{jl}. By the induction hypothesis (4.15.10) one has already an estimate for the second term in (4.15.12)

$$\gamma^{m-k-\alpha-1/2}\|A(0)^{-1}(\partial_x^k\partial_t^\alpha f-\partial_x^k\partial_t^{\alpha+1}u-\gamma\partial_x^k\partial_t^\alpha u)\|_{L^2(\mathbb{R}\times\Omega)} \le CN(u,v). \tag{4.15.14}$$

Next we estimate the terms appearing in the sum (4.15.13) and for this we consider different cases.

Case 1. If $k-j+\alpha-l\le 1$ then one has the estimate $\|\partial_x^{k-j}\partial_t^{\alpha-l}\mathcal{A}(v)\|_{L^\infty(\mathbb{R}\times\Omega)}\le C$, while the terms $\gamma^{m-k-\alpha-1/2}\partial_x^j\partial_t^{l+1}u$ and $\gamma^{m-k-\alpha+1/2}\partial_x^j\partial_t^l u$ can be estimated using the induction hypothesis: Since $j\le k$, $k+\alpha\ge j+l$ and $\gamma\ge 1$

$$\begin{aligned}&\gamma^{m-k-\alpha-1/2}\|\partial_x^{k-j}\partial_t^{\alpha-l}\mathcal{A}(v)\,\partial_x^j\partial_t^l(f-\partial_t u-\gamma u)\|_{L^2(\mathbb{R}\times\Omega)}\\ &\le C\gamma^{m-k-\alpha-1/2}(\|f\|_{H^{j+l}(\mathbb{R}\times\Omega)}+\|\partial_x^j\partial_t^{l+1}u\|_{L^2(\mathbb{R}\times\Omega)}+\gamma\|\partial_x^j\partial_t^l u\|_{L^2(\mathbb{R}\times\Omega)})\\ &\le C\bigg(\frac{1}{\sqrt{\gamma}}\gamma^{m-(j+l)}\|f\|_{H^{j+l}(\mathbb{R}\times\Omega)}+\gamma^{m-(j-1)-(l+1)-1/2}\|\partial_x^j\partial_t^{l+1}u\|_{L^2(\mathbb{R}\times\Omega)}\\ &\qquad+\gamma^{m-(j-1)-l-1/2}\|\partial_x^j\partial_t^l u\|_{L^2(\mathbb{R}\times\Omega)}\bigg) \le\ CN(u,v).\end{aligned}$$

Case 2. If $k-j+\alpha-l=2$ then we first estimate with respect to time and then integrate with respect to space. In the following, for simplicity we write u,v,f for $u(\cdot,x),v(\cdot,x),f(\cdot,x)$, respectively. Using an L^2-L^∞ estimate, the embedding $H^1(\mathbb{R})\hookrightarrow L^\infty(\mathbb{R})$ and $\gamma\ge 1$

$$\begin{aligned}&\gamma^{m-k-\alpha-1/2}\|\partial_x^{k-j}\partial_t^{\alpha-l}\mathcal{A}(v)\,\partial_x^j\partial_t^l(f-\partial_t u-\gamma u)\|_{L^2(\mathbb{R})}\\ &\le C\|v\|_{H^2(\mathbb{R})}\gamma^{m-k-\alpha-1/2}(\|\partial_x^j\partial_t^l f\|_{H^1(\mathbb{R})}+\|\partial_x^j\partial_t^{l+1}u\|_{H^1(\mathbb{R})}+\gamma\|\partial_x^j\partial_t^l u\|_{H^1(\mathbb{R})})\\ &\le\frac{C}{\sqrt{\gamma}}\|v\|_{H^2(\mathbb{R})}(\|f\|_{H^m_\gamma(\mathbb{R})}+\|u\|_{H^m_\gamma(\mathbb{R})})\end{aligned}$$

and integrating with respect to x over Ω and applying the embedding $H^3(\mathbb{R}\times\Omega)\hookrightarrow L^\infty(\Omega;H^2(\mathbb{R}))$

$$\begin{aligned}&\gamma^{m-k-\alpha-1/2}\|\partial_x^{k-j}\partial_t^{\alpha-l}\mathcal{A}(v)\,\partial_x^j\partial_t^l(f-\partial_t u-\gamma u)\|_{L^2(\mathbb{R}\times\Omega)}\\ &\le\frac{C}{\sqrt{\gamma}}\|v\|_{H^3(\mathbb{R}\times\Omega)}(\|f\|_{H^m_\gamma(\mathbb{R}\times\Omega)}+\|u\|_{H^m_\gamma(\mathbb{R}\times\Omega)})\ \le\ CN(u,v).\end{aligned}$$

Case 3. If $k-j+\alpha-l\ge 3$ then $j+l+3\le k+\alpha\le m$ and we have

$$\begin{aligned}&\gamma^{m-k-\alpha-1/2}\|\partial_x^{k-j}\partial_t^{\alpha-l}\mathcal{A}(v)\,\partial_x^j\partial_t^{l+1}u\|_{L^2(\mathbb{R}\times\Omega)}\\ &\le C\|v\|_{H^m(\mathbb{R}\times\Omega)}\gamma^{m-k-\alpha-1/2}\|\partial_x^j\partial_t^{l+1}u\|_{L^\infty(\mathbb{R}\times\Omega)}\\ &\le C\|v\|_{H^m(\mathbb{R}\times\Omega)}\gamma^{m-(k+\alpha)-1/2}\|u\|_{H^{j+l+3}(\mathbb{R}\times\Omega)}\\ &\le C\|v\|_{H^m(\mathbb{R}\times\Omega)}\gamma^{m-(j+l+3)-1/2}\|u\|_{H^{j+l+3}(\mathbb{R}\times\Omega)}\\ &\le\frac{C}{\sqrt{\gamma}}\|v\|_{H^m(\mathbb{R}\times\Omega)}\|u\|_{H^m_\gamma(\mathbb{R}\times\Omega)}\le\frac{C}{\sqrt{\gamma}}N(u,v)\end{aligned}$$

and similar for the other terms $\partial_x^{k-j}\partial_t^{\alpha-l}\mathcal{A}(v)\,\partial_x^j\partial_t^l f$ and $\gamma\partial_x^{k-j}\partial_t^{\alpha-l}\mathcal{A}(v)\,\partial_x^j\partial_t^l u$. Combining the three cases in (4.15.13) one has

$$\gamma^{m-k-\alpha-1/2}\|\partial_x^k\partial_t^\alpha[\mathcal{A}(v)(f-\partial_t u-\gamma u)]\|_{L^2(\mathbb{R}\times\Omega)} \le CN(u,v) \tag{4.15.15}$$

and taking the sum of (4.15.14) and (4.15.15) in (4.15.12) we have

$$\gamma^{m-k-\alpha-1/2}\|\partial_x^{k+1}\partial_t^\alpha u\|_{L^2(\mathbb{R}\times\Omega)} \le CN(u,v)$$

which establishes the induction step.

4.15.3 Weighted-in-Time Estimates

The above estimates give us finally the estimate

$$\begin{aligned}
&\sqrt{\gamma}\|u\|_{H^m_\gamma(\mathbb{R}\times\Omega)} + \|u_{|\partial\Omega}\|_{H^m_\gamma(\mathbb{R})} \\
&\le C\left(\frac{1}{\sqrt{\gamma}}(1+\|v\|_{H^m(\mathbb{R}\times\Omega)})\|L_v^\gamma u\|_{H^m_\gamma(\mathbb{R}\times\Omega)} + \|Bu_{|\partial\Omega}\|_{H^m_\gamma(\mathbb{R})}\right) \\
&\quad + \frac{C}{\sqrt{\gamma}}(1+\|v\|_{H^m(\mathbb{R}\times\Omega)})\|u\|_{H^m_\gamma(\mathbb{R}\times\Omega)}
\end{aligned} \tag{4.15.16}$$

for all $u \in \mathscr{D}(\mathbb{R}\times\overline{\Omega})$ where $C = C(\mathcal{K}, K) > 0$ is independent of u. Choosing γ large enough, the last term on the right hand side of (4.15.16) can be absorbed by the first term on the left hand side and therefore

$$\sqrt{\gamma}\|u\|_{H^m_\gamma(\mathbb{R}\times\Omega)} + \|u_{|\partial\Omega}\|_{H^m_\gamma(\mathbb{R})} \le C\left(\frac{1}{\sqrt{\gamma}}\|L_v^\gamma u\|_{H^m_\gamma(\mathbb{R}\times\Omega)} + \|Bu_{|\partial\Omega}\|_{H^m_\gamma(\mathbb{R})}\right) \tag{4.15.17}$$

where the constant $C > 0$ also depends only on the $W^{1,\infty}$-norm and H^m-norm of v and the compact set $\mathcal{K}$. The passage from (4.15.17) from (4.15.16) by absorption would not be possible if we have the H^m_γ-norm of v in (4.15.16) instead of its H^m-norm.

Replacing u by $e^{-\gamma t}u$, which is still in $\mathscr{D}(\mathbb{R}\times\overline{\Omega})$ provided that u is, noting that $L_v^\gamma(e^{-\gamma t}u) = e^{-\gamma t}L_v u$, and then by a density argument we have the following a priori estimate.

Theorem 4.15.1. *Let $v \in H^m(\mathbb{R}\times\Omega)$ taking values on a compact set $\mathcal{K} \subset \mathcal{U}$, $\|v\|_{W^{1,\infty}(\mathbb{R}\times\Omega)} \le K$ and $\|v\|_{H^m(\mathbb{R}\times\Omega)} \le R$. Then there exist $C_m = C_m(\mathcal{K}, K, R) > 0$ and $\gamma_m = \gamma_m(\mathcal{K}, K, R) \ge 1$ such that for every $\gamma \ge \gamma_m$ and for every $u \in e^{\gamma t}H^{m+1}(\mathbb{R}\times\Omega)$ it holds that*

$$\begin{aligned}
&\sqrt{\gamma}\|e^{-\gamma t}u\|_{H^m_\gamma(\mathbb{R}\times\Omega)} + \|e^{-\gamma t}u_{|\partial\Omega}\|_{H^m_\gamma(\mathbb{R})} \\
&\le C_m\left(\frac{1}{\sqrt{\gamma}}\|e^{-\gamma t}L_v u\|_{H^m_\gamma(\mathbb{R}\times\Omega)} + \|e^{-\gamma t}Bu_{|\partial\Omega}\|_{H^m_\gamma(\mathbb{R})}\right).
\end{aligned} \tag{4.15.18}$$

The proof of Theorem 4.15.1 given above follows the ideas given in the proof of Theorem 9.7 in [**9**]. However, we have a different estimate in (4.15.6). In [**9, p. 252**], the authors seem to use the estimate

$$\|vf\|_{L^2(\Omega;e^{\gamma t}H^m_\gamma(\mathbb{R}))} \le C\|v\|_{L^2(\Omega;H^m(\mathbb{R}))}\|f\|_{L^2(\Omega;e^{\gamma t}H^m_\gamma(\mathbb{R}))}$$

which does not hold in general. We resolved this by estimating in terms of the norm in $H^m_\gamma(\mathbb{R}\times\Omega)$.

The a priori estimate (4.15.18) will be used in pure boundary value problems.

4.16 A PRIORI ESTIMATES IN SOBOLEV SPACES WITH TIME INTERVAL $(-\infty, T]$

While the a priori estimate derived in the previous section is intended to boundary value problems, the a priori estimate in this section is designated to solve homogeneous initial-boundary value problems. All throughout this section $\Omega = (0,1)$.

Suppose that $v \in H^m((-\infty,T]\times\Omega)$ and $u \in \mathscr{D}((-\infty,T]\times\overline{\Omega})$ such that $u_{|t<0} = 0$. Then thanks to (FS) the a priori estimate

$$\begin{aligned}&\|u(t)\|_{L^2(\Omega)} + \sqrt{\gamma}\|u\|_{L^2(\Omega;L^2(-\infty,T])} + \|u_{|\partial\Omega}\|_{L^2(-\infty,T]}\\ &\le C\left(\frac{1}{\sqrt{\gamma}}\|L_v^\gamma u\|_{L^2(\Omega;L^2(-\infty,T])} + \|Bu_{|\partial\Omega}\|_{L^2(-\infty,T]}\right)\end{aligned}$$

holds for all $\gamma \ge \gamma_0(\mathcal{K},K) \ge 1$, see Theorem 4.13.7. The same procedure as in Section 3.1 gives us the inequality

$$\begin{aligned}&\sum_{\alpha=0}^m \gamma^{m-\alpha}\|\partial_t^\alpha u(t)\|_{L^2(\Omega)} + \sqrt{\gamma}\|u\|_{H^m((-\infty,T]\times\Omega)} + \|u_{|\partial\Omega}\|_{H^m_\gamma(-\infty,T]}\\ &\le \frac{C}{\sqrt{\gamma}}(1+\|v\|_{H^m((-\infty,T]\times\Omega)})\|L_v^\gamma u\|_{H^m_\gamma((-\infty,T]\times\Omega)} + C\|Bu_{|\partial\Omega}\|_{H^m_\gamma(-\infty,T]}\\ &\quad + \frac{C}{\sqrt{\gamma}}(1+\|v\|_{H^m((-\infty,T]\times\Omega)})\|u\|_{H^m_\gamma((-\infty,T]\times\Omega)} \;=:\; CN(u,v). \qquad (4.16.1)\end{aligned}$$

We proceed by induction for the pointwise in time estimates for the spatial derivatives. Assume that for k with $k+\alpha \le m$ we have already shown that (the basis step $k=0$ is nothing but the L^2-estimate, which is already given by (4.16.1))

$$\gamma^{m-k-\alpha}\|\partial_x^k\partial_t^\alpha u(t)\|_{L^2(\Omega)} \le CN(u,v), \qquad t\in(-\infty,T].$$

We show that this is true for $k+1$ when $k+1+\alpha \le m$. Recall our formula (4.15.12), and let J denote the first term, i.e., $J := \partial_x^k\partial_t^\alpha[\mathcal{A}(v)(f-\partial_t u - \gamma u)]$. The following weighted Sobolev estimate will be used.

Proposition 4.16.1. *For every $w \in H^1((-\infty,T]\times\Omega)$ and $\gamma > 0$ we have*

$$\|w\|^2_{L^\infty((-\infty,T];L^2(\Omega))} \le \gamma\|w\|^2_{L^2((-\infty,T]\times\Omega)} + \frac{1}{\gamma}\|\partial_t w\|^2_{L^2((-\infty,T]\times\Omega)}. \qquad (4.16.2)$$

Proof. By a standard density argument we may suppose that $w \in \mathscr{D}((-\infty,T]\times\overline{\Omega})$. Let $R_0 < 0$ be such that w vanishes for all $t \le R_0$. For simplicity we assume that w is scalar-valued. Let $R \le 2R_0 - T$ and $\frac{T+R}{2} \le \tau \le T$. Using Young's inequality

$$\begin{aligned}|w(\tau,x)|^2 &= \int_R^\tau \partial_t(|w(t,x)|^2)\,\mathrm{d}t\\ &= 2\int_R^\tau w(t,x)w_t(t,x)\,\mathrm{d}t\\ &\le \gamma\int_R^T |w(t,x)|^2\,\mathrm{d}t + \frac{1}{\gamma}\int_R^T |w_t(t,x)|^2\,\mathrm{d}t.\end{aligned}$$

Letting $R \to -\infty$ we have

$$|w(\tau,x)|^2 \le \gamma\int_{-\infty}^T |w(t,x)|^2\;\mathrm{d}t + \frac{1}{\gamma}\int_{-\infty}^T |w_t(t,x)|^2\,\mathrm{d}t$$

for all $\tau \in (-\infty,T]$ and $x \in \overline{\Omega}$. Integrating the previous inequality over Ω and taking the supremum over all $\tau \in (-\infty,T]$ proves (4.16.2). $\square$

Using (4.16.2) together with the induction hypothesis yields an estimate for the second term in (4.15.12)

$$\gamma^{m-(k+1)-\alpha}\|\partial_x^k\partial_t^\alpha f(t) - \partial_x^k\partial_t^{\alpha+1}u(t) - \gamma\partial_x^k\partial_t^\alpha u(t)\|_{L^2(\Omega)} \le CN(u,v). \quad (4.16.3)$$

As in the computation of mixed derivatives one obtains

$$\begin{aligned} \gamma^{m-k-\alpha-1/2}\|J\|_{L^2((-\infty,T]\times\Omega)} &\le CN(u,v) \\ \gamma^{m-(k+1)-\alpha-1/2}\|\partial_t J\|_{L^2((-\infty,T]\times\Omega)} &\le CN(u,v). \end{aligned}$$

Thus by the weighted Sobolev estimate (4.16.2) we have the estimate

$$\begin{aligned} &\gamma^{m-(k+1)-\alpha}\|J(t)\|_{L^2(\Omega)} \\ &\le C\left(\gamma^{m-(k+1)-\alpha+1/2}\|J\|_{L^2((-\infty,T]\times\Omega)} + \gamma^{m-(k+1)-\alpha-1/2}\|\partial_t J\|_{L^2((-\infty,T]\times\Omega)}\right) \\ &\le CN(u,v) \end{aligned} \quad (4.16.4)$$

Combining (4.16.3) and (4.16.4) proves the induction step.

Therefore we have the full estimate

$$\begin{aligned} &\sum_{|\beta|\le m}\gamma^{m-|\beta|}\|\partial^\beta u(t)\|_{L^2(\Omega)} + \sqrt{\gamma}\|u\|_{H^m_\gamma((-\infty,T]\times\Omega)} + \|u_{|\partial\Omega}\|_{H^m_\gamma(-\infty,T]} \\ &\le \frac{C}{\sqrt{\gamma}}(1+\|v\|_{H^m((-\infty,T]\times\Omega)})\|L_v^\gamma u\|_{H^m_\gamma((-\infty,T]\times\Omega)} + C\|Bu_{|\partial\Omega}\|_{H^m_\gamma(-\infty,T]} \\ &\quad + \frac{C}{\sqrt{\gamma}}(1+\|v\|_{H^m((-\infty,T]\times\Omega)})\|u\|_{H^m_\gamma((-\infty,T]\times\Omega)} \end{aligned}$$

for all $t \in (-\infty, T]$. Now replacing u by $e^{-\gamma t}u$, choosing γ large enough, so that the last term on the right hand side can be absorbed by the second term on the left hand side, and finally using the norm-equivalence

$$\sum_{|\beta|\le m}\gamma^{m-|\beta|}\|\partial^\beta(e^{-\gamma t}u(t))\|_{L^2(\Omega)} \simeq \sum_{|\beta|\le m}\gamma^{m-|\beta|}e^{-\gamma t}\|\partial^\beta u(t)\|_{L^2(\Omega)}.$$

we have the following a priori estimate.

Lemma 4.16.2 (A Priori Estimate in Weighted Sobolev Spaces)**.** *Let $m \ge 3$ be an integer. For each $v \in H^m((-\infty,T)\times\Omega)$ satisfying $\operatorname{ran} v \subset \mathcal{K}$, $\|v\|_{W^{1,\infty}((-\infty,T]\times\Omega)} \le K$ and $\|v\|_{H^m((-\infty,T]\times\Omega)} \le R$ and for all $u \in H^{m+1}((0,T)\times\Omega)$ such that $u_{|t=0} = 0$, there exist $C = C(\mathcal{K},K,R) > 0$ and $\gamma_m(\mathcal{K},K,R) \ge 1$ such that for all $\gamma \ge \gamma_m$ and for all $\tau \in [0,T]$ the following a priori estimate holds*

$$\begin{aligned} &\sum_{|\alpha|\le m}\gamma^{m-|\alpha|}e^{-\gamma\tau}\|\partial^\alpha u(\tau)\|_{L^2(\Omega)} + \sqrt{\gamma}\|e^{-\gamma t}u\|_{H^m_\gamma((0,\tau)\times\Omega)} + \|u_{|\partial\Omega}\|_{H^m_\gamma(0,\tau)} \\ &\le C_m\left(\frac{1}{\sqrt{\gamma}}\|e^{-\gamma t}L_v u\|_{H^m_\gamma((0,\tau)\times\Omega)} + \|e^{-\gamma t}Bu_{|\partial\Omega}\|_{H^m_\gamma(0,\tau)}\right). \end{aligned} \quad (4.16.5)$$

The a priori estimate (4.16.5) is different from those in [**9**] and [**55**] because in (4.16.5) the constants C_m and γ_m depend only on the H^m-norm of v and not on its H^m_γ-norm.

4.17 GAGLIARDO-NIRENBERG TYPE ESTIMATES

For initial boundary value problems with zero initial conditions the a priori estimate (4.16.5) will be used. The next step is to derive an a priori estimate that can be used for problems that are not starting initially from zero. In preparation we borrow the Gagliardo-Nirenberg type estimates in [**55, pp. 69–71**]. In this section, we let $\Omega = (0,1)$.

Theorem 4.17.1 (Gagliardo-Nirenberg)**.** *Let m be a positive integer and $T > 0$. Then there exists $C > 0$, independent of T, such that for all $u \in H^m((-\infty,T)\times\Omega)$ and $1 \le |\alpha| \le m$ we have*

$$\|\partial_x^\alpha u\|_{L^{2m/|\alpha|}((-\infty,T)\times\Omega)} \le C\|u\|_{L^\infty((-\infty,T)\times\Omega)}^{1-|\alpha|/m}\|u\|_{H^m((-\infty,T)\times\Omega)}^{|\alpha|/m}.$$

A similar estimate also holds for $u \in H^m(-\infty,T)$.

The following is a modification of Proposition 4.5.5 in [**55**].

Theorem 4.17.2. *For all $m \in \mathbb{N}$ there exists $C = C(m) > 0$ such that for all $T > 0$ and $\psi \in H^m(0,T)$ and $1 \le j \le m$ we have*

$$\|\psi^{(j)}\|_{L^{2m/j}(0,T)} \le C(K_{m,T}(\psi)^{1-m/j}(\|\psi\|_{H^m(0,T)} + K_{m,T}(\psi))^{m/j} + K_{m,T}(\psi))$$

where

$$K_{m,T}(\psi) = \|\psi\|_{L^\infty(0,T)} + \sum_{i=0}^{m-1}|\psi^{(i)}(0)|.$$

In particular,

$$\|\psi^{(j)}\|_{L^{2m/j}(0,T)} \le C(\|\psi\|_{H^m(0,T)} + K_{m,T}(\psi)).$$

Proof. We adjust the proof in [**55**]. Given $\psi \in H^m(0,T)$, let $\psi_1 \in H^m(\mathbb{R})$ be such that $\psi_1^{(i)}(0) = \psi^{(i)}(0)$ for all $i = 0,\dots,m-1$ and using the fact that the trace operator has a continuous right inverse

$$\|\psi_1\|_{H^m(\mathbb{R})} \le C\sum_{i=0}^{m-1}|\psi_1^{(i)}(0)| = C\sum_{i=0}^{m-1}|\psi^{(i)}(0)|, \tag{4.17.1}$$

where $C > 0$ is independent of ψ. Let $\psi_2 = \psi - \psi_1 \in H^m(0,T)$. Then (4.17.1) and the Sobolev embedding theorem $H^m(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ imply

$$\begin{aligned}\|\psi_2\|_{L^\infty[0,T]} &\le \|\psi\|_{L^\infty[0,T]} + \|\psi_1\|_{L^\infty(\mathbb{R})}\\ &\le \|\psi\|_{L^\infty[0,T]} + C\|\psi_1\|_{H^m(\mathbb{R})} \le CK_{m,T}(\psi)\end{aligned} \tag{4.17.2}$$

and

$$\|\psi_2\|_{H^m(0,T)} \le \|\psi\|_{H^m(0,T)} + \|\psi_1\|_{H^m(0,T)} \le \|\psi\|_{H^m(0,T)} + CK_{m,T}(\psi). \tag{4.17.3}$$

By construction it holds that $\psi_2^{(i)}(0) = 0$ for $i = 0,\dots,m-1$ and therefore extending ψ_2 by 0 for $t < 0$ we have $\psi_2 \in H^m(-\infty,T)$. By the Gagliardo-Nirenberg inequality

$$\|\psi_2^{(j)}\|_{L^{2m/j}(-\infty,T)} \le C\|\psi_2\|_{L^\infty[0,T]}^{1-j/m}\|\psi_2\|_{H^m(0,T)}^{j/m} \tag{4.17.4}$$

$$\|\psi_1^{(j)}\|_{L^{2m/j}(\mathbb{R})} \le C\|\psi_1\|_{L^\infty(\mathbb{R})}^{1-j/m}\|\psi_1\|_{H^m(\mathbb{R})}^{j/m} \le C\|\psi_1\|_{H^m(\mathbb{R})}. \tag{4.17.5}$$

Thus (4.17.2)−(4.17.5) imply that

$$\begin{aligned}
\|\psi^{(j)}\|_{L^{2m/j}(0,T)} &\leq \|\psi_1^{(j)}\|_{L^{2m/j}(0,T)} + \|\psi_2^{(j)}\|_{L^{2m/j}(0,T)} \\
&\leq C\|\psi_2\|_{L^\infty[0,T]}^{1-j/m}\|\psi_2\|_{H^m(0,T)}^{j/m} + C\|\psi_1\|_{H^m(\mathbb{R})} \\
&\leq C(K_{m,T}(\psi)^{1-j/m}(\|\psi\|_{H^m(0,T)} + K_{m,T}(\psi))^{j/m} + K_{m,T}(\psi)).
\end{aligned}$$

This proves the first part. The second part follows immediately using the elementary inequality $a^{1-r}(a+b)^r \leq a+b$ for $a, b \geq 0$ and $0 < r < 1$. □

Theorem 4.17.3. *For all positive integers m there exists $C = C(m) > 0$ such that for all $T > 0$, $u \in H^m((0,T)\times\Omega) \cap L^\infty((0,T)\times\Omega)$ such that $\partial_t^j u_{|t=0} \in H^{m-j}(\Omega)$ for $0 \leq j \leq m-1$ we have*

$$\begin{aligned}
&\|\partial^\alpha u\|_{L^{2m/|\alpha|}((0,T)\times\Omega)} \\
&\leq C(\tilde{K}_{m,T}(u)^{1-|\alpha|/m}(\|u\|_{H^m((0,T)\times\Omega)} + \tilde{K}_{m,T}(u))^{|\alpha|/m} + \tilde{K}_{m,T}(u))
\end{aligned}$$

for $1 \leq |\alpha| \leq m$ where

$$\tilde{K}_{m,T}(u) = \|u\|_{L^\infty((0,T)\times\Omega)} + \sum_{i=0}^{m-1} \|\partial_t^i u(0)\|_{H^{m-i}(\Omega)}.$$

In particular,

$$\|\partial^\alpha u\|_{L^{2m/|\alpha|}((0,T)\times\Omega)} \leq C(\|u\|_{H^m((0,T)\times\Omega)} + \tilde{K}_{m,T}(u)).$$

Proof. The proof is similar as in the previous theorem, see [**55, Proposition 4.5.6**] for the details. □

A function F is said to be a nonlinear function of u of order k if

$$F(u) = \sum_{l=1}^{N} \sum_{|\alpha_1|+\cdots+|\alpha_l|=k} F_{l,\alpha_1,\ldots,\alpha_l}(u)[\partial^{\alpha_1}u, \ldots, \partial^{\alpha_l}u]$$

where $\alpha_i \in \mathbb{N}_0^2$ and $F_{l,\alpha_1,\ldots,\alpha_l}$ are multilinear mappings depending smoothly on u and there exists $(\alpha_1, \ldots, \alpha_l)$ such that $|\alpha_1| + \cdots + |\alpha_l| = k$ and $F_{l,\alpha_1,\ldots,\alpha_l} \neq 0$.

Theorem 4.17.4. *Let m be a positive integer and F be a nonlinear function of order $k \leq m$. There exists $C > 0$, which depends continuously on its argument, such that for all $T > 0$ and $u \in H^m((0,T)\times\Omega) \cap L^\infty((0,T)\times\Omega)$ such that $\partial_t^j u_{|t=0} \in H^{m-j}(\Omega)$ for $0 \leq j \leq m-1$*

$$\|F(u)\|_{L^{2m/k}((0,T)\times\Omega)} \leq C(\tilde{K}_{m,T}(u))(\|u\|_{H^m((0,T)\times\Omega)} + \tilde{K}_{m,T}(u))^{k/m}.$$

In particular,

$$\|F(u)\|_{L^{2m/k}((0,T)\times\Omega)} \leq \tilde{C}(\tilde{K}_{m,T}(u))(\|u\|_{H^m((0,T)\times\Omega)} + 1).$$

where $\tilde{C} \geq 1$. A similar statement holds for $\psi \in H^m(0,T)$ where $m \in \mathbb{N}$.

Proof. For simplicity we assume that u is scalar valued. First note that we have $\|F_{l,\alpha}(u)\|_{L^\infty((0,T)\times\Omega)} \le C(\|u\|_{L^\infty((0,T)\times\Omega)})$. Suppose that $|\alpha_1|+\cdots+|\alpha_l| = k$. Define $p_i = \frac{2m}{|\alpha_i|}$, where we use the convention that $p_i = \infty$ if $\alpha_i = 0$. Then $\sum_{i=1}^l \frac{1}{p_i} = \sum_{i=1}^l \frac{|\alpha_i|}{2m} = \frac{k}{2m}$. By Hölder's inequality and Theorem 4.17.3

$$\begin{aligned}
&\|\partial^{\alpha_1}u\cdots\partial^{\alpha_l}u\|_{L^{2m/k}((0,T)\times\Omega)} \le \|\partial^{\alpha_1}u\|_{L^{p_1}((0,T)\times\Omega)}\cdots\|\partial^{\alpha_l}u\|_{L^{p_l}((0,T)\times\Omega)}\\
&\le \prod_{i=1}^{l} C\tilde{K}_{k,T}(u)^{1-2/p_i}((\|u\|_{H^m((0,T)\times\Omega)} + \tilde{K}_{k,T}(u))^{2/p_i} + \tilde{K}_{k,T}(u)^{2/pi})\\
&\le C^l\tilde{K}_{k,T}(u)^{l-k/m}\prod_{i=1}^{l}((\|u\|_{H^m((0,T)\times\Omega)} + \tilde{K}_{k,T}(u))^{2/p_i} + \tilde{K}_{k,T}(u)^{2/pi})\\
&\le (2C)^l\tilde{K}_{k,T}(u)^{l-k/m}\prod_{i=1}^{l}(\|u\|_{H^m((0,T)\times\Omega)} + \tilde{K}_{k,T}(u))^{2/p_i}\\
&\le C(\tilde{K}_{k,T}(u))(\|u\|_{H^k((0,T)\times\Omega)} + \tilde{K}_{k,T}(u))^{k/m}.
\end{aligned}$$

Taking the sum of all terms we obtain the estimate of the theorem. □

Using classical Sobolev embedding theorems and the identity $u(t) = u(0)+\int_0^t u'(\tau)\,\mathrm{d}\tau$ for a.e. $t\in[0,T]$ and for $u\in W^{1,1}([0,T];X)$ where X is a Banach space, the following estimates can be shown by induction.

Theorem 4.17.5. *Let m be a nonnegative integer and $T>0$. There exists a $C>0$ independent of T such that for all $u\in H^{m+2}((0,T)\times\Omega)$ we have*

$$\|u\|_{W^{m,\infty}((0,T)\times\Omega)} \le \sum_{k=0}^{m}\|\partial_t^k u_{|t=0}\|_{W^{m-k,\infty}(\Omega)} + C\sqrt{T}\|u\|_{H^{m+2}((0,T)\times\Omega)}.$$

Theorem 4.17.6. *Let m be a positive integer. There exists $C>0$ such that for all $T>0$ and $u\in H^m(0,T)$ we have*

$$\|u\|_{H^{m-1}(0,T)} \le C\left(\sum_{i=0}^{m-1}\sqrt{T}|u^{(i)}(0)| + T\|u\|_{H^m(0,T)}\right).$$

Also, there exists $C>0$ such that for all $T>0$ and $u\in H^m((0,T)\times\Omega)$ we have

$$\|u\|_{H^{m-1}((0,T)\times\Omega)} \le C\left(\sum_{i=0}^{m-1}\sqrt{T}\|\partial_t^i u_{|t=0}\|_{H^{m-i-1}(\Omega)} + T\|u\|_{H^m((0,T)\times\Omega)}\right).$$

4.18 REGULARITY OF SOLUTIONS FOR BVP

Thanks to the a priori estimates in Sobolev spaces, we can show additional regularity of weak solutions for the pure boundary value problems under additional smoothness conditions of the data as well. In this section, we assume that the boundary matrices B_0 and B_1 are constant.

Theorem 4.18.1. *Consider the framework of Theorem 4.6.6. Suppose in addition that $v\in H^m(\mathbb{R}\times(0,1))$ for some $m\ge 3$. If $f\in e^{\gamma t}H^m(\mathbb{R}\times(0,1))$ and $g\in e^{\gamma t}H^m(\mathbb{R})$ then the weak solution of the boundary value problem*

$$L_v u = f, \qquad Bu_{|\partial\Omega} = g \tag{4.18.1}$$

lies in $e^{\gamma t}H^m(\mathbb{R}\times(0,1))$ *and satisfies* $u_{|\partial\Omega} \in e^{\gamma t}H^m(\mathbb{R})$ *and the energy estimate*

$$\begin{aligned}&\sqrt{\gamma}\|e^{-\gamma t}u\|_{H^m_\gamma(\mathbb{R}\times(0,1))} + \|e^{-\gamma t}u_{|\partial\Omega}\|_{H^m_\gamma(\mathbb{R})}\\ &\le C_m\left(\frac{1}{\sqrt{\gamma}}\|e^{-\gamma t}f\|_{H^m_\gamma(\mathbb{R}\times(0,1))} + \|e^{-\gamma t}g\|_{H^m_\gamma(\mathbb{R})}\right).\end{aligned} \tag{4.18.2}$$

for every $\gamma \ge \gamma_m$*, where the constants* γ_m *and* C_m *are as in Theorem 4.15.1.*

Proof. We follow the proof in [**9, pp. 281–282**]. Let $(v_j)_j \subset \mathscr{D}(\mathbb{R}\times(0,1))$ such that $v_j \to v$ in $H^m(\mathbb{R}\times(0,1))$ and that for each j the range of v_j lies on a δ-neighborhood of the range of v for some fixed $\delta > 0$. From Theorem 4.10.6 the weak solution of the boundary value problem

$$L_{v_j}u_j = f, \qquad Bu_{j|\partial\Omega} = g \tag{4.18.3}$$

satisfies $u_j \in e^{\gamma t}H^m(\mathbb{R}\times(0,1))$ and $u_{j|\partial\Omega} \in H^m(\mathbb{R})$. Moreover, for every j there exists a sequence $(u_{ji})_i \subset e^{\gamma t}H^{m+1}_\gamma(\mathbb{R}\times(0,1))$ such that $u_{ji} \to u_j$ in $e^{\gamma t}H^m_\gamma(\mathbb{R}\times(0,1))$, $L_{v_j}u_{ji} \to L_{v_j}u_j = f$ in $e^{\gamma t}H^m_\gamma(\mathbb{R}\times(0,1))$ and $u_{ji|\partial\Omega} \to u_{j|\partial\Omega}$ in $e^{\gamma t}H^m_\gamma(\mathbb{R})$. Applying Theorem 4.15.1 to u_{ji} and passing to the limit $i \to \infty$ we obtain the energy estimate

$$\begin{aligned}&\sqrt{\gamma}\|e^{-\gamma t}u_j\|_{H^m_\gamma(\mathbb{R}\times(0,1))} + \|e^{-\gamma t}u_{j|\partial\Omega}\|_{H^m_\gamma(\mathbb{R})}\\ &\le C_m\left(\frac{1}{\sqrt{\gamma}}\|e^{-\gamma t}f\|_{H^m_\gamma(\mathbb{R}\times(0,1))} + \|e^{-\gamma t}g\|_{H^m_\gamma(\mathbb{R})}\right).\end{aligned} \tag{4.18.4}$$

This implies that $(e^{-\gamma t}u_j)_j$ and $(e^{-\gamma t}u_{j|\partial\Omega})_j$ are bounded in $H^m_\gamma(\mathbb{R}\times(0,1))$ and $H^m_\gamma(\mathbb{R})$, respectively. Therefore, up to a subsequence we have $e^{-\gamma t}u_j \rightharpoonup \tilde{u}$ in $H^m_\gamma(\mathbb{R}\times(0,1))$ and $e^{-\gamma t}u_{j|\partial\Omega} \rightharpoonup \tilde{w}$ in $H^m_\gamma(\mathbb{R})$ for some $\tilde{u} \in H^m_\gamma(\mathbb{R}\times(0,1))$ and $\tilde{w} \in H^m_\gamma(\mathbb{R})$.

Now, we deviate the proof from [**9**]. From (4.18.3) the difference $u_j - u_k$ satisfies the boundary value problem

$$L_{v_j}(u_j - u_k) = -(A(v_j) - A(v_k))\partial_x u_k, \qquad B(u_j - u_k)_{|\partial\Omega} = 0. \tag{4.18.5}$$

According to (4.10.4) there exists constants $\gamma_0 \ge 1$ and $C > 0$ both depending only on the range of v_j and $\|v_j\|_{W^{1,\infty}(\mathbb{R}\times(0,1))}$ such that

$$\begin{aligned}&\sqrt{\gamma}\|e^{-\gamma t}(u_j - u_k)\|_{L^2(\mathbb{R}\times\Omega)} + \|e^{-\gamma t}(u_{j|\partial\Omega} - u_{k|\partial\Omega})\|_{L^2(\mathbb{R})}\\ &\le \frac{C}{\sqrt{\gamma}}\|e^{-\gamma t}\partial_x u_k\|_{L^\infty(\mathbb{R}\times(0,1))}\|A(v_j) - A(v_k)\|_{L^2(\mathbb{R}\times(0,1))}\end{aligned} \tag{4.18.6}$$

for every $\gamma \ge \gamma_0$. By construction of the sequence $(v_j)_j$, we can see that γ_0 and C can be made independent of j and the mean value-theorem implies that $A(v_j) \to A(v)$ in $L^2(\mathbb{R}\times(0,1))$. Moreover, the sequence $(e^{-\gamma t}\partial_x u_k)_k$ is a bounded in $L^\infty(\mathbb{R}\times(0,1))$ by the Sobolev Embedding Theorem. Thus, it follows from (4.18.6) that $(e^{-\gamma t}u_j)_j$ and $(e^{-\gamma t}u_{j|\partial\Omega})_j$ are Cauchy sequences in $L^2(\mathbb{R}\times(0,1))$ and $L^2(\mathbb{R})$, respectively.

By interpolation we have $e^{-\gamma t}u_j \to \tilde{u}$ in $H^s(\mathbb{R}\times(0,1))$ and $e^{-\gamma t}u_{j|\partial\Omega} \to \tilde{w}$ in $H^s(\mathbb{R})$ for every $s \in [0,m)$. In particular, the trace theorem implies that $\tilde{w} = \tilde{u}_{|\partial\Omega}$. From (4.18.3)

$$L_{v_j}(e^{-\gamma t}u_j) + \gamma e^{-\gamma t}u_j = e^{-\gamma t}f, \qquad Be^{-\gamma t}u_{j|\partial\Omega} = e^{-\gamma t}g. \tag{4.18.7}$$

Passing to the limit $j \to \infty$ in (4.18.7) gives us

$$L_v\tilde{u} + \gamma\tilde{u} = e^{-\gamma t}f, \qquad B\tilde{u}_{|\partial\Omega} = e^{-\gamma t}g. \tag{4.18.8}$$

Setting $u = e^{\gamma t}\tilde{u} \in e^{\gamma t}H^m(\mathbb{R} \times (0,1))$ we have $u_{|\partial\Omega} \in e^{\gamma t}H^m(\mathbb{R})$ and from (4.18.8), u satisfies (4.18.1). The energy estimate (4.18.2) follows by taking the limit inferior in (4.18.3) and using the inequalities

$$\begin{aligned}\liminf_{j\to\infty} \|e^{-\gamma t}u_j\|_{H^m_\gamma(\mathbb{R}\times(0,1))} &\geq \|e^{-\gamma t}u\|_{H^m_\gamma(\mathbb{R}\times(0,1))}\\ \liminf_{j\to\infty} \|e^{-\gamma t}u_{j|\partial\Omega}\|_{H^m_\gamma(\mathbb{R})} &\geq \|e^{-\gamma t}u_{|\partial\Omega}\|_{H^m_\gamma(\mathbb{R})}.\end{aligned}$$

This completes the proof of the theorem. □

4.19 REGULARITY OF SOLUTIONS FOR IBVP

We would like to extend the regularity results in the previous section to initial-boundary value problems. The first step is to prove additional time regularity in Theorem 4.13.9 in the homogeneous case under additional smoothness assumptions on the frozen coefficient v and on the data f and g. As before we extend the data f and g by zero for negative times and consider the corresponding pure boundary value problem and this enables us to use the results of the previous section. However, we need to extend the frozen coefficient to all times. This is possible thanks to the following lemma.

Lemma 4.19.1. *Let $m \geq 3$ be a positive integer and $v \in H^m((0,T)\times\Omega)$ be such that $\|v\|_{H^m((0,T)\times\Omega)} \leq R$, $\|v\|_{W^{1,\infty}((0,T)\times\Omega)} \leq K$ and the range of v lies on a compact and convex set $\mathcal{K}$ containing 0. Then there exist $\breve{v} \in H^m(\mathbb{R}^2)$ and $(\breve{v}_\epsilon)_{\epsilon>0} \subset \mathscr{C}^\infty(\mathbb{R}^2)$ such that $\breve{v}_{|(0,T)\times\Omega} = v$, $\|\breve{v}_\epsilon - \breve{v}\|_{H^m(\mathbb{R}^2)} \to 0$ as $\epsilon \to 0^+$, and for every $\epsilon > 0$ sufficiently small we have $\|\breve{v}_\epsilon\|_{H^m(\mathbb{R}^2)} \leq C(T,R)$, $\|\breve{v}_\epsilon\|_{W^{1,\infty}(\mathbb{R}^2)} \leq C(K)$ and the range of $\breve{v}_\epsilon$ lies on a δ-neighborhood of $\mathcal{K}$, for a fixed $\delta > 0$.*

Proof. Let $\theta \in \mathscr{C}_0^\infty([0,\infty);[0,1])$ be such that $\theta(0) = 1$ and $\theta^{(j)}(0) = 0$ for every $1 \leq j \leq m-1$. For $a > 0$ define $\theta_a : \mathbb{R} \to [0,T]$ by

$$\theta_a(s) = \begin{cases}\theta(-s), & s < 0,\\ 1, & 0 \leq s \leq a,\\ \theta(s-a), & s > a.\end{cases}$$

By construction $\theta_a \in H^m(\mathbb{R})$. Let $\tilde{v} \in H^m([-T,2T]\times[-1,2])$ be the extension of v using Seeley's reflection argument [**1, p. 84**]. The construction of $\tilde{v}$ implies that $\|\tilde{v}\|_{W^{1,\infty}((-T,2T)\times(-1,2))} \leq C(K)$. Define $\breve{v}(t,x) = \theta_T(t)\theta_1(x)\tilde{v}(t,x)$, where $\tilde{v}$ is extended by zero outside $[-T,2T]\times[-1,2]$. Reducing the support of θ it can be shown that $\breve{v} \in H^m(\mathbb{R}^2)$ and the range of $\breve{v}$ lies on $\delta/2$-neighborhood of $\mathcal{K}$. Let $\breve{v}_\epsilon = \rho_\epsilon \star \breve{v} \in \mathscr{C}^\infty(\mathbb{R}^2)$ where ρ_ϵ is a standard mollifier in the variable (t,x). By definition, $\breve{v} = v$ on $(0,T)\times\Omega$ and $\|\breve{v}_\epsilon - \breve{v}\|_{H^m(\mathbb{R}^2)} \to 0$ as $\epsilon \to 0^+$. The remaning properties can be easily checked using the Sobolev embedding theorem. □

In this section we suppose that the boundary matrices B_0 and B_1 are constant.

4.19.1 The Homogeneous Case

Theorem 4.19.2. *In the framework of Theorem 4.13.3, suppose in addition that the function $v \in H^m((0,T)\times\Omega)$ for some integer $m \geq 3$ and $\|v\|_{H^m((0,T)\times\Omega)} \leq R$. If*

$f \in H^m((0,T)\times\Omega)$ and $g \in H^m(0,T)$ satisfy $(\partial_t^j f)_{|t=0} = 0$ and $(\partial_t^j g)_{|t=0} = 0$ for $0 \le j \le m-1$ then the solution u of the IBVP

$$L_v u = f, \qquad Bu_{|\partial\Omega} = g, \qquad u_{|t=0} = 0 \tag{4.19.1}$$

lies in $CH^m([0,T]\times\Omega)$ with trace $u_{|\partial\Omega} \in H^m(0,T)$ and $(\partial_t^j u)_{|t=0} = 0$ for $0 \le j \le m-1$. Furthermore, there exist $C_m = C_m(\mathcal{K}, K, R, T) > 0$ and $\gamma_m = \gamma_m(\mathcal{K}, K, R, T) \ge 1$ such that for all $\gamma \ge \gamma_m$ and for all $\tau \in [0,T]$ we have

$$\begin{aligned}&\sum_{|\alpha|\le m} \gamma^{m-|\alpha|} e^{-\gamma\tau} \|\partial^\alpha u(\tau)\|_{L^2(\Omega)} + \sqrt{\gamma}\|e^{-\gamma t}u\|_{H^m_\gamma((0,\tau)\times\Omega)} + \|e^{-\gamma t}u_{|\partial\Omega}\|_{H^m_\gamma(0,\tau)} \\ &\le C_m\left(\frac{1}{\sqrt{\gamma}}\|e^{-\gamma t}f\|_{H^m_\gamma((0,\tau)\times\Omega)} + \|e^{-\gamma t}g\|_{H^m_\gamma(0,\tau)}\right).\end{aligned} \tag{4.19.2}$$

Proof. Let $\breve{f} \in H^m(\mathbb{R}\times\Omega)$ and $\breve{g} \in H^m(\mathbb{R})$ be extensions of f and g both vanishing for $t < 0$. Such extensions are possible due to the assumptions on f and g at $t = 0$. Let $\breve{u}$ be the solution of the pure boundary value problem

$$L_{\breve{v}}\breve{u} = \breve{f} \quad \text{in } \mathbb{R}\times\Omega, \qquad B\breve{u}_{|\partial\Omega} = \breve{g} \quad \text{in } \mathbb{R},$$

where $\breve{v}$ is the extension of v in Lemma 4.19.1. From Theorem 4.18.1 this BVP has a unique weak solution $\breve{u} \in L^2(\mathbb{R}\times\Omega)$ with trace $\breve{u}_{|\partial\Omega} \in L^2(\mathbb{R})$. Furthermore $\breve{u} \in H^m(\mathbb{R}\times\Omega)$ and $\breve{u}_{|\partial\Omega} \in H^m(\mathbb{R})$. From the proof of Theorem 4.13.2 $u := \breve{u}_{|[0,T]} \in H^m((0,T)\times\Omega)$ is the solution of the homogeneous IBVP (4.19.1) and it satisfies all the conclusions of the theorem except the energy estimate (4.19.2) and the additional regularity in time, see for instance the proof of Theorem 4.11.4 with Theorem 4.16.2 in place of Theorem 4.11.2. To see this we use the usual weak = strong argument as suggested in in [**9**]. We will do this step because this will reveal some important remarks that are required in the proof of Theorem 4.19.5 below. Let ρ_ϵ be a standard mollifier with respect to t chosen in such a way that $\rho_\epsilon \star \breve{u} =: u_\epsilon$ vanishes for $t < 0$. The notation $R_\epsilon u = \rho_\epsilon \star u$ will also be used. Then $u_\epsilon \in H^m(\Omega; H^{+\infty}(\mathbb{R}))$ where $H^{+\infty}(\mathbb{R}) = \bigcap_{m\in\mathbb{R}} H^m(\mathbb{R})$

The next step is to show additional regularity in x. Note that

$$A_{\breve{v}}^{-1} L_{\breve{v}}\breve{u} = A_{\breve{v}}^{-1}\partial_t\breve{u} + \partial_x\breve{u} = A_{\breve{v}}^{-1}\breve{f}$$

Let $\alpha \in \mathbb{N}_0^2$ be a multiindex with $|\alpha| \le m$. Applying ∂^α to both sides of the latter equality gives

$$A_{\breve{v}}^{-1}\partial_t(\partial^\alpha\breve{u}) + \partial_x(\partial^\alpha\breve{u}) = \partial^\alpha(A_{\breve{v}}^{-1}\breve{f}) + [A_{\breve{v}}^{-1}\partial_t, \partial^\alpha]\breve{u}. \tag{4.19.3}$$

Since the commutator $[A_{\breve{v}}^{-1}\partial_t, \partial^\alpha]$ is of order $|\alpha|$ and $\breve{u} \in H^m(\mathbb{R}\times\Omega)$, it follows $[A_{\breve{v}}^{-1}\partial_t, \partial^\alpha]\breve{u} \in L^2(\mathbb{R}\times\Omega)$. Mollifying both sides of (4.19.3) with respect to time yields

$$A_{\breve{v}}^{-1}\partial_t(\partial^\alpha u_\epsilon) + \partial_x(\partial^\alpha u_\epsilon) = R_\epsilon(\partial^\alpha(A_{\breve{v}}^{-1}\breve{f}) + [A_{\breve{v}}^{-1}\partial_t, \partial^\alpha]\breve{u}) + [A_{\breve{v}}^{-1}\partial_t, R_\epsilon]\partial^\alpha\breve{u}. \tag{4.19.4}$$

Let F_ϵ be the right hand side of (4.19.4). Solving for $\partial_x(\partial^\alpha u_\epsilon)$ shows that $\partial_x(\partial^\alpha u_\epsilon) \in L^2(\mathbb{R}\times\Omega)$. Therefore $u_\epsilon \in H^{m+1}(\mathbb{R}\times\Omega)$. In other words, mollification in time gives additional regularity in time, and together with the PDE one has additional regularity in space.

As $\epsilon \to 0$ it holds that

$$L_{\breve{v}}\partial^\alpha u_\epsilon \to L_{\breve{v}}\partial^\alpha \breve{u}, \qquad \text{in } L^2(\mathbb{R}\times\Omega). \tag{4.19.5}$$

Indeed, we have $R_\epsilon(\partial^\alpha(A_{\breve{v}}^{-1}\breve{f}) + [A_{\breve{v}}^{-1}\partial_t, \partial^\alpha]\breve{u}) \to \partial^\alpha(A_{\breve{v}}^{-1}\breve{f}) + [A_{\breve{v}}^{-1}\partial_t, \partial^\alpha]\breve{u}$ and also $[A_{\breve{v}}^{-1}\partial_t, R_\epsilon]\partial^\alpha\breve{u} \to 0$ both in $L^2(\mathbb{R}\times\Omega)$, where we used Theorem C.1.1 for the latter. Now (4.19.5) follows from

$$[A_{\breve{v}}^{-1}\partial_t, \partial^\alpha]\breve{u} = [A_{\breve{v}}^{-1}L_{\breve{v}}, \partial^\alpha]\breve{u} = A_{\breve{v}}^{-1}L_{\breve{v}}\partial^\alpha\breve{u} - \partial^\alpha(A_{\breve{v}}^{-1}\breve{f})$$

since $[\partial_x, \partial^\alpha]\breve{u} = 0$ and $L_{\breve{v}}\breve{u} = \breve{f}$.

Applying the a priori estimate (4.16.5) to $u_\epsilon - u_{\epsilon'} \in e^{\gamma t}H_\gamma^{m+1}(\mathbb{R}\times\Omega)$ one obtains

$$\begin{aligned}
&\sum_{|\alpha|\le m} \gamma^{m-|\alpha|}e^{-\gamma T}\sup_{\tau\in[0,T]}\|\partial^\alpha(u_\epsilon - u_{\epsilon'})(\tau)\|_{L^2(\Omega)} + \|(u_\epsilon - u_{\epsilon'})_{|\partial\Omega}\|_{H_\gamma^m(0,T)} \\
&\le C_m\left(\frac{1}{\sqrt{\gamma}}\|e^{-\gamma t}L_{\breve{v}}(u_\epsilon - u_{\epsilon'})\|_{H_\gamma^m((0,T)\times\Omega)} + \|e^{-\gamma t}B(u_\epsilon - u_{\epsilon'})_{|\partial\Omega}\|_{H_\gamma^m(0,T)}\right).
\end{aligned} \tag{4.19.6}$$

Since $g_\epsilon = R_\epsilon\breve{g}$ vanishes for $t < 0$ and $B(u_\epsilon)_{|\partial\Omega} = R_\epsilon(B\breve{u}_{|\partial\Omega}) = g_\epsilon$ we have

$$\|e^{-\gamma t}B(u_\epsilon - u_{\epsilon'})_{|\partial\Omega}\|_{H_\gamma^m(0,T)} \le \|e^{-\gamma t}(g_\epsilon - g_{\epsilon'})_{|\partial\Omega}\|_{H_\gamma^m(\mathbb{R})} \to 0$$

as $\epsilon, \epsilon' \to 0$. On the other hand, since $u_\epsilon - u_{\epsilon'}$ vanish for $t < 0$ and the function $t \mapsto e^{-\gamma t}$ is uniformly bounded on compact intervals we have

$$\|e^{-\gamma t}L_{\breve{v}}(u_\epsilon - u_{\epsilon'})\|_{H_\gamma^m((0,T)\times\Omega)} \le C\|A_{\breve{v}}\|_{H_\gamma^m(\mathbb{R}\times\Omega)}\|A_{\breve{v}}^{-1}L_{\breve{v}}(u_\epsilon - u_{\epsilon'})\|_{H_\gamma^m(\mathbb{R}\times\Omega)}.$$

Using commutators we can rewrite

$$\partial^\alpha(A_{\breve{v}}^{-1}L_{\breve{v}}(u_\epsilon - u_{\epsilon'})) = [\partial^\alpha, A_{\breve{v}}^{-1}L_{\breve{v}}](u_\epsilon - u_{\epsilon'}) - A_{\breve{v}}^{-1}L_{\breve{v}}\partial^\alpha(u_\epsilon - u_{\epsilon'}).$$

Because $u_\epsilon \to \breve{u}$ in $H^m(\mathbb{R}\times\Omega)$ and $[\partial^\alpha, A_{\breve{v}}^{-1}L_{\breve{v}}]$ is of order $|\alpha| \le m$, the commutator term on the right hand side tends to zero in $L^2(\mathbb{R}\times\Omega)$ as $\epsilon, \epsilon' \to 0$. On the other hand the second term also tends to zero in $L^2(\mathbb{R}\times\Omega)$ according to (4.19.5). Therefore from (4.19.6) we can see that $(u_{1/j})_j$ and $((u_{1/j})_{|\partial\Omega})_j$ are Cauchy sequences in $CH^m([0,T]\times\Omega)$ and $H^m(0,T)$, respectively, and their limits are u and $u_{|\partial\Omega}$ since $u_{1/j} \to u$ in $CL^2([0,T]\times\Omega)$ and $(u_{1/j})_{|\partial\Omega} \to u_{|\partial\Omega}$ in $L^2(0,T)$.

It remains to establish the energy estimate (4.19.2). First let us note that

$$\partial^\alpha L_{\breve{v}}u_\epsilon = [\partial^\alpha, L_{\breve{v}}]u_\epsilon + L_{\breve{v}}\partial^\alpha u_\epsilon \to [\partial^\alpha, L_{\breve{v}}]\breve{u} + L_{\breve{v}}\partial^\alpha\breve{u} = \partial^\alpha\breve{f}. \tag{4.19.7}$$

in $L^2(\mathbb{R}\times\Omega)$. Thus $L_{\breve{v}}u_\epsilon \to \breve{f}$ in $e^{\gamma t}H_\gamma^m(\mathbb{R}\times\Omega)$. Applying the a priori estimate (4.16.5) to $u_{1/j} \in e^{\gamma t}H_\gamma^{m+1}(\mathbb{R}\times\Omega)$ and letting $j \to \infty$ proves (4.19.2). □

4.19.2 The Non-homogeneous Case

Now we will consider the IBVP with nonzero initial condition. For this one needs compatibility conditions which we are now going to state. Given sufficiently smooth functions f and u_0 define recursively the functions $u_i : \Omega \to \mathbb{R}^n$ by

$$u_i(x) = \partial_t^{i-1}f(0,x) - \sum_{l=0}^{i-1}\binom{i-1}{l}\partial_t^l A(v(0,x))\partial_x u_{i-1-l}(x), \quad x \in \Omega. \tag{4.19.8}$$

The data (u_0, f, g) are said to be compatible up to order p if

$$Bu_{i|\partial\Omega} = \partial_t^i g(0), \qquad i = 0, \ldots, p.$$

By the embedding

$$H^m((0,T) \times \Omega) \hookrightarrow H^{j+1}((0,T); H^{m-j-1}(\Omega)) \hookrightarrow C^j([0,T]; H^{m-j-1}(\Omega))$$

for $0 \leq j \leq m-1$, we have $\partial_t^j v_{|t=0} \in H^{m-j-1}(\Omega)$. However, stronger assumptions are needed for these traces in the general IBVP as included in the following theorem.

Theorem 4.19.3. *Consider the framework of Theorem 4.13.3 and suppose that v satisfies the conditions of Theorem 4.19.2. Suppose in addition that $\partial_t^j v_{|t=0} \in H^{m-j}(\Omega)$ for all $0 \leq j \leq m-1$. If the data*

$$(u_0, f, g) \in H^{m+1/2}(\Omega) \times H^m((0,T) \times \Omega) \times H^m(0,T)$$

is compatible up to order $m-1$ then the initial boundary value problem

$$L_v u = f, \qquad Bu_{|\partial\Omega} = g, \qquad u_{|t=0} = u_0 \tag{4.19.9}$$

has a unique solution $u \in CH^m([0,T] \times \Omega)$ and $u_{|\partial\Omega} \in H^m(0,T)$.

Remark 4.19.4. The proof of this theorem is similar to the proof of Theorem 4.11.4. These is where the additional regularity for u_0 is needed. The proof shows that the solution takes the form $u = u_{a|[0,T]} + u_h$ where $u_a \in H^{m+1}(\mathbb{R} \times \Omega)$ and u_h is a solution of an IBVP with zero initial data. Therefore, according to the proof of Theorem 4.19.2, there exists $(u_j)_j \subset H^{m+1}((0,T) \times \Omega)$ such that

$$\begin{aligned} u_j \to u, &\quad \text{in} \quad CH^m([0,T] \times \Omega) \\ (u_j)_{|\partial\Omega} \to u_{|\partial\Omega}, &\quad \text{in} \quad H^m(0,T) \\ L_v u_j \to L_v u, &\quad \text{in} \quad H^m(0,T). \end{aligned} \tag{4.19.10}$$

The extra regularity imposed on the data u_0 is not necessary since one can have the same result even when it is only in $H^m(\Omega)$. This is the content of the following theorem.

Theorem 4.19.5. *The conclusions of the Theorem 4.19.3 still hold even for initial data $u_0 \in H^m(\Omega)$.*

To prove this theorem one requires the following a priori estimate. This is similar to the one given in Lemma 4.16.2 but with additional terms for the nonzero initial condition.

Lemma 4.19.6. *For every $v \in H^m((0,T) \times \Omega)$ satisfying the conditions in Theorem 4.19.3 and for every $u \in H^{m+1}((0,T) \times \Omega)$ we have*

$$\begin{aligned} &\|u\|_{CH^m([0,T]\times\Omega)} + \|u_{|\partial\Omega}\|_{H^m(0,T)} \\ &\leq C \left(\|L_v u\|_{H^m((0,T)\times\Omega)} + \|Bu_{|\partial\Omega}\|_{H^m(0,T)} + \sum_{i=0}^{m} \|\partial_t^i u_{|t=0}\|_{H^{m-i}(\Omega)} \right) \end{aligned}$$

where $C > 0$ depends only on $T, \mathcal{K}, K, R$ and $\|\partial_t^j v_{|t=0}\|_{H^{m-j}(\Omega)}$ for $0 \leq j \leq m-1$.

Proof. In the following proof $C > 0$ will be a generic constant as in the statement of the lemma independent of $\tau \in [0, T]$. As before, let $f = L_v u$ and $g = Bu_{|\partial\Omega}$. We will use the following a priori estimate

$$\begin{aligned}&\|w(\tau)\|_{L^2(\Omega)} + \frac{1}{\sqrt{\tau}}\|w\|_{L^2((0,\tau)\times\Omega)} + \|w_{|\partial\Omega}\|_{L^2(0,\tau)}\\&\leq C(\|w_{|t=0}\|_{L^2(\Omega)} + \sqrt{\tau}\|L_v w\|_{L^2((0,\tau)\times\Omega)} + \|Bw_{|\partial\Omega}\|_{L^2(0,\tau)})\end{aligned} \tag{4.19.11}$$

which holds for all $\tau \in (0, T]$ and for all $w \in H^1((0,T)\times\Omega)$, where $C = C(\mathcal{K}, K) > 0$. This follows from the a priori estimate (4.13.17) by taking $\gamma = C/\tau$ for some $C = C(T) > 0$. By a standard density argument it is enough to prove the a priori estimate for $u \in \mathscr{D}([0,T]\times\overline{\Omega})$. Applying ∂_t^j for $j = 0, \ldots, m$ to the equality $L_v u = f$ we obtain $L_v\partial_t^j u = f_j := A(v)\partial_t^j(A(v)^{-1}f) - A(v)[\partial_t^j, A(v)^{-1}L_v]u$ and $B(\partial_t^j u)_{|\partial\Omega} = \partial_t^j g$ for $j = 0, \ldots, m$. Taking $w = \partial_t^j u$ in (4.19.11) we have

$$\begin{aligned}&\|\partial_t^j u(\tau)\|_{L^2(\Omega)} + \frac{1}{\sqrt{\tau}}\|\partial_t^j u\|_{L^2((0,\tau)\times\Omega)} + \|\partial_t^j(u_{|\partial\Omega})\|_{L^2(0,\tau)}\\&\leq C(\|\partial_t^j u_{|t=0}\|_{L^2(\Omega)} + \sqrt{\tau}\|f_j\|_{L^2((0,\tau)\times\Omega)} + \|\partial_t^j g\|_{L^2(0,\tau)})\end{aligned} \tag{4.19.12}$$

We are going to estimate each term on the right hand side of this inequality. Expanding the commutator in f_j for $j \geq 1$ we have

$$A(v)[\partial_t^j, A(v)^{-1}L_v]u = A(v)\sum_{1\leq l\leq j} c_{ij}\partial_t^{l-1}(\mathrm{d}A(v)^{-1}\partial_t v)\partial_t^{j-l}(\partial_t u),$$

where $\mathrm{d}A$ is the first order differential of A and c_{ij} are constants. Let us estimate the L^2-norm of each term in the above sum. If $j = 1$ then we immediately have the estimate $\|(\mathrm{d}A(v)^{-1}\partial_t v)\partial_t u\|_{L^2((0,\tau)\times\Omega)} \leq C\|\partial_t u\|_{L^2((0,\tau)\times\Omega)}$. Suppose that $j \geq 2$. Then Hölder's inequality implies that

$$\begin{aligned}&\|\partial_t^{l-1}(\mathrm{d}A(v)^{-1}\partial_t v)\partial_t^{j-l}(\partial_t u)\|_{L^2((0,\tau)\times\Omega)}\\&\leq \|\partial_t^{l-1}(\mathrm{d}A(v)^{-1}\partial_t v)\|_{L^{2(j-1)/(l-1)}((0,\tau)\times\Omega)}\|\partial_t^{j-l}(\partial_t u)\|_{L^{2(j-1)/(j-l)}((0,\tau)\times\Omega)}\end{aligned}$$

Since $\partial_t^{l-1}(\mathrm{d}A(v)^{-1}\partial_t v)$ is a nonlinear function of $\partial_t v$ of order $l-1$ the first factor can be estimated using Theorem 4.17.4 by

$$\|\partial_t^{l-1}(\mathrm{d}A(v)^{-1}\partial_t v)\|_{L^{2(j-1)/(l-1)}((0,\tau)\times\Omega)} \leq C(\tilde{K}_{j-1,\tau}(\partial_t v))(\|\partial_t v\|_{H^{j-1}((0,\tau)\times\Omega)} + 1)$$

On the other hand, the term involving u can also be estimated using Theorem 4.17.3

$$\|\partial_t^{j-l}(\partial_t u)\|_{L^{2(j-1)/(j-l)}((0,\tau)\times\Omega)} \leq C(\|\partial_t u\|_{H^{j-1}((0,\tau)\times\Omega)} + \tilde{K}_{j-1,\tau}(\partial_t u))$$

Theorem 4.17.5 and the Sobolev embedding $H^{k+1}(\Omega) \hookrightarrow W^{k,\infty}(\Omega)$ imply

$$\tilde{K}_{j-1,\tau}(\partial_t u) \leq C\left(\sqrt{\tau}\|u\|_{H^3((0,\tau)\times\Omega)} + \sum_{i=0}^{m-1}\|\partial_t^i u_{|t=0}\|_{H^{m-i}(\Omega)}\right).$$

Furthermore, we have $\|A(v)\partial_t^j(A(v)^{-1}f)\|_{L^2((0,\tau)\times\Omega)} \leq C\|f\|_{H^m((0,T)\times\Omega)}$. Combining all our estimates we deduce that, using $\tau \leq T$,

$$\|f_j\|_{L^2((0,\tau)\times\Omega)} \leq C\left(\|f\|_{H^m((0,T)\times\Omega)} + \|u\|_{H^m((0,\tau)\times\Omega)} + \sum_{i=0}^{m}\|\partial_t^i u_{|t=0}\|_{H^{m-i}(\Omega)}\right).$$

Therefore,

$$\begin{aligned}
&\sum_{j=0}^{m} \|\partial_t^j u(\tau)\|_{L^2(\Omega)} + \|u_{|\partial\Omega}\|_{H^m(0,\tau)} \qquad (4.19.13)\\
&\leq C\left(\|f\|_{H^m((0,T)\times\Omega)} + \|g\|_{H^m(0,T)} + \sum_{i=0}^{m} \|\partial_t^i u_{|t=0}\|_{H^{m-i}(\Omega)} + \|u\|_{H^m((0,\tau)\times\Omega)}\right)
\end{aligned}$$

For convenience we denote by $N(u)$ the term on the right hand side of (4.19.13).

The next step is to estimate the mixed derivatives. We proceed by an induction argument to prove that

$$\|\partial_x^k \partial_t^j u(\tau)\|_{L^2(\Omega)} \leq N(u) \qquad (4.19.14)$$

for all $k + j \leq m$. The basis step $k = 0$ is given by (4.19.13). Before proceeding to the induction step, we prove the estimate in the separate case where $k = j = 1$. The PDE gives us

$$\partial_x \partial_t u(\tau) = \partial_t (A(v(\tau))^{-1} f(\tau)) - \partial_t (A(v(\tau))^{-1}) \partial_t u(\tau) - A(v(\tau))^{-1} \partial_t^2 u(\tau).$$

The estimates on time-derivatives we have shown above and the Sobolev embedding theorem imply

$$\|\partial_x \partial_t u(\tau)\|_{L^2(\Omega)} \leq N(u). \qquad (4.19.15)$$

Now we go to the induction step. Suppose that (4.19.14) is true for k and j such that $k + j \leq m$. The PDE gives us

$$\partial_x^{k+1} \partial_t^j u = \partial_x^k \partial_t^j (A(v)^{-1} f) - \partial_x^k \partial_t^j (A(v)^{-1} \partial_t u)$$

for $k + 1 + j \leq m$ and $k \geq 0$. On one hand, by the Sobolev embedding theorem

$$\|\partial_x^k \partial_t^j (A(v(\tau))^{-1} f(\tau))\|_{L^2(\Omega)} \leq C\|f\|_{H^m((0,T)\times\Omega)}$$

for all $\tau \in [0, T]$. On the other hand, Leibniz's rule gives us

$$\|\partial_x^k \partial_t^j (A(v(\tau))^{-1} \partial_t u(\tau))\|_{L^2(\Omega)} \leq \sum_{l=0}^{k} \sum_{i=0}^{j} c_{li} \|\partial_x^{k-l} \partial_t^{j-i} A(v(\tau))^{-1} \partial_x^l \partial_t^{i+1} u(\tau)\|_{L^2(\Omega)}$$

for some constants c_{li}. Let us consider separate cases. If $k - l + j - i \leq m - 2$ then for all $\tau \in [0, T]$

$$\begin{aligned}
&\|\partial_x^{k-l} \partial_t^{j-i} A(v(\tau))^{-1} \partial_x^l \partial_t^{i+1} u(\tau)\|_{L^2(\Omega)}\\
&\quad \leq \|\partial_x^{k-l} \partial_t^{j-i} A(v(\tau))^{-1}\|_{L^\infty(\Omega)} \|\partial_x^l \partial_t^{i+1} u(\tau)\|_{L^2(\Omega)}\\
&\quad \leq C\|\partial_x^{k-l} \partial_t^{j-i} A(v)^{-1}\|_{H^2((0,T)\times\Omega)} \|\partial_x^l \partial_t^{i+1} u(\tau)\|_{L^2(\Omega)} \leq N(u)
\end{aligned}$$

where the last inequality is due to the induction hypothesis. If $k - l + j - i = m - 1$ then $k + j = m - 1$ and $i = l = 0$ and therefore applying (4.19.15)

$$\begin{aligned}
&\|\partial_x^{k-l} \partial_t^{j-i} A(v(\tau))^{-1} \partial_x^l \partial_t^{i+1} u(\tau)\|_{L^2(\Omega)}\\
&\quad \leq \|\partial_x^{k-l} \partial_t^{j-i} A(v(\tau))^{-1}\|_{L^2(\Omega)} \|\partial_t u(\tau)\|_{L^\infty(\Omega)}\\
&\quad \leq C\|\partial_x^{k-l} \partial_t^{j-i} A(v)^{-1}\|_{H^1((0,T);L^2(\Omega))} (\|\partial_t u(\tau)\|_{L^2(\Omega)} + \|\partial_x \partial_t u(\tau)\|_{L^2(\Omega)})\\
&\quad \leq N(u)
\end{aligned}$$

for all $\tau \in [0, T]$. Taking the sum completes the proof of the induction. Combining the estimates for the time derivatives and the mixed derivatives gives us

$$\sum_{|\beta| \le m} \|\partial^\beta u(\tau)\|_{L^2(\Omega)} + \|u_{|\partial\Omega}\|_{H^m(0,T)} \le N(u). \tag{4.19.16}$$

Squaring this inequality and applying Gronwall's inequality give the estimate stated in the lemma. □

Proof of Theorem 4.19.5. It can be shown that there exists a sequence of more regular functions $(u_0^k)_k \subset H^{m+1/2}(\Omega)$ such that $u_0^k \to u_0$ in $H^m(\Omega)$ and the data (u_0^k, f, g) is still compatible up to order $m-1$ for all k, see for instance [**64**] or the proof of Theorem 4.21.2. Let u_k be the solution of the corresponding initial boundary value problem with data (u_0^k, f, g) given by Theorem 4.19.3. Then the difference $w = u_k - u_j$ satisfies

$$L_v w = 0 \quad \text{in } (0,T) \times \Omega, \qquad Bw_{|\partial\Omega} = 0 \quad \text{in } (0,T), \qquad w_{|t=0} = u_0^k - u_0^j \quad \text{in } \Omega.$$

Then according to Remark 4.19.4 there exists a sequence $w_l \in H^{m+1}((0,T)\times\Omega)$ such that $w_l \to w$ in $CH^m([0,T] \times \Omega)$, $L_v w_l \to 0$ in $H^m((0,T) \times \Omega)$ and $B(w_l)_{|\partial\Omega} \to 0$ in $H^m(0,T)$. Thus applying the a priori estimate in the previous lemma to w_l and passing to the limit $l \to \infty$ we have

$$\begin{aligned} &\|u_k - u_j\|_{CH^m([0,T]\times\Omega)} + \|(u_k)_{|\partial\Omega} - (u_j)_{|\partial\Omega}\|_{H^m(0,T)} \\ &\quad \le C \sum_{i=0}^{m} \|\partial_t^i u_k(0) - \partial_t^i u_j(0)\|_{H^{m-i}(\Omega)}. \end{aligned}$$

However, by recursion we have $\partial_t^i u_k(0) = u_{k,i} \to u_i$ in $H^{m-i}(\Omega)$, where $u_{k,i}$ are the functions defined recursively in (4.19.8) where u_0^k is the initial term. Thus $(u_k)_k$ and $((u_k)_{|\partial\Omega})_k$ are Cauchy sequences in $CH^m([0,T] \times \Omega)$ and $H^m(0,T)$, respectively, and let u and $\tilde{u}$ be their limits. Since $u_k \to u$ in $H^1((0,T) \times \Omega)$, the continuity of the trace operator implies $(u_k)_{|\partial\Omega} \to u_{|\partial\Omega}$ in $L^2(0,T)$ and thus $\tilde{u} = u_{|\partial\Omega}$. Passing to the limit $k \to \infty$ in the IBVP satisfied by u_k we can see that u is the required solution. □

Remark 4.19.7. Given a positive integer k, using Remark 4.19.4, there exists a function $u_k^k \in H^{m+1}((0,T) \times \Omega)$ such that $\|u_k^k - u_k\|_{CH^m([0,T]\times\Omega)} < \frac{1}{k}$ and $\|(u_k^k)_{|\partial\Omega} - (u_k)_{|\partial\Omega}\|_{H^m(0,T)} < \frac{1}{k}$ where u_k is the solution corresponding to the initial data u_0^k in the proof of the previous theorem. By the triangle inequality we have $u_k^k \to u$ in $CH^m([0,T] \times \Omega)$ and $(u_k^k)_{|\partial\Omega} \to u_{|\partial\Omega}$ in $H^m(0,T)$. Moreover, since $L_v u_k^k - L_v u_j^j = F_k - F_j$ where $F_k \to f$ in $H^m((0,T) \times \Omega)$, see (4.19.7) for instance, it follows that $(L_v u_k^k)_k$ is a Cauchy sequence in $H^m((0,T)\times\Omega)$. Since $L_v u_k^k \to L_v u$ in $L^2((0,T)\times\Omega)$ we have $L_v u_k^k \to L_v u$ in $H^m((0,T) \times \Omega)$. This implies that the a priori estimate in Lemma 4.19.6 holds for the solution u of the initial boundary value problem (4.19.9).

4.20 WEAK SOLUTIONS OF A LINEAR HYPERBOLIC PDE-ODE SYSTEM

In this section we prove the existence, uniqueness and regularity of weak solutions to a linear hyperbolic system of partial differential equations coupled with a differential equation at the boundary. We are interested in the L^2-well-posedness of the following system

$$\begin{cases} L_v u(t,x) = f(t,x), \quad 0 < t < T,\ 0 < x < 1, \\ B_0 u(t,0) = g_0(t) + Q_0(t)h(t), \quad 0 < t < T, \\ B_1 u(t,1) = g_1(t) + Q_1(t)h(t), \quad 0 < t < T, \\ h'(t) = H(t)h(t) + G_0(t)u(t,0) + G_1(t)u(t,1) + S(t), \quad 0 < t < T, \\ u(0,x) = u_0(x), \quad 0 < x < 1, \\ h(0) = h_0 \end{cases} \tag{4.20.1}$$

where

$$L_v u(t,x) = \partial_t u(t,x) + A(v(t,x))\partial_x u(x) + R(t,x)u(t,x)$$

and $v \in W^{1,\infty}(Q_T;\mathbb{R}^n)$. All throughout this section we assume that $B_0 \in \mathbb{R}^{p\times n}$, $B_1 \in \mathbb{R}^{(n-p)\times p}$,

$$\begin{aligned} &R \in L^\infty(Q_T;\mathbb{R}^{n\times n}), \quad Q_0 \in L^\infty((0,T);\mathbb{R}^{p\times m}), \quad Q_1 \in L^\infty((0,T);\mathbb{R}^{(n-p)\times m}) \\ &H \in L^\infty((0,T);\mathbb{R}^{m\times m}), \quad G_0, G_1 \in L^\infty((0,T);\mathbb{R}^{m\times n}), \quad S \in L^2((0,T);\mathbb{R}^m). \end{aligned}$$

Furthermore, we suppose that B_0 and B_1 have full ranks and that (FS), (D), and the UKL condition (H5) hold.

Definition 4.20.1. Given $f \in L^2(Q_T)$, $g_0 \in L^2(0,T)$, $g_1 \in L^2(0,T)$, $S \in L^2(0,T)$, $u_0 \in L^2(0,1)$ and $h_0 \in \mathbb{R}^m$, a pair of functions $(u,h) \in L^2(Q_T) \times L^2(0,T)$ is called a *weak solution* of the system (4.20.1) if the variational equality

$$\begin{aligned} &\int_0^T\int_0^1 u(t,x)\cdot L_v^*\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T h(t)\cdot(\eta'(t) + \tilde{H}(t)\eta(t) + Q_1(t)^\top M_1(t)\varphi(t,1) - Q_0(t)^\top M_0(t)\varphi(t,0))\,\mathrm{d}t \\ &= \int_0^T\int_0^1 f(t,x)\cdot\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T g_1(t)\cdot(M_1(t)\varphi(t,1) + (G_1(t)Y_1)^\top\eta(t))\,\mathrm{d}t \\ &+ \int_0^T g_0(t)\cdot(M_0(t)\varphi(t,0) - (G_0(t)Y_0)^\top\eta)\,\mathrm{d}t - \int_0^T S(t)\cdot\eta(t)\,\mathrm{d}t \\ &+ \int_0^1 u_0(x)\cdot\varphi(0,x)\,\mathrm{d}x - h_0\cdot\eta(0) \end{aligned} \tag{4.20.2}$$

where

$$\tilde{H} = (H + G_1Y_1Q_1 + G_0Y_0Q_0)^\top,$$

holds for all $\varphi \in \mathcal{E}^*(Q_T)$ and for all $\eta \in H^1(0,T)$ such that $\varphi(T,\cdot) = 0$, $\eta(T) = 0$, $C_1\varphi_{|x=1} = -(G_1D_1)^\top\eta$ and $C_0\varphi_{|x=0} = (G_0D_0)^\top\eta$.

In Definition 4.20.1, the matrices M_i, Y_i and D_i are those given in Lemma 4.7.3. The definition of a weak solution is obtained by multiplying the system (4.20.1) with appropriate test functions and integrating by parts. The space of test functions in the above definition is denoted by

$$\begin{aligned} W = \{(\varphi,\eta) \in \mathcal{E}^*(Q_T) \times H^1(0,T) : \eta_{|t=T} = 0,\ \varphi_{|t=T} = 0, \\ C_1\varphi_{|x=1} = -(G_1D_1)^\top\eta,\ C_0\varphi_{|x=0} = (G_0D_0)^\top\eta\}. \end{aligned}$$

Because G_0 and G_1 are in L^∞, the functions $(G_1D_1)^\top\eta$ and $(G_0D_0)^\top\eta$ may be only in $L^2(0,T)$ even for $\eta \in H^1(0,T)$. In order for the compatibility conditions $C_1\varphi_{|x=1} = -(G_1D_1)^\top\eta$ and $C_0\varphi_{|x=0} = (G_0D_0)^\top\eta$ to be meaningful, we take the space $\mathcal{E}^*(Q_T)$ to be the space for the first component instead of the space $H^1(Q_T)$ which was used in Definition 4.9.1.

Theorem 4.20.2. *The space W is dense in $L^2(Q_T) \times L^2(0,T)$.*

Proof. Take $(u,h) \in L^2(Q_T) \times L^2(0,T)$ and $\epsilon > 0$. Let $\eta \in H^1(0,T)$ be such that $\eta(T) = 0$ and $\|\eta - h\|_{L^2(0,T)} < \epsilon$. Take $w \in H_0^1(Q_T)$ satisfying $\|u - w\|_{L^2(Q_T)} < \epsilon$. Consider the IBVP

$$L_v^*\psi = 0, \quad C_0\psi_{|x=0} = (G_0D_0)^\top\eta, \quad C_1\psi_{|x=1} = -(G_1D_1)^\top\eta, \quad \psi_{|t=T} = 0. \qquad (4.20.3)$$

This IBVP has a unique solution $\psi \in L^2(Q_T)$ and furthermore $\psi \in \mathcal{E}^*(Q_T)$ according to the dual version of Theorem 4.13.10.

By the absolute continuity of the Lebesgue integral, there exists $\delta = \delta(\epsilon) > 0$ such that if $\mathcal{O} \subset Q_T$ has Lebesgue measure less than or equal to δ then $\|u - \psi\|_{L^2(\mathcal{O})} < \epsilon$. Without loss of generality, we can assume that $\delta < 4T$. Let $\theta \in \mathscr{D}[0,1]$ be such that $0 \le \theta \le 1$ on $[0,1]$, $\theta = 1$ on $(0,\delta/4T) \cup (1-\delta/4T,1)$ and $\theta = 0$ on $(\delta/2T, 1-\delta/2T)$. Define $\varphi = \theta\psi + (1-\theta)w$. Since $\mathcal{E}^*(Q_T)$ is closed under addition and multiplication with smooth functions it holds that $\varphi \in \mathcal{E}^*(Q_T)$. From (4.20.3) and the definition of θ we have $(\varphi,\eta) \in W$. Furthermore,

$$\|u - \varphi\|_{L^2(Q_T)} \le \|\theta\|_{L^\infty(Q_T)}\|u - \psi\|_{L^2(R_{\delta,T})} + \|1-\theta\|_{L^\infty(Q_T)}\|u - w\|_{L^2(Q_T)} < 2\epsilon$$

where $R_{\delta,T} = (0,T) \times ((0,\delta/2T) \cup (1-\delta/2T,1))$. Therefore

$$\|(u,h) - (\varphi,\eta)\|_{L^2(Q_T)\times L^2(0,T)} < \sqrt{5}\epsilon$$

and consequently W is dense in $L^2(Q_T) \times L^2(0,T)$. □

We would like to apply Theorem 4.1.1 to prove the well-posedness of (4.20.1). Therefore the crucial step is to prove an a priori estimate. But first we need to rewrite (4.20.2) in the form (4.1.1). For this purpose, we set $X = e^{-\gamma t}L^2(Q_T) \times e^{-\gamma t}L^2(0,T)$, $Y = \mathcal{E}^*(Q_T) \times H^1(0,T)$ and $Z = e^{-\gamma t}L^2(0,T) \times e^{-\gamma t}L^2(0,T) \times L^2(0,1) \times \mathbb{R}^m$. Define $\Lambda : Y \to X$, $\Psi : Y \to Z$ and $\Phi : Y \to Z$ as follows

$$\begin{aligned}
\Lambda\begin{pmatrix}\varphi\\ \eta\end{pmatrix} &= \begin{pmatrix} L_v^*\varphi \\ \eta' + \tilde{H}\eta + Q_1^\top M_1\varphi_{|x=1} - Q_0^\top M_0\varphi_{|x=0}\end{pmatrix}\\
\Phi\begin{pmatrix}\varphi\\ \eta\end{pmatrix} &= \begin{pmatrix} C_0\varphi_{|x=0} - (G_0D_0)^\top\eta \\ C_1\varphi_{|x=1} + (G_1D_1)^\top\eta \\ \varphi_{|t=T} \\ \eta(T)\end{pmatrix}\\
\Psi\begin{pmatrix}\varphi\\ \eta\end{pmatrix} &= \begin{pmatrix} M_0\varphi_{|x=0} - (G_0Y_0)^\top\eta \\ -(M_1\varphi_{|x=1} + (G_1Y_1)^\top\eta) \\ \varphi_{|t=0} \\ -\eta(0).\end{pmatrix}
\end{aligned}$$

for every $(\varphi,\eta) \in Y$. With these notations, the variational equation (4.20.2) can be rewritten as

$$\begin{aligned}
(e^{-2\gamma t}(u,h), \Lambda(\varphi,\eta))_X &= (e^{-2\gamma t}(f,-S),(\varphi,\eta))_X \\
&\quad + ((e^{-2\gamma t}g_0, e^{-2\gamma t}g_1, u_0, h_0), \Psi(\varphi,\eta))_Z \qquad (4.20.4)
\end{aligned}$$

for all $(\varphi,\eta) \in W = \ker\Phi$.

Theorem 4.20.3. *In the notation of the previous paragraph, there exist $\gamma_0 \geq 1$ and $C > 0$ such that*

$$\gamma\|(\varphi,\eta)\|_X^2 + \|\Psi(\varphi,\eta)\|_Z^2 \leq C\left(\frac{1}{\gamma}\|\Lambda(\varphi,\eta)\|_X^2 + \|\Phi(\varphi,\eta)\|_Z^2\right)$$

holds for all $(\varphi,\eta) \in Y$ and $\gamma \geq \gamma_0$.

Proof. Let $(\varphi,\eta) \in Y$. From the priori estimate (4.13.25) and the triangle inequality it follows that there is a constant $C > 0$ such that

$$\begin{aligned}
&\|\varphi_{|t=0}\|_{L^2(0,1)}^2 + \gamma\|e^{\gamma t}\varphi\|_{L^2(Q_T)}^2 + \|e^{\gamma t}\varphi_{|\partial\Omega}\|_{L^2(0,T)}^2 \\
&+ \|e^{\gamma t}(M_0\varphi_{|x=0} - (G_0Y_0)^\top\eta)\|_{L^2(0,T)}^2 + \|e^{\gamma t}(M_1\varphi_{|x=1} + (G_1Y_1)^\top\eta)\|_{L^2(0,T)}^2 \\
&\leq C\bigg(e^{2\gamma T}\|\varphi_{|t=T}\|_{L^2(0,1)}^2 + \frac{1}{\gamma}\|e^{\gamma t}L_v^*\varphi\|_{L^2(Q_T)}^2 + \|e^{\gamma t}(C_0\varphi_{|x=0} - (G_0D_0)^\top\eta)\|_{L^2(0,T)}^2 \\
&\qquad + \|e^{\gamma t}(C_1\varphi_{|x=1} + (G_1D_1)^\top\eta)\|_{L^2(0,T)}^2 + \|e^{\gamma t}\eta\|_{L^2(0,T)}^2\bigg)
\end{aligned} \tag{4.20.5}$$

for all $\gamma \geq \gamma_0$ where γ_0 is the constant in Theorem 4.13.7. From the a priori estimate (4.2.8) in Theorem 4.2.4 and the triangle inequality we obtain

$$\begin{aligned}
|\eta(0)|^2 + \gamma\|e^{\gamma t}\eta\|_{L^2(0,T)}^2 \;\leq\; & \frac{C}{\gamma}\|e^{\gamma t}(\eta' + \tilde{H}\eta + Q_1^\top M_1\varphi_{|x=1} - Q_0^\top M_0\varphi_{|x=0})\|_{L^2(0,T)}^2 \\
& + \frac{C}{\gamma}\|e^{\gamma t}\varphi_{|\partial\Omega}\|_{L^2(0,T)}^2 + Ce^{2\gamma T}|\eta(T)|^2.
\end{aligned} \tag{4.20.6}$$

From (4.20.5) and (4.20.6) and upon choosing γ_0 large enough, the estimate in the theorem follows after an absorption argument. □

It is now possible to prove the existence and uniqueness of weak solutions of the system (4.20.1).

Theorem 4.20.4. *Let $f \in L^2(Q_T)$, $g_0 \in L^2(0,T)$, $g_1 \in L^2(0,T)$, $S \in L^2(0,T)$, $u_0 \in L^2(0,1)$ and $h_0 \in \mathbb{R}^m$. With the assumptions in the beginning of this section, the system (4.20.1) has a unique weak solution $(u,h) \in L^2(Q_T) \times L^2(0,T)$. Furthermore, $(u,h) \in [CL^2(Q_T) \cap \mathcal{E}(Q_T)] \times H^1(0,T)$ and in particular $u_{|\partial\Omega} \in L^2(0,T)$. The function u is the weak solution of the IBVP*

$$\begin{cases}
L_v u(t,x) = f(t,x), \quad 0 < t < T,\ 0 < x < 1, \\
B_0 u(t,0) = g_0(t) + Q_0(t)h(t), \quad 0 < t < T, \\
B_1 u(t,1) = g_1(t) + Q_1(t)h(t), \quad 0 < t < T, \\
u(0,x) = u_0(x), \quad 0 < x < 1,
\end{cases} \tag{4.20.7}$$

and h is the solution of the ODE

$$\begin{cases}
h'(t) = H(t)h(t) + G_0(t)u(t,0) + G_1(t)u(t,1) + S(t), \quad 0 < t < T, \\
h(0) = h_0
\end{cases} \tag{4.20.8}$$

The weak solution (u,h) satisfies the energy estimate

$$\begin{aligned}
&e^{-2\gamma T}\|u\|_{CL^2(Q_T)}^2 + \gamma\|e^{-\gamma t}u\|_{L^2(Q_T)}^2 + \|e^{-\gamma t}u_{|\partial\Omega}\|_{L^2(0,T)}^2 + \gamma\|e^{-\gamma t}h\|_{L^2(0,T)}^2 \\
&\leq C\left(\|u_0\|_{L^2(0,1)}^2 + |h_0|^2 + \frac{1}{\gamma}\|e^{-\gamma t}f\|_{L^2(Q_T)}^2 + \|e^{-\gamma t}g\|_{L^2(0,T)}^2 + \frac{1}{\gamma}\|e^{-\gamma t}S\|_{L^2(0,T)}^2\right)
\end{aligned}$$

for all $\gamma \geq \gamma_0$ for some $C > 0$ and $\gamma_0 \geq 1$.

Proof. The existence of a weak solution is a direct consequence of Theorem 4.1.1 and Theorem 4.20.3. The next step is to show that if (u, h) is any weak solution of (4.20.1) then u is the weak solution of (4.20.7) and h is the solution of (4.20.8). Suppose that (u, h) is a weak solution of (4.20.1). Taking $\eta = 0$ and $\varphi \in H^1(Q_T)$ with $C\varphi_{|\partial\Omega} = 0$ and $\varphi_{|t=T} = 0$ we have $(\varphi, \eta) \in W$. With this (φ, η) in (4.20.2) we can see that u is the weak solution of the (4.20.7). Therefore from Theorem 4.13.10, $u \in CL^2(Q_T) \cap \mathcal{E}(Q_T)$ and in particular $u_{|\partial\Omega} \in L^2(0, T)$. Moreover, from Remark 4.13.11 and Lemma 4.7.3 u satisfies the variational equation

$$
\begin{aligned}
&\int_0^T \int_0^1 u(t,x) \cdot L_v^* \varphi(t,x) \,\mathrm{d}x \,\mathrm{d}t \\
&= \int_0^T \int_0^1 f(t,x) \cdot \varphi(t,x) \,\mathrm{d}x \,\mathrm{d}t - \int_0^T (g_1(t) + Q_1(t)h(t)) \cdot M_1(t)\varphi(t,1) \,\mathrm{d}t \\
&+ \int_0^T (g_0(t) + Q_0(t)h_0(t)) \cdot M_0(t)\varphi(t,0) \,\mathrm{d}t - \int_0^T N_1 u(t,1) \cdot C_1(t)\varphi(t,1) \,\mathrm{d}t \\
&+ \int_0^T N_0 u(t,0) \cdot C_0(t)\varphi(t,0) \,\mathrm{d}t - \int_0^1 u(T,x) \cdot \varphi(T,x) \,\mathrm{d}x \\
&+ \int_0^1 u_0(x) \cdot \varphi(0,x) \,\mathrm{d}x
\end{aligned}
\tag{4.20.9}
$$

for all $\varphi \in \mathcal{E}^*(Q_T)$.

Given $\eta \in H^1(0, T)$ with $\eta(T) = 0$ consider the IBVP

$$
L_v^* \varphi = 0, \quad C_0 \varphi_{|x=0} = (G_0 D_0)^\top \eta, \quad C_1 \varphi_{|x=1} = -(G_1 D_1)^\top \eta, \quad \varphi_{|t=T} = 0. \tag{4.20.10}
$$

The dual version of Theorem 4.13.10 implies that (4.20.10) has a unique weak solution $\varphi \in L^2(Q_T)$ such that $\varphi \in \mathcal{E}^*(Q_T)$. Thus $(\varphi, \eta) \in W$. From (4.7.3), (4.7.11), (4.20.2) and (4.20.9) we can see that

$$
\begin{aligned}
&\int_0^T h(t) \cdot (\eta'(t) + H(t)^\top \eta(t)) \,\mathrm{d}t \\
&= -h_0 \cdot \eta(0) - \int_0^T (G_0(t)u(t,0) + G_1(t)u(t,1) + S(t)) \cdot \eta(t) \,\mathrm{d}t.
\end{aligned}
\tag{4.20.11}
$$

According to (4.20.11) and Theorem 4.2.5, h is the solution of the ordinary differential equation (4.20.8) and $h \in H^1(0, T)$.

The energy estimate in the statement of the theorem follows from the energy estimate (4.13.28) for u, the energy estimate (4.2.16) for h and an absorption argument. Thus, any weak solution of (4.20.1) satisfies the energy estimate. Consequently, (4.20.1) has a unique weak solution. □

4.21 LINEAR HYPERBOLIC PDE-ODE SYSTEMS WITH CONSTANT COEFFICIENTS

In this section, we show that in the case where the coefficients in (4.20.1) are constant, the weak solution defined in the previous section coincides with the one given by semigroup theory. Consider the weak solution $(u,h) \in C([0,\infty);L^2(0,1)\times\mathbb{R}^m)$ of the system

$$\begin{cases} \partial_t u(t,x) + A\partial_x u(t,x) + Ru(t,x) = 0, \quad t>0,\ 0<x<1, \\ B_0 u(t,0) = Q_0 h(t), \quad t>0, \\ B_1 u(t,1) = Q_1 h(t), \quad t>0, \\ h'(t) = Hh(t) + G_0 u(t,0) + G_1 u(t,1), \quad t>0, \\ u(0,x) = u_0(x), \quad 0<x<1, \\ h(0) = h_0 \end{cases} \tag{4.21.1}$$

The boundary conditions for u and the ODE for h can be viewed as a nonlocal boundary condition for u

$$B_x u(t,x) = Q_x e^{tH} h_0 + \int_0^t Q_x e^{(t-s)H}(G_0 u(s,0) + G_1 u(s,1))\,\mathrm{d}s, \quad x=0,1.$$

This can be derived by using the variation of parameters formula for the differential equation for h and substituting it to the boundary conditions for u. However, we will not treat the boundary conditions in this way.

Let k be a positive integer. For each $u_0 \in H^k(0,1)$ we define

$$u_i = -A\partial_x u_{i-1} - Ru_{i-1}, \qquad i=1,\dots,k. \tag{4.21.2}$$

The data $(u_0,h_0) \in H^k(0,1)\times\mathbb{R}^m$ is said to be *compatible* up to order $k-1$ if

$$B_y u_i(y) = Q_y h_i, \qquad i=0,\dots,k-1 \text{ and } y=0,1, \tag{4.21.3}$$

where

$$h_i = Hh_{i-1} + G_0 u_{i-1}(0) + G_1 u_{i-1}(1), \qquad i=1,\dots,k. \tag{4.21.4}$$

Theorem 4.21.1. *Let $k\in\mathbb{N}$. If the data $(u_0,h_0) \in H^k(0,1)\times\mathbb{R}^m$ is compatible up to order $k-1$ then the weak solution (u,h) of* (4.21.1) *satisfies $(u,h) \in CH^k(Q_T)\times H^{k+1}(0,T)$ and $u_{|\partial\Omega} \in H^k(0,T)$.*

Proof. From Theorem 4.20.4, $h \in H^1(0,T)$ and u is the weak solution of the system

$$\begin{cases} \partial_t u(t,x) + A\partial_x u(t,x) + Ru(t,x) = 0, \quad t>0,\ 0<x<1, \\ B_0 u(t,0) = Q_0 h(t), \quad t>0, \\ B_1 u(t,1) = Q_1 h(t), \quad t>0, \\ u(0,x) = u_0(x), \quad 0<x<1. \end{cases} \tag{4.21.5}$$

From (4.21.3) it can be seen that the data $(u_0, 0, Q_0 h, Q_1 h)$ is compatible up to order 0 for the system (4.21.5). Thus Theorem 4.11.4 implies that $u \in CH^1(Q_T)$ and $u_{|\partial\Omega} \in H^1(0,T)$. On the other hand, h satisfies the ODE

$$\begin{cases} h'(t) = Hh(t) + G_0 u(t,0) + G_1 u(t,1), \quad t>0, \\ h(0) = h_0 \end{cases} \tag{4.21.6}$$

still from Theorem 4.20.4. Since $u_{|\partial\Omega} \in H^1(0,T)$, it follows from (4.21.6) that $h \in H^2(0,T)$. Consequently, Theorem 4.11.4 and (4.21.3) imply that $u \in CH^2(Q_T)$ and $u_{|\partial\Omega} \in H^2(0,T)$. Repeating this process, one eventually arrives at $u \in CH^k(Q_T)$, $u_{|\partial\Omega} \in H^k(0,T)$ and $h \in H^{k+1}(0,T)$. $\square$

Theorem 4.21.2. *Let $k \in \mathbb{N}_0$. If $(u_0, h_0) \in H^k(0,1) \times \mathbb{R}^m$ is compatible up to order $k-1$ when $k \geq 1$, then there exists a sequence $(u_0^\nu)_\nu \subset H^{k+1}(0,1)$ such that (u_0^ν, h_0) is compatible up to order k for each ν and $\|u_0^\nu - u_0\|_{H^k(0,1)} \to 0$.*

Proof. The proof follows the ideas presented in [**64**] for hyperbolic systems. Pick a sequence $(v_\nu)_\nu \subset H^{k+1}(0,1)$ satisfying $v_\nu \to u_0$ in $H^k(0,1)$. Define $u_0^\nu = v_\nu - w_\nu$ where $w_\nu \in H^{k+1}(0,1)$ satisfies $w_\nu \to 0$ in $H^k(0,1)$ and to be constructed below. The compatibility conditions for u_0^ν is given by

$$B_y w_{\nu,i}(y) = B_y v_{\nu,i}(y) - Q_y h_{\nu,i}, \qquad 0 \leq i \leq k,\ y = 0,1, \tag{4.21.7}$$

where

$$\begin{aligned} w_{\nu,0} &= w_\nu, \qquad v_{\nu,0} = v_\nu, \qquad h_{\nu,0} = h_0, \\ w_{\nu,i} &= -A\partial_x w_{\nu,i-1} - Rw_{\nu,i-1}, \quad 1 \leq i \leq k+1 \\ v_{\nu,i} &= -A\partial_x v_{\nu,i-1} - Rv_{\nu,i-1}, \quad 1 \leq i \leq k+1 \\ h_{\nu,i} &= Hh_{\nu,i-1} + G_0(v_{\nu,i-1}(0) - w_{\nu,i-1}(0)) + G_1(v_{\nu,i-1}(1) - w_{\nu,i-1}(1)),\ 1 \leq i \leq k \end{aligned}$$

The compatibility conditions (4.21.7) can be rewritten as

$$B_y w_\nu(y) = B_y v_\nu(y) - Q_y h_0 \tag{4.21.8}$$

$$\begin{aligned} B_y A^i \partial_x^i w_\nu(y) = B_y A^i \partial_x^i v_\nu(y) + \ell_{y,i}(&h_0, v_\nu - w_\nu, \ldots, \partial_x^{i-1} v_\nu - \partial_x^{i-1} w_\nu, \\ &v_\nu(0) - w_\nu(0), v_\nu(1) - w_\nu(1), \ldots, \partial_x^{i-1} v_\nu(0) - \partial_x^{i-1} w_\nu(0), \\ &\partial_x^{i-1} v_\nu(1) - \partial_x^{i-1} w_\nu(1)) \end{aligned} \tag{4.21.9}$$

for $y = 0,1$ and $i = 1, \ldots, k$, where $\ell_{y,i}$ is linear in all its arguments.

Recall from Lemma 4.7.1 that exits a matrix Y_y such that $B_y Y_y = I$ where I is the identity matrix I_p if $y = 0$ and I_{n-p} if $y = 1$. Consider the following equations

$$w_\nu(y) = Y_y(B_y v_\nu(y) - Q_y h_0) \tag{4.21.10}$$

$$\begin{aligned} \partial_x^i w_\nu(y) = A^{-i} Y_y(B_y A^i \partial_x^i v_\nu(y) + \ell_{y,i}(&h_0, v_\nu - w_\nu, \ldots, \partial_x^{i-1} v_\nu - \partial_x^{i-1} w_\nu, \\ &v_\nu(0) - w_\nu(0), v_\nu(1) - w_\nu(1), \ldots, \partial_x^{i-1} v_\nu(0) - \partial_x^{i-1} w_\nu(0), \\ &\partial_x^{i-1} v_\nu(1) - \partial_x^{i-1} w_\nu(1))) \end{aligned} \tag{4.21.11}$$

for $y = 0,1$ and $i = 1, \ldots, k$. By multiplying $B_y A$ and $B_y A^i$ to both sides of (4.21.10) and (4.21.11), respectively, we obtain (4.21.8) and (4.21.9), respectively. For this reason we construct w_ν that satisfies (4.21.10) and (4.21.11) in addition to the property $w_\nu \to 0$ in $H^k(0,1)$.

For $i = 0, \ldots k$ and $\nu \in \mathbb{N}$, let $\sigma_{\nu,i}(y)$ denote the right hand side of (4.21.10) and (4.21.11). Since $v_\nu \to u_0$ and $w_\nu \to 0$ both in $H^k(0,1)$, we have $\partial_x^i v_\nu(y) \to \partial_x^i u_0(y)$ and $\partial_x^i w_\nu(y) \to 0$ for all $0 \leq i \leq k-1$ by the Sobolev embedding. Thus, by the compatibility conditions for (u_0, h) we have $\sigma_{\nu,i}(y) \to 0$ for $0 \leq i \leq k-1$ and $y = 0,1$. Now given $(\sigma_{\nu,0}(0), \sigma_{\nu,0}(1), \ldots, \sigma_{\nu,k-1}(0), \sigma_{\nu,k-1}(1), 0, 0) \in \mathbb{R}^{2n \times (k+1)}$ there exists $\tilde{v}_\nu \in H^{k+1}(0,1)$ such that $\partial_x^i \tilde{v}_\nu(y) = \sigma_{\nu,i}(y)$ for $0 \leq i \leq k-1$, $\partial_x^k \tilde{v}_\nu(y) = 0$ and

$$\|\tilde{v}_\nu\|_{H^{k+1}(0,1)} \leq C \sum_{i=0}^{k-1} (|\sigma_{\nu,i}(0)| + |\sigma_{\nu,i}(1)|) \to 0 \tag{4.21.12}$$

for some $C > 0$ independent of ν. Define $w_\nu = \tilde{v}_\nu + \tilde{w}_\nu$ where $\tilde{w}_\nu \in H^{k+1}(0,1)$ satisfies $\partial_x^i \tilde{w}_\nu(y) = 0$ for $0 \leq i \leq k-1$, $\partial_x^k \tilde{w}_\nu(y) = \sigma_{\nu,k}(y)$, and $\|\tilde{w}_\nu\|_{H^k(0,1)} \to 0$.

Then w_ν satisfies the desired properties $w_\nu \to 0$ in $H^k(0,1)$ and $\partial_x^i w_\nu(y) = \sigma_{\nu,i}(y)$ for $0 \le i \le k$ and $y = 0, 1$.

Thus the last step is to construct the function $\tilde{w}_\nu$. Set $c_\nu = \sigma_{\nu,k}(0)$. Because it is enough to consider each component of c_ν separately, we may assume without loss of generality that c_ν is scalar. Let us consider the two cases $|c_\nu| \le 1$ and $|c_\nu| > 1$ separately. Suppose that $|c_\nu| \le 1$. Let $\phi \in \mathscr{D}(\mathbb{R})$ be such that $\phi(x) = 1$ for $|x| \le \epsilon$ for some $\epsilon > 0$ small enough and supp $\phi \subset [-1, 1]$. Define

$$\psi_\nu(x) = \frac{x^k}{k!}\phi(\nu x)c_\nu.$$

Then by Leibniz' formula we have for $1 \le j \le k$

$$\partial_x^j \psi_\nu(x) = \sum_{i=0}^{j} \binom{j}{i} \frac{x^{k-i}}{(k-i)!} \nu^{j-i} \partial_x^{j-i} \phi(\nu x) c_\nu. \tag{4.21.13}$$

It can be seen from (4.21.13) that $\partial_x^j \psi_\nu(0) = 0$ for $1 \le j \le k-1$ and $\partial_x^k \psi_\nu(0) = c_\nu$. Moreover, using the change of variable $y = \nu x$ we obtain

$$\begin{aligned}
\|\partial_x^j \psi_\nu\|_{L^2(\mathbb{R})}^2 &\le C(k) \sum_{i=0}^{j} \int_{\mathbb{R}} |x|^{2(k-i)} \nu^{2(j-i)} |\partial_x^{j-i} \phi(\nu x)|^2 |c_\nu|^2 \, \mathrm{d}x \\
&= C(k) \sum_{i=0}^{j} \int_{\mathbb{R}} |y|^{2(k-i)} \nu^{2(j-k)} |\partial_x^{j-i} \phi(y)|^2 \, \frac{\mathrm{d}y}{\nu} \\
&\le \frac{C(k)}{\nu} \sum_{i=0}^{j} \int_{\mathbb{R}} |y|^{2(k-i)} |\partial_x^{j-i} \phi(y)|^2 \, \mathrm{d}y \;\le\; \frac{C(k,\phi)}{\nu}
\end{aligned}$$

for $0 \le j \le k$.

If $|c_\nu| > 1$ then we take

$$\psi_\nu(x) = \frac{x^k}{k!}\phi(|c_\nu|^2 \nu x)c_\nu.$$

For $1 \le j \le k$, applying Leibniz' rule yields

$$\partial_x^j \psi_\nu(x) = \sum_{i=0}^{j} \binom{j}{i} \frac{x^{k-i}}{(k-i)!} (|c_\nu|^2\nu)^{j-i} \partial_x^{j-i} \phi(|c_\nu|^2 \nu x) c_\nu. \tag{4.21.14}$$

From (4.21.14) we obtain $\partial_x^j \psi_\nu(0) = 0$ for $1 \le j \le k-1$, $\partial_x^k \psi^\nu(0) = c_\nu$ and

$$\begin{aligned}
\|\partial_x^j \psi_\nu\|_{L^2(\mathbb{R})}^2 &\le C(k) \sum_{i=0}^{j} \int_{\mathbb{R}} |x|^{2(k-i)} (|c_\nu|^2\nu)^{2(j-i)} |\partial_x^{j-i} \phi(|c_\nu|^2 \nu x)|^2 |c_\nu|^2 \, \mathrm{d}x \\
&= C(k) \sum_{i=0}^{j} \int_{\mathbb{R}} |y|^{2(k-i)} (|c_\nu|^2\nu)^{2(j-k)} |\partial_x^{j-i} \phi(y)|^2 \, \frac{\mathrm{d}y}{\nu} \\
&\le \frac{C(k)}{\nu} \sum_{i=0}^{j} \int_{\mathbb{R}} |y|^{2(k-i)} |\partial_x^{j-i} \phi(y)|^2 \, \mathrm{d}y \;\le\; \frac{C(k,\phi)}{\nu}
\end{aligned}$$

since $j - k \le 0$ and $|c_\nu|^2 \nu > 1$. Therefore in any case we have $\|\psi_\nu\|_{H^k(\mathbb{R})} \le C(k,\phi)\nu^{-1/2}$.

For $\sigma_{\nu,k}(1)$ we can also do the same construction by replacing ϕ by a smooth function that is equal to 1 in an ϵ-neigborhood of $x = 1$. By taking the sum of the functions ψ_ν constructed for $x = 0$ and $x = 1$ and choosing ϵ small enough so that their supports do not intersect we obtain an appropriate $\tilde{w}_\nu$ satisfying all the required properties. $\square$

Using a diagonalization argument, the following result can be shown.

Corollary 4.21.3. *For every $(u_0, h_0) \in L^2(0,1) \times \mathbb{R}^m$ and $k \in \mathbb{N}$, there exists a sequence $(u_0^\nu)_\nu \subset H^k(0,1)$ such that (u_0^ν, h_0) is compatible up to order $k-1$ and $\|u_0^\nu - u_0\|_{L^2(0,1)} \to 0$.*

For each $t \geq 0$, define the operator $\mathcal{T}(t) : L^2(0,1) \times \mathbb{R}^m \to L^2(0,1) \times \mathbb{R}^m$ by

$$\mathcal{T}(t)(u_0, h_0) = (u(t,\cdot), h(t)), \qquad t \geq 0,\ (u_0, h_0) \in L^2(0,1) \times \mathbb{R}^m,$$

where (u, h) is the unique weak solution of the system (4.21.1). The linearity of $\mathcal{T}(t)$ follows from the linearity of the system (4.21.1) and the uniqueness of weak solutions. The boundedness follows from the energy estimate in Theorem 4.20.4. Also, $\mathcal{T}(0) = I$ and $(\mathcal{T}(t))_{t\geq 0}$ is strongly continuous since $(u, h) \in C([0,T]; L^2(0,1) \times \mathbb{R}^m)$ for any $T > 0$. Finally, since the system (4.21.1) is autonomous, $(\mathcal{T}(t))_{t\geq 0}$ satisfies the semigroup property.

The goal is to determine the generator of the $\mathcal{C}_0$-semigroup $(\mathcal{T}(t))_{t\geq 0}$, which we denote by $\mathcal{A}$. A candidate generator is given by the linear operator $\tilde{\mathcal{A}} : D(\tilde{\mathcal{A}}) \to L^2(0,1) \times \mathbb{R}^m$ defined by

$$\tilde{\mathcal{A}}\begin{pmatrix} u \\ h \end{pmatrix} = \begin{pmatrix} -Au_x - Ru \\ Hh + G_0 u(0) + G_1 u(1) \end{pmatrix} \tag{4.21.15}$$

where

$$D(\tilde{\mathcal{A}}) = \{(u,h) \in H^1(0,1) \times \mathbb{R}^m : B_0 u(0) = Q_0 h, B_1 u(1) = Q_1 h.\}$$

To prove that $\mathcal{A} = \tilde{\mathcal{A}}$ we proceed using the method in [**19**] applied to delay equations. This requires the following three steps: (1) characterize the resolvent $R(\lambda, \mathcal{A})$, (2) show that $\lambda I - \tilde{\mathcal{A}}$ is injective and (3) the resolvent of $\mathcal{A}$ and $\tilde{\mathcal{A}}$ at λ coincide. It is sufficient to prove these three steps for large enough λ.

Step 1. Suppose that $(u_0, h_0) \in H^1(0,1) \times \mathbb{R}^m$ satisfies the compatibility condition up to order 0, in other words, $(u_0, h_0) \in \mathcal{D}(\tilde{A})$. Then $u \in CH^1(Q_T)$ and $h \in H^2(0,T)$ from Theorem 4.21.1. For $\lambda > \omega_0$, where ω_0 is the growth bound of $\mathcal{T}(t)$, the resolvent of $\mathcal{A}$ at λ is given by the Laplace transform of the semigroup $\mathcal{T}(t)$, i.e.,

$$R(\lambda, \mathcal{A})(u_0, h_0) = \int_0^\infty e^{-\lambda t} \mathcal{T}(t)(u_0, h_0)\,\mathrm{d}t = \int_0^\infty e^{-\lambda t}(u(t,\cdot), h(t))\,\mathrm{d}t.$$

Define $w : (0,1) \to \mathbb{R}^n$ and $g \in \mathbb{R}^m$ by

$$\begin{aligned} w(x) &= \int_0^\infty e^{-\lambda t} u(t,x)\,\mathrm{d}t \\ g &= \int_0^\infty e^{-\lambda t} h(t)\,\mathrm{d}t \end{aligned}$$

so that $R(\lambda, \mathcal{A})(u_0, h_0) = (w, g)$.

Because $\partial_x : H^1(0,1) \to L^2(0,1)$ is a closed operator, $u \in C([0,T]; H^1(0,1))$ and $t \mapsto e^{-\lambda t} u_x(t,\cdot)$ is integrable for $\lambda > \gamma_1$ according to (4.11.5), (4.2.16) and (4.2.17), we can interchange differentiation and integration to obtain

$$w'(x) = \int_0^\infty e^{-\lambda t} u_x(t,x)\,\mathrm{d}t,$$

see [**34, Theorem 3.7.12**] and [**23, Chap. II, Theorem 6**]. Thus we take $\lambda > \max(\omega_0, \gamma_0, \gamma_1)$. Integrating by parts

$$\begin{aligned}\lambda w(x) &= -\, e^{-\lambda t} u(t,x)\Big|_{t=0}^{t=\infty} + \int_0^\infty e^{-\lambda t} u_t(t,x)\,\mathrm{d}t \\ &= u_0(x) - \int_0^\infty e^{-\lambda t}(A u_x(t,x) + R u(t,x))\,\mathrm{d}t \\ &= u_0(x) - A w'(x) - R w(x). \end{aligned} \tag{4.21.16}$$

Because we already know that $w \in L^2(0,1)$, (4.21.16) implies that $w \in H^1(0,1)$. Furthermore, for $y = 0, 1$ we have

$$B_y w(y) = \int_0^\infty e^{-\lambda t} B_y u(t,y)\,\mathrm{d}t = \int_0^\infty e^{-\lambda t} Q_y h(t)\,\mathrm{d}t = Q_y g.$$

Similarly,

$$\lambda g = Hg + h_0 + G_0 w(0) + G_1 w(1).$$

Therefore the resolvent of $\mathcal{A}$ at $\lambda > \max(\omega_0, \gamma_0, \gamma_1)$ is given by $R(\lambda)(u_0, h_0) = (w, g)$, for $(u_0, h_0) \in \mathcal{D}(\tilde{A})$, where w and g satisfy the system

$$\begin{cases} A w'(x) + (\lambda I_n + R) w(x) = u_0(x) \\ B_0 w(0) = Q_0 g \\ B_1 w(1) = Q_1 g \\ (\lambda I_m - H) g = h_0 + G_0 w(0) + G_1 w(1) \end{cases} \tag{4.21.17}$$

and in particular $(w, g) \in D(\tilde{\mathcal{A}})$.

Step 2. In this step we wish to show that $\lambda I - \tilde{A}$ is injective for sufficiently large λ. However, we only consider the case where $R = 0$ and $H = 0$ in this step. Let us denote the operator $\tilde{\mathcal{A}}$ by $\mathcal{A}_0$ when $R = 0$ and $H = 0$. We even prove the stronger property that $\lambda I - \mathcal{A}_0$ is bijective for λ large enough. Given $(u_0, h_0) \in L^2(0,1) \times \mathbb{R}^m$ we show that there exists a unique $(w, g) \in D(\mathcal{A}_0)$ such that $(\lambda I - \mathcal{A}_0)(w, g) = (u_0, h_0)$. This is equivalent to the system

$$\begin{cases} A w'(x) + \lambda w(x) = u_0(x) \\ B_0 w(0) = Q_0 g \\ B_1 w(1) = Q_1 g \\ \lambda g = h_0 + G_0 w(0) + G_1 w(1). \end{cases} \tag{4.21.18}$$

Thus w satisfies the two-point boundary value problem

$$\begin{cases} A w'(x) + w(x) = u_0(x) \\ \lambda B_0 w(0) = Q_0(h_0 + G_0 w(0) + G_1 w(1)) \\ \lambda B_1 w(1) = Q_1(h_0 + G_0 w(0) + G_1 w(1)). \end{cases} \tag{4.21.19}$$

Therefore to show that there exists a unique (w, g) satisfying (4.21.18) it is enough to prove that the two-point boundary value problem (4.21.19) has a unique solution.

Due to the assumption on the matrix A, there exists an invertible matrix T such that $T^{-1}AT = \Lambda$ where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. By rearranging the columns of T we can assume without loss of generality that $\lambda_1 \leq \cdots \leq \lambda_{n-p} < 0 < \lambda_{n-p+1} \leq \cdots \lambda_n$. Let $v = T^{-1}w$, $v_0 = T^{-1}u_0$ and $\tilde{B}_y = B_y T$ for $y = 0, 1$. Then (4.21.19) is equivalent to

$$\begin{cases} \lambda v + \Lambda v_x = v_0 \\ \lambda \tilde{B}_0 v(0) = Q_0 h_0 + Q_0 G_0 T v(0) + Q_0 G_1 T v(1) \\ \lambda \tilde{B}_1 v(1) = Q_1 h_0 + Q_0 G_0 T v(0) + Q_1 G_1 T v(1) \end{cases} \tag{4.21.20}$$

Note that $(\Lambda, \tilde{B})$ still satisfies the uniform Lopatinskii condition. Thus $\tilde{B}_0$ is injective on the unstable subspace of Λ which is $\{0\}^{n-p} \oplus \mathbb{R}^p$, while $\tilde{B}_1$ is injective on the stable subspace of Λ which is $\mathbb{R}^{n-p} \oplus \{0\}^p$. We will decompose a vector v in $\mathbb{R}^n$ by $v = \binom{v^-}{v^+}$ where $v^- \in \mathbb{R}^{n-p}$ and $v^+ \in \mathbb{R}^p$. Partitioning $\tilde{B}_0 = (\tilde{B}_0^- \ \tilde{B}_0^+)$ we have

$$\tilde{B}_0 v(0) = \tilde{B}_0^- v^-(0) + \tilde{B}_0^+ v^+(0). \tag{4.21.21}$$

where $\tilde{B}_0^+ \in \mathbb{R}^{p \times p}$ and $\tilde{B}_0^- \in \mathbb{R}^{p \times (n-p)}$. The matrix $\tilde{B}_0^+$ is invertible and so from (4.21.21) the boundary condition at $x = 0$ in (4.21.20) can be written as

$$(\lambda I_p + R_1) v^+(0) = (\lambda R_2 + R_3) v^-(0) + R_4 v^-(1) + R_5 v^+(1) + R_6 h_0 \tag{4.21.22}$$

for some matrices R_i. Similarly, the boundary condition at $x = 1$ is equivalent to

$$(\lambda I_{n-p} + S_1) v^-(1) = (\lambda S_2 + S_3) v^+(1) + S_4 v^-(0) + S_5 v^+(0) + S_6 h_0 \tag{4.21.23}$$

for some matrices S_i.

By the variation of parameters formula, the function v in (4.21.20) is given by

$$v(x) = e^{-x\lambda\Lambda^{-1}} \begin{pmatrix} c^- \\ c^+ \end{pmatrix} + \int_0^x e^{-(x-y)\lambda\Lambda^{-1}} \Lambda^{-1} v_0(y) \, \mathrm{d}y \tag{4.21.24}$$

and from (4.21.22) and (4.21.23) the vectors c^- and c^+ satisfy the equations

$$\begin{cases} (\lambda I_p + R_1) c^+ = (\lambda R_2 + R_3) c^- + R_4 (e^{-\lambda(\Lambda^-)^{-1}} c^- + d^-) \\ \qquad\qquad + R_5 (e^{-\lambda(\Lambda^+)^{-1}} c^+ + d^+) + R_6 h_0 \\ (\lambda I_{n-p} + S_1)(e^{-\lambda(\Lambda^-)^{-1}} c^- + d^-) = (\lambda S_2 + S_3)(e^{-\lambda(\Lambda^+)^{-1}} c^+ + d^+) \\ \qquad\qquad + S_4 c^- + S_5 c^+ + S_6 h_0 \end{cases} \tag{4.21.25}$$

where $\Lambda^- = \mathrm{diag}(\lambda_1, \ldots, \lambda_{n-p})$, $\Lambda^+ = \mathrm{diag}(\lambda_{n-p+1}, \ldots, \lambda_n)$ and

$$d = \int_0^1 e^{-(1-y)\lambda\Lambda^{-1}} \Lambda^{-1} v_0(y) \, \mathrm{d}y. \tag{4.21.26}$$

The system (4.21.25) can be written in matrix form as

$$\begin{pmatrix} R_5 e^{-\lambda(\Lambda^+)^{-1}} - R_1 - \lambda I_p & \lambda R_2 + R_3 + R_4 e^{-\lambda(\Lambda^-)^{-1}} \\ (\lambda S_2 + S_3) e^{-\lambda(\Lambda^+)^{-1}} + S_5 & S_4 - (\lambda I_{n-p} + S_1) e^{-\lambda(\Lambda^-)^{-1}} \end{pmatrix} \begin{pmatrix} c^+ \\ c^- \end{pmatrix}$$
$$= \begin{pmatrix} -R_6 h_0 + R_7 d \\ -S_6 h_0 + S_7(\lambda) d \end{pmatrix}. \tag{4.21.27}$$

Therefore to show that (4.21.20) has a unique solution, we must prove that the 2×2 matrix on the left hand side of (4.21.27) is invertible. To prove this, we need the following result in linear algebra.

Lemma 4.21.4. *Let A, B, C and D be $m \times m$, $m \times (n-m)$, $(n-m) \times m$ and $(n-m) \times (n-m)$ matrices, respectively. If A and $D - CA^{-1}B$ are invertible then the block matrix*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{4.21.28}$$

is invertible.

Proof. Since A is invertible, we can express the block matrix as a product of a lower triangular matrix and an upper triangular matrix as follows

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I_m & O_{m\times(n-m)} \\ CA^{-1} & I_{n-m} \end{pmatrix} \begin{pmatrix} A & B \\ O_{(n-m)\times m} & D - CA^{-1}B \end{pmatrix}. \tag{4.21.29}$$

The lower triangular matrix in (4.21.29) is clearly invertible. The upper triangular matrix in (4.21.29) is invertible as well because A and $D - CA^{-1}B$ are invertible. Therefore the block matrix (4.21.28) is invertible. □

For sufficiently large $\lambda > 0$, the matrix

$$\Xi_\lambda = \lambda^{-1}(R_5 e^{-\lambda(\Lambda^+)^{-1}} - R_1) - I_p.$$

is invertible and so $\lambda\Xi_\lambda$ is invertible. Consider the matrix

$$\begin{aligned} \Sigma_\lambda &= S_4 - (\lambda I_{n-p} + S_1)e^{-\lambda(\Lambda^-)^{-1}} \\ &- [(\lambda S_2 + S_3)e^{-\lambda(\Lambda^+)^{-1}} + S_5]\lambda^{-1}\Xi_\lambda^{-1}[\lambda R_2 + R_3 + R_4 e^{-\lambda(\Lambda^-)^{-1}}]. \end{aligned}$$

It can be seen that the matrix

$$\begin{aligned} &\lambda^{-1}\Sigma_\lambda e^{\lambda(\Lambda^-)^{-1}} = \lambda^{-1}(S_4 e^{\lambda(\Lambda^-)^{-1}} - S_1) - I_{n-p} \\ &- [(S_2 + \lambda^{-1}S_3)e^{-\lambda(\Lambda^+)^{-1}} + \lambda^{-1}S_5]\Xi_\lambda^{-1}[R_2 e^{\lambda(\Lambda^-)^{-1}} + \lambda^{-1}R_3 e^{\lambda(\Lambda^-)^{-1}} + \lambda^{-1}R_4] \end{aligned}$$

is invertible for large $\lambda > 0$. Consequently Σ_λ is invertible for sufficiently large $\lambda > 0$. Therefore from Lemma 4.21.4, the system (4.21.27) has a unique solution $(c^+ \; c^-)$ and so (4.21.20) has a unique solution v. As a result, (4.21.17) has a unique solution (w, g).

From (4.21.24), (4.21.26) and (4.21.27) there exists a constant $C_\lambda > 0$ such that

$$\|w\|_{L^2(0,1)} = \|Tv\|_{L^2(0,1)} \le C_\lambda(\|u_0\|_{L^2(0,1)} + |h_0|).$$

The last equation in (4.21.18) together with (4.21.24), (4.21.26) and (4.21.27) imply that

$$|g| \le C_\lambda(\|u_0\|_{L^2(0,1)} + |h_0|)$$

for some $C_\lambda > 0$. Therefore $R(\lambda, \mathcal{A}_0) \in \mathcal{L}(L^2(0,1) \times \mathbb{R}^m)$ so that $\mathcal{A}_0$ has a nonempty resolvent. Hence $\mathcal{A}_0$ is closed.

Step 3. In this step we show that the resolvents of $\mathcal{A}$ and $\mathcal{A}_0$ at λ are the same for sufficiently large λ. Let $(u_0, h_0) \in D(\mathcal{A}_0)$. From (4.21.17) and (4.21.18) we have

$$(\lambda I - \mathcal{A}_0)R(\lambda, \mathcal{A})(u_0, h_0) = (\lambda I - \mathcal{A}_0)(w, g) = (u_0, h_0).$$

Thus $(\lambda I - \mathcal{A}_0)R(\lambda, \mathcal{A}) = I$ in $D(\mathcal{A}_0)$. Since $R(\lambda, \mathcal{A}) \in \mathcal{L}(L^2(0,1) \times \mathbb{R}^m)$, $\mathcal{A}_0$ is closed and $D(\mathcal{A}_0)$ is dense in $L^2(0,1) \times \mathbb{R}^m$ according to Corollary 4.21.3, we have $(\lambda I - \mathcal{A}_0)R(\lambda, \mathcal{A}) = I$ in $L^2(0,1) \times \mathbb{R}^m$.

Let $z \in D(\mathcal{A}_0)$ and $y = R(\lambda, \mathcal{A})(\lambda I - \mathcal{A}_0)z$. Then $(\lambda I - \mathcal{A}_0)y = (\lambda I - \mathcal{A}_0)z$. Since $\lambda I - \mathcal{A}_0$ is injective for sufficiently large $\lambda > 0$ it follows that $y = z$ and hence $R(\lambda, \mathcal{A})(\lambda I - \mathcal{A}_0)z = z$ for all $z \in D(\mathcal{A}_0)$. Therefore $R(\lambda, \mathcal{A}_0) = R(\lambda, \mathcal{A})$ and also the domain of $\mathcal{A}$ is $D(\mathcal{A}_0)$. Since

$$\begin{aligned} \lambda I - \mathcal{A} &= (\lambda I - \mathcal{A}_0)R(\lambda, \mathcal{A}_0)(\lambda I - \mathcal{A}) \\ &= (\lambda I - \mathcal{A}_0)R(\lambda, \mathcal{A})(\lambda I - \mathcal{A}) \ = \ \lambda I - \mathcal{A}_0. \end{aligned}$$

we conclude that $\mathcal{A} = \mathcal{A}_0$.

Now let us turn to the general case where R and H are not necessarily zero. We can write the operator $\tilde{\mathcal{A}}$ defined by (4.21.15) as $\tilde{\mathcal{A}} = \mathcal{A}_0 + \mathcal{B}$ where $\mathcal{A}_0 : D(\tilde{\mathcal{A}}) \to L^2(0,1) \times \mathbb{R}^m$ and $\mathcal{B} : L^2(0,1) \times \mathbb{R}^m \to L^2(0,1) \times \mathbb{R}^m$ are given by

$$\begin{aligned} \mathcal{A}_0\begin{pmatrix} u \\ h \end{pmatrix} &= \begin{pmatrix} -Au_x \\ G_0u(0) + G_1u(1) \end{pmatrix} \\ \mathcal{B}\begin{pmatrix} u \\ h \end{pmatrix} &= \begin{pmatrix} -Ru \\ Hh \end{pmatrix}. \end{aligned}$$

Since $\mathcal{A}_0$ is closed and $\mathcal{B}$ is bounded, $\tilde{\mathcal{A}}$ is closed. We know from above that $\mathcal{A}_0$ generates a $\mathcal{C}_0$-semigroup on $L^2(0,1) \times \mathbb{R}^m$. It follows from the bounded perturbation theorem of semigroups that $\tilde{\mathcal{A}}$ generates a $\mathcal{C}_0$-semigroup on $L^2(0,1) \times \mathbb{R}^m$. Therefore $\lambda I - \tilde{\mathcal{A}}$ is invertible for sufficiently large $\lambda > 0$. Similar arguments as in Step 3 show that $\mathcal{A} = \tilde{\mathcal{A}}$.

Therefore, the solution of the system (4.21.1) given by semigroup theory coincides with the weak solution given in Definition 4.20.1. An alternative way of proving that the weak and semigroup solutions are the same is to prove that the operator $\tilde{\mathcal{A}}$ generates a $\mathcal{C}_0$-semigroup. For initial data in $\mathcal{D}(\tilde{A}^2)$ we have a classical solution and so we can multiply the system with the appropriate test functions and use integration by parts to show that the semigroup solution is the weak solution. By the density of $\mathcal{D}(\tilde{A}^2)$ in $L^2(0,1) \times \mathbb{R}^m$, this also true for every initial data in $L^2(0,1) \times \mathbb{R}^m$, see Section 3.3. However, proving that $\tilde{\mathcal{A}}$ is a generator is a difficult task, specifically it is hard to show that $\tilde{\mathcal{A}} - \lambda I$ is dissipative for some $\lambda \in \mathbb{R}$.

If (u, h) is the weak solution of (4.21.1) then $u_{|\partial\Omega} \in L^2(0,T)$ and $h \in H^1(0,T)$ for every $T > 0$ according to Theorem 4.20.4. These properties are called *hidden regularity*. Note that these cannot be obtained directly from standard semigroup methods because in general the solution given by semigroup theory only satisfies $(u, h) \in C([0,\infty); L^2(0,1) \times \mathbb{R}^m)$. In the literature, hidden regularity property for weak solutions of partial differential equations were established using Fourier analysis and multiplier methods.

If we define the operator $\mathcal{C} : D(\mathcal{A}) \to \mathbb{R}^s$ by

$$\mathcal{C}(u_0, h_0) = J\begin{pmatrix} u_0(0) \\ u_0(1) \end{pmatrix}$$

where $D(\mathcal{A})$ is the domain of the generator $\mathcal{A}$ of the semigroup $(\mathcal{T}(t))_{t\geq 0}$ defined above and $J \in \mathbb{R}^{s\times 2n}$, then $\mathcal{C}$ is an admissible observation operator for $(\mathcal{T}(t))_{t\geq 0}$. Indeed, the direct inequality (B.3.5) follows immediately from the energy estimate in Theorem 4.20.4.

4.22 EXAMPLES

Example 4.22.1 (Linearized Flow in an Elastic Tube)**.** The two tank model in Chapter 3 can be put in the form (4.21.1). It can be easily checked that all the properties that are required in Theorem 4.20.4 are satisfied. The hidden regularity on the velocity component was shown using methods in control theory. This was established by proving the direct inequality using the Fourier representation of the semigroup, cf. Remark 3.5.8. In this section we have shown this with a different method and in addition we also established that the cross-section admits traces in L^2 and the level heights are in H^1.

Example 4.22.2 (Wave Equations with Oscillator Boundary Conditions)**.** Consider the one-dimensional undamped wave equation with oscillator boundary conditions, [**6, 39**]

$$\begin{cases} \partial_{tt}\psi(t,x) - \partial_{xx}\psi(t,x) = 0, \quad t > 0,\ 0 < x < \ell, \\ \psi_x(t,0) = -\delta_0'(t), \quad t > 0, \\ \psi_x(t,\ell) = \delta_\ell'(t), \quad t > 0, \\ m_0\delta_0''(t) + d_0\delta_0'(t) + k_0\delta_0(t) = -\rho\partial_t\psi(t,0), \quad t > 0, \\ m_\ell\delta_\ell''(t) + d_\ell\delta_\ell'(t) + k_\ell\delta_\ell(t) = -\rho\partial_t\psi(t,\ell), \quad t > 0, \\ \psi(0,x) = \psi_0(x), \quad 0 < x < \ell, \\ \partial_t\psi(0,x) = \psi_1(x), \qquad 0 < x < \ell, \\ \delta_i(0) = \delta_i^0, \quad i = 0,\ell, \\ \delta_i'(0) = v_i^0, \quad i = 0,\ell. \end{cases} \tag{4.22.1}$$

The system (4.22.1) models the velocity potential ψ of the acoustics in a homogeneous fluid with nominal density ρ contained in a wave guide of length ℓ and terminated by oscillators. In this model it is assumed that the fluid does not penetrate the surface of the oscillators.

As in Ito and Propst [**39**], we introduce the variables $\phi^- = \frac{1}{2}(\partial_t\psi + \partial_x\psi)$, $\phi^+ = \frac{1}{2}(\partial_t\psi - \partial_x\psi)$, $v_0 = \delta_0'$ and $v_\ell = \delta_\ell'$. The system (4.22.1) can be put in the form (4.21.1) as

$$\begin{cases} \partial_t\phi^-(t,x) - \partial_x\phi^-(t,x) = 0, \quad t > 0,\ 0 < x < \ell, \\ \partial_t\phi^+(t,x) + \partial_x\phi^+(t,x) = 0, \quad t > 0,\ 0 < x < \ell, \\ \phi^-(t,0) - \phi^+(t,0) = -v_0(t), \quad t > 0, \\ \phi^-(t,\ell) - \phi^+(t,\ell) = v_\ell(t), \quad t > 0, \\ \delta_0'(t) = v_0(t), \quad t > 0, \\ \delta_\ell'(t) = v_\ell(t), \quad t > 0, \\ v_0'(t) = -\frac{d_0}{m_0}v_0(t) - \frac{k_0}{m_0}\delta_0(t) - \frac{\rho}{m_0}(\phi^-(t,0) + \phi^+(t,0)), \quad t > 0, \\ v_\ell'(t) = -\frac{d_\ell}{m_\ell}v_\ell(t) - \frac{k_\ell}{m_\ell}\delta_\ell(t) - \frac{\rho}{m_\ell}(\phi^-(t,\ell) + \phi^+(t,\ell)), \quad t > 0, \\ \phi^-(0,x) = \phi_0^-(x), \quad 0 < x < \ell, \\ \phi^+(0,x) = \phi_0^+(x), \quad 0 < x < \ell, \\ \delta_i(0) = \delta_i^0, \quad i = 0,\ell, \\ v_i(0) = v_i^0, \quad i = 0,\ell, \end{cases} \tag{4.22.2}$$

where $\phi_0^- = \frac{1}{2}(\psi_1 + \psi_0')$ and $\phi_0^+ = \frac{1}{2}(\psi_1 - \psi_0')$. It can be checked that all the requirements in Theorem 4.20.4 are satisfied by the system (4.22.2). Therefore for every $(\phi_0^-, \phi_0^+, \delta_0, \delta_\ell, v_0, v_\ell) \in L^2(0,\ell)^2 \times \mathbb{R}^4$ the system (4.22.2) has a unique weak solution $(\phi^-, \phi^+, \delta_0, \delta_\ell, v_0, v_\ell) \in C([0,\infty); L^2(0,\ell)^2 \times \mathbb{R}^4)$ and it satisfies $\phi^\pm(\cdot,0), \phi^\pm(\cdot,\ell) \in L^2(0,T)$ and $\delta_0, \delta_\ell, v_0, v_\ell \in H^1(0,T)$ for every $T > 0$. The well-posedness of (4.22.2) was established in [**39**] using semigroup methods. Here, we improved this result by showing that the solutions admit traces in L^2 and that the oscillator components are more regular.

Example 4.22.3 (Wave Equations with Exponential Memory Kernel)**.** The next example is the normalized damped wave equation with memory boundary conditions [**62**]

$$\begin{cases} \partial_{tt}\phi(t,x) - \partial_{xx}\phi(t,x) + \partial_t\phi(t,x) = 0, \quad t>0,\ 0<x<1, \\ (a_0 \star \phi_t(\cdot,0))(t) - \phi_x(t,0) = 0, \quad t>0, \\ (a_1 \star \phi_t(\cdot,1))(t) + \phi_x(t,1) = 0, \quad t>0, \\ \phi(0,x) = \phi_0(x), \quad 0<x<1, \\ \phi_t(0,x) = \phi_1(x), \quad 0<x<1. \end{cases} \tag{4.22.3}$$

where $a \star u$ is the convolution

$$(a \star u)(t) = \int_0^t a(t-s)u(s)\,\mathrm{d}s.$$

Suppose that the kernels a_0 and a_1 take the form $a_0(t) = \kappa_0 e^{\alpha_0 t}$ and $a_1(t) = \kappa_1 e^{\alpha_1 t}$ for some nonzero real numbers $\kappa_0, \kappa_1, \alpha_0, \alpha_1$. Introducing the state vector

$$(u,v,h,g)(t) = \left(\phi_t(t,\cdot), \phi_x(t,\cdot), \int_0^t e^{\alpha_0(t-s)}\phi_t(s,0)\,\mathrm{d}s, \int_0^t e^{\alpha_1(t-s)}\phi_t(s,1)\,\mathrm{d}s\right)$$

at time t, the system (4.22.3) can be written in the form of (4.21.1) as

$$\begin{cases} \partial_t u(t,x) - \partial_x v(t,x) + u(t,x) = 0, \quad t>0,\ 0<x<1, \\ \partial_t v(t,x) - \partial_x u(t,x) = 0, \quad t>0,\ 0<x<1, \\ v(t,0) = \kappa_0 h(t), \quad t>0, \\ v(t,1) = -\kappa_1 g(t), \quad t>0, \\ h'(t) = \alpha_0 h(t) + u(t,0), \quad t>0, \\ g'(t) = \alpha_1 g(t) + u(t,1), \quad t>0, \\ u(0,x) = u_0(x), \quad 0<x<1, \\ v(0,x) = v_0(x), \quad 0<x<1, \\ h(0) = h_0, \\ g(0) = g_0. \end{cases} \tag{4.22.4}$$

where $u_0 = \phi_1$, $v_0 = \phi_0'$ and $h_0 = g_0 = 0$. The conditions for Theorem 4.20.4 are satisfied by the system (4.22.4). Thus, for each initial data $(u_0, v_0, h_0, g_0) \in L^2(0,1)^2 \times \mathbb{R}^2$ the system (4.22.4) admits a unique weak solution $(u,v,h,g) \in C([0,\infty); L^2(0,1)^2 \times \mathbb{R}^2)$, and moreover, $u(\cdot,0), v(\cdot,0), u(\cdot,1), v(\cdot,1) \in L^2(0,T)$ and $h, g \in H^1(0,T)$ for every $T > 0$.

Part II

NONLINEAR SYSTEMS

5

LOCAL EXISTENCE AND BLOW-UP CRITERION FOR NONLINEAR PDE-ODE SYSTEMS

The aim of this chapter is to obtain a well-posedness result for a hyperbolic system of first order quasilinear partial differential equations in the bounded interval $\Omega = (0,1)$ with dynamic boundary conditions

$$\begin{cases} u_t(t,x) + A(u(t,x))u_x(t,x) = f(u(t,x)), & 0 < t < T,\ 0 < x < 1, \\ B_0 u(t,0) = b_0(p_0(t), h(t)), & 0 < t < T, \\ B_1 u(t,1) = b_1(p_1(t), h(t)), & 0 < t < T, \\ h'(t) = H(h(t), q(t), u(t,0), u(t,1)), & 0 < t < T, \\ u(0,x) = u_0(x), & 0 < x < 1, \\ h(0) = h_0. \end{cases} \tag{5.0.1}$$

The unknown state variables are $u : [0,T] \times [0,1] \to \mathbb{R}^n$ and $h : [0,T] \to \mathbb{R}^d$ taking values in the open and convex sets $\mathcal{U}$ and $\mathcal{H}$, respectively. We assume for simplicity that $0 \in \mathcal{U}$ and $0 \in \mathcal{H}$. This is not restrictive since one can shift a general problem to this case. The coefficients appearing in (5.0.1) are assumed to have the following properties. The flux matrix $A : \mathcal{U} \to \mathbb{R}^{n\times n}$ and the source term $f : \mathcal{U} \to \mathbb{R}^n$ are both infinitely differentiable. The boundary matrices $B_0 \in \mathbb{R}^{p\times n}$ and $B_1 \in \mathbb{R}^{(p-n)\times n}$ are of full rank, where p is the number of incoming characteristics from the left boundary, or equivalently, the number of positive eigenvalues of the flux matrix. According to the diagonalizability assumption (D) in Chapter 4, $n-p$ is the number of incoming characteristics from the right boundary. This assumption further implies that we are in the non-characteristic case. It should be noted that unlike in multidimensions, cf. [**9, Chapter 11**], for which the boundary matrix should be of constant maximal rank along the boundary, in the case of one space dimension the boundary matrices can have different ranks. However, the sum of their ranks should be the same as the number of components of u.

The boundary data p_0, p_1, and q are given by $p_0 : [0,T] \to \mathbb{R}^{n_0}$, $p_1 : [0,T] \to \mathbb{R}^{n_1}$, $q : [0,T] \to \mathbb{R}^{n_2}$, while $b_0 : \mathbb{R}^{n_0} \times \mathcal{H} \to \mathbb{R}^p$, $b_1 : \mathbb{R}^{n_1} \times \mathcal{H} \to \mathbb{R}^{n-p}$ and $H : \mathcal{H} \times \mathbb{R}^{n_2+2n} \to \mathbb{R}^d$. Again for simplicity we assume that b_0, b_1 and H are all infinitely differentiable. If b_0 and b_1 are independent of h then (5.0.1) includes systems of balance laws that are decoupled from the h-dynamics. If H is independent of h then (5.0.1) includes balance laws with nonlocal boundary conditions of the form

$$B_y u(t,y) = b_y\left(p_y(t), \int_0^t H(q(s), u(s,0), u(s,1))\,\mathrm{d}s\right), \qquad 0 < t < T,\ y = 0, 1.$$

We assume that $f(0) = 0$, $H(0) = 0$, and $b(0) = 0$. Again these are not restrictions since one may consider affine shifts of the state spaces. Other assumptions, for example on the initial and boundary data, will be stated later. According to our hypotheses, we include the case of non-symmetric fluxes with symmetrizers. The diagonalizability assumption though would give us a new diagonal system through a change of variables, and thus the flux matrix will be trivially symmetric. However, the cost of this diagonalization would be that the boundary matrices will be time-dependent. For this reason, we do not diagonalize the system.

One possible generalization of (5.0.1) is to consider nonlinear boundary conditions, e.g. $B(u, h) = 0$ where B satisfies the condition $B(0) = 0$. To deal with the nonlinearity, one first studies the linearized problem. The linearized boundary condition takes the form $\tilde{B}(v, g)u = \tilde{g}$ for which the boundary matrix $\tilde{B}$ depends on t through the frozen coefficients v and g. We shall not pursue this generalization and consider the simpler case where the boundary matrices are constant. Regarding time-dependent boundary matrices we refer to [**9, Chapter 9**]. We believe that the method used in this thesis will work on these types of problems.

Systems of the form (5.0.1) occur in multiscale blood flow models [**14, 27, 65, 66, 67, 68**] and in valveless pumping [**13, 60, 63**]. Our well-posedness results are based on Sobolev spaces. The motivation for studying the well-posedness in Sobolev spaces, rather than spaces of continuous functions [**27, 47, 48**], lies in the later study of global-in-time existence of smooth solutions for which energy estimates formulated in Sobolev norms are used. The presence of a damping term, the bounded space domain and the ODE boundary conditions will not cause much technical difficulty, we will address methods on how to treat them. Broadly speaking, we will follow the frameworks in Benzoni-Gavage and Serre [**9**] and Métivier [**55**] to prove our result.

However, there will be differences specifically when it comes to the full nonlinear PDE-ODE system where an appropriate linearization and a modified a priori estimate will be used. Recent results regarding the mixing of conservation laws and balance laws with ODEs on the boundary, but with another notion of solutions and on a semi-infite interval, are given in [**11**] and [**12**], respectively.

As in the linear case, to prove the existence of solutions in Sobolev spaces, the initial and boundary data should be compatible. These compatibility conditions are given in Section 5.1. Using the well-posedness theory in Section 4.19 and a Picard iteration scheme, the local-in-time well posedness of (5.0.1) is discussed in Section 5.2. In the event that the local solution cannot be continued for all times, a blow-up criterion will be proved in Section 5.3. To close this chapter, some examples that have the form (5.0.1) will be given in Section 5.4.

5.1 COMPATIBILITY CONDITIONS

The existence of smooth solutions requires, and also implies, *compatibility conditions* between the initial data and the boundary data. These are additional constraints for the initial and boundary data. The compatibility conditions are obtained by (a) formally differentiating the PDE with respect to time, (b) evaluate the time derivatives at $t = 0$ and use the initial data to compute the spatial derivatives and (c) differentiate the boundary conditions, use the information in (b) and evaluate them along the boundary. The result in (c) will be the compatibility conditions.

Suppose that u and h are C^p-functions satisfying $\partial_t u + A(u)\partial_x u = f(u)$ in $(t,x) \in (0,T)\times\Omega$ and $\dot{h} = H(h, q, u_{|\partial\Omega})$ in $t \in (0,T)$, respectively. Then by Leibniz's rule

$$\partial_t^i u = -\sum_{l=0}^{i-1} \binom{i-1}{l} \partial_t^l (A(u))\partial_x \partial_t^{i-1-l} u + \partial_t^{i-1} f(u), \qquad i = 1,\dots,p.$$

The terms $\partial_t^l(A(u))$ and $\partial_t^{i-1} f(u)$ can be expanded with the aid of Faá di Bruno's formula. If u is continuous up to the boundary then

$$B_y u(0,y) = b_y(p_y(0), h(0)), \qquad y = 0,1.$$

In general, if u is C^i up to the boundary then we must have

$$B_y \partial_t^i u(0,y) = D_t^i b_y(p_y(t), h(t))_{|t=0}, \qquad y = 0,1.$$

We can use Faá di Bruno's formula to expand the right hand term and then use the ODE satisfied by h. Thus, we are led to the following definitions. Given a sufficiently smooth function $u_0 : \Omega \to \mathbb{R}^n$ with values in $\mathcal{U}$, recursively define $u_i : \Omega \to \mathbb{R}^n$ as

$$\begin{aligned}
u_1 &= -A(u_0)\partial_x u_0 + f(u_0) \\
u_i &= -\sum_{l=0}^{i-1}\sum_{k=1}^{l}\sum_{l_1+\dots+l_k=l}\binom{i-1}{l} c_{l_1,\dots,l_k}(\mathrm{d}^k A)(u_0)[u_{l_1},\dots,u_{l_k}]\partial_x u_{i-1-l} \\
&\quad -A(u_0)\partial_x u_{i-1} + \sum_{k=1}^{i-1}\sum_{l_1+\dots+l_k=i-1} c_{l_1,\dots,l_k}(\mathrm{d}^k f)(u_0)[u_{l_1},\dots,u_{l_k}], \\
&\quad \text{for } i = 2,\dots,p
\end{aligned} \tag{5.1.1}$$

where $\mathrm{d}^k F$ denotes the kth order differential of a smooth function F viewed as multilinear form. Here, $c_{l_1,\dots,l_k}$ are nonnegative coefficients which depend only on i. Given $h_0 \in \mathcal{H}$ define $\eta = (h_0, q(0), u_0(0), u_0(1))$,

$$\begin{aligned}
h_1 &= H(\eta) \\
h_i &= \sum_{k=1}^{i-1}\sum_{l_1+\dots+l_k=i-1} c_{l_1,\dots,l_k}(\mathrm{d}^k H)(\eta)[z_{l_1},\dots,z_{l_k}], \qquad \text{for } i = 2,\dots,p-1.
\end{aligned} \tag{5.1.2}$$

where $z_j = (h_j, q^{(j)}(0), u_j(0), u_j(1))^\top$ and the u_j are defined according to (5.1.1). For $y = 0,1$, define

$$\begin{aligned}
C_{y,0} &= b_y(p_y(0), h_0) \\
C_{y,i} &= \sum_{k=1}^{i}\sum_{l_1+\dots+l_k=i} c_{l_1,\dots,l_k}(\mathrm{d}^m b_y)(p_y(0), h_0)[w_{l_1,y},\dots,w_{l_k,y}]
\end{aligned}$$

where $w_{k,y} = (p_y^{(k)}(0), h_k)^\top$. With these notations we are now in position to state the necessary compatibility conditions.

(CC_m) Let $m \geq 1$ be an integer and $T > 0$. The data

$$(u_0, h_0, p, q) \in H^m(0,1)\times\mathcal{H}\times H^m(0,T)\times H^m(0,T)$$

are said to be compatible up to order $m-1$ if $B_y u_i(y) = C_{y,i}$ for all $i = 0,\dots,m-1$ and $y = 0,1$.

We are going to state the regularity properties of the functions u_i, $i = 1, \ldots, m$, defined in (5.1.1) for a given $u_0 \in H^m(\Omega)$.

Lemma 5.1.1. *Let $s \geq 1$ be an integer. Let $u_0 \in H^s(\Omega)$ such that the range of u_0 lies in a compact subset $\mathcal{K}$ of $\mathcal{U}$ and $u_1, \ldots, u_s$ be defined as in (5.1.1). Then $u_i \in H^{s-i}(\Omega)$ for all $1 \leq i \leq s$. Moreover, there exist continuous functions $C_i : [0, \infty) \to [0, \infty)$ such that*

$$\|u_i\|_{H^{s-i}(\Omega)} \leq C_i(\|u_0\|_{H^s(\Omega)}), \qquad 1 \leq i \leq s. \tag{5.1.3}$$

Proof. We follow the proof in [**9, pp. 322–323**] and proceed by strong induction on i. In this proof, all Sobolev spaces are defined in $\Omega = (0, 1)$. By redefining A and f in (5.0.1) outside a neighborhood of $\mathcal{K}$ one can assume without loss of generality that A and f are $\mathscr{C}^\infty$ on $\mathbb{R}^n$. From the assumption that $f(0) = 0$ we have $f(u_0) \in H^s$ by Proposition 4.14.3. We rewrite

$$A(u_0)\partial_x u_0 = (A(u_0) - A(0))\partial_x u_0 + A(0)\partial_x u_0$$

Propostion 4.14.3 can now be applied so that $A(u_0) - A(0) \in H^s$, since Ω is bounded. Thus $(A(u_0) - A(0))\partial_x u_0 \in H^{s-1}$ by Proposition 4.14.1. Moreover we have

$$\begin{aligned} \|A(u_0)\partial_x u_0\|_{H^{s-1}} &\leq C\|A(u_0) - A(0)\|_{H^s}\|\partial_x u_0\|_{H^{s-1}} + |A(0)|\|\partial_x u_0\|_{H^{s-1}} \\ &\leq C(\|u_0\|_{L^\infty})\|u_0\|_{H^s}\|\partial_x u_0\|_{H^{s-1}} + |A(0)|\|\partial_x u_0\|_{H^{s-1}} \\ &\leq C(\|u_0\|_{H^s}) \end{aligned}$$

by the Sobolev embedding $H^s \hookrightarrow L^\infty$. The H^{s-1}-norm of $f(u_0)$ can be estimated similarly. Thus $u_1 \in H^{s-1}$ and (4.14.1) holds for $i = 1$.

Now suppose that for $1 \leq i \leq s$ we have $u_k \in H^{s-k}$ and $\|u_k\|_{H^{s-k}} \leq C_k(\|u_0\|_{H^s})$ holds for $k = 0, 1, \ldots, i-1$. We show that $u_i \in H^{s-i}$ and (5.1.3) holds. A similar argument as above yields $A(u_0)\partial_x u_{i-1} \in H^{s-i}$. The triple sum in u_i contains terms of the form

$$\varrho(u_0)u_{l_1,j_1} \cdots u_{l_k,j_k}\partial_x u_{i-1-l,\sigma} \tag{5.1.4}$$

where $l_1 + \cdots + l_k = l$ for $k = 1, \ldots, l$, with $l = 1, \ldots, i-1$ and for some $\varrho \in \mathscr{C}^\infty$. Here u_{l_1,j_1} denotes the j_1th component of the vector u_{l_1}. By the induction hypothesis $u_{l_1,j_1} \in H^{s-l_1}, \ldots, u_{l_k,j_k} \in H^{s-l_k}$, $\partial_x u_{i-1-l,\sigma} \in H^{s-(i-1-l)} \subset H^{s-i+l}$ and $\varrho(u_0) \in H^s$. Since

$$\min(s, s-l_1, \ldots, s-l_k, s-i+l) \geq \min(s-l, s-i+1) = s-i+1.$$

and since $ks \geq s > 1/2$

$$s + (s-l_1) + \cdots + (s-l_k) + (s-i+l) = (k+2)s - i > s - i + 1/2$$

it follows from the remark succeeding Proposition 4.14.1 that (5.1.4) lies in H^{s-i}. Similarly, the double sum in u_i contains terms of the from

$$\vartheta(u_0)u_{l_1,j_1} \cdots u_{l_k,j_k} \tag{5.1.5}$$

where $l_1 + \cdots + l_k = i-1$ for some $\vartheta \in \mathscr{C}^\infty$. Because

$$\min(s, s-l_1, \ldots, s-l_k) \geq s-(i-1) = s-i+1$$

and

$$s + (s-l_1) + \cdots + (s-l_k) = (k+1)s - (i-1) > s-i+1/2$$

the terms of the form (5.1.5) belong to H^{s-i}. Collecting all our observations, we obtain that $u_i \in H^{s-i}$. The estimate $\|u_i\|_{H^{s-i}(\Omega)} \leq C_i(\|u_0\|_{H^s(\Omega)})$ can be shown from the definition of u_i, the induction hypothesis, and (4.14.1). $\square$

5.2 LOCAL-IN-TIME EXISTENCE

Now we are ready to state and prove one of the main results of this chapter.

Theorem 5.2.1 (Local Existence). *Let $m \geq 3$, $T_0 > 0$ and $(u_0, h_0, p, q) \in H^m(\Omega) \times \mathcal{H} \times H^m(0, T_0) \times H^m(0, T_0)$. Assume that the range of u_0 lies in a compact and convex set $\mathcal{K}_0 \subset \mathcal{K}_1 \subset \mathcal{U}$, $h_0 \in \mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{H}$ where $\mathcal{K}_1$ and $\mathcal{G}_1$ are also compact and convex sets containing neighborhoods of $\mathcal{K}_0$ and $\mathcal{G}_0$, respectively, and moreover $\|u_0\|_{H^m(\Omega)} \leq M$. Suppose that (FS), (D), (UKL) and (CC_m) hold. Then there exists $T \in (0, T_0)$ depending only on $(\mathcal{K}_1, \mathcal{G}_1, M)$ such that the nonlinear system* (5.0.1) *has a unique solution $(u, h) \in CH^m([0,T] \times \Omega) \times H^m(0,T)$. Furthermore, $u_{|\partial\Omega} \in H^m(0,T)$ and consequently $h \in H^{m+1}(0,T)$.*

Proof. The proof is a Picard iteration scheme using the linear well-posedness theory of Chapter 4.

Step 1. Existence of initial functions for the iteration scheme. In this step we find $v \in CH^m([0,T_0] \times \Omega)$ such that $\partial_t^j v(0) = u_j$ for all $0 \leq j \leq m-1$. The following construction is inspired by [**20, 71**]. Let $g \in H^m(0,T_0)$ be such that $\partial_t^j g(0) = h_j$ for all $0 \leq j \leq m-1$ where h_j are the constants defined from (5.1.2) and $\|g\|_{H^m(0,T_0)} \leq C \sum_{j=0}^{m-1} |h_j|$. This is possible by the trace theorem. Consider the initial-value boundary value problem

$$v_t + A(u_0)v_x = f(u_0) + G, \qquad Bv_{|\partial\Omega} = b(p,g), \qquad v(0) = u_0 \tag{5.2.1}$$

for some $G \in H^m((0,T_0) \times \Omega)$ to be specified below. The local existence result Theorem 4.19.5 for linear systems shows that the system (5.2.1) has a unique solution $v \in CH^m([0,T_0] \times \Omega)$ with $v_{|\partial\Omega} \in H^m(0,T_0)$ *provided* that the data $(u_0, f(u_0) + G, b(p,g))$ is compatible up to order $m-1$ for the linear system (5.2.1). To ensure this, let v_j for $0 \leq j \leq m-1$ be $\partial_t^j v_{|t=0}$ that is obtained from (5.2.1) by formal differentiation. Similarly, let $\tilde{v}_j$ be $\partial_t^j \tilde{v}_{|t=0}$ that is obtained from

$$\tilde{v}_t + A(u_0)\tilde{v}_x = f(u_0), \qquad \tilde{v}(0) = u_0 \tag{5.2.2}$$

by differentiating formally. The equation $v_j = u_j$ holds if

$$\partial_t^j G(0) = u_j - \tilde{v}_j \in H^{m-j}(\Omega) \subset H^{m-1-j+1/2}(\Omega), \qquad 0 \leq j \leq m-1. \tag{5.2.3}$$

By the trace theorem there exists $G \in H^m((0,T_0) \times \Omega)$ such that (5.2.3) hold and

$$\|G\|_{H^m((0,T_0)\times\Omega)} \leq C(\|u_0\|_{H^m(\Omega)}) \tag{5.2.4}$$

for some continuous function $C : [0,\infty) \to [0,\infty)$. This estimate follows from the trace theorem and a result similar to Lemma 5.1.1 applied to the PDEs (5.2.1) and (5.2.2). Since $B_y v_j(0,y) = B_y u_j(0,y) = C_{y,j}$ for $y = 0,1$ and $0 \leq j \leq m-1$, due to the compatibility condition for the nonlinear system, it follows that the data $(u_0, f(u_0) + G, b(p,g))$ is compatible up to order $m-1$ for the linear system (5.2.1).

Step 2. An invariant set. Let $R, K, T > 0$. Define $V^m_{T,K,R}$ to be a subset of $CH^m([0,T] \times \Omega) \times H^m(0,T)$ such that $(v,g) \in V^m_{T,K,R}$ if and only if

(V1) Compatibility: $\partial_t^j v_{|t=0} = u_j$ for all $0 \leq j \leq m-1$ and $\partial_t^j g(0) = h_j$ for all $0 \leq j \leq m-1$ where u_j and h_j are defined by (5.1.1) and (5.1.2)

(V2) Range condition: $\operatorname{ran}(v,g) \subset \mathcal{K}_1 \times \mathcal{G}_1$

(V3) $W^{1,\infty}$-bound: $\|v\|_{W^{1,\infty}((0,T)\times\Omega)} + \|g\|_{W^{1,\infty}(0,T)} \le K$

(V4) H^m-bound: $\|v\|_{H^m((0,T)\times\Omega)} + \|v_{|\partial\Omega}\|_{H^m(0,T)} + \|g\|_{H^m(0,T)} \le R$.

Consider the function $(v, g) \in CH^m([0,T_0]\times\Omega)\times H^m(0,T_0)$ constructed in the previous step. By construction of g we already know that $\|g\|_{H^m(0,T_0)} \le C(\mathcal{G}_1, M)$. According to Remark 4.19.7

$$\begin{aligned}&\|v\|_{CH^m([0,T_0]\times\Omega)} + \|v_{|\partial\Omega}\|_{H^m(0,T_0)}\\ &\le C\left(\|f(u_0)+G\|_{H^m((0,T_0)\times\Omega)} + \|b(p,g)\|_{H^m(0,T_0)} + \sum_{i=0}^{m}\|\partial_t^i v_{|t=0}\|_{H^{m-i}(\Omega)}\right)\end{aligned}$$

where C depends on the range of u_0, which lies in $\mathcal{K}_0$, and on $\|u_0\|_{H^m((0,T_0)\times\Omega)} \le C(T_0, M)$. From this, it can be seen that

$$\|v\|_{H^m((0,T_0)\times\Omega)} + \|v_{|\partial\Omega}\|_{H^m(0,T_0)} \le C(\mathcal{K}_1, \mathcal{G}_1, M) =: R_1$$

where we removed the explicit dependence of C on T_0 since it is fixed from the beginning. By Theorem 4.17.5 and the PDE (5.2.1)

$$\|v\|_{W^{1,\infty}((0,T_0)\times\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + \|f(u_0) + G(0,\cdot) - A(u_0)\partial_x u_0\|_{L^\infty(\Omega)} + \sqrt{T_0}R_1.$$

Applying the Sobolev embedding theorem and (5.2.4) we have $\|v\|_{W^{1,\infty}((0,T_0)\times\Omega)} \le C(R_1, M)$. One can do the same procedure for the $W^{1,\infty}$-norm of g. Hence

$$\|v\|_{W^{1,\infty}((0,T_0)\times\Omega)} + \|g\|_{W^{1,\infty}(0,T_0)} \le C(\mathcal{K}_1, \mathcal{G}_1, M) =: K_1.$$

Finally, for the range condition, Theorem 4.17.5 and $v_{|t=0} = u_0$ imply that $\|v - u_0\|_{L^\infty((0,T)\times\Omega)} \le TR_1$. Therefore there exists $T_1 = T_1(R_1) > 0$ such that the range of v lies in $\mathcal{K}_1$ for all $T \in (0, T_1]$. Using the same argument, it can be shown that the range of g also lies in $\mathcal{G}_1$ for all $T \in (0,T_1]$ by reducing T_1 if necessary. Therefore $V^m_{T,K,R}$ is nonempty for all $K \ge K_1$, $R \ge R_1$ and for $T \in (0, T_1]$ for some $T_1 = T_1(\mathcal{K}_1, \mathcal{G}_1, M) > 0$.

We will show that there exist $K > K_1$, $R = R(K) > R_1$ and $T = T(R) > 0$ such that given $(v, g) \in V^m_{T,K,R}$ the solution of the system

$$\begin{cases} u_t + A(v)u_x = f(v), & t>0,\ 0<x<1,\\ Bu_{|\partial\Omega} = b(p,h), & t>0,\\ h' = H(g, q, v_{|\partial\Omega}), & t>0,\\ u_{|t=0} = u_0, & 0<x<1,\\ h(0) = h_0 \end{cases} \tag{5.2.5}$$

satisfies $(u,h) \in V^m_{T,K,R}$. Let us verify the regularity of (u,h). Note that $\partial_t^j v \in CH^{m-j}([0,T]\times\Omega)$ it follows that $\partial_t^j v \in C^{m-j-1}([0,T]\times\Omega) \subset C([0,T]\times[0,1])$ for all $0 \le j \le m-1$. Therefore

$$\partial_t^j (v_{|\partial\Omega})_{|t=0} = (\partial_t^j v)_{|\{t=0\}\times\partial\Omega} = (\partial_t^j v_{|t=0})_{|\partial\Omega} = u_{j|\partial\Omega}, \qquad 0 \le j \le m-1.$$

Together with (V1) it can be shown that the compatibility conditions are satisfied by (u,h). Since

$$h(t) = h_0 + \int_0^t H(g(s), q(s), v_{|\partial\Omega}(s))\,\mathrm{d}s$$

we have $h \in H^{m+1}(0,T)$ and therefore $u \in CH^m([0,T]\times\Omega)$ with $u_{|\partial\Omega} \in H^m(0,T)$ according to Theorem 4.19.5 . Furthermore, u and h satisfies (V1) since v and g satisfy the same property. Thus by Theorem 4.17.5

$$\|u\|_{W^{1,\infty}([0,T]\times\Omega)} + \|h\|_{W^{1,\infty}(0,T)} \le C(\mathcal{K}_1, M) + R\sqrt{T}.$$

Take $K = 2\max(K_1, C(\mathcal{K}_1, M))$. Letting $T = T(R, \mathcal{K}_1, \mathcal{G}_1, M) > 0$ small enough, condition (V3) is satisfied by (u,h). A similar argument using the same Theorem 4.17.5 implies that (u,h) satisfies (V2) by reducing T if necessary. It remains to prove that (u,h) also satisfies (V4). Indeed, as in [**55**], one can prove the following additional a priori estimate

$$\|u\|_{H^m([0,T]\times\Omega)} + \|u_{|\partial\Omega}\|_{H^m(0,T)} + \|h\|_{H^m(0,T)} \le R \tag{5.2.6}$$

for some $R = R(K) > R_1$. The proof of this estimate is straightforward but lengthy. For this reason we postpone its proof. In summary, $V^m_{T,K,R}$ is invariant under the map $(v,g) \mapsto (u,h)$ where (u,h) solves (5.2.5) for some $T, K, R > 0$.

Step 3. Existence and Higher regularity. Let $V = V^m_{T,K,R}$ where the parameters T, K, and R are those given in the previous step. Let $(u^0, h^0) \in V$ be given and for each nonnegative integer k, define (u^{k+1}, h^{k+1}) recursively to be the solution of

$$\begin{cases} u_t^{k+1} + A(u^k)u_x^{k+1} = f(u^k), & t > 0,\ 0 < x < 1, \\ Bu^{k+1} = b(p, h^{k+1}), & t > 0, \\ (h^{k+1})' = H(h^k, q, u^k_{|\partial\Omega}), & t > 0, \\ u^{k+1}_{|t=0} = u_0, & 0 < x < 1, \\ h^{k+1}(0) = h_0 \end{cases} \tag{5.2.7}$$

Note that the boundary condition in (5.2.7) depends on h^{k+1} which is possible because h^{k+1} does not depend on u^{k+1} and at the same time couples the PDE to the ODE. Then according to Step 2, $(u^{k+1}, h^{k+1}) \in V$ for all $k = 1, 2, \ldots$ Thus $(u^k, (u^k)_{|\partial\Omega}, h^k)$ is bounded in $H^m((0,T)\times\Omega)\times H^m(0,T)\times H^m(0,T)$ and one can extract a weakly convergent subsequence. By compact embedding and by extracting an appropriate subsequence $(u^k, (u^k)_{|\partial\Omega}, h^k)$ converges in $L^2((0,T)\times\Omega)\times L^2(0,T)\times L^2(0,T)$ and let $(u, \tilde{u}, h)$ be the limit. The limit is necessarily in $H^m((0,T)\times\Omega)\times H^m(0,T)\times H^m(0,T)$. Since $(u^k, (u^k)_{|\partial\Omega}, h^k)$ is bounded in $H^m((0,T)\times\Omega)\times H^m(0,T)\times H^m(0,T)$, by interpolation theory for Sobolev spaces, $(u^k, (u^k)_{|\partial\Omega}, h^k) \to (u, \tilde{u}, h)$ in $H^s((0,T)\times\Omega)\times H^s(0,T)\times H^s(0,T)$ for all $s \in [0,m)$. The continuity of the trace operator implies that $(u^k)_{|\partial\Omega} \to u_{|\partial\Omega}$ in $L^2(0,T)$ and therefore $u_{|\partial\Omega} = \tilde{u}$ by uniqueness of limits in $L^2(0,T)$. By passing to the L^2-limit in the system satisfied by (u^k, h^k), we can see that the pair (u,h) satisfies the nonlinear system (5.0.1). Note that $\partial_t^j u_{|t=0} = u_j \in H^{m-j}(\Omega)$ for $0 \le j \le m-1$ from Lemma 5.1.1. Finally, Theorem 4.19.5 implies the additional regularity $u \in CH^m([0,T]\times\Omega)$.

Step 4. Uniqueness. Let (u_1, h_1) and (u_2, h_2) be two solutions of the system (5.0.1) on the time interval $[0,T]$. Introducing the variables $w = u_1 - u_2$ and $\eta = h_1 - h_2$ we have the system

$$\begin{cases} L_{u_1}w = f(u_1) - f(u_2) - (A(u_1) - A(u_2))\partial_x u_2, & 0 < t < T,\ 0 < x < 1 \\ Bw_{|\partial\Omega} = b(p, h_1) - b(p, h_2), & 0 < t < T, \\ \eta' = H(h_1, q, u_{1|\partial\Omega}) - H(h_2, q, u_{2|\partial\Omega}), & 0 < t < T, \\ w_{|t=0} = 0, & 0 < x < 1, \\ \eta_{|t=0} = 0. \end{cases}$$

Let $\mathcal{K} \times \mathcal{G} \subset \mathcal{U} \times \mathcal{H}$ be a compact set both containing the ranges of (u_1, h_1) and (u_2, h_2) and let $K > 0$ be such that the $W^{1,\infty}$-norms of (u_1, h_1) and (u_2, h_2) are bounded above by K. According to (4.19.11), there exists $C = C(\mathcal{K}, K) > 0$ such that for all $0 < \tau \leq T$

$$\begin{aligned} \|w\|^2_{CL^2([0,\tau]\times\Omega)} + \|w_{|\partial\Omega}\|^2_{L^2(0,\tau)} \ &\leq \ C\tau\|f(u_1) - f(u_2)\|^2_{L^2((0,\tau)\times\Omega)} \\ + \, C\tau\|(A(u_1) - A(u_2))\partial_x u_2\|^2_{L^2((0,\tau)\times\Omega)} &+ \ C\|b(p, h_2) - b(p, h_1)\|^2_{L^2(0,\tau)} \end{aligned} \tag{5.2.8}$$

By the mean value theorem

$$\|b(p, h_1) - b(p, h_2)\|^2_{L^2(0,\tau)} \leq C\|\eta\|^2_{L^2(0,\tau)}. \tag{5.2.9}$$

A similar argument proves that

$$\begin{aligned} &\|f(u_1) - f(u_2)\|^2_{L^2((0,\tau)\times\Omega)} + \|(A(u_1) - A(u_2))\partial_x u_2\|^2_{L^2((0,\tau)\times\Omega)} \\ &\leq C\|w\|^2_{L^2((0,\tau)\times\Omega)} \leq C\tau\|w\|^2_{CL^2([0,\tau]\times\Omega)}. \end{aligned} \tag{5.2.10}$$

The differential equation for η gives us the following pointwise estimate

$$|\eta(t)|^2 \leq C\tau(\|\eta\|^2_{L^2(0,\tau)} + \|w_{|\partial\Omega}\|^2_{L^2(0,\tau)}), \qquad t \in [0, \tau].$$

Integrating the last inequality and choosing $\tau = \tau(\mathcal{K}, K) > 0$ small enough

$$\|\eta\|^2_{L^2(0,\tau)} \leq \frac{C\tau^2}{1 - C\tau^2}\|w_{|\partial\Omega}\|^2_{L^2(0,\tau)}. \tag{5.2.11}$$

From (5.2.8)–(5.2.9) and reducing $\tau > 0$ if necessary it can be seen that $w = 0$ on $[0, \tau]$ and from (5.2.11) $\eta = 0$ as well on $[0, \tau]$. Repeating the process on intervals of the form $[k\tau, (k + 1)\tau]$ for positive integers k shows that $w = 0$ and $\eta = 0$ on $[0, T]$ and therefore the uniqueness of solutions. □

Now we prove the estimate (5.2.6) used in the third step of the proof of the previous theorem. The proof of this estimate is similar to the proof of Lemma 4.19.6, however, the difference is that the source terms appearing on the PDE and the boundary condition now depend on the frozen coefficients v and g. From the proof of Lemma 4.19.6 we already have the estimate

$$\begin{aligned} &\frac{1}{\sqrt{T}}\|u\|_{L^2(\Omega;H^m(0,T))} + \|u_{|\partial\Omega}\|_{H^m(0,T)} \\ &\leq C\left(\sum_{j=1}^m \|\partial_t^j u_{|t=0}\|^2_{L^2(\Omega)} + \sqrt{T}\sum_{j=1}^m \|f_j\|_{L^2((0,T)\times\Omega)} + \|b(p, h)\|_{H^m(0,T)}\right) \end{aligned} \tag{5.2.12}$$

for all $T \in (0, T_0]$, where $f_j = A(v)\partial_t^j(A(v)^{-1}f(v)) - A(v)[\partial_t^j, A(v)^{-1}L_v]u$. For the rest of the proof C will denote a positive constant depending only on $T_0, K, \mathcal{K}_1, \mathcal{G}_1, M$ $\|p\|_{H^m(0,T_0)}$, $\|q\|_{H^m(0,T_0)}$, and is independent on R and T. The commutator has been estimated uniformly in T in the proof of Lemma 4.19.6. Let us consider the first term of f_j. Note that it is a nonlinear function of order at most m and thus by Theorem 4.17.4 we have

$$\|A(v)\partial_t^j(A(v)^{-1}f(v))\|_{L^2((0,T)\times\Omega)} \leq C(\|v\|_{H^m((0,T)\times\Omega)} + 1)$$

Because $(u,h) \in V^m_{T,K,R}$ we have $\partial_t^j u_{|t=0} = u_j$ for all $0 \le j \le m-1$. Using this in (5.2.12) and recalling Lemma 5.1.1 we have

$$\begin{aligned}&\frac{1}{\sqrt{T}}\|u\|_{L^2(\Omega;H^m(0,T))} + \|u_{|\partial\Omega}\|_{H^m(0,T)} \qquad (5.2.13)\\ &\le C(1+\sqrt{T}\|v\|_{H^m((0,T)\times\Omega)} + \sqrt{T}(1+R)\|u\|_{H^m((0,T)\times\Omega)} + \|b(p,h)\|_{H^m(0,T)})\end{aligned}$$

where R is a positive constant to be chosen below. Next we will estimate the boundary terms on the right hand side of (5.2.12). By Theorem 4.17.4 once more

$$\|b(p,h)\|_{H^m(0,T)} \le C(K_{m,T}(p,h))(\|p\|_{H^m(0,T)} + \|h\|_{H^m(0,T)} + 1).$$

The fact that $(u,h) \in V$ implies that $h^{(j)}(0) = h_j$ for all $0 \le j \le m-1$. The differential equation $h' = H(q,g,v_{|\partial\Omega})$ for h gives us the estimate

$$\|h\|_{H^m(0,T)} \le C(K_{m-1,T}(q,g,v_{|\partial\Omega}))(\|q\|_{H^{m-1}(0,T)} + \|g\|_{H^{m-1}(0,T)} + 1)$$

With these, together with Theorem 4.17.6 we have

$$\|b(p,h)\|_{H^m(0,T)} \le C(T\|g\|_{H^m(0,T)} + 1). \qquad (5.2.14)$$

Using (5.2.14) in (5.2.13) we have

$$\begin{aligned}&\frac{1}{\sqrt{T}}\|u\|_{L^2(\Omega;H^m(0,T))} + \|u_{|\partial\Omega}\|_{H^m(0,T)} \qquad (5.2.15)\\ &\le C(1+\sqrt{T}\|v\|_{H^m((0,T)\times\Omega)} + \sqrt{T}(1+R)\|u\|_{H^m((0,T)\times\Omega)} + T\|g\|_{H^m(0,T)}).\end{aligned}$$

It remains to estimate the mixed derivatives. As usual we proceed by an induction argument. Suppose that $\|\partial_x^l\partial_t^j u\|_{L^2((0,T)\times\Omega)} \le N(u)$ for all $l = 0,1\ldots,k-1$ and j such that $l+j \le m$, where $N(u)$ is the right hand side of (5.2.15). Let k and j be integers such that $k+j \le m$. The PDE implies that

$$\partial_x^k\partial_t^j u = \partial_x^{k-1}\partial_t^j(A(v)^{-1}f(v)) - \partial_x^{k-1}\partial_t^j(A(v)^{-1}\partial_t u).$$

The first term on the right hand side is a nonlinear function of v of order at most $m-1$, and therefore using Theorem 4.17.4, Theorem 4.17.6 and (V1) we have

$$\|\partial_x^{k-1}\partial_t^j(A(v)^{-1}f(v))\|_{L^2((0,T)\times\Omega)} \le C(T\|v\|_{H^m((0,T)\times\Omega)} + 1).$$

We can expand the second term using Leibniz's rule and estimate each term in the sum. Let $0 \le l \le k-1$ and $0 \le j \le i$. If $l+i \le m-3$ then Theorem 4.17.4 implies

$$\begin{aligned}&\|\partial_x^{k-1-l}\partial_t^{j-i}(A(v)^{-1})\partial_x^l\partial_t^{i+1}u\|_{L^2((0,T)\times\Omega)}\\ &\le \|\partial_x^{k-1-l}\partial_t^{j-i}(A(v)^{-1})\|_{L^2((0,T)\times\Omega)}\|\partial_x^l\partial_t^{i+1}u\|_{L^\infty((0,T)\times\Omega)}\\ &\le C(1+\|v\|_{H^{m-1}((0,T)\times\Omega)})\|u\|_{W^{m-2}((0,T)\times\Omega)}.\end{aligned}$$

According to Theorem 4.17.6 we have

$$\begin{aligned}\|u\|_{W^{m-2}((0,T)\times\Omega)} &\le \sum_{k=0}^{m-2}\|\partial_t^k u_{|t=0}\|_{W^{m-2-k}(\Omega)} + C\sqrt{T}\|u\|_{H^m((0,T)\times\Omega)}\\ &\le C\sum_{k=0}^{m-2}\|\partial_t^k u_{|t=0}\|_{H^{m-k-1}(\Omega)} + C\sqrt{T}\|u\|_{H^m((0,T)\times\Omega)}\end{aligned}$$

Thus $\|\partial_x^{k-1-l}\partial_t^{j-i}(A(v)^{-1})\partial_x^l\partial_t^{i+1}u\|_{L^2((0,T)\times\Omega)} \leq N(u)$. Suppose that $l+i=m-2, m-1$ then $k-1-l+j-i=1,0$. Thus we can have a standard $L^\infty - L^2$ estimate to obtain

$$\|\partial_x^{k-1-l}\partial_t^{j-i}(A(v)^{-1})\partial_x^l\partial_t^{i+1}u\|_{L^2((0,T)\times\Omega)} \leq C\|\partial_x^l\partial_t^{i+1}u\|_{L^2((0,T)\times\Omega)} \leq N(u)$$

where the last inequality is due to the induction hypothesis. This completes the proof of the induction step. Therefore we have

$$\begin{aligned}&\left(\frac{1}{\sqrt{T}} - C\sqrt{T}(1+R)\right)\|u\|_{H^m((0,T)\times\Omega)} + \|u_{|\partial\Omega}\|_{H^m(0,T)}\\ &\leq C(1+\sqrt{T}\|v\|_{H^m((0,T)\times\Omega)} + \sqrt{T}\|g\|_{H^m(0,T)}) \leq C(1+\sqrt{T}R).\end{aligned}$$

Choosing $R = \max(5C, R_1)$ where C is the constant in the last inequality and choosing $T = T(R) > 0$ small enough so that $\frac{1}{\sqrt{T}} - C\sqrt{T}(1+R) > \frac{1}{2}$ and $\sqrt{T}R < 1$ finally proves (5.2.6).

5.3 BLOW-UP CRITERION

We prove the following standard blow-up criterion for first order quasilinear PDEs. The idea of the proof is the following. Boundedness in $W^{1,\infty}$ of the local solution implies boundedness in H^m, which can be further improved to show boundedness in CH^m. If this is known, then a standard argument shows that the solution can be extended.

Theorem 5.3.1 (Blow-up Criterion in Finite Time)**.** *Let $(u,h) \in CH^m([0,T]\times\Omega)\times H^m(0,T)$ be a solution of* (5.0.1) *having a trace $u_{|\partial\Omega} \in H^m(0,T)$, where $m \geq 3$ is an integer, and T^* be the maximal time of existence. If $T^* < \infty$ then the range of (u,h) on $[0,T]\times[0,1]$ leaves every compact subset of $\mathcal{U}\times\mathcal{H}$ as $T \to T^*$, i.e. for every compact set $\mathcal{K}\times\mathcal{G}$ in $\mathcal{U}\times\mathcal{H}$ there exists $\epsilon > 0$ and $(t,x) \in (0, T^*-\epsilon]\times[0,1]$ such that $(u(t,x), h(t)) \notin \mathcal{K}\times\mathcal{G}$, or*

$$\limsup_{t\uparrow T^*} \|\partial_x u(t)\|_{L^\infty[0,1]} = \infty.$$

Proof. Suppose that the range of (u,h) on $[0,T]\times\Omega$ lies in a compact subset $\mathcal{K}_0\times\mathcal{G}_0$ of $\mathcal{U}\times\mathcal{H}$ and $\|u\|_{W^{1,\infty}([0,T]\times[0,1])} \leq K_0$ for some constant $K_0 > 0$ and $(u,h) \in CH^m([0,T]\times\Omega)\times H^m(0,T)$ for all $T \in (0,T^*)$. We show that there exists a $\tau > 0$ such that the solution can be extended to a solution $(u,h) \in CH^m([0,T^*+\tau]\times\Omega)\times H^m(0,T^*+\tau)$ satisfying $u_{|\partial\Omega} \in H^m(0,T^*+\tau)$.

Step 1. Uniform boundedness in $CH^m\times H^m$. The following estimates are again in the same spirit as before, but now, the frozen coefficients are the solutions of the PDE. For completeness we include their proof. According to (4.13.17), we have for all $u \in H^1((0,T)\times\Omega)$ and for all $\gamma \geq \gamma_0$

$$\begin{aligned}&\sqrt{\gamma}\|u\|_{L^2((0,T)\times\Omega)} + \|u_{|\partial\Omega}\|_{L^2(0,T)}\\ &\leq C\left(\frac{1}{\sqrt{\gamma}}\|L_u u\|_{L^2((0,T)\times\Omega)} + \|Bu_{|\partial\Omega}\|_{L^2(0,T)} + \|u_{|t=0}\|_{L^2(\Omega)}\right).\end{aligned}$$

for some constants $C > 0$ and $\gamma_0 \geq 1$ depending only on $(\mathcal{K}_0, \mathcal{G}_0, K_0)$. Applying this estimate to $\partial_t^j u$, for $j = 0, 1, \ldots, j$ where $k = 0, 1, \ldots, m$ we have

$$\begin{aligned}
&\sqrt{\gamma}\|u\|_{L^2(\Omega;H^k(0,T))} + \|u_{|\partial\Omega}\|_{H^k(0,T)} \\
&\leq C\left(\frac{1}{\sqrt{\gamma}}\sum_{j=0}^{k}\|f_j\|_{L^2((0,T)\times\Omega)} + \|b(p,h)\|_{H^k(0,T)} + 1\right)
\end{aligned}$$

where $f_j = A(u)\partial_t^j(A(u)^{-1}f(u)) - A(u)[\partial_t^j, A(u)^{-1}L_u]u$. For $j \geq 1$, f_j is a nonlinear function of $\partial_t u$ of order at most $j-1$. Thus, using Theorem 4.17.4 we have

$$\|f_j\|_{L^2((0,T)\times\Omega)} \leq C(\|\partial_t u\|_{H^{j-1}((0,T)\times\Omega)} + 1) \leq C(\|u\|_{H^j((0,T)\times\Omega)} + 1).$$

The case of $f_0 = f(u)$ can be done merely by the mean-value theorem. On the other hand, by a similar argument we also have $\|b(p,h)\|_{H^k(0,T)} \leq C(\|h\|_{H^k(0,T)} + 1)$. The differential equation for h gives us $\|h\|_{L^2(0,T)} \leq C$ and $\|h\|_{H^k(0,T)} \leq C(\|h\|_{H^{k-1}(0,T)} + \|u_{|\partial\Omega}\|_{H^{k-1}(0,T)} + 1)$ for $1 \leq k \leq m$. Combining all of these in a recursive manner, we obtain

$$\sqrt{\gamma}\|u\|_{L^2(\Omega;H^m(0,T))} + \|u_{|\partial\Omega}\|_{H^m(0,T)} + \|h\|_{H^m(0,T)} \leq C\left(\frac{1}{\sqrt{\gamma}}\|u\|_{H^m((0,T)\times\Omega)} + 1\right).$$

From the PDE we note that $\partial_x u = A(u)^{-1}f(u) - A(u)^{-1}\partial_t u$. Therefore $\partial_x^j\partial_t^k u$ can be written in terms of derivatives of u with respect to t only, and is a nonlinear function of u of order at most $k+j$. Fixing $x \in \Omega$, we apply Theorem 4.17.4 to the function $u(\cdot, x) \in H^m(0,T)$ to obtain

$$\|\partial_x^j\partial_t^k u(\cdot,x)\|_{L^2(0,T)} \leq C(\|u(\cdot,x)\|_{H^m(0,T)} + 1).$$

Integrating over the bounded domain Ω yields

$$\|\partial_x^j\partial_t^k u\|_{L^2((0,T)\times\Omega)} \leq C(\|u\|_{L^2(\Omega;H^m(0,T))} + 1).$$

Combining this with our estimates above and choosing γ large enough we have

$$\|u\|_{H^m((0,T)\times\Omega)} + \|h\|_{H^m(0,T)} \leq C, \qquad \text{for all } 0 < T < T^*. \tag{5.3.1}$$

for some constant $C > 0$ independent of $T \in (0, T^*)$.

Let $\varphi \in \mathscr{D}(\mathbb{R})$ be a cut-off function such that $\varphi(t) = 0$ if $t \leq T^*/4$ and $\varphi(t) = 1$ if $t \geq T^*/2$. Multiplying the system (5.0.1) by this cut-off function we have the new homogeneous system for $w = \varphi u$ and $g = \varphi h$

$$\begin{cases}
w_t + A(u)w_x = \varphi f(u) + \dot{\varphi}u, & 0 < t < T,\ 0 < x < 1, \\
Bw_{|\partial\Omega} = \varphi b(p,h), & 0 < t < T, \\
g' = \varphi H(h, q, u_{|\partial\Omega}) + \dot{\varphi}h, & 0 < t < T, \\
w_{|t=0} = 0, & 0 < x < 1, \\
g_{|t=0} = 0.
\end{cases} \tag{5.3.2}$$

Applying the energy estimates for the initial boundary value problem with homogeneous data (4.19.2) together with the previous result (5.3.1) shows that there exists an $M > 0$ such that

$$\|u\|_{CH^m([0,T]\times\Omega)} + \|h\|_{H^m(0,T)} \leq M, \qquad \text{for all } 0 < T < T^*.$$

Step 2. Extension. According to the previous step there exist an $M > 0$ and a sequence $(t_n)_n \subset (0,T)$ such that $t_n \to T^*$ and $\|u(t_n)\|_{H^m} + |h(t_n)| \le M$ for all n. Consider the initial boundary value problem

$$\begin{cases} v_t + A(v)v_x = f(v), & t>0,\ 0<x<1, \\ Bv_{|\partial\Omega} = b(p,g), & t>0, \\ g' = H(g,q,v_{|\partial\Omega}), & t>0, \\ v_{|t=0} = u(t_n), & 0<x<1, \\ g_{|t=0} = h(t_n). \end{cases} \tag{5.3.3}$$

The local existence result Theorem 5.2.1 implies that the exists $\tau > 0$, depending only on M and in some neighborhoods of $\mathcal{K}_0$ and $\mathcal{G}_0$ but independent of n, such that (5.3.3) has a unique solution on $[0,\tau]$. Choose n large enough so that $t_n + \tau > T^*$. Then the functions w and η defined by

$$(w,\eta)(t) = \begin{cases} (u,h)(t), & 0 \le t \le t_n \\ (v,g)(t-t_n), & t_n \le t \le t_n+\tau, \end{cases}$$

lies in $CH^m([0,t_n+\tau]\times\Omega)\times H^m(0,t_n+\tau)$ since (u,h) and (v,g) must coincide in $[t_n,(t_n+T^*)/2]$ by uniqueness. Thus (w,η) satisfies (5.0.1). Therefore the solution can be extended up to the time $t_n + \tau > T^*$. This completes the proof of the theorem. □

5.4 EXAMPLES

In this section we cite some examples that fit in the general system (5.0.1).

5.4.1 Flow in an elastic tube revisited

Consider the following system modelling the velocity v of an incompressible fluid contained in an elastic tube of length ℓ, cross-section a that is connected to a tank at each end having cross-section a_T and level height h_0, h_ℓ, respectively,

$$\begin{cases} a_t(t,x) + v(t,x)a_x(t,x) + a(t,x)v_x(t,x) = 0, \quad 0<t<T,\ 0<x<\ell, \\ v_t(t,x) + \dfrac{\kappa^2 a_x(t,x)}{\sqrt{a(t,x)}} + v(t,x)v_x(t,x) = -\beta v(t,x), \quad 0<t<T,\ 0<x<\ell, \\ a_T h_0'(t) = -a(t,0)v(t,0), \quad 0<t<T, \\ a_T h_\ell(t) = a(t,\ell)v(t,\ell), \quad 0<t<T, \\ a(t,0) = a_0(1+p_0(t)+bh_0(t))^2, \quad 0<t<T, \\ a(t,\ell) = a_0(1+p_\ell(t)+bh_\ell(t))^2, \quad 0<t<T, \end{cases} \tag{5.4.1}$$

see (2.6.5). Here a_0 is the rest cross-sectional area of the tube, $b,\kappa > 0$ are parameters incorporating the material properties of the tube and $\beta \ge 0$ is a parameter modeling linear tube friction. The tanks are subjected from above to external forcing pressures represented by p_0 and p_ℓ. Letting $u = (u_1,u_2) = (a,v)$, $h = (h_1,h_2) = (h_0,h_\ell)$, and $p = (p_1,p_2) = (p_0,p_\ell)$ we can transform (5.4.1) into (5.0.1) with

$$A(u) = \begin{pmatrix} u_2 & u_1 \\ \kappa^2 u_1^{-\frac{1}{2}} & u_2 \end{pmatrix}, \quad f(u) = \begin{pmatrix} 0 \\ -\beta u \end{pmatrix}, \quad B_0 = B_\ell = (1\ \ 0),$$

$$b(p,h) = \begin{pmatrix} a_0(1+p_1+bh_1)^2 \\ a_0(1+p_2+bh_2)^2 \end{pmatrix}, \quad H(h,u,w) = \begin{pmatrix} -\frac{1}{a_T}u_1u_2 \\ \frac{1}{a_T}w_1w_2 \end{pmatrix}.$$

The eigenvalues of the flux matrix $A(u)$ are given by $\lambda(u) = u_2 - \kappa u_1^{\frac{1}{4}}$ and $\mu(u) = u_2 + \kappa u_1^{\frac{1}{4}}$ with corresponding eigenvectors

$$e_\lambda(u) = \begin{pmatrix} u_1 \\ -\kappa u_1^{\frac{1}{4}} \end{pmatrix}, \quad e_\mu(u) = \begin{pmatrix} u_1 \\ \kappa u_1^{\frac{1}{4}} \end{pmatrix},$$

respectively. Let $\tilde{\mathcal{U}} = \{(u_1,u_2) \in \mathbb{R}^2 : u_1 > 0, |u_2| < \kappa u_1^{\frac{1}{4}}\}$. It follows that $A(w)$ has one negative and one positive eigenvalue for every $w \in \tilde{\mathcal{U}}$. Thus $E^s(A(w)) = \operatorname{span}\{e_\lambda(w)\}$ and $E^u(A(w)) = \operatorname{span}\{e_\mu(w)\}$. The estimate $\|e_\mu(w)\| \le C\|B_0 e_\mu(w)\|$ is equivalent to

$$u_1 \le \kappa^{-4}(C^2-1)^2 u_1^4. \tag{5.4.2}$$

Let $\tilde{\mathcal{U}}_\epsilon = \{w \in \tilde{\mathcal{U}} : \operatorname{dist}(w, \partial\tilde{\mathcal{U}}) > \epsilon\}$ for $\epsilon > 0$. By continuity it can be seen from (5.4.2) that there exists $C_\epsilon > 1$ such that $\|e_\mu(w)\| \le C_\epsilon\|B_0 e_\mu(w)\|$ for all $w \in \tilde{\mathcal{U}}_\epsilon$. By positive homogeneity of the norm it follows that $\|V\| \le C_\epsilon\|B_0 V\|$ for all $V \in E^u(A(w))$ and for all $w \in \tilde{\mathcal{U}}_\epsilon$. Similarly, $\|V\| \le C_\epsilon\|B_\ell V\|$ for all $V \in E^s(A(w))$ for all $w \in \tilde{\mathcal{U}}_\epsilon$. Therefore the uniform Kreiss-Lopatinskiĭ condition holds for $w \in \tilde{\mathcal{U}}_\epsilon$.

It remains to verify Friedrichs symmetrizability. It can be easily seen that the matrix

$$S(w) = \begin{pmatrix} \kappa^2 u_1^{-\frac{3}{2}} & 0 \\ 0 & 1 \end{pmatrix}$$

is a Friedrichs symmetrizer of the system. For $R > 0$ define $\mathcal{U} = \{w \in \tilde{\mathcal{U}}_\epsilon : \|w\| < R\}$. It is clear that there exists $\alpha = \alpha(\epsilon, R) > 0$ such that $S(w) \ge \alpha I_2$ for all $w \in \mathcal{U}$. Therefore if the the initial data for the system (5.4.1) and the boundary data p satisfy the conditions of Theorem 5.2.1 then (5.4.1) has a unique solution $(a, v, h_0, h_\ell) \in CH^m([0,T]\times[0,\ell])^2 \times H^{m+1}(0,T)^2$ for some $T > 0$. Moreover, if the maximal time $T^* > 0$ of existence is finite then either the range of (a, v, h_0, h_ℓ) leaves every compact set of $\mathcal{U} \times \mathbb{R}^2$ or

$$\limsup_{t\to T^*} \left(\|\partial_x a(t)\|_{L^\infty[0,\ell]} + \|\partial_x v(t)\|_{L^\infty[0,\ell]}\right) = \infty.$$

5.4.2 Multiscale blood flow model

Consider the following system [**27, 67**]

$$\begin{cases} a_t(t,x) + q_x(t,x) = 0, \\ q_t(t,x) + \left(\dfrac{q(t,x)^2}{a(t,x)}\right)_x + \dfrac{1}{\rho} a(t,x) p_x(t,x) = -8\pi\rho\nu \dfrac{q(t,x)}{a(t,x)} \end{cases} \tag{5.4.3}$$

with $0 < t < T$ and $0 < x < \ell$. This models the flow rate q of the blood in a vessel of cross-section a and length ℓ. The pressure p is given by the constitutive law

$$p = \frac{\sqrt{\pi} hE}{a_0(1-\sigma^2)}(\sqrt{a} - \sqrt{a_0}). \tag{5.4.4}$$

All the parameters are positive and they represent various physical quantities depicting the properties of the blood and the vessel. Here, a_0, E, h, σ denote the rest cross-section, Young's modulus, thickness and Poisson coefficient of the vessel wall, respectively, whereas ρ is the blood density and ν is the kinematic blood viscosity.

To have a more realistic description of the cardiovascular system, lumped parameter models based on ordinary differential equations were introduced. These ODEs can be derived by linearizing and integrating the hyperbolic models with respect to space. Following [**27**] we have

$$\begin{aligned} \dot{y}_0(t) &= A_0 y_0(t) + r_{H0}(t, y_0(t)) + s_0(t, y_0(t)) && (5.4.5)\\ \dot{y}_\ell(t) &= A_\ell y_\ell(t) + r_{H\ell}(t, y_\ell(t)) + s_\ell(t, y_\ell(t)) && (5.4.6) \end{aligned}$$

where $y_0(t), y_\ell(t) \in \mathbb{R}^m$, A_0, A_ℓ are $m \times m$ matrices and $r_{H0}, r_{H\ell}, s_0, s_\ell$ are source terms. The coupling of the hyperbolic PDE (5.4.3) and the ODEs (5.4.5) and (5.4.6) is done by imposing the pressure at the boundaries to be equal to a specific entry of the ODE, i.e.,

$$p(t,0) = y_{0i}(t), \qquad p(t,\ell) = y_{\ell j}(t) \tag{5.4.7}$$

for some $1 \le i, j \le m$. Writing the system in terms of a and q only by using the constitutive law (5.4.4) it can be shown as in the previous example that (5.4.3)−(5.4.6) can be written in the form (5.0.1) and satisfies (FS), (D) and (UKL) with appropriate $\mathcal{U}$. Alternatively, one can diagonalize the system as in [**27**], and thus Friedrichs symmetrizability is easily checked. The boundary matrices will be transformed, however, the UKL condition is preserved. This can be verified in the same manner as in the previous example and for this reason we omit the details.

5.4.3 **1-Tank model**

Consider a 1-D tank of length ℓ filled with inviscid incompressible irrotational fluid which is subjected by a horizontal force. Then using the Saint-Venant equation one can derive the following system [**16**]

$$\begin{cases} H_t(t,x) + v(t,x)H_x(t,x) + H(t,x)v_x(t,x) = 0, \quad 0<t<T,\ 0<x<\ell,\\ v_t(t,x) + gH_x(t,x) + v(t,x)v_x(t,x) = -u(t), \quad 0<t<T,\ 0<x<\ell,\\ v(t,0) = v(t,L) = 0, \quad 0<t<T,\\ \dot{s}(t) = u(t), \quad 0<t<T,\\ \dot{D}(t) = s(t), \quad 0<t<T, \end{cases} \tag{5.4.8}$$

where g is the gravitational force, H is the height of the fluid in the tank, v is the referential horizontal velocity of water, s is the horizontal velocity of the tank, D is the horizontal displacement of the tank and u is the horizontal acceleration of the tank in the absolute referential and is viewed as the control. Note that the PDE part is not of the same form as the PDE part in (5.0.1), but instead, it is of the form

$$u_t(t,x) + A(u(t,x))u_x(t,x) = F(t,x).$$

The results given in the previous sections extend to the case where there is an extra source term F on the right hand side of the PDE part.

6

GLOBAL EXISTENCE AND NONLINEAR STABILITY

We know already from Chapter 5 that the two-tank model (2.6.5) has at least a local-in-time smooth solution. Can we extend this smooth solution to all positive times? It is known that in general, quasilinear systems do not have global-in-time solutions and blow-up in finite time may occur. So at the very least, sufficient conditions should be given to guarantee that a global smooth solution exists. In the event that this global solution exists, what can we expect about its long-time behavior? We have seen that for the linearized version of the system, the solution tends to the steady state exponentially fast. Can we expect the same result for the original nonlinear system (2.6.5)? One might expect that this is true for dynamics near the steady state, i.e., the nonlinear system behaves *like* the linear system if the data is close enough to the steady state. With the results of Chapter 5 together with energy and entropy methods, we will show in the present chapter that as long as a smooth data is close enough to the steady state, (2.6.5) has a global-in-time solution and this solution tends to the steady state.

To simplify notation, we rename the parameters in (2.6.5). The system (2.6.5) can be rewritten as

$$\begin{cases} A_t + uA_x + Au_x = 0, & t>0,\ 0<x<\ell, \\ u_t + \kappa^2 A^{-\frac{1}{2}} A_x + uu_x = -\beta u, & t>0,\ 0<x<\ell, \\ A_T h_0'(t) = -A(t,0)u(t,0), & t>0, \\ A_T h_\ell'(t) = A(t,\ell)u(t,\ell), & t>0, \\ A(t,0) = (a_0 + bh_0(t))^2, & t>0, \\ A(t,\ell) = (a_\ell + bh_\ell(t))^2, & t>0, \\ A(0,x) = A^0(x), \qquad u(0,x) = u^0(x), & 0<x<1, \\ h_0(0) = h_0^0, \qquad h_\ell(0) = h_\ell^0 \end{cases} \tag{6.0.1}$$

where

$$\kappa^2 = \frac{sE}{2\rho r_0\sqrt{A_0}}, \qquad \beta = \frac{8\pi\mu}{\rho A_0}, \qquad b = \frac{r_0\rho g\sqrt{A_0}}{sE},$$
$$a_0 = \sqrt{A_0}\left(1 + \frac{r_0 p_{f0}}{sE}\right), \qquad a_\ell = \sqrt{A_0}\left(1 + \frac{r_0 p_{f\ell}}{sE}\right),$$

The main result of this chapter will be stated in Section 6.1. The energy method is used to prove the existence of global solutions for (6.0.1). In deriving the energy estimates we shall make use of entropies. Relative entropies and entropy-entropy flux pairs relevant to the proof of the main result will be tackled in Section 6.2. The relative entropy gives an entropy dissipation identity which will be useful in zero order estimates [**33**]. The entropy-entropy flux pairs on the other hand are used in deriving first order and second order estimates, cf. [**68**]. These estimates will be proved in Section 6.3. The proof of the global existence using energy estimates will be provided in Section 6.4. Thanks to the energy estimates, it immediately follows that the solution tends to the equilibrium in $H^1 \times H^1 \times \mathbb{R}^2$. With respect to the norm of $L^2 \times L^2 \times \mathbb{R}^2$ it will be shown in Section 6.5 that this convergence is exponential.

6.1 STATEMENT OF THE MAIN RESULT

The volume of the fluid inside the tube and the tanks at time $t \geq 0$ is given by

$$V(t) = \int_0^\ell A(t,x)\,\mathrm{d}x + A_T h_0(t) + A_T h_\ell(t). \tag{6.1.1}$$

If (A, u, h_0, h_ℓ) is a smooth solution of (6.0.1) on $[0, T]$ then $V(t)$ is conserved on $[0, T]$, i.e., $V(t) = V(0)$ for all $t \in [0, T]$. This can be seen immediately by taking the derivative of V and using the first, third and fourth equations in (6.0.1). In this chapter, by a smooth solution we mean that each state component is at least continuously differentiable. The equilibrium state of (6.0.1) is given by $(A_e, 0, h_{0e}, h_{\ell e})$ where

$$A_e = (a_0 + bh_{0e})^2 = (a_\ell + bh_{\ell e})^2. \tag{6.1.2}$$

For a given fixed volume and with the assumption that the pressures p_{f0} or $p_{f\ell}$ are given (not too large), the equilibrium is uniquely determined. Indeed, if V_0 denotes the fixed volume then we have $V_0 = A_e\ell + A_T h_{0e} + A_T h_{\ell e}$. The latter equality together with (6.1.2) provide explicit expressions for A_e, h_{0e} and $h_{\ell e}$ in terms of V_0.

In Chapter 5, the mth order compatibility condition of the initial data wass defined and the following local-in-time existence result and blow-up criterion was shown.

Theorem 6.1.1 (Local Existence and Blow-up Criterion). *Let $(A^0, u^0, h_0^0, h_\ell^0) \in H^m(0,\ell) \times H^m(0,\ell) \times \mathbb{R}^2$ be compatible up to order $m-1$ for some integer $m \geq 3$. Suppose that the range of (A^0, u^0) lies on a compact and convex subset of $\mathcal{U} := \{(A,u) \in (0,\infty) \times \mathbb{R} : |u| < \kappa A^{1/4}\}$. Then there exists $T > 0$ such that* (6.0.1) *has a unique solution (A, u, h_0, h_ℓ) such that $A, u \in \cap_{k=0}^m C^{m-k}([0,T]; H^m(0,\ell))$ and $h_0, h_\ell \in H^{m+1}(0,T)$. Furthermore, if the maximal time T^* of existence is finite then either (A, u, h_0, h_ℓ) leaves every compact set of $\mathcal{U} \times \mathbb{R}^2$ or*

$$\lim_{t \uparrow T^*} (\|A_x(t)\|_{L^\infty[0,\ell]} + \|u_x(t)\|_{L^\infty[0,\ell]}) = +\infty.$$

If the maximal time is finite, the first scenario is typical for ODEs while the second one is called shock formation. For the first one, the state approaches the boundary of $\mathcal{U}$ and as a result the flux matrix will become singular. In the region $\mathcal{U}$, there is one negative eigenvalue and one positive eigenvalue for the flux matrix and the flow in this case is subsonic. On the other hand, the shock formation is a typical behavior for first order quasilinear PDEs where waves are compressed within finite time and therefore wave profiles can have arbitrary large slope. However, for data close enough to an equilibrium state and with dissipative terms these will not happen. This assertion with regard to (6.0.1) is the main result of this chapter.

Theorem 6.1.2 (Global Existence)**.** *In the framework of Theorem 6.1.1, there exists* $\delta_0 > 0$ *such that if* $E_0 := \|A^0 - A_e\|_{H^2}^2 + \|u^0\|_{H^2}^2 + |h_0^0 - h_{0e}|^2 + |h_\ell^0 - h_{\ell e}|^2 \leq \delta_0$ *then there is a unique global solution* (A, u, h_0, h_ℓ) *of* (6.0.1) *such that*

$$A, u \in C([0,\infty); H^2(0,\ell)) \cap C^1([0,\infty); H^1(0,\ell)), \quad h_0, h_\ell \in C^2[0,\infty),$$

and

$$\sup_{t\geq 0} (\|A(t) - A_e\|_{H^2}^2 + \|u(t)\|_{H^2}^2 + |h_0(t) - h_{0e}|^2 + |h_\ell(t) - h_{\ell e}|^2) \\ + \int_0^\infty \|A_x(t)\|_{H^1}^2 + \|u(t)\|_{H^1}^2 \, \mathrm{d}t \;\leq\; CE_0$$

for some $C > 0$.

6.2 ENTROPY-ENTROPY FLUX PAIRS

Entropies of the system (6.0.1) can be obtained by solving a wave equation as shown in the following. For a more general result of a similar model and in the case of $\beta = 0$ we refer to the paper of Lions, Perthame and Tadmor [**51**].

Proposition 6.2.1. *Let* $\eta \in C^2((0,\infty)\times\mathbb{R}) \cap C^1([0,\infty)\times\mathbb{R})$ *satisfy the wave equation*

$$\frac{\partial^2 \eta}{\partial A^2}(A,u) = \kappa^2 A^{-\frac{3}{2}} \frac{\partial^2 \eta}{\partial u^2}(A,u), \qquad \text{in } (0,\infty)\times\mathbb{R}. \tag{6.2.1}$$

Then any smooth functions A and u satisfying the first two equations in (6.0.1) *also satisfy the* entropy dissipation identity

$$\frac{\partial}{\partial t}\eta(A,u) + \frac{\partial}{\partial x} q(A,u) = -\beta u \frac{\partial}{\partial u}\eta(A,u), \qquad \text{in } (0,\infty)\times\mathbb{R}, \tag{6.2.2}$$

where $q \in C^2((0,\infty)\times\mathbb{R})$ *is given by*

$$q(A,u) = \int_0^u v\eta_u(A,v) + A\eta_A(A,v)\,\mathrm{d}v + \int_0^A \kappa^2 a^{-\frac{1}{2}} \eta_u(a,0)\,\mathrm{d}a. \tag{6.2.3}$$

Proof. The regularity of q stated above follows immediately from the regularity of η. Since u and A satisfy the first two equations in (6.0.1), the PDE (6.2.2) is equivalent to

$$u_x(q_u - u\eta_u - A\eta_A) + A_x(q_A - \kappa^2 A^{-\frac{1}{2}}\eta_u - u\eta_A) = 0. \tag{6.2.4}$$

The first term vanishes due to the construction of q since $q_u = u\eta_u + A\eta_A$. We show that the second term also vanishes. Differentiating the latter equality with respect to A and using (6.2.1) we have

$$q_{Au} = q_{uA} = u\eta_{uA} + \eta_A + A\eta_{AA} = (u\eta_A + \kappa^2 A^{-\frac{1}{2}}\eta_u)_u. \tag{6.2.5}$$

Integrating (6.2.5) twice, first with respect to u and then with respect to A, we have

$$q(A,u) = \int_0^A u\eta_A(a,u) + \kappa^2 a^{-\frac{1}{2}}\eta_u(a,u)\,\mathrm{d}a + F(A) \tag{6.2.6}$$

for some function F. Taking $u = 0$ in (6.2.3) and (6.2.6) shows that $F \equiv 0$. Thus, differentiating (6.2.6) with respect to A shows that the second term in (6.2.4) is identically zero. Hence (6.2.4) is satisfied and so is (6.2.2). □

The function η is called an *entropy* and q is the corresponding *entropy flux*. The entropy dissipation identity (6.2.2) is commonly called a *companion law* to the first two equations in (6.0.1). Let $\eta_p = a_1 u + a_2 A + a_3 uA + a_4$ where the a_i's are constants. Notice that the wave equation is invariant under perturbations of the form η_p, i.e., if η satisfies (6.2.1) then so does $\eta + \eta_p$.

A common entropy of the above system is

$$\eta(A,u) = \frac{1}{2}Au^2 + \frac{4}{3}\kappa^2 A^{\frac{3}{2}},$$

called the *mechanical energy* and it is strictly convex in the variables $(A, Au) \in (0,\infty)\times\mathbb{R}$. This particular entropy satisfies the boundary conditions $\eta(0,u) = 0$ and $\eta_A(0,u) = \frac{1}{2}u^2$. Such entropies are called weak entropies [**51**]. However, for our purpose we will modify this entropy. We want an entropy η_0 such that $\eta_0(A_e, 0) = 0$ and $D\eta_0(A_e,0) = (0,0)$. This can be done by choosing

$$\begin{aligned}
\eta_0(A,u) &= \eta(A,u) - \eta(A_e,0) - (D\eta(A_e,0),(A-A_e,u)) \\
&= \frac{1}{2}Au^2 + \frac{4}{3}\kappa^2(A^{\frac{3}{2}} - A_e^{\frac{3}{2}}) - 2\kappa^2 A_e^{\frac{1}{2}}(A - A_e). \qquad (6.2.7)
\end{aligned}$$

In the literature, η_0 is referred to as the relative entropy with respect to the state $(A_e, 0)$. Notice that the difference of the mechanical energy η and its modified version η_0 is a function of the form η_p stated above. By invariance, η_0 also satisfies the wave equation (6.2.1) and therefore if (A,u) satisfies the first two equations in (6.0.1), η_0 also satisfies the entropy dissipation identity (6.2.2) with the corresponding entropy flux

$$q_0(A,u) = \frac{1}{2}Au^3 + 2\kappa^2(A^{\frac{1}{2}} - A_e^{\frac{1}{2}})uA. \qquad (6.2.8)$$

obtained from (6.2.3). Moreover, η_0 is also strictly convex in the variables (A, uA). This entropy-entropy flux pair will be used in the next section to obtain zero order estimates. By a second order Taylor expansion we can see that there exist constants $c_K, C_K > 0$ such that

$$c_K(|uA|^2 + |A - A_e|^2) \le \eta_0(A,u) \le C_K(|uA|^2 + |A - A_e|^2) \qquad (6.2.9)$$

for every $(A,u) \in K$ where $K \subset (0,\infty)\times\mathbb{R}$ is a compact set containing $(A_e,0)$. Thus the relative entropy serves as a distance between the smooth solutions of the system and the constant equilibrium state.

The next step is to develop entropy-entropy flux pairs to deal with first order and second order estimates as done by Ruan et al. [**68**]. This will be done using an appropriate diagonal form of the system. The eigenvalues of the associated flux matrix of (6.0.1) are $\tilde\lambda = u - \kappa A^{\frac{1}{4}}$ and $\tilde\mu = u + \kappa A^{\frac{1}{4}}$. Multiplying the first two equations in (6.0.1) by $(\kappa A^{-\frac{3}{4}}, 1)$ and by $(\kappa A^{-\frac{3}{4}}, -1)$ we obtain a diagonal system

$$\begin{aligned}
\tilde w_t + \tilde\lambda(\tilde w,\tilde z)\tilde w_x &= \frac{\beta}{2}(\tilde z - \tilde w) \\
\tilde z_t + \tilde\mu(\tilde w,\tilde z)\tilde z_x &= -\frac{\beta}{2}(\tilde z - \tilde w)
\end{aligned}$$

where $\tilde w = -u + 4\kappa A^{\frac{1}{4}}$, $\tilde z = u + 4\kappa A^{\frac{1}{4}}$, $\tilde\lambda = -\frac{5}{8}\tilde w + \frac{3}{8}\tilde z$ and $\tilde\mu = -\frac{3}{8}\tilde w + \frac{5}{8}\tilde z$. If (A,u) is close to the equilibrium state $(A_e, 0)$ then (w,z) is close to $(4\kappa A_e^{\frac{1}{4}}, 4\kappa A_e^{\frac{1}{4}})$. With this in mind, we shall consider the shifted Riemann invariants $w = \tilde{w} - 4\kappa A_e^{\frac{1}{4}}$ and $z = \tilde{z} - 4\kappa A_e^{\frac{1}{4}}$ so that

$$w = -u + 4\kappa(A^{\frac{1}{4}} - A_e^{\frac{1}{4}}), \qquad z = u + 4\kappa(A^{\frac{1}{4}} - A_e^{\frac{1}{4}}). \tag{6.2.10}$$

Therefore the state variables (A, u) and the shifted Riemann invariants (w, z) are related by

$$u = \frac{1}{2}(z - w), \qquad A^{\frac{1}{4}} - A_e^{\frac{1}{4}} = \frac{1}{8\kappa}(z + w). \tag{6.2.11}$$

Using the Riemann invariants, the system (6.0.1) can be written in diagonal form

$$\begin{cases} w_t + \lambda(w,z)w_x = \dfrac{\beta}{2}(z - w), & t > 0,\ 0 < x < \ell, \\ z_t + \mu(w,z)z_x = -\dfrac{\beta}{2}(z - w), & t > 0,\ 0 < x < \ell, \\ h_0'(t) = -\theta(w(t,0), z(t,0))(z(t,0) - w(t,0)), & t > 0, \\ h_\ell'(t) = \theta(w(t,\ell), z(t,\ell))(z(t,\ell) - w(t,\ell)), & t > 0, \\ z(t,0) + w(t,0) = \zeta_0(h_0(t))(h_0(t) - h_{0e}), & t > 0, \\ z(t,\ell) + w(t,\ell) = \zeta_\ell(h_\ell(t))(h_\ell(t) - h_{\ell e}), & t > 0, \end{cases} \tag{6.2.12}$$

where the coefficient functions are given by

$$\begin{aligned} \lambda(w,z) &= -\frac{5}{8}w + \frac{3}{8}z - \frac{1}{4}C_e, \qquad C_e = 4\kappa A_e^{\frac{1}{4}} && (6.2.13) \\ \mu(w,z) &= -\frac{3}{8}w + \frac{5}{8}z + \frac{1}{4}C_e && (6.2.14) \\ \theta(w,z) &= \frac{1}{2^{13}\kappa^4 A_T}(w + z + 2C_e)^4 && (6.2.15) \\ \zeta_k(h) &= b(\sqrt{a_k + bh} + \sqrt{a_k + bh_{ke}})^{-1}, \qquad k = 0, \ell. && (6.2.16) \end{aligned}$$

Differentiating the first two equations in (6.2.12) with respect to x once and twice we have

$$\begin{aligned} (\partial_x^k w)_t + \lambda(w,z)(\partial_x^k w)_x &= F_k && (6.2.17) \\ (\partial_x^k z)_t + \mu(w,z)(\partial_x^k z)_x &= G_k && (6.2.18) \end{aligned}$$

for $k = 1, 2$ where

$$\begin{aligned} F_1 &= -\lambda_x w_x + \frac{\beta}{2}(z_x - w_x) && (6.2.19) \\ G_1 &= -\mu_x z_x - \frac{\beta}{2}(z_x - w_x) && (6.2.20) \\ F_2 &= -2\lambda_x w_{xx} - \lambda_{xx} w_x + \frac{\beta}{2}(z_{xx} - w_{xx}) && (6.2.21) \\ G_2 &= -2\mu_x z_{xx} - \mu_{xx} z_x - \frac{\beta}{2}(z_{xx} - w_{xx}). && (6.2.22) \end{aligned}$$

Consider differentiable functions $\phi_k = \phi_k(t, x, W)$ and $\psi_k = \psi_k(t, x, Z)$ for $k = 1, 2$. Using the equation (6.2.17) we have for a smooth solution (w, z) of the system (6.2.12),

$$\begin{aligned}
&\partial_t \phi_k(t, x, \partial_x^k w(t, x)) + \partial_x(\lambda(t, x)\phi_k(t, x, \partial_x^k w(t, x))) \\
&= \phi_{kt}(t, x, \partial_x^k w(t, x)) + \phi_{kW}(t, x, \partial_x^k w(t, x))\partial_t(\partial_x^k w(t, x)) \\
&\quad + \lambda_x(t, x)\phi_k(t, x, \partial_x^k w(t, x)) + \lambda(t, x)\phi_{kx}(t, x, \partial_x^k w(t, x)) \\
&\quad + \lambda(t, x)\phi_{kW}(t, x, \partial_x^k w(t, x))\partial_x(\partial_x^k w(t, x)) \\
&= \phi_{kt}(t, x, \partial_x^k w(t, x)) + \lambda_x(t, x)\phi_k(t, x, \partial_x^k w(t, x)) + \lambda(t, x)\phi_{kx}(t, x, \partial_x^k w(t, x)) \\
&\quad + \phi_{kW}(t, x, \partial_x^k w(t, x))F_k(t, x) \qquad (6.2.23)
\end{aligned}$$

for $k = 1, 2$. Similarly, using (6.2.18) we get

$$\begin{aligned}
&\partial_t \psi_k(t, x, \partial_x^k z(t, x)) + \partial_x(\mu(t, x)\psi_k(t, x, \partial_x^k z(t, x))) \\
&= \psi_{kt}(t, x, \partial_x^k z(t, x)) + \mu_x(t, x)\psi_k(t, x, \partial_x^k z(t, x)) + \mu(t, x)\psi_{kx}(t, x, \partial_x^k z(t, x)) \\
&\quad + \psi_{kZ}(t, x, \partial_x^k z(t, x))G_k(t, x) \qquad (6.2.24)
\end{aligned}$$

for $k = 1, 2$. Subtracting (6.2.23) from (6.2.24) we obtain the partial differential equation

$$\partial_t(\psi_k - \phi_k) + \partial_x(\mu\psi_k - \lambda\phi_k) = M_k(\psi_k, \phi_k) \qquad (6.2.25)$$

where

$$\begin{aligned}
M_k(\psi_k, \phi_k) \;&=\; (\psi_{kt} - \phi_{kt}) + (\mu_x\psi_k - \lambda_x\phi_k) + (\mu\psi_{kx} - \lambda\phi_{kx}) \\
&\quad + (\psi_{kZ}G_k - \phi_{kW}F_k). \qquad (6.2.26)
\end{aligned}$$

Integrating (6.2.25) over $[0, t] \times [0, \ell]$ and using Fubini's theorem we have

$$\begin{aligned}
&\int_0^\ell \eta_k(t, x) - \eta_k(0, x)\,\mathrm{d}x + \int_0^t q_k(\tau, \ell) - q_k(\tau, 0)\,\mathrm{d}\tau \\
&= \int_0^t \int_0^\ell M_k(\psi_k, \phi_k)\,\mathrm{d}x\,\mathrm{d}\tau \qquad (6.2.27)
\end{aligned}$$

where

$$\begin{aligned}
\eta_k(t, x) \;&=\; \psi_k(t, x, \partial_x^k w(t, x)) - \phi_k(t, x, \partial_x^k w(t, x)) \\
q_k(t, x) \;&=\; \mu(t, x)\psi_k(t, x, \partial_x^k w(t, x)) - \lambda(t, x)\phi_k(t, x, \partial_x^k w(t, x)).
\end{aligned}$$

The point is that solutions (w, z) of (6.2.12) that are sufficiently smooth satisfy (6.2.27) for $k = 1, 2$. Equation (6.2.27) will be of great importance in deriving the energy estimates. This is done by choosing appropriate functions ψ and ϕ such that the term M_k will be, in some sense, dominated by the velocity u or its derivatives.

6.3 ENERGY ESTIMATES

For $T > 0$ define the solution space

$$X_T = (C([0, T]; H^2(0, \ell)^2) \cap C^1([0, T]; H^1(0, \ell)^2) \cap C^2([0, T]; L^2(0, \ell)^2)) \times C^2[0, T]^2.$$

By using classical embedding results we can see that X_T is continuously embedded in $C^1([0, T] \times [0, \ell])^2 \times C^2[0, T]^2$. All throughout this section (A, u, h_0, h_ℓ) will be a smooth solution to the system on the time interval $[0,T]$, provided that such solution exists on such interval. Define the energy functionals $N_k : [0,\infty) \to [0,\infty)$ for $k = 0,1,2$ by

$$\begin{aligned} N_k^2(t) &= \sup_{\tau\in[0,t]} (\|u(\tau)\|_{H^k}^2 + \|A^{\frac{1}{4}}(\tau) - A_e^{\frac{1}{4}}\|_{H^k}^2 + |h_0(\tau) - h_{0e}|^2 + |h_\ell(\tau) - h_{\ell e}|^2) \\ &\quad + \int_0^t \|u(\tau)\|_{H^k}^2 + k\|(A^{\frac{1}{4}})_x(\tau)\|_{H^{k-1}}^2 \,\mathrm{d}\tau. \end{aligned}$$

In the following estimates, and C_δ and $C_{i\delta}$ will denote generic positive constants that depend on the system parameters and may depend on $\delta > 0$, and

$$C_\delta \text{ and } C_{i\delta} \text{ remain bounded as long as } \delta \text{ stays on a bounded set in } (0,\infty). \tag{6.3.1}$$

Before we proceed we state the following equivalence of norms of the state variables u, A and the Riemann invariants

$$2\|\partial_x^k u(t)\|_{L^2}^2 + 32\kappa^2\|\partial_x^k(A^{\frac{1}{4}}(t) - A_e^{\frac{1}{4}})\|_{L^2}^2 = \|\partial_x^k w(t)\|_{L^2}^2 + \|\partial_x^k z(t)\|_{L^2}^2. \tag{6.3.2}$$

for $k = 0,1,2$ and for $t \in [0,T]$. This follows immediately from the identity $2w^2 + 2z^2 = (z-w)^2 + (z+w)^2$ in $\mathbb{R}$ and the transformations given in (6.2.11). This norm equivalence will be used in converting an estimate involving the Riemann invariants into an estimate involving the state variables and vice versa. Furthermore, if $0 < \delta < A_e$ then $|A - A_e| \le \delta$ implies that

$$C_{1\delta}|A - A_e| \le |A^{\frac{1}{4}} - A_e^{\frac{1}{4}}| \le C_{2\delta}|A - A_e|. \tag{6.3.3}$$

This can be seen from the elementary identity $A - A_e = (A^{\frac{1}{4}} - A_e^{\frac{1}{4}})(A^{\frac{1}{4}} + A_e^{\frac{1}{4}})(A^{\frac{1}{2}} + A_e^{\frac{1}{2}})$ whenever $A, A_e > 0$.

6.3.1 **Zero Order Estimates**

Lemma 6.3.1 (Zero Order Estimate)**.** *There exist $\delta > 0$ and $C_\delta > 0$ such that for any solution $(A,u,h_0,h_\ell) \in X_T$ satisfying $N_2^2(T) \le \delta$ also satisfies the energy estimate*

$$N_0^2(t) \le C_\delta \left(N_0^2(0) + \sup_{\tau\in[0,t]} \|u(\tau)\|_{H^1} \int_0^t \|u(\tau)\|_{H^1}^2 \,\mathrm{d}\tau \right) \tag{6.3.4}$$

for all $t \in [0,T]$.

Proof. Recall that η_0 and q_0 given in (6.2.7) and (6.2.8), respectively, satisfy the entropy dissipation identity (6.2.2). Integrating (6.2.2) over $[0,t]\times[0,\ell]$ and using Fubini's Theorem yield

$$\begin{aligned} &\int_0^\ell \eta_0(A(t,x),u(t,x)) - \eta_0(A(0,x),u(0,x)) \,\mathrm{d}x \\ &+ \int_0^t q_0(A(\tau,\ell),u(\tau,\ell)) - q_0(A(\tau,0),u(\tau,0)) \,\mathrm{d}\tau = -\beta \int_0^t \int_0^\ell (Au^2)(\tau,x) \,\mathrm{d}x \,\mathrm{d}\tau. \end{aligned} \tag{6.3.5}$$

Let us estimate the left hand side of (6.3.5) from below and its right hand side from above. According to (6.2.9) and (6.3.3) it holds that, choosing $\delta > 0$ sufficiently small,

$$\begin{aligned} &\int_0^\ell \eta_0(A(t,x),u(t,x)) - \eta_0(A(0,x),u(0,x)) \,\mathrm{d}x \\ &\ge C_\delta(\|(uA)(t)\|_{L^2}^2 + \|A^{\frac{1}{4}}(t) - A_e^{\frac{1}{4}}\|_{L^2}^2 - \|(uA)(0)\|_{L^2}^2 - \|A^{\frac{1}{4}}(0) - A_e^{\frac{1}{4}}\|_{L^2}^2) \end{aligned} \tag{6.3.6}$$

Using (6.1.2) and the last four equations of (6.0.1) in (6.2.8) we have

$$\begin{aligned}
q_0(A(\tau,\ell),u(\tau,\ell)) &= \frac{1}{2}(Au^3)(\tau,\ell) + 2A_T\kappa^2 b(h_\ell(\tau)-h_{\ell e})h'_\ell(\tau) \\
q_0(A(\tau,0),u(\tau,0)) &= \frac{1}{2}(Au^3)(\tau,0) - 2A_T\kappa^2 b(h_0(\tau)-h_{0e})h'_0(\tau).
\end{aligned}$$

Plugging these in the second integral in (6.3.5) and using the Sobolev embedding theorem we have

$$\begin{aligned}
&\int_0^t q_0(A(\tau,\ell),u(\tau,\ell)) - q_0(A(\tau,0),u(\tau,0))\,\mathrm{d}\tau \\
&\geq C(|h_0(t)-h_{0e}|^2 + |h_\ell(t)-h_{\ell e}|^2 - |h_0^0 - h_{0e}|^2 - |h_\ell^0 - h_{\ell e}|^2) \qquad (6.3.7)\\
&\quad - C_\delta \sup_{\tau\in[0,t]} \|u(\tau)\|_{H^1} \int_0^t \|u(\tau)\|_{H^1}^2\,\mathrm{d}\tau
\end{aligned}$$

Moreover, the Sobolev embedding theorem again implies that

$$-\beta\int_0^t\int_0^\ell (Au^2)(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau \leq -\beta C_\delta \int_0^t \|u(\tau)\|_{L^2}^2\,\mathrm{d}\tau. \qquad (6.3.8)$$

Now it can be seen that (6.3.4) follows from (6.3.5)–(6.3.8) and the fact that the L^2- norm of $(uA)(t)$ and $u(t)$ are equivalent for each t provided that $\delta > 0$ is small enough. □

6.3.2 First Order Estimates

The next step is to derive estimates involving the spatial derivatives of the state components u and $A^{\frac{1}{4}}$. To this end we recall two classical inequalities frequently used in deriving estimates. The first one is Young's inequality: For each real numbers a, b and $\epsilon > 0$ we have $ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$. The second one is the following modified Sobolev embedding.

Proposition 6.3.2. *Let $a < b$. For every $\vartheta > 0$ there exists $C(a,b,\vartheta) > 0$ such that*

$$\|u\|_{L^\infty(a,b)}^2 \leq \vartheta\|u_x\|_{L^2(a,b)}^2 + C(a,b,\vartheta)\|u\|_{L^2(a,b)}^2 \qquad (6.3.9)$$

for all $u \in H^1(a,b)$.

Proof. Let $a \leq x_0 \leq \frac{a+b}{2}$. Consider the linear multiplier $m(x) = \frac{2}{b-x_0}(x-x_0) - 1$ satisfying $\|m\|_{L^\infty[x_0,b]} = 1$. Thus

$$\begin{aligned}
|u(x_0)|^2 + |u(b)|^2 &= \int_{x_0}^b (mu^2)_x\,\mathrm{d}x = \frac{2}{b-x_0}\int_{x_0}^b u^2\,\mathrm{d}x + 2\int_{x_0}^b muu_x\,\mathrm{d}x \\
&\leq \vartheta\|u_x\|_{L^2(x_0,b)}^2 + \left(\frac{4}{a+b} + \frac{1}{\vartheta}\right)\|u\|_{L^2(x_0,b)}^2
\end{aligned}$$

where we use Young's inequality in the last step. A similar process can be done for the case $\frac{a+b}{2} \leq x_0 \leq b$, now using the multiplier $n(x) = \frac{2}{x_0-a}(x-x_0)+1$ and integration over $[a, x_0]$. These estimates imply (6.3.9). □

The proposition is useful when dealing with higher order estimates. For example, in obtaining estimates for z_x and w_x we will put a small factor, if necessary, to these terms, but the drawback is the occurrence of a large factor to lower order terms. However, this will not cause problems when we have already derived estimates for the lower order terms, specifically, the one given in Lemma 6.3.1.

Lemma 6.3.3 (First Order Estimate)**.** *There exist $\delta > 0$ and $C_\delta > 0$ such that for any solution $(A, u, h_0, h_\ell) \in X_T$ satisfying $N_2^2(T) \leq \delta$ we have*

$$\|u_x(t)\|_{L^2}^2 + \|(A^{\frac{1}{4}})_x(t)\|_{L^2}^2 + \int_0^t \|u_x(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \;\leq\; C_\delta N_1^2(0) \tag{6.3.10}$$

$$+ \, C_\delta \sup_{\tau\in[0,t]} (\|u(\tau)\|_{H^2} + \|A^{\frac{1}{4}}(\tau) - A_e^{\frac{1}{4}}\|_{H^2}) \int_0^t \|u(\tau)\|_{H^1}^2 + \|(A^{\frac{1}{4}})_x(\tau)\|_{L^2}^2 \,\mathrm{d}\tau$$

for all $t \in [0, T]$.

Proof. To prove the lemma we will utilize the system satisfied by the (shifted) Riemann invariants (6.2.12). Let us consider the entropy $\eta_1 = \psi_1 - \phi_1$ where

$$\begin{aligned} \psi_1(t, x, Z) &= \theta(w(t, x), z(t, x))\mu(t, x)Z^2 \\ \phi_1(t, x, W) &= \theta(w(t, x), z(t, x))\lambda(t, x)W^2. \end{aligned}$$

We will estimate each integral in (6.2.27) with these particular functions.

Suppose that $N_2^2(T) \leq \delta$. If $\delta > 0$ is sufficiently small then there exist positive constants $C_{i\delta}$ such that $C_{1\delta} \leq \zeta_k(h_k(t)) \leq C_{2\delta}$ for $k = 0, \ell$, $-C_{3\delta} \leq \lambda(t, x) \leq -C_{4\delta}$, $C_{5\delta} \leq \mu(t, x) \leq C_{6\delta}$ and $C_{7\delta} \leq \theta(w(t, x), z(t, x)) \leq C_{8\delta}$ for all $(t, x) \in [0, T] \times [0, \ell]$. We shall use these properties all throughout without mentioning them anymore.

We estimate each of the integrals on the left hand side of (6.2.27) from below and estimate the integral on the right hand side from above. For ease of reading, we divide the process into three steps. To make the terms more compact we also introduce the variable $V = (w, z)$.

Step 1. Estimate from below. The preceding remarks about θ, λ and μ show that

$$C_{1\delta}(w_x^2(t, x) + z_x^2(t, x)) \leq \eta_1(t, x) \leq C_{2\delta}(w_x^2(t, x) + z_x^2(t, x)) \tag{6.3.11}$$

for all $(t, x) \in [0, T] \times [0, \ell]$. Thus

$$\int_0^\ell \eta_1(t, x) - \eta_1(0, x) \,\mathrm{d}x \geq C_\delta(\|V_x(\tau)\|_{L^2}^2 - \|V_x(0)\|_{L^2}^2). \tag{6.3.12}$$

Next, we deal with boundary terms. Let us note the identity

$$\begin{aligned} q_1 &= \theta(w, z)((\mu z_x)^2 - (\lambda w_x)^2) \\ &= \theta(w, z)\left(\left(-z_t - \frac{\beta}{2}(z - w)\right)^2 - \left(-w_t + \frac{\beta}{2}(z - w)\right)^2\right) \\ &= \theta(w, z)(z_t^2 - w_t^2 + \beta(z_t + w_t)(z - w)) \end{aligned}$$

obtained from the first two equations in (6.2.12). Each term of the above equality is evaluated at either $(t, 0)$ and (t, ℓ). Consider the case where it is evaluated at $(t, 0)$. Differentiating the fifth equation in (6.2.12) and using the third equation we arrive at

$$\begin{aligned} z_t(t, 0) + w_t(t, 0) &= [\zeta_0'(h_0(t))(h_0(t) - h_{0e}) + \zeta_0(h_0(t))]h_0'(t) && (6.3.13) \\ &= -S_1(t)(z(t, 0) - w(t, 0)). && (6.3.14) \end{aligned}$$

where $S_1(t) = \theta(w(t, 0), z(t, 0))[\zeta_0'(h_0(t))(h_0(t) - h_{0e}) + \zeta_0(h_0(t))]$. Thus we have

$$\begin{aligned} -q_1(t, 0) &= -\theta(w(t, 0), z(t, 0))(z_t^2(t, 0) - w_t^2(t, 0)) \\ &\quad - \beta\theta(w(t, 0), z(t, 0))S_1(t)(z(t, 0) - w(t, 0))^2 =: \Psi_1(t) + \Psi_2(t). \end{aligned} \tag{6.3.15}$$

Using the estimate in Propostion 6.3.2, the Sobolev embedding theorem and the equality $2u = z - w$ we have

$$\int_0^t \Psi_2(\tau)\,\mathrm{d}\tau \geq -C_\delta \vartheta \int_0^t \|u_x(\tau)\|_{L^2}^2\,\mathrm{d}\tau - C_{\delta,\vartheta}\int_0^t \|u(\tau)\|_{L^2}^2\,\mathrm{d}\tau. \tag{6.3.16}$$

Differentiating the third equation in (6.2.12) gives

$$\begin{aligned} h_0''(t) \;=\; &-\theta_1(w(t,0),z(t,0))(z_t(t,0)+w_t(t,0))(z(t,0)-w(t,0)) \\ &-\theta(w(t,0),z(t,0))(z_t(t,0)-w_t(t,0)) \end{aligned} \tag{6.3.17}$$

where $\theta_1(w,z) = \frac{1}{2^{11}\kappa A_T}(w+z+2C_e)^3$. Multiplying the left hand side of (6.3.13) with the right hand side of (6.3.17), rearranging the terms and then using (6.3.14) we obtain

$$\Psi_1(t) = S_2(t)(z(t,0)-w(t,0))^3 + \frac{1}{2}S_3(t)\frac{\mathrm{d}}{\mathrm{d}t}|h'(t)|^2. \tag{6.3.18}$$

where $S_2(t) = \theta_1(w(t,0),z(t,0))S_1^2(t)$ and $S_3(t) = \zeta_0'(h_0(t))(h_0(t)-h_{0e}) + \zeta_0(h_0(t))$. Let us integrate (6.3.18) from 0 to t. The first term of the integral can be estimated as follows

$$\int_0^t S_2(t)(z(t,0)-w(t,0))^3 \geq -C_\delta \sup_{\tau\in[0,t]}\|u(\tau)\|_{H^1}\int_0^t \|u(\tau)\|_{H^1}^2\,\mathrm{d}\tau. \tag{6.3.19}$$

For the remaining term we integrate by parts, use the the third equation in (6.0.1), apply the Sobolev embedding and Proposition 6.3.2 to obtain

$$\begin{aligned} \frac{1}{2}\int_0^t S_3(\tau)\frac{\mathrm{d}}{\mathrm{d}t}|h'(\tau)|^2\,\mathrm{d}\tau &= \frac{1}{2}S_3(t)|h_0'(t)|^2 - \frac{1}{2}S_3(0)|h_0'(0)|^2 \\ &\quad - \frac{1}{2}\int_0^t [\zeta_0''(h_0(\tau))(h_0(\tau)-h_{0e}) + 2\zeta_0'(h_0(\tau))]h_0'(\tau)^3\,\mathrm{d}\tau \\ &\geq -C_\delta\Big(\vartheta\|u_x(t)\|_{L^2}^2 + C_\vartheta\|u(t)\|_{L^2}^2 + \|u(0)\|_{H^1}^2 \\ &\qquad + \sup_{\tau\in[0,t]}\|u(\tau)\|_{H^1}\int_0^t \|u(\tau)\|_{H^1}^2\,\mathrm{d}\tau\Big). \end{aligned} \tag{6.3.20}$$

Therefore, (6.3.15) and the inequalities (6.3.16), (6.3.19) and (6.3.20) give us the estimate

$$\begin{aligned} -\int_0^t q_1(\tau,0)\,\mathrm{d}\tau \;&=\; \int_0^t \Psi_1(\tau)\,\mathrm{d}\tau + \int_0^t \Psi_2(\tau)\,\mathrm{d}\tau \\ \geq -C_\delta\Big(&\vartheta\|u_x(t)\|_{L^2}^2 + \vartheta\int_0^t \|u_x(\tau)\|_{L^2}^2\,\mathrm{d}\tau + C_\vartheta\|u(t)\|_{L^2}^2 + C_\vartheta\int_0^t \|u(\tau)\|_{L^2}^2\,\mathrm{d}\tau \\ &+ \|u(0)\|_{H^1}^2 + \sup_{\tau\in[0,t]}\|u(\tau)\|_{H^1}\int_0^t \|u(\tau)\|_{H^1}^2\,\mathrm{d}\tau\Big). \end{aligned}$$

In an analogous manner we can obtain the same form of estimate from below for the integral $\int_0^t q_1(\tau,\ell)\,\mathrm{d}\tau$. Combining the estimates that we have obtained so far, we have the following estimate from below for the left hand side of (6.2.27)

$$\begin{aligned} &\int_0^\ell \eta_1(t,x) - \eta_1(0,x)\,\mathrm{d}x + \int_0^t q_1(\tau,\ell) - q_1(\tau,0)\,\mathrm{d}\tau \\ &\geq C_\delta\Big((1-\vartheta)\|V_x(t)\|_{L^2}^2 - \vartheta\int_0^t \|u_x(\tau)\|_{L^2}^2\,\mathrm{d}\tau - C_\vartheta\|V(t)\|_{L^2}^2 \\ &\qquad - C_\vartheta\int_0^t \|u(\tau)\|_{L^2}^2\,\mathrm{d}\tau - \|V(0)\|_{H^1}^2 - \sup_{\tau\in[0,t]}\|u(\tau)\|_{H^1}\int_0^t \|u(\tau)\|_{H^1}^2\,\mathrm{d}\tau\Big). \end{aligned} \tag{6.3.21}$$

Step 2. Estimate from above. First we will express the derivative of the eigenvalues λ and μ with respect to t in terms of the Riemann invariants w and z. A straightforward calculation and application of the two PDEs in (6.2.12) gives us

$$\begin{aligned}\mu_t &= -\frac{3C_e}{32}w_x - \frac{5C_e}{32}z_x - \frac{\beta}{2}(z-w) + R_1\\ \lambda_t &= -\frac{5C_e}{32}w_x - \frac{3C_e}{32}z_x - \frac{\beta}{2}(z-w) + R_2\end{aligned}$$

where $R_k = c_{k1}ww_x + c_{k2}zw_x + c_{k3}wz_x + c_{k4}zz_x$, $k = 1, 2$, for some constants c_{kj}. Therefore, each term of μ_t and λ_t contains at least one factor among $z - w, w_x, z_x$. Consequently, the same is true for w_t and z_t according to the PDE and in turn for $\theta_t(w, z) = \theta_1(w, z)(w_t + z_t)$. This observation is important because we want to avoid the term $\int_0^t \|A^{\frac{1}{4}}(\tau) - A_e^{\frac{1}{4}}\|_{L^2}\,\mathrm{d}\tau$ which is not present in the energy functional N_2.

Now the first three pairs appearing in (6.2.26) for $k = 1$ are given by

$$\begin{aligned}\psi_{1t} - \phi_{1t} &= (\theta_t\mu + \theta\mu_t)z_x^2 - (\theta_t\lambda + \theta\lambda_t)w_x^2\\ \mu_x\psi_1 - \lambda_x\phi_1 &= \theta\mu\mu_x z_x^2 - \theta\lambda\lambda_x w_x^2\\ \mu\psi_{1x} - \lambda\phi_{1x} &= \mu(\theta_x\mu + \theta\mu_x)z_x^2 - \lambda(\theta_x\lambda + \theta\lambda_x)w_x^2.\end{aligned}$$

From the previous remarks we notice that the factors of z_x^2 and w_x^2 appearing on the right hand sides of the last three equations are polynomials of degree at least 1 in z, w, z_x, w_x. Applying the Sobolev embedding theorem for these factors and then taking the supremum over $[0, t]$ we have

$$\begin{aligned}&\int_0^t\int_0^\ell (\psi_{1t} - \phi_{1t}) + (\mu_x\psi_1 - \lambda_x\phi_1) + (\mu\psi_{1x} - \lambda\phi_{1x})\,\mathrm{d}\tau\,\mathrm{d}x\\ &\qquad \le C_\delta \sup_{\tau\in[0,t]} \|V(\tau)\|_{H^2} \int_0^t \|V_x(\tau)\|_{L^2}^2\,\mathrm{d}\tau \qquad (6.3.22)\end{aligned}$$

The last term in M_1 is more delicate since it contains second order terms. Indeed, we have

$$\begin{aligned}\psi_{1Z}G_1 - \phi_{1W}F_1 &= 2\theta\mu z_x G_1 - 2\theta\lambda w_x F_1\\ &= 2\theta\mu z_x\left(-\mu_x z_x - \frac{\beta}{2}(z_x - w_x)\right) - 2\theta\lambda w_x\left(-\lambda_x w_x + \frac{\beta}{2}(z_x - w_x)\right)\\ &= -\frac{\theta_c C_e\beta}{4}(z_x - w_x)^2 + R_3 \qquad (6.3.23)\end{aligned}$$

where $\theta_c > 0$ is the constant term of θ. Here R_3 are terms of degree at least 3 that contain either z_x^2, w_x^2, or $w_x z_x$. Hence

$$\begin{aligned}\int_0^t\int_0^\ell \psi_{1Z}G_1 - \phi_{1W}F_1\,\mathrm{d}\tau\,\mathrm{d}x &\le -\tilde{C}\int_0^t \|u_x(\tau)\|_{L^2}^2\,\mathrm{d}\tau \qquad (6.3.24)\\ &\quad + C_\delta \sup_{\tau\in[0,t]} \|V(\tau)\|_{H^2}\int_0^t \|V_x(\tau)\|_{L^2}^2\,\mathrm{d}\tau\end{aligned}$$

where $\tilde{C} = \frac{\theta_c C_e\beta}{4} > 0$, if $\beta > 0$, independent of δ. Adding (6.3.22) and (6.3.24) we arrive at

$$\begin{aligned}\int_0^t\int_0^\ell M_1(\psi_1, \phi_1)\,\mathrm{d}\tau\,\mathrm{d}x &\le -\tilde{C}\int_0^t \|u_x(\tau)\|_{L^2}^2\,\mathrm{d}\tau \qquad (6.3.25)\\ &\quad + C_\delta \sup_{\tau\in[0,t]} \|V(\tau)\|_{H^2}\int_0^t \|V_x(\tau)\|_{L^2}^2\,\mathrm{d}\tau.\end{aligned}$$

Step 3. Let us combine the estimates obtained from Step 1 and Step 2. Choosing $\vartheta > 0$ small enough so that $\tilde{C} - C_\delta \vartheta > 0$ we have

$$\begin{aligned} \|V_x(t)\|_{L^2}^2 + \int_0^t \|u_x(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \ &\le \ C_\delta \|V(t)\|_{L^2}^2 + C_\delta \|V(0)\|_{H^1}^2 \\ &+ C_\delta \int_0^t \|u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau + C_\delta \sup_{\tau \in [0,t]} \|V(\tau)\|_{H^2} \int_0^t \|V_x(\tau)\|_{L^2}^2 + \|u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau. \end{aligned} \tag{6.3.26}$$

We can use Lemma 6.3.1 to bound the first and third terms on the right hand side of (6.3.26) from above. Consequently, (6.3.10) follows from (6.3.26), (6.3.4) and (6.3.2). □

To complete the estimate for the energy functional N_1 we need the following additional estimate.

Lemma 6.3.4. *There exist $\delta > 0$ and $C_\delta > 0$ such that for any solution $(A, u, h_0, h_\ell) \in X_T$ satisfying $N_2^2(T) \le \delta$ we have*

$$\begin{aligned} \int_0^t \|(A^{\frac14})_x(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \ &\le \ C_\delta N_1^2(0) \\ &+ C_\delta \sup_{\tau \in [0,t]} (\|u(\tau)\|_{H^2} + \|A^{\frac14}(\tau) - A_e^{\frac14}\|_{H^2}) \int_0^t \|u(\tau)\|_{H^1}^2 + \|(A^{\frac14})_x(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \end{aligned} \tag{6.3.27}$$

for all $t \in [0, T]$.

Proof. The proof of the lemma is basically the same as the proof of Lemma 6.3.3 and the main difference is the particular choice of the entropy appearing in (6.2.27). In the current situation we consider the entropy $\tilde{\eta}_1 = \tilde{\psi}_1 - \tilde{\phi}_1$ with corresponding entropy flux $\tilde{q}_1 = \mu \tilde{\psi}_1 - \lambda \tilde{\phi}_1$ where

$$\begin{aligned} \tilde{\phi}_1(t, x, W) \quad &= \quad \frac{\theta(w(t,x), z(t,x))}{\lambda(t,x)} \left(\lambda(t,x) W - \frac{\beta}{2}(z(t,x) - w(t,x)) \right)^2 \\ \tilde{\psi}_1(t, x, Z) \quad &= \quad \frac{\theta(w(t,x), z(t,x))}{\mu(t,x)} \left(\mu(t,x) Z + \frac{\beta}{2}(z(t,x) - w(t,x)) \right)^2 \end{aligned}$$

Let $F_0 = \frac{\beta}{2}(z - w)$. Using Young's inequality

$$\begin{aligned} \tilde{\eta}_1 \quad &= \quad \theta \mu^{-1} \left(\mu^2 z_x^2 + 2\mu F_0 z_x + F_0^2 \right) - \theta \lambda^{-1} (\lambda^2 w_x^2 - 2\lambda F_0 w_x + F_0^2) \\ &= \quad \theta \left(\mu z_x^2 - \lambda w_x^2 + 2F_0 z_x + 2F_0 w_x + (\mu^{-1} - \lambda^{-1}) F_0^2 \right) \\ &\ge \quad C_\delta (w_x^2 + z_x^2) - C_\delta (\epsilon z_x^2 + 2\epsilon^{-1} F_0^2 + \epsilon w_x^2) + C_\delta F_0^2 \\ &\ge \quad C_\delta (w_x^2 + z_x^2) - C_\delta (w^2 + z^2) \end{aligned}$$

for some $\epsilon \in (0, 1)$ small enough. Similarly, $\tilde{\eta}_1 \le C_\delta (w_x^2 + z_x^2 + w^2 + z^2)$. Thus

$$\int_0^\ell \tilde{\eta}_1(t, x) - \tilde{\eta}_1(0, x) \,\mathrm{d}x \ge C_\delta (\|V_x(t)\|_{L^2}^2 - \|V(t)\|_{L^2}^2 - \|V(0)\|_{H^1}^2). \tag{6.3.28}$$

From (6.2.12), (6.3.18), (6.3.19) and (6.3.20) and according to the statement following (6.3.20) we immediately get

$$\begin{aligned} \int_0^t \tilde{q}_1(\tau, \ell) - \tilde{q}_1(\tau, 0) \,\mathrm{d}\tau &= - \int_0^t \theta(w(\tau, 0), z(\tau, 0)) (z_\tau^2(\tau, 0) - w_\tau^2(\tau, 0)) \,\mathrm{d}\tau \\ \ge - C_\delta \Bigg(\vartheta \|u_x(t)\|_{L^2}^2 &+ C_\vartheta \|u(t)\|_{L^2}^2 + \|u(0)\|_{H^1}^2 \\ &+ \sup_{\tau \in [0,t]} \|u(\tau)\|_{H^1} \int_0^t \|u(\tau)\|_{H^1}^2 \,\mathrm{d}\tau \Bigg). \end{aligned} \tag{6.3.29}$$

The remaining task is to obtain estimates from above. As in the previous lemma, we need to look carefully at each pair appearing in $\tilde{M}_1$ since some of them contain terms of degree only 2. For the rest of the proof R_i will denote terms that are degree at least 3 and contain at least two factors among $z-w, w_x, z_x$. Note that using (6.2.12) we have

$$z_t - w_t = -\frac{C_e}{4}(z_x + w_x) - \beta(z-w) + \hat{R}_0 \tag{6.3.30}$$

where $\hat{R}_0 = c_1 w w_x + c_2 z w_x + c_3 w z_x + c_4 z z_x$ for some constants c_i. Thus have

$$\begin{aligned}
\tilde{\psi}_{1t} - \tilde{\phi}_{1t} &= (\mu z_x + F_0)^2 \frac{\mu\theta_t - \theta\mu_t}{\mu^2} + \frac{2\theta}{\mu}(\mu z_x + F_0)(\mu_t z_x + F_{0t}) \\
&- (\lambda w_x - F_0)^2 \frac{\lambda\theta_t - \theta\lambda_t}{\lambda^2} - \frac{2\theta}{\lambda}(\lambda w_x - F_0)(\lambda_t z_x - F_{0t}) \\
&= 2\theta\left(z_x + w_x + \left(\frac{1}{\mu} - \frac{1}{\lambda}\right)F_0\right)F_{0t} + \frac{1}{\lambda^2\mu^2}R_4 \\
&= -\frac{C_e\beta\theta}{4}(z_x + w_x)^2 - \beta^2\theta(z_x + w_x)(z-w) \\
&- \frac{C_e\beta^2\theta}{8}\left(\frac{1}{\mu} - \frac{1}{\lambda}\right)(z_x + w_x)(z-w) - \frac{\beta^3\theta}{2}\left(\frac{1}{\mu} - \frac{1}{\lambda}\right)(z-w)^2 \\
&+ \frac{1}{\lambda^2\mu^2}R_5
\end{aligned}$$

By Young's inequality and the Sobolev embedding theorem we have

$$\tilde{\psi}_{1t} - \tilde{\phi}_{1t} \le \left(-\frac{\theta_c C_e \beta}{4} + C_\delta \epsilon\right)(z_x + w_x)^2 + C_{\delta,\epsilon}(z-w)^2 + \frac{1}{\lambda^2\mu^2}R_5. \tag{6.3.31}$$

For the second pair we can see that

$$\mu_x \tilde{\psi}_1 - \lambda_x \tilde{\phi}_1 = \frac{\theta}{\mu}\mu_x(\mu z_x + F_0)^2 - \frac{\theta}{\lambda}\lambda_x(\lambda w_x - F_0)^2 = \frac{1}{\lambda\mu}R_6. \tag{6.3.32}$$

The third pair can be computed as in the first pair and we get

$$\begin{aligned}
\mu\tilde{\psi}_{1x} - \lambda\tilde{\phi}_{1x} &= (\mu z_x + F_0)^2 \frac{\mu\theta_x - \theta\mu_x}{\mu} + 2\theta(\mu z_x + F_0)(\mu_x z_x + F_{0x}) \\
&- (\lambda w_x - F_0)^2 \frac{\lambda\theta_x - \theta\lambda_x}{\lambda} - 2\theta(\lambda w_x - F_0)(\lambda_x w_x - F_{0x}) \\
&= 2\theta\left(\left(\mu z_x + \frac{\beta}{2}(z-w)\right) + \left(\lambda w_x - \frac{\beta}{2}(z-w)\right)\right)\frac{\beta}{2}(z_x - w_x) \\
&+ \frac{1}{\lambda\mu}R_7 \\
&= \frac{\theta_c C_e \beta}{4}(z_x - w_x)^2 + \frac{1}{\lambda\mu}R_8
\end{aligned} \tag{6.3.33}$$

Finally, for the last pair we use (6.2.19) and (6.2.20) to obtain

$$\begin{aligned}
\tilde{\psi}_{1Z}G_1 - \tilde{\phi}_{1W}F_1 &= \frac{2\theta}{\mu}(\mu z_x + F_0)\mu G_1 - \frac{2\theta}{\lambda}(\lambda w_x - F_0)\lambda F_1 \\
&= 2\theta\left(\frac{C_e}{4}z_x + \frac{\beta}{2}(z-w) + \hat{R}_1\right)\left(-\mu_x z_x - \frac{\beta}{2}(z_x - w_x)\right) \\
&- 2\theta\left(-\frac{C_e}{4}w_x - \frac{\beta}{2}(z-w) + \hat{R}_2\right)\left(-\lambda_x w_x + \frac{\beta}{2}(z_x - w_x)\right) \\
&= -\frac{\theta_c C_e \beta}{4}(z_x - w_x)^2 + R_9.
\end{aligned} \tag{6.3.34}$$

where $\hat{R}_1$, $\hat{R}_2$ are of degree 2 and have the same form as $\hat{R}_0$.

Taking the sum of (6.3.31)–(6.3.34), choosing $\epsilon > 0$ small enough so that $\tilde{C}_1 = \frac{\theta_c C_e \beta}{4} - C_\delta \epsilon > 0$, using the Sobolev embedding theorem and the transformations (6.2.11) we obtain

$$\begin{aligned}\int_0^t \int_0^\ell \tilde{M}_1(\tilde{\psi}_1, \tilde{\phi}_1)\,\mathrm{d}x\,\mathrm{d}\tau \quad \leq \quad & -\tilde{C}_1 \int_0^t \|(A^{\frac{1}{4}})_x(\tau)\|_{L^2}^2\,\mathrm{d}\tau \\ & + C_\delta \sup_{\tau\in[0,t]} \|V(\tau)\|_{H^2} \int_0^t \|V_x(\tau)\|_{L^2}^2 + \|u(\tau)\|_{L^2}^2\,\mathrm{d}\tau.\end{aligned} \tag{6.3.35}$$

Now it can be seen that (6.3.27) follows from (6.3.28), (6.3.29), (6.3.35), Lemma 6.3.1, and from the equivalence of norms in (6.3.2). □

Remark 6.3.5. It is worth pointing out that by an appropriate modification of the entropy-entropy flux pair we saw in the proof of Lemma 6.3.4 that the term u_x^2, or equivalently $(z_x - w_x)^2$, which appears on the right hand side of (6.2.27) cancels when adding (6.3.33) and (6.3.34). Moreover it was replaced by a term involving $(A^{\frac{1}{4}})_x^2$, or equivalently $(z_x + w_x)^2$. The appearance of $(A^{\frac{1}{4}})_x^2$ is precisely what we want in order to prove Lemma 6.3.4. This observation will also be used in the following two lemmas.

6.3.3 **Second Order Estimates**

Before we proceed in obtaining estimates for the second spatial derivatives of the state variables, we will derive some identities from the two PDEs in the diagonal system (6.2.12). In the following, we concentrate on the linear terms and state only the properties of the higher degree terms. Differentiating the first equation in (6.2.12) with respect to t we get

$$\lambda w_{xt} = -w_{tt} - \lambda_t w_x + \frac{\beta}{2}(z_t - w_t). \tag{6.3.36}$$

However, we note from (6.2.17) for $k = 1$ that

$$\lambda w_{tx} = -\lambda^2 w_{xx} + \lambda F_1. \tag{6.3.37}$$

Thus, according to (6.3.36), (6.3.37) and (6.2.19) we have

$$w_{tt} = \lambda^2 w_{xx} + \frac{\beta}{2}(z_t - w_t) - \frac{\beta\lambda}{2}(z_x - w_x) + \lambda\lambda_x w_x - \lambda_t w_x. \tag{6.3.38}$$

In a similar way we have the equation for z_{tt}

$$z_{tt} = \mu^2 z_{xx} - \frac{\beta}{2}(z_t - w_t) + \frac{\beta\mu}{2}(z_x - w_x) + \mu\mu_x z_x - \mu_t z_x. \tag{6.3.39}$$

Taking the derivative with respect to x of both sides of (6.3.30) we have

$$z_{tx} - w_{tx} = -\frac{C_e}{4}(z_{xx} + w_{xx}) - \beta(z_x - w_x) + \hat{R}_3 \tag{6.3.40}$$

where $\hat{R}_3 = \sum_{j+k=2} c_{jk}(\partial_x^j w)(\partial_x^k z)$ for some constants c_{jk}. Subtracting (6.3.38) from (6.3.39) and using (6.3.30) we have

$$z_{tt} - w_{tt} = \frac{C_e^2}{16}(z_{xx} - w_{xx}) + \frac{\beta C_e}{2} z_x + \beta^2(z - w) + \hat{R}_4 \tag{6.3.41}$$

where $\hat{R}_4$ are terms of degree at least 2 and contain at least one factor among $z-w, w_x, z_x, z_{xx}, w_{xx}$, however, each term has at most one factor among w_{xx}, z_{xx}.

Lemma 6.3.6 (Second Order Estimate)**.** *There exist $\delta > 0$ and $C_\delta > 0$ such that for any solution $(A, u, h_0, h_\ell) \in X_T$ satisfying $N_2^2(T) \le \delta$ it holds that*

$$\begin{aligned} \|u_{xx}(t)\|_{L^2}^2 + \|(A^{\frac{1}{4}})_{xx}(t)\|_{L^2}^2 + \int_0^t \|u_{xx}(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \ &\le\ C_\delta N_2^2(0) \qquad (6.3.42) \\ + C_\delta \sup_{\tau\in[0,t]} (\|u(\tau)\|_{H^2} + \|A^{\frac{1}{4}}(\tau) - A_e^{\frac{1}{4}}\|_{H^2}) \int_0^t \|u(\tau)\|_{H^2}^2 + \|(A^{\frac{1}{4}})_x(\tau)\|_{H^1}^2 \,\mathrm{d}\tau & \end{aligned}$$

for all $t \in [0, T]$.

Proof. Again we will proceed in the same manner, now with the entropy $\eta_2 = \psi_2 - \phi_2$ where

$$\begin{aligned} \psi_2(t,x,Z) &= \frac{\theta(w,z)}{\mu}\left(\mu^2 Z - \frac{\beta}{2}(z_t - w_t) + \frac{\beta\mu}{2}(z_x - w_x) + \mu\mu_x z_x - \mu_t z_x\right)^2 \\ \phi_2(t,x,W) &= \frac{\theta(w,z)}{\lambda}\left(\lambda^2 W + \frac{\beta}{2}(z_t - w_t) - \frac{\beta\lambda}{2}(z_x - w_x) + \lambda\lambda_x w_x - \lambda_t w_x\right)^2. \end{aligned}$$

We estimate (6.2.27) with these particular functions and as before we divide the procedure in three steps, namely, the derivation of estimates of the left hand side of (6.2.27) from below, estimates of the right hand side of (6.2.27) from above and finally to combine the two.

Step 1. Estimate from below. For brevity let us set

$$\tilde{N} = -\frac{\beta}{2}(z_t - w_t) + \frac{\beta\mu}{2}(z_x - w_x) + \mu\mu_x z_x - \mu_t z_x \qquad (6.3.43)$$

$$\tilde{P} = \frac{\beta}{2}(z_t - w_t) - \frac{\beta\lambda}{2}(z_x - w_x) + \lambda\lambda_x w_x - \lambda_t w_x. \qquad (6.3.44)$$

Using Young's inequality we have, for $\delta > 0$ small enough,

$$\begin{aligned} \psi_2(t,x,z_{xx}(t,x)) &= \theta\mu^{-1}(\mu^4 z_{xx}^2 + 2\mu^2 z_{xx}\tilde{N} + \tilde{N}^2) \\ &\ge \theta\mu^3 z_{xx}^2 - \theta\mu(\epsilon z_{xx}^2 + C_\epsilon \tilde{N}^2) + \theta\mu^{-1}\tilde{N}^2 \\ &= (\theta\mu^3 - \theta\mu\epsilon) z_{xx}^2 - (\theta\mu C_\epsilon - \theta\mu^{-1})\tilde{N}^2. \end{aligned}$$

for every $\epsilon > 0$, we removed the arguments (t, x) on the right hand sides for simplicity. Using the definition of $\tilde{N}_2$ and replacing the term $z_t - w_t$ by the right hand side of (6.3.30) we can see that

$$\tilde{N}(t,x)^2 \le C_\delta(w(t,x)^2 + z(t,x)^2 + w_x(t,x)^2 + z_x(t,x)^2).$$

This follows immediately from the fact that $\tilde{N}$ consists of terms that are at least degree 1 in w, z, w_x, z_x and so $\tilde{N}^2$ will have at least degree 2 terms in these variables. Then we retain two factors and take the supremum of the rest, employing the Sobolev embedding theorem to estimate the supremum and finally use the assumption that $N_2^2(T) \le \delta$, for $\delta > 0$ small enough.

Now, choosing $\epsilon > 0$ sufficiently small we have

$$\psi_2(t,x,z_{xx}(t,x)) \ge C_\delta z_{xx}^2(t,x) - C_\delta(|V(t,x)|^2 + |V_x(t,x)|^2). \qquad (6.3.45)$$

for all $(t,x) \in [0,T] \times [0,\ell]$. Recall that $V = (w,z)$. Similarly, we have the upper bound

$$\psi_2(t,x,z_{xx}(t,x)) \leq C_\delta z_{xx}^2(t,x) + C_\delta(|V(t,x)|^2 + |V_x(t,x)|^2). \tag{6.3.46}$$

for all $(t,x) \in [0,T] \times [0,\ell]$. Doing the same process with ϕ_2 and recalling that λ is negative for small enough $\delta > 0$ we have

$$-C_\delta w_{xx}^2 - C_\delta(|V|^2 + |V_x|^2) \leq \phi_2 \leq -C_\delta w_{xx}^2 + C_\delta(|V|^2 + |V_x|^2) \tag{6.3.47}$$

From (6.3.45)–(6.3.47) we have

$$\int_0^\ell \eta_2(t,x) - \eta_2(0,x)\,\mathrm{d}x \geq C_\delta(\|V_{xx}(\tau)\|_{L^2}^2 - \|V(0)\|_{H^2}^2). \tag{6.3.48}$$

According to (6.3.38) and (6.3.39) we have

$$-\int_0^t q_2(\tau,0)\,\mathrm{d}\tau = -\int_0^t \theta(w(\tau,0),z(\tau,0))(z_{\tau\tau}^2(\tau,0) - w_{\tau\tau}^2(\tau,0))\,\mathrm{d}\tau. \tag{6.3.49}$$

Let us use the boundary conditions to rewrite the integrand in terms of w, z and their first derivatives with respect to x. For convenience, the functions in the following discussions are to be evaluated at $(t,0)$ or t, or with other variables representing time, where they make sense. First, we notice from (6.3.13) that

$$z_t + w_t = S(h_0)\theta(w,z)(z-w) \tag{6.3.50}$$

where $S(h_0) = -\zeta_0'(h_0)(h_0 - h_{0e}) - \zeta_0(h_0)$. Let

$$p_1(w,z,w_x,z_x) = -\frac{C_e}{4}(z_x + w_x) - \beta(z-w) + \hat{R}_0 \tag{6.3.51}$$

and from (6.3.30) we have $z_t - w_t = p_1(w,z,w_x,z_x)$. Using (6.3.50) in (6.3.17) yields

$$\begin{aligned} h_0'' &= -S(h_0)\theta_1(w,z)\theta(w,z)(z-w)^3 - \theta(w,z)p_1(w,z,w_x,z_x) \\ &=: p_2(w,z,w_x,z_x). \end{aligned} \tag{6.3.52}$$

Taking the derivative of both sides of (6.3.13) gives us

$$\begin{aligned} z_{tt} + w_{tt} &= [\zeta_0''(h_0)(h_0 - h_{0e}) + 2\zeta_0'(h_0)](h_0')^2 + [\zeta_0'(h_0)(h_0 - h_{0e}) + \zeta_0(h_0)]h_0'' \\ &=: S_1(h_0)(h_0')^2 + S_2(h_0)h_0''. \end{aligned} \tag{6.3.53}$$

Thus, (6.3.52) implies that

$$\begin{aligned} z_{tt} + w_{tt} &= S_1(h_0)\theta(w,z)^2(z-w)^2 + S_2(h_0)p_2(w,z,w_x,z_x) \\ &=: p_3(w,z,w_x,z_x). \end{aligned} \tag{6.3.54}$$

We also take the derivative of (6.3.17) and apply (6.3.50) and (6.3.54) to obtain

$$\begin{aligned} h_0^{(3)} &= -\theta_2(w,z)(z_t + w_t)^2(z-w) - \theta_1(w,z)(z_{tt} + w_{tt})(z-w) \\ &\quad -2\theta_1(w,z)(z_t + w_t)(z_t - w_t) - \theta(w,z)(z_{tt} - w_{tt}) \\ &=: p_4(w,z,w_x,z_x) - \theta(w,z)(z_{tt} - w_{tt}) \end{aligned} \tag{6.3.55}$$

where $\theta_2(w,z) = \frac{12}{A_T}(w+z+2C_e)^2$ and

$$\begin{aligned} p_4(w,z,w_x,z_x) &= -S(h_0)^2\theta_2(w,z)\theta(w,z)^2(z-w)^3 \\ &\quad - \theta_1(w,z)(z-w)p_3(w,z,w_x,z_x) \\ &\quad - 2\theta_1(w,z)S(h_0)\theta(w,z)(z-w)p_1(w,z,w_x,z_x). \end{aligned} \tag{6.3.56}$$

Note that p_1, p_2 and p_3 contain terms that are degree at least 1 and have at least one factor among $z-w, w_x, z_x$ while p_4 has terms with degree at least 2 that contain at least two factors among $z-w, w_x, z_x$. Moreover, we note that each S_i is bounded as long as its arguments stay on a bounded subset of $(0,\infty)$, which is the case due to the assumption that $|h_0(t)-h_{0e}|^2 \leq \delta$ for small enough $\delta > 0$.

From (6.3.53), (6.3.54) and (6.3.55) we can now rewrite (6.3.49) as

$$\begin{aligned} -\int_0^t q_2(\tau,0)\,\mathrm{d}\tau &= \int_0^t (h_0^{(3)} - p_4(w,z,w_x,z_x))(S_1(h_0)(h_0')^2 + S_2(h_0)h_0'')\,\mathrm{d}\tau \\ &= \int_0^t S_1(h_0)(h_0')^2 h_0^{(3)}\,\mathrm{d}\tau + \frac{1}{2}\int_0^t S_2(h_0)\frac{\mathrm{d}}{\mathrm{d}t}|h_0''|^2\,\mathrm{d}\tau \\ &\quad - \int_0^t p_4(w,z,w_x,z_x)p_3(w,z,w_x,z_x)\,\mathrm{d}\tau \\ &=: J_1 + J_2 + J_3. \end{aligned}$$

Integrating by parts and using (6.3.52)

$$\begin{aligned} J_1 &= S_1(h_0(\tau))h_0'(\tau)^2 h_0''(\tau)\Big|_{\tau=0}^{\tau=t} - \int_0^t S_1'(h_0)(h_0')^3 h_0'' + 2S_1(h_0)h_0'(h_0'')^2\,\mathrm{d}\tau \\ &= S_1(h_0(\tau))\theta(w,z)^2(z-w)^2 p_2(w,z,w_x,z_x)\Big|_{\tau=0}^{\tau=t} \\ &\quad + \int_0^t S_1'(h_0)\theta(w,z)^3(z-w)^3 p_2 + 2S_1(h_0)\theta(w,z)(z-w)p_2^2\,\mathrm{d}\tau. \end{aligned}$$

Applying Proposition 6.3.2 to the terms having either $z_x(\tau,0)$ or $w_x(\tau,0)$ appearing in the first term of the above last expression and using the Sobolev embedding theorem for the rest we obtain the inequality

$$\begin{aligned} J_1 &\geq -C_\delta\vartheta\|V_{xx}(t)\|_{L^2}^2 - C_{\delta,\vartheta}\|V(t)\|_{H^1}^2 - C_\delta\|V(0)\|_{H^2}^2 \\ &\quad - C_\delta \sup_{\tau\in[0,t]}\|V(\tau)\|_{H^2}\int_0^t \|V_x(\tau)\|_{H^1}^2 + \|u(\tau)\|_{L^2}^2\,\mathrm{d}\tau \end{aligned}$$

In the above computations it is important to note the properties of p_2.

In a similar way we can integrate by parts and use the same techniques to obtain

$$\begin{aligned} J_2 &\geq -C_\delta\vartheta\|V_{xx}(t)\|_{L^2}^2 - C_{\delta,\vartheta}\|V(t)\|_{H^1}^2 - C_\delta\|V(0)\|_{H^2}^2 \\ &\quad - C_\delta \sup_{\tau\in[0,t]}\|V(\tau)\|_{H^2}\int_0^t \|V_x(\tau)\|_{H^1}^2 + \|u(\tau)\|_{L^2}^2\,\mathrm{d}\tau. \end{aligned}$$

Furthermore, invoking the properties of p_3 and p_4 we have

$$J_3 \geq -C_\delta \sup_{\tau\in[0,t]}\|V(\tau)\|_{H^2}\int_0^t \|V_x(\tau)\|_{H^1}^2 + \|u(\tau)\|_{L^2}^2\,\mathrm{d}\tau.$$

Adding the lower bounds for J_1, J_2 and J_3 gives us a lower bound of $-\int_0^t q_2(\tau,0)\,\mathrm{d}\tau$, which has essentially the form of the lower bound for J_1. We can repeat the same process for $\int_0^t q_2(\tau,\ell)\,\mathrm{d}\tau$ and obtain a lower bound having the same form as stated above. With these we finally obtain

$$\begin{aligned}\int_0^t q_2(\tau,\ell)-q_2(\tau,0)\,\mathrm{d}\tau \geq &- C_\delta\vartheta\|V_{xx}(t)\|_{L^2}^2 - C_{\delta,\vartheta}\|V(t)\|_{H^1}^2 - C_\delta\|V(0)\|_{H^2}^2\\ &- C_\delta \sup_{\tau\in[0,t]}\|V(\tau)\|_{H^2}\int_0^t \|V_x(\tau)\|_{H^1}^2+\|u(\tau)\|_{L^2}^2\,\mathrm{d}\tau. \qquad (6.3.57)\end{aligned}$$

Inequalities (6.3.48) and (6.3.57) give us the desired estimate from below.

Step 2. Estimate from above. In this step R_i will denote terms of degree at least 3 containing at least two factors among $z-w, w_x, z_x, z_{xx}, w_{xx}$ and containing at most two among z_{xx}, w_{xx}. First, we have

$$\begin{aligned}\psi_{2t}-\phi_{2t} =& (\mu^2 z_{xx}+\tilde{N})^2\frac{\mu\theta_t-\theta\mu_t}{\mu^2}+\frac{2\theta}{\mu}\bigg(\mu^2 z_{xx}-\frac{\beta}{2}(z_t-w_t)+\frac{\beta\mu}{2}(z_x-w_x)\\ &+\mu\mu_x z_x-\mu_t z_x\bigg)\bigg(2\mu\mu_t z_{xx}-\frac{\beta}{2}(z_{tt}-w_{tt})+\frac{\beta}{2}\mu_t(z_x-w_x)+\frac{\beta\mu}{2}(z_{tx}-w_{tx})\\ &+(\mu\mu_x z_x-\mu_t z_x)_t\bigg)-(\lambda^2 w_{xx}+\tilde{P})^2\frac{\lambda\theta_t-\theta\lambda_t}{\lambda^2}-\frac{2\theta}{\lambda}\bigg(\lambda^2 w_{xx}+\frac{\beta}{2}(z_t-w_t)\\ &-\frac{\beta\lambda}{2}(z_x-w_x)+\lambda\lambda_x w_x-\lambda_t w_x\bigg)\bigg(2\lambda\lambda_t w_{xx}+\frac{\beta}{2}(z_{tt}-w_{tt})-\frac{\beta}{2}\lambda_t(z_x-w_x)\\ &-\frac{\beta\lambda}{2}(z_{tx}-w_{tx})+(\lambda\lambda_x w_x-\lambda_t w_x)_t\bigg)\\ =& -\theta\beta(\mu z_{xx}+\lambda w_{xx})(z_{tt}-w_{tt})+\frac{\theta\beta^2}{2}\left(\frac{1}{\mu}-\frac{1}{\lambda}\right)(z_t-w_t)(z_{tt}-w_{tt})\\ &+\theta\beta(\mu^2 z_{xx}+\lambda^2 w_{xx})(z_{tx}-w_{tx})+\frac{\theta\beta^2}{2}(\mu-\lambda)(z_x-w_x)(z_{tx}-w_{tx})+\frac{R_1}{\lambda^2\mu^2}\\ =:& I_1+I_2+I_3+I_4+\frac{R_1}{\lambda^2\mu^2}. \qquad (6.3.58)\end{aligned}$$

Consider each I_i. According to (6.3.41) and Young's inequality we have

$$\begin{aligned}I_1 &= -\frac{\theta_c C_e\beta}{4}(z_{xx}-w_{xx})\left(\frac{C_e^2}{16}(z_{xx}-w_{xx})+\frac{\beta C_e}{2}z_x+\beta^2(z-w)\right)+R_2\\ &\leq \left(-\frac{\theta_c\beta C_e^3}{64}+C_\delta\epsilon\right)(z_{xx}-w_{xx})^2+C_{\delta,\epsilon}(z_x^2+(z-w)^2)+R_2 \qquad (6.3.59)\end{aligned}$$

Also, from (6.3.30) and (6.3.41)

$$\begin{aligned}I_2 &= \frac{\theta_c\beta^2}{2}\left(\frac{1}{\mu}-\frac{1}{\lambda}\right)\left(-\frac{C_e}{4}(z_x+w_x)-\beta(z-w)\right)\bigg(\frac{C_e^2}{16}(z_{xx}-w_{xx})\\ &\quad+\frac{\beta C_e}{2}z_x+\beta^2(z-w)\bigg)+R_3\\ &\leq C_\delta\epsilon(z_{xx}-w_{xx})^2+C_{\delta,\epsilon}(z_x^2+w_x^2+(z-w)^2)+R_3. \qquad (6.3.60)\end{aligned}$$

From (6.3.40) we see that

$$\begin{aligned}I_3 &= \frac{\theta_c\beta C_e^2}{16}(z_{xx}+w_{xx})\left(-\frac{C_e}{4}(z_{xx}+w_{xx})-\beta(z_x-w_x)\right)+R_4\\ &= -\frac{\theta_c\beta C_e^3}{64}(z_{xx}+w_{xx})^2-\frac{\theta_c\beta^2 C_e^2}{16}(z_{xx}+w_{xx})(z_x-w_x)+R_4 \qquad (6.3.61)\end{aligned}$$

and

$$\begin{aligned} I_4 &= \frac{\theta_c \beta^2 C_e}{4}(z_x - w_x)\left(-\frac{C_e}{4}(z_{xx}+w_{xx}) - \beta(z_x - w_x)\right) + R_5 \\ &= -\frac{\theta_c \beta^2 C_e^2}{16}(z_{xx}+w_{xx})(z_x - w_x) - \frac{\theta_c \beta^3 C_e}{4}(z_x - w_x)^2 + R_5 \end{aligned} \tag{6.3.62}$$

Adding (6.3.59)–(6.3.62) we have

$$\begin{aligned} \psi_{2t} - \phi_{2t} \leq\ & \left(-\frac{\theta_c \beta C_e^3}{64} + C_\delta \epsilon\right)(z_{xx} - w_{xx})^2 - \frac{\theta_c \beta C_e^3}{64}(z_{xx}+w_{xx})^2 \\ & - \frac{\theta_c \beta^2 C_e^2}{8}(z_{xx}+w_{xx})(z_x - w_x) + C_{\delta,\epsilon}(z_x^2 + w_x^2 + (z-w)^2) \\ & + C_\delta (z_x - w_x)^2 + \frac{R_7}{\lambda^2 \mu^2} \end{aligned} \tag{6.3.63}$$

It can be checked that

$$\mu_x \psi_2 - \lambda_x \phi_2 = \frac{1}{\lambda \mu} R_8. \tag{6.3.64}$$

Similarly for the third pair we have

$$\begin{aligned} &\mu \psi_{2x} - \lambda \phi_{2x} = (\mu^2 z_{xx} + \tilde{N})^2 \frac{\mu \theta_x - \theta \mu_x}{\mu} + 2\theta\Bigg(\mu^2 z_{xx} - \frac{\beta}{2}(z_t - w_t) \\ &+ \frac{\beta\mu}{2}(z_x - w_x) + \mu\mu_x z_x - \mu_t z_x\Bigg)\Bigg(2\mu\mu_x z_{xx} - \frac{\beta}{2}(z_{tx} - w_{tx}) + \frac{\beta}{2}\mu_x(z_x - w_x) \\ &+ \frac{\beta\mu}{2}(z_{xx} - w_{xx}) + (\mu\mu_x z_x - \mu_t z_x)_x\Bigg) - (\lambda^2 w_{xx} + \tilde{P})^2 \frac{\lambda\theta_x - \theta\lambda_x}{\lambda} \\ &- 2\theta\Bigg(\lambda^2 w_{xx} + \frac{\beta}{2}(z_t - w_t) - \frac{\beta\lambda}{2}(z_x - w_x) + \lambda\lambda_x w_x - \lambda_t w_x\Bigg)\Bigg(2\lambda\lambda_x w_{xx} \\ &+ \frac{\beta}{2}(z_{tx} - w_{tx}) - \frac{\beta}{2}\lambda_x(z_x - w_x) - \frac{\beta\lambda}{2}(z_{xx} - w_{xx}) + (\lambda\lambda_x w_x - \lambda_t w_x)_x\Bigg) \\ &= -\theta\beta(\mu^2 z_{xx} + \lambda^2 w_{xx})(z_{tx} - w_{tx}) - \frac{\theta\beta^2}{2}(\mu - \lambda)(z_x - w_x)(z_{tx} - w_{tx}) \\ &+ \theta\beta(\mu^3 z_{xx} + \lambda^3 w_{xx})(z_{xx} - w_{xx}) - \frac{\theta\beta^2}{2}(\mu - \lambda)(z_t - w_t)(z_{xx} - w_{xx}) \\ &+ \frac{\theta\beta^2}{2}(\mu^2 - \lambda^2)(z_x - w_x)(z_{xx} - w_{xx}) + \frac{R_9}{\lambda^2\mu^2} \\ &=: I_5 + I_6 + I_7 + I_8 + I_9 + \frac{R_9}{\lambda^2\mu^2}. \end{aligned} \tag{6.3.65}$$

From (6.3.30), (6.3.40) and Young's inequality we have

$$\begin{aligned} I_5 &= \frac{-\theta_c \beta C_e^2}{16}(z_{xx}+w_{xx})\left(-\frac{C_e}{4}(z_{xx}+w_{xx}) - \beta(z_x - w_x)\right) + R_{10} \\ &= \frac{\theta_c \beta C_e^3}{64}(z_{xx}+w_{xx})^2 + \frac{\theta_c \beta^2 C_e^2}{16}(z_{xx}+w_{xx})(z_x - w_x) + R_{10} \end{aligned} \tag{6.3.66}$$

$$\begin{aligned}
I_6 &= -\frac{\theta_c\beta^2C_e}{4}(z_x-w_x)\left(-\frac{C_e}{4}(z_{xx}+w_{xx})-\beta(z_x-w_x)\right)+R_{11}\\
&= \frac{\theta_c\beta^2C_e^2}{16}(z_{xx}+w_{xx})(z_x-w_x)+\frac{\theta_c\beta^3C_e}{4}(z_x-w_x)^2+R_{11} &(6.3.67)\\
I_7 &= \frac{\theta_c\beta C_e^3}{64}(z_{xx}-w_{xx})^2+R_{12} &(6.3.68)\\
I_8 &= -\frac{\theta_c\beta^2C_e}{8}(z_t-w_t)(z_{xx}-w_{xx})+R_{13}\\
&= -\frac{\theta_c\beta^2C_e}{8}(z_{xx}-w_{xx})\left(-\frac{C_e}{4}(z_x+w_x)-\beta(z-w)\right)+R_{13}\\
&\leq C_\delta\epsilon(z_{xx}-w_{xx})^2+C_{\delta,\epsilon}((z_x+w_x)^2+(z-w)^2) &(6.3.69)\\
I_9 &= R_{14}. &(6.3.70)
\end{aligned}$$

The last equation is due to the fact that the terms in $\mu^2-\lambda^2$ are of degree at least 1. Therefore from (6.3.66)–(6.3.70) we have

$$\begin{aligned}
\psi_{2x}-\phi_{2x} \leq{}& \left(\frac{\theta_c\beta C_e^3}{64}+C_\delta\epsilon\right)(z_{xx}-w_{xx})^2+\frac{\theta_c\beta C_e^3}{64}(z_{xx}+w_{xx})^2 &(6.3.71)\\
&+\frac{\theta_c\beta^2C_e^2}{8}(z_{xx}+w_{xx})(z_x-w_x)+C_{\delta,\epsilon}((z_x+w_x)^2+(z-w)^2)\\
&+C_\delta(z_x-w_x)^2+R_{15}
\end{aligned}$$

Finally for the last pair in M_2 we use (6.2.21) and (6.2.22) to obtain

$$\begin{aligned}
\psi_{2Z}G_2-\phi_{2W}F_2 &= \frac{2\theta}{\mu}(\mu^2z_{xx}+\tilde{N})\mu^2\left(-\frac{\beta}{2}(z_{xx}-w_{xx})-2\mu_xz_x-\mu_{xx}z_x\right)\\
&\quad -\frac{2\theta}{\lambda}(\lambda^2w_{xx}+\tilde{P})\lambda^2\left(\frac{\beta}{2}(z_{xx}-w_{xx})-2\lambda_xw_x-\lambda_{xx}w_x\right)\\
&= \theta_c\beta(-\mu^3z_{xx}(z_{xx}-w_{xx})-\lambda^3w_{xx}(z_{xx}-w_{xx}))+R_{16}\\
&= -\frac{\theta_c\beta C_e^3}{64}(z_{xx}-w_{xx})^2+R_{17} &(6.3.72)\\
&=: I_{10}+R_{17}.
\end{aligned}$$

Adding (6.3.63), (6.3.64), (6.3.71), (6.3.72), choosing $\epsilon>0$ small enough so that $\tilde{C}_2=\frac{\theta_c\beta C_e^3}{64}-C_\delta\epsilon>0$, where the first term is independent of δ and ϵ, using the Sobolev embedding for the terms R_i and finally invoking (6.2.11) yields

$$\begin{aligned}
&\int_0^t\int_0^\ell M_2(\psi_2,\phi_2)\,\mathrm{d}x\,\mathrm{d}\tau\\
&\leq -\tilde{C}_2\int_0^t\|u_{xx}(\tau)\|_{L^2}^2\,\mathrm{d}\tau+C_\delta\left(\int_0^t\|u(\tau)\|_{H^1}^2\,\mathrm{d}\tau+\int_0^t\|(A^{\frac14})_x(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right.\\
&\qquad\left.+\sup_{\tau\in[0,t]}\|V(\tau)\|_{H^2}\int_0^t\|V_x(\tau)\|_{H^1}^2+\|u(\tau)\|_{L^2}^2\,\mathrm{d}\tau\right). \qquad (6.3.73)
\end{aligned}$$

Step 3. The estimate (6.3.42) immediately follows from (6.3.48), (6.3.57), (6.3.73), Lemmas 6.3.1–6.3.4, (6.3.2) and by choosing $\vartheta>0$ in Proposition 6.3.2 sufficiently small enough. □

As in the case of first order estimates, we shall also need the following estimate in order to complete an estimate for the full energy functional N_2.

Lemma 6.3.7. *There exist $\delta > 0$ and $C_\delta > 0$ such that for any solution $(A, u, h_0, h_\ell) \in X_T$ satisfying $N_2^2(T) \le \delta$ it holds that*

$$\begin{aligned} \int_0^t \|(A^{\frac{1}{4}})_{xx}(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \ &\le \ C_\delta N_2^2(0) \qquad (6.3.74) \\ &+ C_\delta \sup_{\tau\in[0,t]} (\|u(\tau)\|_{H^2} + \|A^{\frac{1}{4}}(\tau) - A_e^{\frac{1}{4}}\|_{H^2}) \int_0^t \|u(\tau)\|_{H^2}^2 + \|(A^{\frac{1}{4}})_x(\tau)\|_{H^1}^2 \,\mathrm{d}\tau \end{aligned}$$

for all $t \in [0, T]$.

Proof. We modify the entropy of the previous lemma. We consider the entropy $\tilde{\eta}_2 = \tilde{\psi}_2 - \tilde{\phi}_2$ with corresponding entropy flux $\tilde{q}_2 = \mu\tilde{\psi}_2 - \lambda\tilde{\phi}_2$ where

$$\begin{aligned} \tilde{\psi}_2(t, x, Z) &= \frac{\theta}{\mu}\left(\mu^2 Z + \frac{\beta\mu}{2}(z_x - w_x) + \mu\mu_x z_x - \mu_t z_x\right)^2 \\ \tilde{\phi}_2(t, x, W) &= \frac{\theta}{\lambda}\left(\lambda^2 W - \frac{\beta\lambda}{2}(z_x - w_x) + \lambda\lambda_x w_x - \lambda_t w_x\right)^2. \end{aligned}$$

Doing the same process as in the first step of the Lemma 6.3.6 we can show that

$$\int_0^\ell \tilde{\eta}_2(t, x) - \tilde{\eta}_2(0, x) \,\mathrm{d}x \ge C_\delta(\|V_{xx}(\tau)\|_{L^2}^2 - \|V(0)\|_{H^2}^2). \qquad (6.3.75)$$

Using (6.3.39) and (6.3.38), a simple computation gives us

$$\begin{aligned} \tilde{q}_2 &= \theta(w, z)((z_{tt} + (\beta/2)(z_t - w_t))^2 - (w_{tt} - (\beta/2)(z_t - w_t))^2) \\ &= \theta(w, z)(z_{tt}^2 - w_{tt}^2) + \beta\theta(w, z)(z_{tt} + w_{tt})(z_t - w_t) \\ &= q_2 + \beta\theta(w, z)p_3(w, z, w_x, z_x)p_1(w, z, w_x, z_x). \qquad (6.3.76) \end{aligned}$$

where q_2 is the entropy flux in the previous lemma and p_1 and p_3 are defined by (6.3.51) and (6.3.54), respectively. A straightforward calculation gives

$$\begin{aligned} p_3(w, z, w_x, z_x)p_1(w, z, w_x, z_x) &= -\theta(w, z)p_1^2(w, z, w_x, z_x) + \hat{R}_3 \\ &\ge - C_\delta(z_x + w_x)^2 - C_\delta(z - w)^2 + \hat{R}_4 \end{aligned}$$

where $\hat{R}_3$ and $\hat{R}_4$ are terms of degree at least 3 and contain at least two factors among $z - w, w_x, z_x$. By the estimate Proposition 6.3.2 and (6.3.2) we have

$$\begin{aligned} &\int_0^t \beta(\theta p_3 p_1)(\tau, \ell) - \beta(\theta p_3 p_1)(\tau, 0) \,\mathrm{d}\tau \\ &\ge - C_\delta\vartheta \int_0^t \|(A^{\frac{1}{4}})_{xx}(\tau)\|_{L^2}^2 \,\mathrm{d}\tau - C_{\delta,\vartheta} \int_0^t \|(A^{\frac{1}{4}})_x(\tau)\|_{L^2}^2 + \|u(\tau)\|_{H^1}^2 \,\mathrm{d}\tau \\ &- C_\delta \sup_{\tau\in[0,t]} \|V(\tau)\|_{H^2} \int_0^t \|V_x(\tau)\|_{H^1}^2 + \|u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau. \qquad (6.3.77) \end{aligned}$$

Integrating (6.3.76) from 0 to t and using (6.3.57) and (6.3.77) we have

$$\begin{aligned} \int_0^t \tilde{q}_2(\tau, \ell) - \tilde{q}_2(\tau, 0) \,\mathrm{d}\tau &\ge - C_\delta\vartheta\|V_{xx}(t)\|_{L^2}^2 - C_{\delta,\vartheta}\|V(t)\|_{H^1}^2 - C_\delta\|V(0)\|_{H^2}^2 \\ &- C_\delta\vartheta \int_0^t \|(A^{\frac{1}{4}})_{xx}(\tau)\|_{L^2}^2 \,\mathrm{d}\tau - C_{\delta,\vartheta} \int_0^t \|(A^{\frac{1}{4}})_x(\tau)\|_{L^2}^2 + \|u(\tau)\|_{H^1}^2 \,\mathrm{d}\tau \\ &- C_\delta \sup_{\tau\in[0,t]} \|V(\tau)\|_{H^2} \int_0^t \|V_x(\tau)\|_{H^1}^2 + \|u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau. \qquad (6.3.78) \end{aligned}$$

Observe that the deviation of ψ_2 and ϕ_2 from $\tilde{\psi}_2$ and $\tilde{\phi}_2$, respectively, is that the former terms contain $\frac{\beta}{2}(z_t - w_t)$ while the latter terms do not. This means that $\tilde{M}_2$ will consist of the same terms as M_2 but without those that stem from $\frac{\beta}{2}(z_t - w_t)$. Thus, crossing out the terms that appear due to the said extra term, a careful analysis in the second step of the proof of Lemma 6.3.6 shows that

$$\tilde{M}_2 = I_3 + I_4 + I_7 + I_9 + I_{10} + \frac{R_{18}}{\lambda^2\mu^2}$$

where R_{18} is again terms of degree at least 3 containing at least two factors among $z - w, w_x, z_x, z_{xx}, w_{xx}$ and contains at most two among z_{xx}, w_{xx}. Therefore we have, according to Young's inequality,

$$\begin{aligned}\tilde{M}_2 &\leq -\frac{\theta_c\beta C_e^2}{64}(z_{xx}+w_{xx})^2 - \frac{\theta_c\beta^2 C_e^2}{8}(z_{xx}+w_{xx})(z_x - w_x) + R_{19}\\ &\leq -\tilde{C}_3(z_{xx}+w_{xx})^2 + C(z_x - w_x)^2 + R_{19}.\end{aligned}$$

for some $\tilde{C}_3 > 0$. With the same explanations as above we have

$$\int_0^t \int_0^\ell \tilde{M}_2(\tilde{\psi}_2, \tilde{\phi}_2)\,\mathrm{d}x\,\mathrm{d}\tau \leq -\tilde{C}_3 \int_0^t \|(A^{\frac{1}{4}})_{xx}(\tau)\|_{L^2}^2\,\mathrm{d}\tau \tag{6.3.79}$$
$$+ C_\delta\bigg(\int_0^t \|u_x(\tau)\|_{L^2}^2\,\mathrm{d}\tau + \sup_{\tau\in[0,t]} \|V(\tau)\|_{H^2} \int_0^t \|V_x(\tau)\|_{H^1}^2 + \|u(\tau)\|_{L^2}^2\,\mathrm{d}\tau\bigg)$$

From (6.3.75), (6.3.78), (6.3.79), choosing $\vartheta > 0$ in Proposition 6.3.2 small enough and using Lemmas 6.3.1–6.3.6, the estimate (6.3.74) follows. □

6.4 PROOF OF THE GLOBAL EXISTENCE AND STABILITY IN $H^1 \times H^1 \times \mathbb{R}^2$

An immediate consequence of the results in the previous section is the following estimate for the energy N_2.

Corollary 6.4.1. *Let $T > 0$ be such that* (6.0.1) *has a solution that belongs to X_T. Then there is a $\delta > 0$ such that if $N_2^2(T) \leq \delta$ then $N_2^2(t) \leq C_\delta(N_2^2(0) + N_2^3(t))$ for all $t \in [0, T]$ and for some $C_\delta > 0$ independent of T. In particular, there exists a $\delta > 0$ such that if $N_2^2(T) \leq \delta$ then $N_2^2(T) \leq \tilde{C}_\delta N_2^2(0)$ for some $\tilde{C}_\delta > 0$ independent of T.*

Proof. According to Lemmas 6.3.1, 6.3.3–6.3.7, there is a $\delta > 0$ such that $N_2^2(t) \leq C_\delta(N_2^2(0) + N_2^3(t))$ for all $t \in [0, T]$ whenever $N_2^2(T) \leq \delta$. In particular, $N_2^2(T) \leq C_\delta(N_2^2(0) + \sqrt{\delta}N_2^2(T))$. Since (6.3.1) holds, one may choose $\delta > 0$ small enough so that $\tilde{C}_\delta := C_\delta(1 - C_\delta\sqrt{\delta})^{-1} > 0$ and thus $N_2^2(T) \leq \tilde{C}_\delta N_2^2(0)$. □

Proof of Theorem 6.1.2. The proof is standard, however, we include it here for completeness. According to Corollary 6.4.1 we have a $\delta > 0$ such that $N_2^2(T) \leq \tilde{C}_\delta N_2^2(0)$ for some $\tilde{C}_\delta > 0$ whenever $N_2^2(T) \leq \delta$. Take $\delta_0 = \min(\delta/(2\tilde{C}_\delta), \delta/4) > 0$. Suppose that the maximal time of existence $T^* > 0$ is finite. Then either (A, u) leaves every compact subset of $\mathcal{U}$ or $\|(A_x, u_x)(t)\|_{L^\infty[0,\ell]} \to \infty$ as $t \uparrow T^*$. Classical embedding results imply that

$$\|(A, u) - (A_e, 0)\|_{W^{1,\infty}([0,t]\times[0,\ell])^2} \leq C_\delta N_2(t).$$

In any case, by continuity there exists $0 < T_1 < T^*$ such that $N_2^2(T_1) = \frac{\delta}{2}$ and $N_2^2(t) > \frac{\delta}{2}$ for all $t \in (T_1, T_1 + \epsilon)$ where $\epsilon > 0$ and $T_1 + \epsilon < T^*$. Because $N_2^2(T_1) < \delta$, there exists $T_2 \in (T_1, T_1+\epsilon)$ satisfying $N_2^2(T_2) \le \delta$. Corollary 6.4.1 implies that $N_2^2(T_2) \le \tilde{C}_\delta N_2^2(0) \le \frac{\delta}{2}$, which is a contradiction. Therefore we must have $T^* = +\infty$ and this proves that a global-in-time solution exists. Furthermore, we have the estimate $N_2^2(t) \le \tilde{C}_\delta N_2^2(0)$ for all $t \ge 0$. □

By applying the PDEs, the estimate in Theorem 6.1.2 implies the following estimate on the time-derivatives of the state.

Corollary 6.4.2. *In the situation of Theorem 6.1.2. there exists a $C_\delta > 0$ such that*

$$\sup_{t\ge 0} (\|A_t(t)\|_{H^1}^2 + \|A_{tt}(t)\|_{L^2}^2 + \|u_t(t)\|_{H^1}^2 + \|u_{tt}(t)\|_{L^2}^2)$$
$$+ \int_0^\infty (\|A_\tau(\tau)\|_{H^1}^2 + \|A_{\tau\tau}(\tau)\|_{L^2}^2 + \|u_\tau(\tau)\|_{H^1}^2 + \|u_{\tau\tau}(\tau)\|_{L^2}^2 \,\mathrm{d}\tau) \le C_\delta E_0.$$

Now we are ready to prove the following asymptotic behaviour of the solutions.

Theorem 6.4.3 (Asymptotic Stability)**.** *In the framework of Theorem 6.1.2 we have*

$$\lim_{t\to\infty} (\|A(t) - A_e\|_{H^1(0,\ell)} + \|u(t)\|_{H^1(0,\ell)} + |h_0(t) - h_{0e}| + |h_\ell(t) - h_{\ell e}|) = 0. \quad (6.4.1)$$

Proof. As functions of time $\|u(\cdot)\|_{H^1(0,\ell)}^2$ and $\|A_x(\cdot)\|_{L^2(0,\ell)}^2$ belong to $W^{1,1}(0,\infty)$ according to Theorem 6.1.2 and Corollary 6.4.1 . Hence

$$\lim_{t\to\infty} (\|u(t)\|_{H^1(0,\ell)} + \|A_x(t)\|_{L^2(0,\ell)}) = 0 \quad (6.4.2)$$

Using a Gagliardo-Nirenberg-Moser interpolation, see [**74**], we have

$$\|A(t) - A_e\|_{L^\infty(0,\ell)} \le C_\ell \|\partial_x A(t)\|_{L^2(0,\ell)}^{1/2} \|A(t) - A_e\|_{L^2(0,\ell)}^{1/2}.$$

Theorem 6.1.2 implies that $\|A(t) - A_e\|_{L^2(0,\ell)}$ is uniformly bounded in $t \in [0,\infty)$ and thus from (6.4.2) we get $\|A(t) - A_e\|_{L^\infty(0,\ell)} \to 0$ as $t \to \infty$. In particular, this implies that $\|A(t) - A_e\|_{L^2(0,\ell)} \to 0$, $A(t,0) \to A_e$ and $A(t,\ell) \to A_e$ as $t \to \infty$. The latter two further imply that $h_0(t) \to h_{0e}$ and $h_\ell(t) \to h_{\ell e}$ as $t \to \infty$. Combining these with (6.4.2) we obtain (6.4.1). □

The decay rate at which the state converges to the equilibrium can be shown to be exponential, however, if one uses the norm in $L^2(0,\ell)^2 \times \mathbb{R}^2$. This is the goal of the next section.

6.5 EXPONENTIAL CONVERGENCE TO THE EQUILIBRIUM IN $L^2(0,\ell)^2 \times \mathbb{R}^2$

The exponential stability result for (6.0.1) is based on linear stability and treating the higher order terms as perturbation of the linearized system. The basic ingredients are the exponential stability derived from semigroup theory, the variation of parameters formula and interpolation estimates. However, care should be taken since the linearization yields a nontrivial kernel and therefore stability for the linearized problem is only possible in a factor space. The smallness of the data and the order of nonlinearity play an important role in the proof, specifically the applicability of a Gronwall-type estimate. In this way the decay rate for the nonlinear system is the same as the decay rate for the linearized system.

First, we revisit the stability result in Chapter 3. Define the following constants

$$\alpha = \frac{\kappa^2}{\sqrt{A_e}}, \qquad \gamma = 2b(a_0 + bh_{0e}) = 2b(a_\ell + bh_{\ell e}).$$

Let $\mathcal{X} = L^2(0,\ell)^2 \times \mathbb{R}^2$ be equipped with the weighted norm

$$\|(A, u, h_0, h_\ell)\|_{\mathcal{X}}^2 = \frac{1}{A_e}\|A\|_{L^2(0,\ell)}^2 + \frac{1}{\alpha}\|u\|_{L^2(0,\ell)}^2 + \frac{\gamma A_T}{A_e}(|h_0|^2 + |h_\ell|^2).$$

Consider the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{X}$ with domain $\mathcal{D}(\mathcal{A}) = \{(A, u, h_0, h_\ell) \in H^1(0,\ell)^2 \times \mathbb{R}^2 : A(0) = \gamma h_0, A(\ell) = \gamma h_\ell\}$ defined by

$$\mathcal{A}\begin{pmatrix} A \\ u \\ h_0 \\ h_\ell \end{pmatrix} = \begin{pmatrix} -A_e u_x \\ -\alpha A_x - \beta u \\ -\frac{A_e}{A_T} u(0) \\ \frac{A_e}{A_T} u(\ell) \end{pmatrix}.$$

This operator is obtained by linearizing the system (6.0.1) including its boundary conditions about the equilibrium state $(A_e, 0, h_{0e}, h_{\ell e})$. The operator $\mathcal{A}$ has a non-trivial kernel $\mathcal{N}(\mathcal{A}) = \{c(\gamma, 0, 1, 1) : c \in \mathbb{R}\}$. The orthogonal complement $\mathcal{N}(\mathcal{A})^\perp$ of $\mathcal{N}(\mathcal{A})$ coincides with the kernel of the *volume functional* $\mathcal{V} : \mathcal{X} \to \mathbb{R}$

$$\mathcal{V}(A, u, h_0, h_\ell) = \int_0^\ell A(x)\,\mathrm{d}x + A_T h_0 + A_T h_\ell.$$

In the following theorem $\sigma(\mathcal{A})$ will denote the spectrum of $\mathcal{A}$, which consists of eigenvalues since the operator is discrete. For the proof and explicit values of σ and k we refer to Chapter 3.

Theorem 6.5.1. *The operator $\mathcal{A}$ is a discrete spectral operator that generates a strongly continuous group $T(t)$, $t \in \mathbb{R}$, on $\mathcal{X}$. If $\beta > 0$ then there exists $M \geq 1$ such that*

$$\|T(t)\|_{\mathcal{L}(\mathcal{N}(\mathcal{A})^\perp)} \leq M(1 + t^k)e^{-\sigma t}, \qquad t \geq 0,$$

where $\sigma = -\sup_{\lambda \in \sigma(\mathcal{A})} \Re\lambda > 0$ and k is either 0 or 1.

To use this result for the nonlinear system (6.0.1), we need further tools. The first one is the following Gronwall-type lemma whose proof can be found in [**26**].

Lemma 6.5.2. *Let $u \in \mathrm{Lip}([0,\infty), \mathbb{R}_+)$ and suppose that for some $C > 0$*

$$u(t) \leq C(1 + t^k)e^{-\sigma t}u(0) + C\int_0^t (1 + (t-s)^k)e^{-\sigma(t-s)}u(s)^\varrho\,\mathrm{d}s, \quad t \geq 0,$$

for some $\sigma > 0$, $\varrho > 1$ and nonnegative integer k. Then there exists $\epsilon > 0$ and $C > 0$ such that if $u(0) < \epsilon$ then

$$u(t) \leq C(1 + t^k)e^{-t\sigma}, \qquad t \geq 0.$$

The next tool is a simple interpolation estimate obtained from the well-known Gagliardo-Nirenberg inequality, see [**74**] for example.

Theorem 6.5.3 (Gagliardo-Nirenberg)**.** *Let m be a positive integer. There exists $C_\ell > 0$ such that for all $u \in H^m(0,\ell)$ and $j \leq m$ we have*

$$\|u^{(j)}\|_{L^{2m/j}(0,\ell)} \leq C_\ell \|u\|_{L^\infty(0,\ell)}^{1-j/m}\|u\|_{H^m(0,\ell)}^{j/m}.$$

As a consequence, we have the following estimate.

Corollary 6.5.4. *There exists $C > 0$ such that for all $u \in H^2(0,\ell)$ it holds that*

$$\|u_x\|_{L^\infty(0,\ell)} \le C\|u\|^{7/8}_{H^2(0,\ell)}\|u\|^{1/8}_{L^2(0,\ell)}.$$

Proof. Using the Gagliardo-Nirenberg-Moser estimate in [**74**], Hölder's inequality and Theorem 6.5.3 with $m = 2$ and $j = 1$ we have, for generic constants $C > 0$,

$$\begin{aligned}
\|u_x\|_{L^\infty(0,\ell)} &\le C\|u_{xx}\|^{1/2}_{L^2(0,\ell)}\|u_x\|^{1/2}_{L^2(0,\ell)}\\
&\le C\|u_{xx}\|^{1/2}_{L^2(0,\ell)}\|u_x\|^{1/2}_{L^4(0,\ell)}\\
&\le C\|u_{xx}\|^{1/2}_{L^2(0,\ell)}(\|u\|^{1/2}_{L^\infty(0,\ell)}\|u\|^{1/2}_{H^2(0,\ell)})^{1/2}\\
&\le C\|u_{xx}\|^{1/2}_{L^2(0,\ell)}(\|u_x\|^{1/4}_{L^2(0,\ell)}\|u\|^{1/4}_{L^2(0,\ell)}\|u\|^{1/2}_{H^2(0,\ell)})^{1/2}.
\end{aligned}$$

This clearly implies the estimate given in the corollary. □

Now we are in position to prove the following stability result.

Theorem 6.5.5 (Exponential Stability)**.** *Consider the framework of Theorem 6.1.2. There exists $\delta_0 > 0$ such that if $E_0 \le \delta_0$ then the solution of* (6.0.1) *satisfies*

$$\|A(t) - A_e\|_{L^2(0,\ell)} + \|u(t)\|_{L^2(0,\ell)} + |h_0(t) - h_{0e}| + |h_\ell(t) - h_{\ell e}| \le C(1 + t^k)e^{-\sigma t}$$

for all $t \ge 0$ and for some constant $C = C(E_0) > 0$. The constants k and σ are those of Theorem 6.5.1.

Proof. Let $z = (B, v, \eta_0, \eta_\ell) = (A - A_e, u, h_0 - h_{0e}, h_\ell - h_{\ell e})$ denote the deviation of the state from the equilibrium. The system (6.0.1) can be rewritten in terms of the deviations as

$$\begin{cases}
B_t = -A_e v_x - (A - A_e)u_x - uA_x,\\
v_t = -\alpha B_x - \beta v + \alpha A^{-\frac12}(A^{\frac12} + A_e^{\frac12})^{-1}(A - A_e)A_x - uu_x,\\
\eta_0'(t) = -\frac{A_e}{A_T}v(t,0) - \frac{1}{A_T}(A(t,0) - A_e)u(t,0),\\
\eta_\ell'(t) = \frac{A_e}{A_T}v(t,\ell) + \frac{1}{A_T}(A(t,\ell) - A_e)u(t,\ell),\\
B(t,0) = \gamma\eta_0(t) + b^2(h_0(t) - h_{0e})^2,\\
B(t,\ell) = \gamma\eta(t) + b^2(h(t) - h_{\ell e})^2.
\end{cases}$$

In order to use the results for abstract homogeneous linear time-invariant systems via semigroup theory, we consider a new state variable $w := z - (\phi, 0, 0, 0)$ where

$$\phi(t,x) = \frac{\ell - x}{\ell}b^2(h_0(t) - h_{0e})^2 + \frac{x}{\ell}b^2(h_\ell(t) - h_{\ell e})^2.$$

This is introduced in order to compensate for the nonlinearity in the boundary conditions. It is easy to see that $w(t) \in \mathcal{D}(\mathcal{A})$ for all $t \ge 0$ and it satisfies the system

$$\dot w(t) = \mathcal{A}w(t) + F(t), \qquad t > 0, \tag{6.5.1}$$

where

$$F(t) = \begin{pmatrix} -(A(t)-A_e)u_x(t) - u(t)A_x(t) - \phi_t(t) \\ \alpha A(t)^{-\frac{1}{2}}(A(t)^{\frac{1}{2}} + A_e^{\frac{1}{2}})^{-1}(A(t)-A_e)A_x(t) - u(t)u_x(t) - \alpha\phi_x(t) \\ -\frac{1}{A_T}(A(t,0)-A_e)u(t,0) \\ \frac{1}{A_T}(A(t,\ell)-A_e)u(t,\ell). \end{pmatrix}$$

Because $u \in C^1([0,\infty); H^1(0,\ell))$ it follows that $uu_x \in C^1([0,\infty); L^2(0,\ell))$. Using the regularity of A, u, h_0 and h_ℓ stated in Theorem 6.1.2 together with a similar argument as in the previous statement one can show that $F \in C^1([0,\infty); \mathcal{X})$. A standard result in semigroup theory, see [**61, Section 4.2**] for example, shows that (6.5.1) has a unique solution in $\mathcal{X}$ and it is given by the variation of parameters formula

$$w(t) = T(t)w(0) + \int_0^t T(t-s)F(s)\,\mathrm{d}s. \tag{6.5.2}$$

By uniqueness, this function w must coincide with the function $z - (\phi, 0, 0, 0)$ above.

Since the semigroup $T(t)$ is exponentially stable only in $\mathcal{N}(\mathcal{A})^\perp$, we will decompose the solution w into two parts. First decompose F as a sum $F = F_1 + (F_2)_t$ where $F_2 = (-\phi, 0, 0, 0)$. By construction, $F_1(s) \in \mathcal{N}(\mathcal{A})^\perp$ for all $s \geq 0$. This can be easily seen since $F_1(s)$ lies in the kernel of $\mathcal{V}$ for all $s \geq 0$. Let $\Pi : \mathcal{X} \to \mathcal{N}(\mathcal{A})$ be the orthogonal projection of $\mathcal{X}$ onto $\mathcal{N}(\mathcal{A})$. Conservation of volume implies that $\mathcal{V}(A_0^0, u^0, h_0^0, h_\ell^0) = \mathcal{V}(A_e, 0, h_{0e}, h_{\ell e})$ or equivalently $z(0) \in \mathcal{N}(\mathcal{A})^\perp$. Furthermore, we have $F_1(s) + (I - \Pi)(F_2)_t(s) \in \mathcal{N}(\mathcal{A})^\perp$ for all $s \geq 0$. We write

$$w(t) = w_1(t) + w_2(t)$$

where

$$w_1(t) = T(t)(z(0) + (I-\Pi)F_2(0)) + \int_0^t T(t-s)(F_1(s) + (I-\Pi)(F_2)_t(s))\,\mathrm{d}s$$

$$w_2(t) = T(t)\Pi F_2(0) + \int_0^t T(t-s)\Pi(F_2)_t(s)\,\mathrm{d}s.$$

Because $T(t)\Pi = \Pi$ and $\Pi(F_2)_t(s) = (\Pi F_2(s))_t$ we actually have $w_2(t) = \Pi F_2(t)$. Using (6.5.2) and Theorem 6.5.1 we have

$$\begin{aligned} \|w(t)\|_{\mathcal{X}} \;\leq\; & M(1+t^k)e^{-\sigma t}\|z(0) + (I-\Pi)F_2(0)\|_{\mathcal{X}} + \|\Pi F_2(t)\|_{\mathcal{X}} \\ & + M\int_0^t (1+(t-s)^k)e^{-\sigma t}\|F_1(s) + (I-\Pi)(F_2)_t(s)\|_{\mathcal{X}}\,\mathrm{d}s. \end{aligned} \tag{6.5.3}$$

The next task is to estimate each term of (6.5.3) in terms of the norm $\|z(t)\|_{\mathcal{X}}$ of the deviation $z(t)$. Since $\|I - \Pi\|_{\mathcal{L}(\mathcal{X})} \leq 1$ it holds that for all $t \geq 0$

$$\|(I-\Pi)F_2(t)\|_{\mathcal{X}} \leq C\|\phi(t)\|_{L^2(0,\ell)} \leq C\|z(t)\|_{\mathcal{X}}^2 \leq CE_0^{1/2}\|z(t)\|_{\mathcal{X}} \tag{6.5.4}$$

for some $C > 0$ independent of E_0. Similarly, for all $t \geq 0$

$$\|w(t)\|_{\mathcal{X}} = \|z(t) + F_2(t)\|_{\mathcal{X}} \geq (1 - CE_0^{1/2})\|z(t)\|_{\mathcal{X}}. \tag{6.5.5}$$

From Corollary 6.5.4 we obtain

$$\begin{aligned} \|u(t)u_x(t)\|_{L^2(0,\ell)} &\leq \|u(t)\|_{L^2(0,\ell)}\|u_x(t)\|_{L^\infty(0,\ell)} \\ &\leq C\|u(t)\|_{H^2(0,\ell)}^{7/8}\|u(t)\|_{L^2(0,\ell)}^{9/8} \leq CE_0^{7/16}\|z(t)\|_{L^2(0,\ell)}^{9/8}. \end{aligned}$$

The other terms in the first and second rows of F_1 can be estimated similarly. Now we estimate the third and fourth rows of F_1. By Sobolev embedding we have

$$|(A(t,y) - A_e)u(t,y)| \leq C(\|(A(t) - A_e)u(t)\|_{L^2(0,\ell)} + \|[(A(t) - A_e)u(t)]_x\|_{L^2(0,\ell)}),$$

for $y = 0, \ell$. Expanding the term $[(A(t) - A_e)u(t)]_x = A_x(t)u(t) + (A(t) - A_e)u_x(t)$, it can be seen that each term can be estimated in the same manner as we estimated $u(t)u_x(t)$ above. For the first term, we apply the Gagliardo-Nirenberg-Moser interpolation once more to get

$$\begin{aligned} \|(A(t) - A_e)u(t)\|_{L^2(0,\ell)} &\leq \|A(t) - A_e\|_{L^2(0,\ell)}\|u(t)\|_{L^\infty(0,\ell)} \\ &\leq C\|A(t) - A_e\|_{L^2(0,\ell)}\|u_x(t)\|^{1/2}_{L^2(0,\ell)}\|u(t)\|^{1/2}_{L^2(0,\ell)} \\ &\leq C(E_0)\|z(t)\|^{3/2}_{\mathcal{X}} \leq C(E_0)\|z(t)\|^{9/8}_{\mathcal{X}}. \end{aligned}$$

Combining all of our estimates yields

$$\|F_1(t)\|_{\mathcal{X}} \leq C(E_0)\|z(t)\|^{9/8}_{\mathcal{X}}. \tag{6.5.6}$$

The next step is to estimate $\|(1 - \Pi)(F_2)_t(t)\|_{\mathcal{X}}$. Using the differential boundary conditions, the derivative of ϕ with respect to t is given by

$$\begin{aligned} \phi_t(t,x) = &- 2A_T b^2\ell^{-1}(\ell - x)(h_0(t) - h_{0e})A(t,0)u(t,0) \\ &+ 2A_T b^2\ell^{-1}x(h_\ell(t) - h_{\ell e})A(t,\ell)u(t,\ell) \end{aligned}$$

and by interpolation we can estimate its L^2-norm by

$$\begin{aligned} \|\phi_t(t)\|_{L^2(0,\ell)} &\leq C(|h_0(t) - h_{0e}| + |h_\ell(t) - h_{\ell e}|)\|A(t)\|_{L^\infty(0,\ell)}\|u(t)\|_{L^\infty(0,\ell)} \\ &\leq CE_0^{1/2}(|h_0(t) - h_{0e}| + |h_\ell(t) - h_{\ell e}|)\|A(t)\|^{1/2}_{L^2(0,\ell)}\|u(t)\|^{1/2}_{L^2(0,\ell)} \\ &\leq C(E_0)\|z(t)\|^{9/8}_{\mathcal{X}}. \end{aligned}$$

Consequently,

$$\|(1 - \Pi)(F_2)_t(t)\|_{\mathcal{X}} \leq C(E_0)\|z(t)\|^{9/8}_{\mathcal{X}}. \tag{6.5.7}$$

Using (6.5.4), (6.5.5), (6.5.6), (6.5.7) in (6.5.3) we have

$$\begin{aligned} \|z(t)\|_{\mathcal{X}} \leq \frac{MC(E_0)}{1 - CE_0^{1/2}}\Bigg((1 + t^k)e^{-\sigma t}\|z(0)\|_{\mathcal{X}} \\ + \int_0^t (1 + (t-s)^k)e^{-\sigma t}\|z(s)\|^{9/8}_{\mathcal{X}}\,\mathrm{d}s\Bigg) \end{aligned} \tag{6.5.8}$$

whenever $CE_0^{1/2} \leq C\delta_0^{1/2} < 1$.

Finally, we check the Lipschitz continuity of the map $t \mapsto \|z(t)\|_{\mathcal{X}}$. From the continuity equation, it holds that

$$\begin{aligned} &|\|A(t) - A_e\|_{L^2(0,\ell)} - \|A(s) - A_e\|_{L^2(0,\ell)}| \\ &\quad\leq \|A(t) - A(s)\|_{L^2(0,\ell)} \\ &\quad\leq \left\|\int_t^s u(\tau)A_x(\tau) + A(\tau)u_x(\tau)\,\mathrm{d}\tau\right\|_{L^2(0,\ell)} \\ &\quad\leq |t-s|\max_{\tau\geq 0}\|u(\tau)A_x(\tau) + A(\tau)u_x(\tau)\|_{L^2(0,\ell)} \\ &\quad\leq C|t-s|\max_{\tau\geq 0}(\|u(\tau)\|_{H^1(0,\ell)}\|A_x(\tau)\|_{L^2(0,\ell)} + \|A(\tau)\|_{H^1(0,\ell)}\|u_x(\tau)\|_{L^2(0,\ell)}) \\ &\quad\leq C(E_0)|t-s|. \end{aligned}$$

for all $s, t \geq 0$. The same estimate can be obtained for u and h_0, h_ℓ using the momentum equation and the ODE boundary conditions, respectively. Therefore $\|z(\cdot)\|_{\mathcal{X}} \in \operatorname{Lip}([0,\infty), \mathbb{R}_+)$. The result now easily follows from (6.5.8) and the Gronwall-type estimate Lemma 6.5.2. $\square$

Part III

APPENDICES

A

SEMIGROUPS AND RIESZ SPECTRAL OPERATORS

In this section, we state the basic facts in the theory of strongly continuous semigroups of bounded linear operators and in particular those that have Riesz spectral generators. We restrict ourselves to the case of Hilbert spaces. All throughout this chapter, unless otherwise stated, we let X be a complex Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm is $\| \cdot \|$.

A.1 STRONGLY CONTINUOUS SEMIGROUPS

A family $(T(t))_{t\geq 0}$ of bounded linear operators in X is called a *strongly continuous semigroup* or $\mathcal{C}_0$*-semigroup* if $T(0) = I$, $T(t)T(s) = T(t+s)$ for all $t, s \geq 0$ (semigroup property) and $\|T(t)x - x\| \to 0$ as $t \to 0^+$ for every $x \in X$ (strong continuity). If $t \geq 0$ is replaced by $t \in \mathbb{R}$ then the family is called a $\mathcal{C}_0$*-group*. The *infinitesimal generator*, or *generator* in short, of a $\mathcal{C}_0$-semigroup $(T(t))_{t\geq 0}$ is the linear operator $A : D(A) \to X$, where

$$D(A) = \left\{ z \in X : \lim_{t\to 0^+} \frac{1}{t}(T(t)z - z) \text{ exists in } X \right\},$$

defined by

$$Az = \lim_{t\to 0^+} \frac{1}{t}(T(t)z - z).$$

The generator of a $\mathcal{C}_0$-semigroup is necessarily closed and its domain $D(A)$ is dense in X. If $(T(t))_{t\geq 0}$ is a $\mathcal{C}_0$-semigroup with generator A then the notation e^{tA} is also used for $T(t)$. It is well known that for a $\mathcal{C}_0$-semigroup $(T(t))_{t\geq 0}$ there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$ for $t \geq 0$.

If $z_0 \in D(A)$ then $z(t) := e^{tA}z_0$ solves the initial-value problem in X

$$\begin{cases} \dot{z}(t) = Az(t), & t > 0, \\ z(0) = z_0. \end{cases} \tag{A.1.1}$$

Semigroup theory is therefore a suitable tool in studying the well-posedness of the initial-value problem (A.1.1). This is also the motivation in writing the semigroup $T(t)$ as e^{tA} because $e^{tA}z_0$ is precisely the solution of (A.1.1) in the finite-dimensional case. Later in Appendix B we also consider nonhomogeneous differential equations.

In applications, the operator A is the one that is known and the question is whether it generates a strongly continuous semigroup. The two main generation theorems are the Hille-Yosida Theorem and the Lumer-Phillips Theorem. However, we only state here the Lumer-Phillips Theorem in reflexive Banach spaces, see e.g. [**25, Corollary III.3.20**]

Theorem A.1.1 (Lumer-Phillips)**.** *A closed linear operator $A : D(A) \to X$ in a reflexive Banach space X generates a $\mathcal{C}_0$-semigroup of contractions if and only if A is dissipative, i.e., $\Re\langle Az, z\rangle \leq 0$ for every $z \in D(A)$, and $\mathcal{R}(\lambda I - A) = X$ for some $\lambda > 0$.*

Let $A : D(A) \to X$ be a linear operator in X with dense domain. Suppose that $z \in X$ satisfies $|\langle Aw, z\rangle| \leq C\|w\|$ for some $C \geq 0$ and for all $w \in D(A)$. For such z, define the linear operator $\ell_z : X \to \mathbb{C}$ by

$$\ell_z w = \lim_{n\to\infty} \langle Aw_n, z\rangle$$

where $(w_n)_{n\in\mathbb{N}} \subset D(A)$ and $w_n \to w$ in X. It is clear that the limit on the right hand side exists and is independent of the sequence used to approach w. By assumption, $\ell_z \in X'$ and it is the unique bounded linear functional defined in X such that $\ell_z w = \langle Aw, z\rangle$ for all $w \in D(A)$. By the Riesz Representation Theorem there exists a unique $y \in X$ such that

$$\ell_z w = \langle w, y\rangle, \qquad \forall\, w \in X.$$

Define the operator $A^* : D(A^*) \to X$, where

$$D(A^*) = \{z \in X : \exists\, C \geq 0 \text{ such that } |\langle Aw, z\rangle| \leq C\|w\| \ \forall w \in D(A)\}$$

by $A^* z = y$ where y is the Riesz representor of ℓ_z. The operator A^* is called the *adjoint* of A. By definition, we have

$$\langle Aw, z\rangle = \langle w, A^* z\rangle, \qquad \forall\, w \in D(A),\ z \in D(A^*).$$

If $A : D(A) \to X$ is a densely defined closed linear operator then its adjoint is also a densely defined closed linear operator. If A generates a $\mathcal{C}_0$-semigroup then A^* generates a $\mathcal{C}_0$-semigroup as well and $e^{tA^*} = (e^{At})^*$ where the star on the right hand side denotes the adjoint of a bounded linear operator.

A densely defined operator is called *skew adjoint* if $A^* = -A$. Using Lumer-Phillips Theorem the following generation theorem for skew-adjoint operators can be shown. For a proof, see [**25, Theorem 3.24**].

Theorem A.1.2 (Stone)**.** *A closed linear operator $A : D(A) \to X$ generates a $\mathcal{C}_0$-group of unitary operators if and only if A is skew-adjoint.*

A.2 PART OF GENERATORS AND INVARIANT SUBSPACES OF SEMIGROUPS

In certain situations, it is also important to look at the restriction of an operator to a subspace. Let V be a subspace of X. The *part* of an operator $A : D(A) \to X$ in V is the operator $A_V : D(A_V) \to X$, where $D(A_V) = \{z \in D(A) \cap V : Av \in V\}$, defined by $A_V z = Az$. In other words, A_V is the restriction of A to $D(A_V)$. A subspace V is said to be *invariant* under a linear operator $T : X \to X$ if $TV \subset V$ and it is said to be *invariant* under a family $(T_i)_{i\in I}$ of linear operators if V is invariant under T_i for all $i \in I$. Invariant subspaces of semigroups and parts of generators are related in the following theorem. The proof can be seen in [**77, Proposition 2.4.4**].

Theorem A.2.1. *Let X be a Hilbert space and $V \subset X$ with continuous embedding. Suppose that A generates a strongly continuous semigroup in X. If V is invariant under $(e^{tA})_{t\geq 0}$ and $((e^{tA})_{|V})_{t\geq 0}$ is strongly continuous in V then $((e^{tA})_{|V})_{t\geq 0}$ is a $\mathcal{C}_0$-semigroup with generator A_V. On the other hand, if A_V generates a $\mathcal{C}_0$-semigroup then $e^{tA_V} = (e^{tA})_{|V}$ for all $t \geq 0$.*

By taking V to be a closed subspace of X we have the following corollary.

Corollary A.2.2. *Suppose that $A : D(A) \to X$ generates a $\mathcal{C}_0$-semigroup and Y is a closed subspace of X that is invariant under $(e^{tA})_{t\geq 0}$. Then $A_Y : D(A_Y) \to Y$ generates a $\mathcal{C}_0$-semigroup in Y and $e^{tA_Y} = (e^{tA})_{|Y}$ for all $t \geq 0$. In particular, if A generates a semigroup of contractions then so is A_Y.*

The following theorem states that the adjoint of the part of A in Y is the same as the part of the adjoint of A in Y, whenever Y satisfies certain properties.

Theorem A.2.3. *Let $A : D(A) \to X$ generate a $\mathcal{C}_0$-semigroup and Y be a closed subspace of X that is invariant under $(e^{tA})_{t\geq 0}$ and $(e^{tA^*})_{t\geq 0}$. Then $(A^*)_Y = (A_Y)^*$.*

Proof. From Corollay A.2.2, $e^{tA}|_Y = e^{tA_Y}$ and $e^{tA^*}|_Y = e^{t(A^*)_Y}$ for all $t \geq 0$. For each $w, z \in Y$, using the fact that $(e^{tA})^* = e^{tA^*}$, we have

$$\langle w, (e^{tA_Y})^* z\rangle = \langle e^{tA_Y} w, z\rangle = \langle e^{tA} w, z\rangle = \langle w, e^{tA^*} z\rangle = \langle w, e^{t(A^*)_Y} z\rangle.$$

Therefore $(e^{tA_Y})^* = e^{t(A^*)_Y}$ for all $t \geq 0$. The generator of $((e^{tA_Y})^*)_{t\geq 0}$ is $(A_Y)^*$ while the generator of $(e^{t(A^*)_Y})_{t\geq 0}$ is $(A^*)_Y$, and since the generator is uniquely determined by the semigroup we conclude that $(A^*)_Y = (A_Y)^*$. □

These results can be extended to the case of groups.

Theorem A.2.4. *Suppose that A generates a $\mathcal{C}_0$-group on X and Y is a closed subspace of X invariant under the group $(e^{tA})_{t\in\mathbb{R}}$. Then A_Y generates a $\mathcal{C}_0$-group on Y and $e^{tA_Y} = (e^{tA})_{|Y}$ for all $t \in \mathbb{R}$. If the group generated by A is unitary in X, then the group generated by A_Y is unitary in Y.*

Proof. The first conclusion follows from the fact that Y is invariant under the semigroups $(e^{tA})_{t\geq 0}$ and $(e^{-tA})_{t\geq 0}$ and we have $e^{tA}|_Y = e^{tA_Y}$ and $e^{-tA}|_Y = e^{-tA_Y}$ for all $t \geq 0$. Suppose that A generates a unitary group on X. Then by Stone's Theorem A is skew-adjoint. Also, Y is invariant under $(e^{tA^*})_{t\geq 0}$ since $e^{tA^*} = e^{-tA}$ for $t \geq 0$. From Corollary A.2.2, A_Y is a generator of a $\mathcal{C}_0$-semigroup on Y and hence $A_Y : D(A_Y) \to Y$ is closed and $D(A_Y)$ is dense in Y. Because A is skew-adjoint, $Az \in Y$ if and only if $A^*z \in Y$. Therefore $D((A^*)_Y) = D(A_Y)$ and from Theorem A.2.3

$$(A_Y)^* z = (A^*)_Y z = A^* z = -Az = -A_Y z$$

for all $z \in D(A_Y)$, that is, $(A_Y)^* = -A_Y$. Thus A_Y is skew-adjoint and therefore it generates a unitary group in Y according to Stone's Theorem. □

A.3 RIESZ BASES AND RIESZ SPECTRAL OPERATORS

Let X be a separable Hilbert space and $(e_n)_{n\in\mathbb{N}}$ be an of orthonormal basis in X. A sequence $(z_n)_{n\in\mathbb{N}}$ in X is called a *Riesz basis* in X if there exists an invertible bounded linear operator $Q : X \to X$ such that $Qz_n = e_n$ for every positive integer n. Two sequences $(w_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X are said to be *biorthogonal* if $\langle w_n, y_m\rangle = \delta_{nm}$, where δ_{nm} is the Kronecker delta symbol. If $(z_n)_{n\in\mathbb{N}}$ is a Riesz basis in X then $(z_n)_{n\in\mathbb{N}}$ and $(Q^*Qz_n)_{n\in\mathbb{N}}$ are biorthogonal.

Just like orthonormal bases, Riesz bases can be used to express an element in X as a Fourier series. Indeed, each $z \in X$ can be uniquely written as

$$z = \sum_{n\in\mathbb{N}} \langle z, Q^*Qz_n\rangle z_n \tag{A.3.1}$$

and there exist constants $C \geq c > 0$ independent of z such that

$$c\|z\|^2 \leq \sum_{n\in\mathbb{N}} |\langle z, Q^*Qz_n\rangle|^2 \leq C\|z\|^2.$$

The series in (A.3.1) is called the Fourier series representation of z with respect to the Riesz basis $(z_n)_{n\in\mathbb{N}}$ and $\langle z, Q^*Qz_n\rangle$ are the Fourier coefficients. Fourier series with respect to a Riesz basis and square summable sequences are closely related. In fact, $(a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{C})$ if and only if $\sum_{n\in\mathbb{N}} a_n z_n \in X$. For the proofs of these statements, see Young [**81**].

The resolvent set and the spectrum of a closed linear operator A are denoted by $\rho(A)$ and $\sigma(A)$, respectively. If $\lambda \in \rho(A)$ then $R(\lambda, A) := (\lambda I - A)^{-1}$ is called a *resolvent* of A. An element $z \in X$ is called a *generalized eigenvector* of A if there exist $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$ such that $(\lambda I - A)^m z = 0$. If $m = 1$ then z is simply called an *eigenvector* and λ is the corresponding *eigenvalue*. The point spectrum of A, denoted by $\sigma_p(A)$, is the set of all eigenvalues of A.

An operator A is called *Riesz spectral* if it has a Riesz basis consisting of generalized eigenvectors. Let A be a Riesz spectral operator and $\{z_{n,m} : n \in \mathbb{N},\ m = 1, \dots, m_n\}$ be a Riesz basis consisting of generalized eigenvectors where $z_{n,1}$ is an eigenvector associated with the eigenvalue λ_n of A and $(\lambda_n I - A)z_{n,k} = z_{n,k-1}$ for $k = 2, \dots, m_n$. Suppose that $|\lambda_n| \to \infty$ as $n \to \infty$, $(\Re\lambda_n)_{n\in\mathbb{N}}$ is bounded from above and $(m_n)_{n\in\mathbb{N}}$ is bounded. If in addition, A generates a $\mathcal{C}_0$-semigroup $(T(t))_{t\geq 0}$ then $T(t)$ can be written as a Fourier series. To see this, we express an element $z \in X$ by its Fourier representation

$$z = \sum_{n\in\mathbb{N}} \sum_{m=1}^{m_n} \langle z, \tilde{z}_{n,m}\rangle z_{n,m}$$

where $\tilde{z}_{n,m} = Q^*Qz_{n,m}$, and use the continuity of the operator $T(t)$ to obtain

$$T(t)z = \sum_{n\in\mathbb{N}} \sum_{m=1}^{m_n} \langle z, \tilde{z}_{n,m}\rangle T(t) z_{n,m}.$$

The next task is to determine the action of the semigroup on the generalized eigenvectors $z_{n,m}$ for each n and m. Let us start in the case of eigenvectors. Because $z_{n,1} \in D(A)$ it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t} T(t)z_{n,1} = T(t)Az_{n,1} = \lambda_n T(t) z_{n,1}.$$

Therefore $w_{n,1}(t) := T(t)z_{n,1}$ solves the initial-value problem

$$\begin{cases} \dot{w}(t) = \lambda_n w(t), & t > 0, \\ w(0) = z_{n,1}. \end{cases}$$

Therefore $T(t)z_{n,1} = e^{\lambda_n t} z_{n,1}$. We prove by induction that

$$T(t)z_{n,k} = \sum_{j=0}^{k-1} \frac{(-t)^j}{j!} e^{\lambda_n t} z_{n,k-j}, \qquad k = 1, \dots, m_n. \tag{A.3.2}$$

The basis step $k = 1$ has been already shown above. Suppose that (A.3.2) is true for $k \leq m_n - 1$. We show that it also true when k is replaced by $k+1$. By differentiation

$$\frac{\mathrm{d}}{\mathrm{d}t} T(t)z_{n,k+1} = T(t)Az_{n,k+1} = \lambda_n T(t)z_{n,k+1} - T(t)z_{n,k}.$$

Thus $w_{n,k+1}(t) = T(t)z_{n,k+1}$ solves the initial-value problem

$$\begin{cases} \dot{w}(t) = \lambda_n w(t) - T(t)z_{n,k}, & t > 0, \\ w(0) = z_{n,k+1}. \end{cases}$$

Using the variation of parameters formula and the induction hypothesis we have

$$\begin{aligned} w_{n,k+1}(t) &= e^{\lambda_n t} z_{n,k+1} - \int_0^t e^{\lambda_n (t-s)} T(s) z_{n,k} \, \mathrm{d}s \\ &= e^{\lambda_n t} z_{n,k+1} + \sum_{j=0}^{k-1} \frac{(-t)^{j+1}}{(j+1)!} e^{\lambda_n t} z_{n,k-j} \\ &= \sum_{j=0}^{k} \frac{(-t)^j}{j!} e^{\lambda_n t} z_{n,k+1-j}. \end{aligned}$$

This proves the induction step. Therefore, $T(t)$ has the Fourier representation

$$T(t)z = \sum_{n \in \mathbb{N}} \sum_{m=1}^{m_n} \sum_{j=0}^{m-1} \frac{(-t)^j}{j!} e^{\lambda_n t} \langle z, \tilde{z}_{n,m} \rangle z_{n,m-j}.$$

If $(\Re \lambda_n)_{n \in \mathbb{N}}$ is both bounded from above and below then it follows from this representation that A generates a $\mathcal{C}_0$-group.

A sufficient condition for a sequence of generalized eigenvectors to form a Riesz basis in a separable Hilbert space is the following improvement of the theorem of Guo [**29, Theorem 6.3**] by Xu and Weiss [**79, Theorem 2.4**]. The proof is based on properties of discrete spectral operators [**24**] and Bari's Theorem [**81**]. Recall that a sequence of vectors is called linearly independent if every finite subsequence is linearly independent.

Theorem A.3.1 (Guo-Xu-Weiss)**.** *Let A be a densely defined linear operator in a separable Hilbert space X with a nonempty resolvent set and compact resolvents. Suppose that $(z_n)_{n \in \mathbb{N}}$ is a Riesz basis in X and $(w_n)_{n \in \mathbb{N}}$ is a sequence of linearly independent generalized eigenvectors of A such that for some $m \in \mathbb{N}_0$, $(z_{n+m})_{n \in \mathbb{N}}$ and $(w_n)_{n \in \mathbb{N}}$ are quadratically close in the sense that*

$$\sum_{n \in \mathbb{N}} \|w_n - z_{n+m}\|^2 < \infty.$$

Then there exist generalized eigenvectors $y_1, \ldots, y_m$ of A such that $(y_n)_{n=1}^m \cup (w_n)_{n \in \mathbb{N}}$ is a Riesz basis in X.

By taking $m = 0$ in the preceding theorem, we see that a sequence of linearly independent generalized eigenvectors of A that is quadratically close to a Riesz basis in X is a Riesz basis in X as well.

B

ABSTRACT BOUNDARY CONTROL SYSTEMS

All throughout this chapter, X will denote a complex Hilbert space. To distinguish the inner products and norms between different Hilbert spaces, they are usually denoted with a subscript. For example, the inner product and the corresponding norm in X are denoted by $\langle\cdot,\cdot\rangle_X$ and $\|\cdot\|_X$, respectively. Our main references in this chapter are Salamon [**70**] and Tucsnak and Weiss [**77**] .

B.1 GELFAND TRIPLES

Let V and X be Hilbert spaces such that V is a dense subset of X and the embedding $V \subset X$ is continuous. The functional defined on X

$$\|z\|_* = \sup_{v\in V\setminus\{0\}} \frac{|\langle z,v\rangle_X|}{\|v\|_V}, \qquad \forall\, z \in X,$$

is a norm on X. Denote by V^* the completion of X with respect to the norm $\|\cdot\|_*$. The map $J : V^* \to V'$, where V' is the space of bounded conjugate linear functionals on V, defined by

$$\langle Jz, v\rangle_{V'\times V} = \lim_{n\to\infty} \langle z_n, v\rangle_X, \quad \forall\, z \in V^*,\ v \in V,$$

where $(z_n)_{n\in\mathbb{N}}$ is a sequence in X such that $\|z_n - z\|_* \to 0$, is a well-defined isometric isomorphism. By identifying elements of V^* and V' through the isomorphism J, we have

$$V \subset X \subset V'$$

with continuous and dense embeddings. Such triple is called a *Gelfand triple*. The space V^* is called the *dual* of V with respect to the *pivot space* X and $\|\cdot\|_*$ is the *dual norm*. Finally, we have

$$\langle z,v\rangle_{V'\times V} = \langle z,v\rangle_X, \quad \forall\, z \in X,\ v \in V.$$

Assume that $A : D(A) \to X$ is a densely defined operator in X with nonempty resolvent. Then A is closed and its adjoint $A^* : D(A^*) \to X$ is also a densely defined closed operator with nonempty resolvent. If $D(A^*)$ is equipped with the graph norm, then $D(A^*)$ is a Hilbert space, $A^* \in \mathcal{L}(D(A^*), X)$ and we have the Gelfand triple $D(A^*) \subset X \subset D(A^*)'$. Taking the adjoint of A^* as a bounded linear operator and identifying X with its dual, we have $A_e := (A^*)^* \in \mathcal{L}(X, D(A^*)')$. The operator A_e is an extension of A. Indeed, if $z \in D(A)$ then for all $w \in D(A^*)$

$$\langle A_e z, w\rangle_{D(A^*)'\times D(A^*)} = \langle z, A^* w\rangle_X = \langle Az, w\rangle_X = \langle Az, w\rangle_{D(A^*)'\times D(A^*)}.$$

Therefore $A_e z = Az$ for all $z \in D(A)$.

It can be shown that if A generates a $\mathcal{C}_0$-semigroup in X then A_e generates a $\mathcal{C}_0$-semigroup in $D(A^*)'$ and $(e^{tA_e})_{|X} = e^{tA}$ for all $t \geq 0$, [**77, Proposition 2.10.4**].

B.2 NONHOMOGENEOUS INITIAL VALUE PROBLEMS

Let $A : D(A) \to X$ be a generator of a $\mathcal{C}_0$-semigroup in X and A_e be the extension of A constructed in the previous section. Suppose that $f \in L^1_{\mathrm{loc}}([0,\infty); D(A^*)')$ and $z_0 \in D(A^*)'$. Consider the following nonhomegeneous initial value problem in the extended space $D(A^*)'$

$$\begin{cases} \dot{z}(t) = A_e z(t) + f(t), \quad t > 0, \\ z(0) = z_0. \end{cases} \tag{B.2.1}$$

A function $z \in L^1_{\mathrm{loc}}([0,\infty); X) \cap C([0,\infty); D(A^*)')$ is called a *weak solution* in $D(A^*)'$ of (B.2.1) if for every $w \in D(A^*)$ we have

$$\langle z(t) - z_0, w\rangle_{D(A^*)' \times D(A^*)} = \int_0^t \langle z(s), A^* w\rangle_X + \langle f(s), w\rangle_{D(A^*)' \times D(A^*)} \,\mathrm{d}s$$

for every $t \geq 0$. If z is a weak solution, then

$$z(t) = z_0 + \int_0^t A_e z(s) + f(s)\,\mathrm{d}s, \quad t \geq 0, \tag{B.2.2}$$

where the integral is computed with respect to the norm of $D(A^*)'$. As a consequence, z is absolutely continuous with values in $D(A^*)$ and has a derivative, computed with respect to the norm of $D(A^*)$, for a.e. $t \geq 0$ and it is given by the integrand in (B.2.2).

Weak solutions defined above is adapted from Tucsnak and Weiss [**77**]. This concept is stronger than the one in the literature due to the additional local integrability condition with respect to the norm in X, see [**19, Definition 3.1.6**] for example. If z is a weak solution of (B.2.2) in $D(A^*)'$ then necessarily it is given by the variation of parameters formula

$$z(t) = e^{tA_e} z_0 + \int_0^t e^{(t-s)A_e} f(s)\,\mathrm{d}s. \tag{B.2.3}$$

In particular, weak solutions are unique. The function z defined by (B.2.3) is called the *mild solution* of (B.2.1). Thus, every weak solution is a mild solution. However, the converse it not necessarily true.

A sufficient condition for the existence of a weak solution of (B.2.1) is that $z_0 \in X$ and $f \in H^1_{\mathrm{loc}}([0,\infty); D(A^*)')$ and, moreover, the weak solution has the regularity $z \in C([0,\infty); X) \cap C^1([0,\infty); D(A^*)')$, see [**77, Theorem 4.1.6**]. In other words, for initial data in X the trajectories lie in X even though the forcing functions have values in the extended space $D(A^*)'$ and the differential equation is posed in $D(A^*)'$.

B.3 CONTROL AND OBSERVATION OPERATORS

In this section, we are interested in the existence of a weak solution in $D(A^*)'$ for the initial-value problem

$$\begin{cases} \dot{z}(t) = A_e z(t) + Bu(t) + F(t), \quad t > 0, \\ z(0) = z_0, \end{cases} \tag{B.3.1}$$

where $z_0 \in X$, $u \in L^2_{\text{loc}}([0,\infty);U)$, U is a Hilbert space, $B \in \mathcal{L}(U, D(A^*)')$ and $F \in H^1_{\text{loc}}([0,\infty); D(A^*)')$. The function u is called an *input*, U is the *input space* and B is called a *control operator.*

Formally, if (B.3.1) has a weak solution in $D(A^*)'$ then it must be given by the variation of parameters formula

$$z(t) = e^{tA} z_0 + \int_0^t e^{sA_e} Bu(s)\, \mathrm{d}s + \int_0^t e^{sA_e} F(s)\, \mathrm{d}s.$$

With this observation, we are led to the following definition. Given an operator $B \in \mathcal{L}(U, D(A^*)')$ and $\tau \geq 0$ we define the *controllability map* $\Phi_\tau \in \mathcal{L}(L^2([0,\infty);U), D(A^*)')$ by

$$\Phi_\tau u = \int_0^\tau e^{sA_e} Bu(s)\, \mathrm{d}s.$$

The control operator $B \in \mathcal{L}(U, D(A^*)')$ is said to be an *admissible* for $(e^{tA})_{t\geq 0}$ if ran $\Phi_\tau \subset X$ for some $\tau > 0$. For admissible control operators, the associated operator Φ_τ can be regarded as a bounded operator into X. For a proof, see [**77, Proposition 4.2.2**].

Proposition B.3.1. *If $B \in \mathcal{L}(U, D(A^*)')$ is an admissible control operator for $(e^{tA})_{t\geq 0}$ then $\Phi_t \in \mathcal{L}(L^2([0,\infty);U), X)$ for every $t \geq 0$.*

Admissibility of the control operator is sufficient for the existence and uniqueness of a weak solution in $D(A^*)'$ for the problem (B.3.1) as stated in the succeeding theorem.

Theorem B.3.2. *Let $B \in \mathcal{L}(U, D(A^*)')$ be an admissible control operator for $(e^{tA})_{t\geq 0}$. For every $z_0 \in X$, $u \in L^2_{\text{loc}}([0,\infty);U)$ and $F \in H^1_{\text{loc}}([0,\infty), D(A^*)')$,*

$$z(t) = e^{tA} z_0 + \Phi_t u + \int_0^t e^{sA_e} F(s)\, \mathrm{d}s.$$

is the unique weak solution of (B.3.1) *and it satisfies*

$$z \in C([0,\infty); X) \cap H^1_{\text{loc}}([0,\infty); D(A^*)').$$

Proof. Consider the initial-value problems

$$\begin{cases} \dot{y}(t) = A_e y(t) + Bu(t), & t > 0, \\ y(0) = z_0, \end{cases} \tag{B.3.2}$$

and

$$\begin{cases} \dot{w}(t) = A_e w(t) + F(t), & t > 0, \\ w(0) = 0. \end{cases} \tag{B.3.3}$$

Then according to [**77, Proposition 4.2.5**], $y(t) = e^{tA} z_0 + \Phi_t u$ is the weak solution of (B.3.2) and $y \in C([0,\infty); X) \cap H^1_{\text{loc}}([0,\infty); D(A^*)')$. According to Appendix B.2, (B.3.3) has a weak solution $w \in C([0,\infty); X) \cap C^1([0,\infty); D(A^*)')$ and it is given by the mild solution

$$w(t) = \int_0^t e^{sA_e} F(s)\, \mathrm{d}s$$

It is easy to see that $z = y + w$ is the weak solution in $D(A^*)'$ of (B.3.1) and has the desired regularity. □

Let Y be a Hilbert space and $C \in \mathcal{L}(D(A), Y)$ where $D(A)$ is equipped with the graph norm. Consider the system

$$\begin{cases} \dot{z}(t) = Az(t), & t > 0, \\ y(t) = Cz(t), & t > 0, \\ z(0) = z_0, \end{cases} \tag{B.3.4}$$

where $z_0 \in D(A)$. Here, y is called the *output*, Y is the *output space* and C is an *observation operator*. The output y can be easily solved and it is given by $y(t) = Ce^{tA}z_0$.

For each $\tau \geq 0$, define the *observability map* $\Psi_\tau \in \mathcal{L}(D(A), L^2([0,\infty); Y))$ by

$$(\Psi_\tau z)(t) = \mathbf{1}_{[0,\tau]}(t) Ce^{tA}z, \qquad t \geq 0,$$

where $\mathbf{1}_{[0,\tau]}$ is the indicator function of $[0,\tau]$. The observation operator $C \in \mathcal{L}(D(A), Y)$ is said to be *admissible* for $(e^{tA})_{t\geq 0}$ if there exist $\tau > 0$ and $M_\tau \geq 0$ such that

$$\int_0^\tau \|Ce^{tA}z\|_Y^2 \,\mathrm{d}t \leq M_\tau \|z\|_X^2, \qquad \forall\, z \in D(A). \tag{B.3.5}$$

Therefore, C is admissible for the semigroup generated by A if and only if the operator Ψ_τ can be extended to a bounded linear operator $\Psi_\tau^e \in \mathcal{L}(X, L^2([0,\infty); Y))$. By density of $D(A)$ in X, this extension is unique. The definition of admissibility of observation operators is independent of the time $\tau > 0$. This is the content of the following proposition. A proof can be found in [**77, Proposition 4.3.2**].

Proposition B.3.3. *If $C \in \mathcal{L}(D(A), Y)$ is an admissible observation operator for $(e^{tA})_{t\geq 0}$ then Ψ_t has a unique extension $\Psi_t^e \in \mathcal{L}(X, L^2([0,\infty); Y))$ for every $t \geq 0$.*

Admissibility of control and observation operators are dual to each other. More, precisely we have the following theorem. For a proof, see [**77, Theorem 4.4.3**].

Theorem B.3.4. *The operator $B \in \mathcal{L}(U, D(A^*))$ is an admissible control operator for the semigroup generated by A if and only if $B^* \in \mathcal{L}(D(A^*), U)$ is an admissible observation operator for the semigroup generated by A^*.*

This means that the set-theoretic condition ran $\Phi_\tau \subset X$ is equivalent to the algebraic condition

$$\int_0^\tau \|B^* e^{tA^*} z\|_U^2 \,\mathrm{d}t \leq M_\tau \|z\|_X^2, \qquad \forall\, z \in D(A^*), \tag{B.3.6}$$

for some $M_\tau > 0$ independent of z. For concrete systems, the inequality (B.3.6) is easier to verify. For example, if the semigroup $(e^{tA^*})_{t\geq 0}$ can be expressed as a Fourier series then (B.3.6) can be verified using tools from nonharmonic Fourier analysis.

Let $B \in \mathcal{L}(U, D(A^*)')$ be an admissible control operator for the semigroup generated by A. The pair (A, B) is said to be *exactly controllable in time* $\tau > 0$ if ran $\Phi_\tau = X$. Exact controllability is equivalent to the statement that for every $z_0, z_1 \in X$ there exists $u \in L^2((0,\tau); U)$ such that the weak solution of the system

$$\begin{cases} \dot{z}(t) = A_e z(t) + Bu(t), & 0 < t < \tau, \\ z(0) = z_0, \end{cases} \tag{B.3.7}$$

in $D(A^*)'$ satisfies $z(\tau) = z_1$. The pair (A, B) is said to be *approximately controllable in time* $\tau > 0$ if ran Φ_τ is dense in X. This is equivalent to the following: For every $z_0, z_1 \in X$ and for every $\epsilon > 0$ there exists $u \in L^2((0,\tau); U)$ such that the weak solution of (B.3.7) satisfies $\|z(\tau) - z_1\|_X < \epsilon$.

Suppose that $C \in \mathcal{L}(D(A), X)$ is an admissible observation operator for the $\mathcal{C}_0$-semigroup $(e^{tA})_{t\geq 0}$. The pair (A, C) is said to be *exactly observable in time* $\tau > 0$ if there exists $m_\tau > 0$ such that

$$\int_0^\tau \|Ce^{tA}z\|_Y^2 \,\mathrm{d}t \geq m_\tau \|z\|_X^2, \qquad \forall\, z \in D(A). \tag{B.3.8}$$

The inequality (B.3.8) holds if and only if Ψ_τ is bounded from below. Therefore exact observability is equivalent to the statement that any initial state $z_0 \in X$ can be recovered continuously through the observation $y(t) = Cz(t)$, $0 \leq t \leq \tau$, through $z_0 = (\Psi_\tau^* \Psi_\tau)^{-1} \Psi_\tau^* y$. The pair (A, C) is said to be *approximately observable in time* $\tau > 0$ if ker $\Psi_\tau = \{0\}$. Approximate observability means that the the only initial data with zero output in $[0, \tau]$ is the zero initial data.

The controllability and observability concepts defined above are dual to each other, see [**77, Theorem 11.2.1**]

Theorem B.3.5. *Assume that $B \in \mathcal{L}(U, D(A^*)')$ is an admissible control operator for the semigroup generated by A and $\tau > 0$. The pair (A, B) is exactly controllable in time τ if and only if (A^*, B^*) is exactly observable in time τ. The pair (A, B) is approximately controllable in time τ if and only if (A^*, B^*) is approximately observable in time τ.*

B.4 NONHOMOGENEOUS BOUNDARY CONTROL SYSTEMS

Let Z, X and U be Hilbert spaces and $Z \subset X$ with continuous embedding. Consider the abstract initial-boundary value problem in X

$$\begin{cases} \dot{z}(t) = Lz(t) + F(t), \quad 0 < t < T, \\ Gz(t) = u(t), \quad 0 < t < T, \\ z(0) = z_0, \end{cases} \tag{B.4.1}$$

where $z_0 \in X$, $F \in H^1((0,T); X)$, $u \in L^2((0,T); U)$, $L \in \mathcal{L}(Z, X)$ and $G \in \mathcal{L}(Z, U)$. The spaces Z, X and U are called the *solution space*, *state space* and *input space*, respectively. The system (B.4.1) arises in the control of partial differential equations where the control acts on the whole or a part of the boundary and in ordinary differential equations with delay in their input, see Salamon [**70**] for examples.

Definition B.4.1. Let $L \in \mathcal{L}(Z, X)$ and $G \in \mathcal{L}(Z, U)$. The pair (L, G) is called a *boundary control system* if ker G is dense in X, ran $G = U$ and there exists $\beta > 0$ such that ker $(\beta I - L) \cap$ ker $G = \{0\}$ and $(\beta I - L)(\ker G) = X$.

If (L, G) is a boundary control system then the linear operator $A : D(A) \to X$, where $D(A) = \ker G$, defined by $Az = Lz$ is a densely defined operator on X and $A \in \mathcal{L}(D(A), X)$. Thus $\beta \in \rho(A)$ by the Banach Inverse Theorem. Therefore A is closed and from Appendix B.1 the operator A admits a unique extension $A_e \in \mathcal{L}(X, D(A^*)')$. For the proof of the following theorem, see [**77, Proposition 10.1.2**].

Theorem B.4.2. *Let (L, G) be a boundary control system. Then there exists a unique operator $B \in \mathcal{L}(D(A^*)', U)$, called the control operator associated with (L, G), such that*

$$L = A_e + BG.$$

For boundary control systems, the IBVP (B.4.1) in X can be written as a pure IVP in the extended space $D(A^*)'$. Using Theorem B.4.2, the following theorem can be shown, see [**77, Remark 10.1.4**] for the homogeneous case, i.e. $F = 0$.

Theorem B.4.3. *Let (L, G) be a boundary control system and $F \in H^1((0, T); X)$. A function $z \in C^1([0, T]; X) \cap C([0, T]; Z)$ satisfies the abstract initial boundary value problem* (B.4.1) *in X if and only if it satisfies the following pure initial value problem in $D(A^*)'$*

$$\begin{cases} \dot{z}(t) = A_e z(t) + Bu(t) + F(t), & 0 < t < T, \\ z(0) = z_0, \end{cases} \tag{B.4.2}$$

where B is the control operator associated with (L, G).

Sufficient conditions for the existence of solution of (B.4.1) are given in the following theorem. The proof of this theorem is similar to the homogeneous case in [**77, Proposition 10.1.8**].

Theorem B.4.4. *Let (L, G) be a boundary control system such that A generates a $\mathcal{C}_0$-semigroup in X and the corresponding control operator B is admissible for $(e^{tA})_{t \geq 0}$. Then for every $T > 0$, $z_0 \in Z$, $u \in H^1((0, T); U)$ and $F \in H^1((0, T); X)$ such that $Gz_0 = u(0)$, the system* (B.4.1) *has a unique solution $z \in C^1([0, T]; X) \cap C([0, T]; Z)$.*

C

PSEUDODIFFERENTIAL AND PARADIFFERENTIAL CALCULUS

In this section we present a short survey of results in pseudodifferential and paradifferential calculus that will be needed in deriving symmetrizers for boundary value problems. For the proofs and details we refer the readers to [**2, 4, 9, 10, 15, 17, 36, 54, 57, 56, 74**].

C.1 PSEUDODIFFERENTIAL OPERATORS

One of the motivations of studying pseudodifferential calculus is due to the observation that a partial differential operator can be written in terms of the Fourier transform through an appropriate symbol. To illustrate this, consider the differential operator

$$L = \sum_{|\alpha| \le m} a_\alpha(x) \partial^\alpha$$

where $a_\alpha \in \mathscr{C}_b^\infty(\mathbb{R}^d)$. Using the properties of the Fourier transform we obtain

$$\begin{aligned} (Lu)(x) &= \sum_{|\alpha| \le m} a_\alpha(x) \partial^\alpha u(x) = \sum_{|\alpha| \le m} a_\alpha(x) [\mathscr{F}^{-1}((i\,\cdot)^\alpha \mathscr{F} u)](x) \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x,\xi) \mathscr{F} u(\xi)\, \mathrm{d}\xi \end{aligned}$$

for every $u \in \mathscr{S}(\mathbb{R}^d)$, where

$$a(x,\xi) = \sum_{|\alpha| \le m} (i\xi)^\alpha a_\alpha(x) \tag{C.1.1}$$

and $\mathscr{F} u$ is the Fourier transform of u. Thus the differential operator L can be written in terms of the Fourier transform and the function a, called the *symbol* of L. Pseudodifferential calculus aims to generalize this to symbols that are not necessarily polynomial in ξ.

Let $m \in \mathbb{R}$ and $N \in \mathbb{N}$. Denote by S^m the set of all $a \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}^{N\times N})$ such that for all $\alpha, \beta \in \mathbb{N}_0^d$ there is $C_{\alpha,\beta} > 0$ such that

$$\sup_{x\in\mathbb{R}^d} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|^2)^{(m-|\beta|)/2}.$$

The elements of S^m are called *symbols* of order m. In light of the motivation discussed above, we call the variable x the *Fourier variable* and ξ its associated *frequency*. This is a special class of the more general type of symbols $S^m_{\varrho,\delta}$, $0 \le \delta \le \varrho \le 1$, in [**36**].

The set $S^m_{\varrho,\delta}$ consists of all $a \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}^{N\times N})$ with the property that for every $\alpha, \beta \in \mathbb{N}_0^d$ there is $C_{\alpha,\beta} > 0$ such that

$$\sup_{x\in\mathbb{R}^d} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|^2)^{(m+\delta|\alpha|-\varrho|\beta|)/2}.$$

Indeed, we have $S^m = S^m_{1,0}$. We let $S^{-\infty} = \bigcap_{m\in\mathbb{R}} S^m$ and $S^\infty = \bigcup_{m\in\mathbb{R}} S^m$.

If $n \leq m$ then $S^n \subset S^m$. If $a \in S^m$ and $b \in S^n$ then $ab \in S^{m+n}$ and $\partial_x^\alpha \partial_\xi^\beta a \in S^{m-|\beta|}$. In other words, the product of two symbols is again a symbol having an order equal to the sum of the orders of the symbol. Likewise, differentiation with respect to the frequency ξ reduces the order of the symbol by the same order as that of the differentiation.

Let us cite some examples of symbols. The function a defined by (C.1.1) is a symbol of order m. These symbols are called *differential symbols*. It is clear that $\lambda^m(\xi) := (1+|\xi|^2)^{m/2} \in S^m$ for any $m \in \mathbb{R}$ and these are called *Sobolev symbols*. It can be checked that $\mathscr{S}(\mathbb{R}^d) \subset S^{-\infty}$.

Our last example deals with homogeneous functions. A function $f : \mathbb{R}^d \to \mathbb{C}^N$ is said to be *homogeneous degree* m if $f(t\xi) = t^m f(\xi)$ for all $\xi \neq 0$ and $t > 0$. If f is differentiable at all points except at 0 and it is homogeneous degree m then $\partial_j f$ is also homogeneous of degree $m-1$ for all $1 \leq j \leq d$. To see this, define $g(s) = f(t(\xi + se_j))$ and $h(s) = t^m f(\xi + se_j)$, where e_j denotes the canonical unit vector in $\mathbb{R}^d$ in the jth direction. Because f is homogeneous, $g(s) = f(s)$ whenever $\xi \neq 0$, $t > 0$ and $|s|$ is small enough. The chain rule implies that $g'(0) = \nabla f(t\xi) \cdot (te_j) = t\partial_j f(t\xi)$ and $h'(0) = t^m \nabla f(\xi) \cdot e_j = t^m \partial_j f(\xi)$. Hence $\partial_j f(t\xi) = t^{m-1}\partial_j f(\xi)$, which proves that $\partial_j f$ is homogeneous degree $m-1$.

Assume that $a \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d \setminus \{0\}; \mathbb{C}^{N\times N})$ is bounded as well as all derivatives with respect to x and homogeneous degree m in ξ. Then there exists $\tilde{a} \in S^m$ such that $a = \tilde{a}$ in $|\xi| \geq 1$. Indeed, let us introduce a frequency cut-off function $\chi \in \mathscr{D}(\mathbb{R}^d)$ such that χ vanishes in a neighborhood of 0 and $\chi = 1$ for $|\xi| \geq 1$. Then the function

$$\tilde{a}(x,\xi) = \chi(\xi)a(x,\xi)$$

satisfies the requirement. If χ_1 is another frequency cut-off function which vanishes in a neighborhood of 0 and is equal to 1 for $|\xi| \geq 1$ and $\tilde{a}_1 = \chi_1 a$ then $\tilde{a} = a = \tilde{a}_1$ for $|\xi| \geq 1$ so that $\tilde{a} - \tilde{a}_1 \in \mathscr{D}(\mathbb{R}^d) \subset S^{-\infty}$. Therefore $\tilde{a}$ is unique modulo $S^{-\infty}$. We say that a property of a symbol is unique modulo S^m for some $m \in \mathbb{R} \cup \{\pm\infty\}$ if for any other symbol having the property their difference is a symbol in S^m.

For each $a \in S^m$, the operator $\mathrm{Op}(a) : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ defined by

$$(\mathrm{Op}(a)u)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x,\xi) \mathscr{F}u(\xi)\, \mathrm{d}\xi$$

is well-defined and continuous. The map $\mathrm{Op} : S^m \to \mathscr{B}(\mathscr{S}(\mathbb{R}^d))$ is injective. Here, $\mathscr{B}(\mathscr{S}(\mathbb{R}^d))$ denotes the space of continuous linear operators from $\mathscr{S}(\mathbb{R}^d)$ into itself. The operator $\mathrm{Op}(a)$ is called a *pseudodifferential operator of order* m and a is called its *symbol*.

For each $a \in S^m$ it can be shown that $\mathrm{Op}(a)^* \in \mathrm{Op}(S^m)$. However, the symbol of $\mathrm{Op}(a)^*$ is not the same as the adjoint a^* of a but it differs from a^* by a lower order symbol. More precisely, $\mathrm{Op}(a)^* - \mathrm{Op}(a^*) = \mathrm{Op}(b)$ for some $b \in S^{m-1}$. With this information, the operator $\mathrm{Op}(a)$ which is originally defined in $\mathscr{S}(\mathbb{R}^d)$ can be extended to $\mathscr{S}'(\mathbb{R}^d)$ by duality. The map $\mathrm{Op}(a) : \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ is defined by

$$\langle \mathrm{Op}(a)u, \varphi\rangle_{\mathscr{S}'(\mathbb{R}^d)\times\mathscr{S}(\mathbb{R}^d)} = \langle u, \mathrm{Op}(a)^*\varphi\rangle_{\mathscr{S}'(\mathbb{R}^d)\times\mathscr{S}(\mathbb{R}^d)}, \quad u \in \mathscr{S}'(\mathbb{R}^d),\ \varphi \in \mathscr{S}(\mathbb{R}^d).$$

This extension maps Sobolev spaces into Sobolev spaces continuously, i.e., $\mathrm{Op}(a) \in \mathcal{L}(H^s(\mathbb{R}^d); H^{s-m}(\mathbb{R}^d))$ for every $s \in \mathbb{R}$ whenever $a \in S^m$. In particular, operators associated with symbols of negative order are regularizing.

Given two operators F and G, we define the commutator $[F, G] = FG - GF$ whenever the products FG and GF are well-defined. For pseudo-differential operators corresponding to mollifiers, the following result will be used. For a proof, see [**9, Theorem C.14**].

Theorem C.1.1. *Let $\rho_\epsilon(x) = \epsilon^{-d}\rho(x/\epsilon)$ be a mollifier with the properties $\rho \in \mathscr{D}(\mathbb{R}^d)$, $\rho \geq 0$ and $\int_{\mathbb{R}^d} \rho = 1$. Let $R_\epsilon = \mathrm{Op}(\mathscr{F}\rho_\epsilon)$. For all $a \in W^{1,\infty}(\mathbb{R}^d)$, $u \in L^2(\mathbb{R}^d)$ and $j = 1, \dots, d$ there exists $C > 0$ independent of u, a, and $\epsilon \in (0, 1)$ such that*

$$\|[R_\epsilon, a\partial_j]u\|_{L^2(\mathbb{R}^d)} \leq C\|a\|_{W^{1,\infty}(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)} \tag{C.1.2}$$

and

$$\lim_{\epsilon \to 0} \|[R_\epsilon, a\partial_j]u\|_{L^2(\mathbb{R}^d)} = 0. \tag{C.1.3}$$

The previous theorem is used in proving that weak solutions of boundary value problems are strong solutions. This is done thanks to the regularizing operator R_ϵ. However, we also need an analogous result for the generalized trace of weak solutions, which have less regularity than the solution. For this, we need the following generalization of Theorem C.1.1: If $a \in W^{1,\infty}(\mathbb{R}^d)$ and $u \in H^{-1}(\mathbb{R}^d)$ then

$$\lim_{\epsilon \to 0} \|[R_\epsilon, a]u\|_{L^2(\mathbb{R}^d)} = 0. \tag{C.1.4}$$

Indeed, given $u \in H^{-1}(\mathbb{R}^d)$ there exist $k_j \in \mathbb{C}$ and $u_j \in L^2(\mathbb{R}^d)$ for $j = 0, 1, \dots, d$ such that

$$u = k_0 u_0 + \sum_{j=1}^{d} k_j \partial_j u_j.$$

Since $au_0, u_0 \in L^2(\mathbb{R}^d)$ we have $[R_\epsilon, a]u_0 = R_\epsilon(au_0) - aR_\epsilon u_0 \to 0$ in $L^2(\mathbb{R}^d)$. For each j the commutator $[R_\epsilon, a]\partial_j u_j$ can be rewritten as follows

$$\begin{aligned}
[R_\epsilon, a]\partial_j u_j &= R_\epsilon(a\partial_j u_j) - aR_\epsilon\partial_j u_j \\
&= R_\epsilon(a\partial_j u_j) - a\partial_j(R_\epsilon u_j) - a(R_\epsilon\partial_j u_j - \partial_j(R_\epsilon u_j)) \\
&= [R_\epsilon, a\partial_j]u_j - a[R_\epsilon, \partial_j]u_j.
\end{aligned}$$

Because the constant identity matrix I_N is in $W^{1,\infty}(\mathbb{R}^d)$ and $u_j \in L^2(\mathbb{R}^d)$, according to Theorem C.1.1 we have

$$\|[R_\epsilon, a]\partial_j u_j\|_{L^2} \leq \|[R_\epsilon, a\partial_j]u_j\|_{L^2} + \|a\|_{L^\infty}\|[R_\epsilon, \partial_j]u_j\|_{L^2} \to 0, \quad \text{as } \epsilon \to 0.$$

Taking the sum for j we obtain (C.1.4).

C.2 PSEUDODIFFERENTIAL OPERATORS WITH PARAMETER

In deriving a priori estimates for boundary value problems with smooth coefficients, the weighted Lebesgue spaces $L^2(\mathbb{R} \times (0, 1); e^{-\gamma t}\,\mathrm{d}t\,\mathrm{d}x)$ with $\gamma \geq 1$ will be used. For this reason we need a parameter version of the pseudodifferential calculus that was introduced above.

Without confusion, we use the same notation S^m to denote the set of all functions $a : \mathbb{R}^d \times \mathbb{R}^d \times [1, \infty) \to \mathbb{C}^{N\times N}$ with $a(\cdot, \cdot, \gamma) \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}^{N\times N})$ and for every $\gamma \geq 1$ and for each $\alpha, \beta \in \mathbb{N}_0^d$ there exists $C_{\alpha,\beta} > 0$ such that

$$\sup_{x\in\mathbb{R}^d} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi, \gamma)| \leq C_{\alpha,\beta}(\gamma^2 + |\xi|^2)^{(m-|\beta|)/2}.$$

The space S^m is a Fréchet space by taking the best constants $C_{\alpha,\beta}$ in the above inequality as seminorms.

A family $\{P^\gamma\}_{\gamma\geq\gamma_0} \subset \mathrm{Op}(S^m)$, where $\gamma_0 \geq 1$, is said to be of *order m* if for each $s \in \mathbb{R}$ there exists $C_s > 0$ such that

$$\sup_{\gamma\geq\gamma_0} \|P^\gamma\|_{\mathcal{L}(H^s_\gamma(\mathbb{R}^d);H^{s-m}_\gamma(\mathbb{R}^d))} \leq C_s.$$

If $\{P^\gamma\}_{\gamma\geq\gamma_0}$ is a family of order $m < 0$ then we have

$$\|P^\gamma u\|_{L^2(\mathbb{R}^d)} \leq C\gamma^m\|u\|_{L^2(\mathbb{R})}. \tag{C.2.1}$$

This estimate is important in absorption arguments, cf. [**9, Remark C.2**].

Given $a \in S^m$ the operator $\mathrm{Op}^\gamma(a) : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ defined by

$$\mathrm{Op}^\gamma(a) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x, \xi, \gamma)\mathscr{F}u(\xi)\,\mathrm{d}\xi$$

after extending to $\mathscr{S}'(\mathbb{R}^d)$ defines a bounded linear operator from $H^s_\gamma(\mathbb{R}^d)$ to $H^{s-m}_\gamma(\mathbb{R}^d)$ for each $s \in \mathbb{R}$ and for a fixed $\gamma \geq 1$. The following theorem states that for each fixed $s \in \mathbb{R}$ the operator norms of $\mathrm{Op}^\gamma(a)$ are uniformly bounded in γ.

Theorem C.2.1. *Let $m, n \in \mathbb{R}$, $a \in S^m$ and $b \in S^n$.*

1. *$\{\mathrm{Op}^\gamma(a)\}_{\gamma\geq1}$ is a family of order m.*
2. *$\{\mathrm{Op}^\gamma(a)^* - \mathrm{Op}^\gamma(a^*)\}_{\gamma\geq1}$ is a family of order $m - 1$.*
3. *$\{\mathrm{Op}^\gamma(a)\,\mathrm{Op}^\gamma(b) - \mathrm{Op}^\gamma(ab)\}_{\gamma\geq1}$ is a family of order $m + n - 1$*
4. *if either a or b is scalar-valued then $\{[\mathrm{Op}^\gamma(a), \mathrm{Op}^\gamma(b)] - \mathrm{Op}^\gamma([a, b])\}_{\gamma\geq1}$ is a family of order $m + n - 1$.*

Next, we state two parameter versions of the Garding's inequality.

Theorem C.2.2 (Garding's Inequality)**.** *Suppose that $a \in S^{2m}$ satisfies $2\Re a(x, \xi, \gamma) \geq \alpha(\gamma^2 + |\xi|^2)^m I_N$ for some $\alpha > 0$ and for all $(x, \xi, \gamma) \in \mathbb{R}^d \times \mathbb{R}^d \times [1, \infty)$. Then for every $\vartheta \in (0, \alpha)$ there exists $\gamma_0 = \gamma_0(\vartheta) \geq 1$ such that*

$$\Re\langle \mathrm{Op}^\gamma(a)u, u\rangle_{H^{-m}_\gamma(\mathbb{R}^d)\times H^m_\gamma(\mathbb{R}^d)} \geq \vartheta\|u\|^2_{H^m_\gamma(\mathbb{R}^d)}$$

for all $\gamma \geq \gamma_0$ and $u \in H^m_\gamma(\mathbb{R}^d)$.

Theorem C.2.3 (Sharp Garding's Inequality)**.** *If $a \in S^{2m}$ satisfies $2\Re a(x, \xi, \gamma) \geq 0$ for all $(x, \xi, \gamma) \in \mathbb{R}^d \times \mathbb{R}^d \times [1, \infty)$ then there exist $C > 0$ and $\gamma_0 \geq 1$ such that*

$$\Re\langle \mathrm{Op}^\gamma(a)u, u\rangle_{H^{-m}_\gamma(\mathbb{R}^d)\times H^m_\gamma(\mathbb{R}^d)} \geq -C\|u\|^2_{H^{m-\frac{1}{2}}_\gamma(\mathbb{R}^d)}$$

for every $\gamma \geq \gamma_0$ and $u \in H^m_\gamma(\mathbb{R}^d)$.

C.3 PARADIFFERENTIAL OPERATORS WITH PARAMETER

The concepts introduced in the previous section apply to problems with smooth coefficients. These definitions can be also extended to symbols with limited regularity in the variable x and such formulations are useful for problems with coefficients having limited regularity.

We begin by a defining a symbol. Given $m \in \mathbb{R}$ and $k \in \mathbb{N}_0$ we denote by Γ_k^m the set of all functions $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}^{N\times N}$ satisfying the following properties.

1. $a(x, \cdot, \gamma) \in \mathscr{C}^\infty(\mathbb{R}^d)$ for almost all $x \in \mathbb{R}^d$ and for every $\gamma \geq 1$
2. For every $(\xi, \gamma, \beta) \in \mathbb{R}^d \times [1, \infty) \times \mathbb{N}_0^d$, $\partial_\xi^\beta a(\cdot, \xi, \gamma) \in W^{k,\infty}(\mathbb{R}^d)$ and there exists $C_\beta > 0$ such that

$$\|\partial_\xi^\beta a(\cdot, \xi, \gamma)\|_{W^{k,\infty}(\mathbb{R}^d)} \leq C_\beta(\gamma^2 + |\xi|^2)^{(m-|\beta|)/2}.$$

It is clear that $S^m \subset \Gamma_k^m$ for every $k \in \mathbb{N}_0$. The elements of Γ_k^m are called *symbols of order m and regularity k*.

In contrast to symbols in S^m, symbols in Γ_k^m, in general, are not associated with bounded operators between Sobolev spaces. However, this is possible for the class of symbols in Γ_k^m with some spectral properties. Let Σ_k^m be the set of all $a \in \Gamma_k^m$ such that for some $\epsilon \in (0, 1)$ independent of (ξ, γ) we have

$$\operatorname{supp} \mathscr{F}(a(\cdot, \xi, \gamma)) \subset B(0; \epsilon(\gamma^2 + |\xi|^2)^{\frac{1}{2}}) \tag{C.3.1}$$

for all $\xi \in \mathbb{R}^d$ and $\gamma \geq 1$. The symbols in Σ_k^m are necessarily $\mathscr{C}^\infty$ with respect to x, see [**35, Theorem 7.1.14**] for example.

Symbols in Σ_k^m can be associated with bounded operators between weighted Sobolev spaces. For the proof of the following theorem, we refer to [**17**].

Theorem C.3.1. *For all $a \in \Sigma_k^m$, $k \geq 0$, one can associate a family of operators $\{\mathrm{Op}^\gamma(a)\}_{\gamma\geq 1}$, where $\mathrm{Op}^\gamma(a) : \mathscr{F}^{-1}(\mathscr{E}'(\mathbb{R}^d)) \to \mathscr{C}_b^\infty(\mathbb{R}^d)$ is defined by*

$$(\mathrm{Op}^\gamma(a)u)(x) = \frac{1}{(2\pi)^d}\langle e^{ix\cdot} a(x, \cdot, \gamma), \mathscr{F}u\rangle_{\mathscr{C}^\infty(\mathbb{R}^d)\times\mathscr{E}'(\mathbb{R}^d)}$$

This definition of $\mathrm{Op}^\gamma(a)$ coincides with the one defined when $a \in S^m$. For all $s \in \mathbb{R}$ and $\gamma \geq 1$, $\mathrm{Op}^\gamma(a)$ extends in a unique way into an element in $\mathcal{L}(H_\gamma^s(\mathbb{R}^d), H_\gamma^{s-m}(\mathbb{R}^d))$ and there exists $C_s > 0$ such that

$$\sup_{\gamma\geq 1} \| \mathrm{Op}^\gamma(a)\|_{\mathcal{L}(H_\gamma^s(\mathbb{R}^d), H_\gamma^{s-m}(\mathbb{R}^d))} \leq C_s.$$

It can be shown that $\Sigma_k^m \subset S_{1,1}^m$ where $S_{1,1}^m$ is the class of symbols defined in Appendix C.1. For a proof see [**9**] or [**17**].

The next step is how to obtain a symbol in Σ_k^m from a given symbol in Γ_k^m. We can do this by multiplying an appropriate cut-off function on the Fourier side so that the spectral condition (C.3.1) is satisfied. The idea is to cut-off the higher frequencies associated to the variable of limited regularity, i.e., with respect to the variable x.

A function $\chi \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d \times [1, \infty); [0, \infty))$ is called an *admissible frequency cut-off function* if there exist $0 < \epsilon_1 < \epsilon_2 < 1$ satisfying

$$\chi(\eta, \xi, \gamma) = \begin{cases} 1, & \text{if } |\eta| \leq \epsilon_1(\gamma^2 + |\xi|^2)^{\frac{1}{2}} \\ 0, & \text{if } |\eta| \geq \epsilon_2(\gamma^2 + |\xi|^2)^{\frac{1}{2}} \end{cases}$$

and if for all $\alpha, \beta \in \mathbb{N}_0^d$ there exists $C_{\alpha,\beta} > 0$ such that

$$|\partial_\eta^\alpha \partial_\xi^\beta \chi(\eta, \xi, \gamma)| \leq C_{\alpha,\beta}(\gamma^2 + |\xi|^2)^{-(|\alpha|+|\beta|)/2}, \qquad \forall (\xi, \gamma) \in \mathbb{R}^d \times [1, \infty). \tag{C.3.2}$$

An example of an admissible frequency cut-off function based on the parameter version of the Littlewood-Paley decomposition can be found [**9, p. 489**].

By taking the convolution of the inverse Fourier transform of an admissible frequency cut-off function and a symbol in Γ_k^m one obtains a symbol in Σ_k^m. This is the content of the following proposition, see [**17**] for a proof.

Proposition C.3.2. *Let χ be an admissible cut-off function. The operator R^χ : $\Gamma_k^m \to \Sigma_k^m$ given by*

$$(R^\chi(a))(\cdot, \xi, \gamma) = \mathscr{F}^{-1}(\chi(\cdot, \xi, \gamma)) \star a(\cdot, \xi, \gamma) \tag{C.3.3}$$

is well-defined and

$$R^\chi(\Gamma_k^m) \subset \Sigma_k^m = \{a \in \Gamma_k^m : \operatorname{supp} \mathscr{F}(a(\cdot, \xi, \gamma)) \subset B(0; \epsilon_2(\gamma^2 + |\xi|^2)^{\frac{1}{2}}\}.$$

If $k \geq 1$ then $a - R^\chi(a) \in \Gamma_{k-1}^{m-1}$ for all $a \in \Gamma_k^m$. In particular, if χ_1 and χ_2 are two admissible frequency cut-off functions then $R^{\chi_1}(a) - R^{\chi_2}(a) \in \Gamma_{k-1}^{m-1}$ for every $a \in \Gamma_k^m$ with $k \geq 1$.

Suppose that χ is an admissible frequency cut-off function. For each element $a \in \Gamma_k^m$ we define the operator $T_a^{\chi,\gamma}$ for $\gamma \geq 1$ by

$$T_a^{\chi,\gamma} = \mathrm{Op}^\gamma(R^\chi(a)).$$

The operator $T_a^{\chi,\gamma}$ is called a *paradifferential operator* with parameter $\gamma \geq 1$ associated with the symbol a and the cut-off function χ. For each $b \in W^{k,\infty}(\mathbb{R}^d)$ with $k \geq 1$ one can show that

$$T_{i^{|\alpha|}\xi^\alpha b}^{\chi,\gamma} = T_b^{\chi,\gamma} \partial^\alpha. \tag{C.3.4}$$

In deriving a priori estimates for partial differential operators with coefficients that are at least Lipschitz, it is enough to consider their paradifferential version and use the following error estimate in [**17**].

Theorem C.3.3. *There exists $C > 0$ such that for all $a \in W^{1,\infty}(\mathbb{R}^d)$, $u \in L^2(\mathbb{R}^d)$ and $\gamma \geq 1$ we have*

$$\|a\partial_j u - T_a^{\chi,\gamma} \partial_j u\|_{L^2(\mathbb{R}^d)} + \|au - T_a^{\chi,\gamma} u\|_{H_\gamma^1(\mathbb{R}^d)} \leq C\|a\|_{W^{1,\infty}(\mathbb{R}^d)} \|u\|_{L^2(\mathbb{R}^d)}.$$

Finally, we also have the following results similar to pseudodifferential operators with parameter.

Theorem C.3.4. *For all $a \in \Gamma_1^m$ and $b \in \Gamma_1^n$ we have $ab \in \Gamma_1^{m+n}$. Moreover, $\{T_a^{\chi,\gamma}\}_{\gamma \geq 1}$, $\{(T_a^{\chi,\gamma})^* - T_{a^*}^{\chi,\gamma}\}_{\gamma \geq 1}$, $\{T_{ab}^{\chi,\gamma}\}_{\gamma \geq 1}$ and $\{T_a^{\chi,\gamma} T_b^{\chi,\gamma} - T_{ab}^{\chi,\gamma}\}_{\gamma \geq 1}$ are families of paradifferential operators of orders m, $m-1$, $m+n$, and $m+n-1$, respectively.*

Theorem C.3.5 (Garding's Inequality)**.** *Assume that $a \in \Gamma_1^{2m}$ satisfies $2\Re a(x, \xi, \gamma) \geq \alpha(\gamma^2 + |\xi|^2)^m I_N$ for some $\alpha > 0$ and for all $(x, \xi, \gamma) \in \mathbb{R}^d \times \mathbb{R}^d \times [1, \infty)$. Then there exist $\gamma_0 \geq 1$ and $C > 0$ such that for all $\gamma \geq \gamma_0$ and $u \in H_\gamma^m(\mathbb{R}^d)$ we have*

$$\Re \langle T_a^{\chi,\gamma} u, u \rangle_{H_\gamma^{-m}(\mathbb{R}^d) \times H_\gamma^m(\mathbb{R}^d)} \geq \frac{\alpha}{4} \|u\|_{H_\gamma^m(\mathbb{R}^d)}^2.$$

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