BOUNDARY FEEDBACK STABILIZATION OF THE EULER-BERNOULLI BEAM WITH GLOBAL OR LOCAL KELVIN-VOIGT AND VISCOUS DAMPING

by

Gilbert R. Peralta

a master's thesis submitted to the

Department of Mathematics and Computer Science
College of Science
University of the Philippines Baguio
Baguio City

as partial fulfillment of the
requirements for the degree of
Master of Science in Mathematics

November 2009

This is to certify that this Master's Thesis entitled **"Boundary Feedback Stabilization of the Euler-Bernoulli Beam with Local or Global Kelvin-Voigt and Viscous Damping"**, prepared and submitted by **Gilbert R. Peralta** to fulfill part of the requirements for the degree of **Master of Science in Mathematics**, was successfully defended and approved in November 7, 2009.

RICARDO C.H. DEL ROSARIO, PH.D.
Thesis Adviser

PRISCILLA SUPNET-MACANSANTOS, PH.D.
Thesis Co-Adviser

TEOFINA A. RAPANUT, PH.D.
Thesis Reader

The Department of Mathematics and Computer Science endorses the acceptance of this Master's Thesis as partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

RIZAVEL CORSINO-ADDAWE, M.SC.
Chair, Department of Mathematics
and Computer Science

This Mater's Thesis is hereby officially accepted as partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

WILFREDO V. ALANGUI, M.SC.
Dean, College of Science

Abstract

Boundary Feedback Stabilization of the Euler-Bernoulli Beam with Local or Global Kelvin-Voigt and Viscous Damping

Gilbert R. Peralta
University of the Philippines Baguio

Adviser:
Ricardo C.H. del Rosario, Ph.D.
Genome Institute of Singapore

Co-Adviser:
Priscilla Supnet-Macansantos, Ph.D.
University of the Philippines Baguio

We analyze a boundary feedback system of a class of non-homogeneous cantilever Euler-Bernoulli beam with Kelvin-Voigt and viscous damping. We consider global damping where the bending moment is controlled by the linear feedback of angular velocity and the shear force is controlled by the linear feedback of velocity. The exponential stability of the system will be established and in the process we also show that the non-real spectrum of the infinitesimal generator lies in its point spectrum. Furthermore, the exponential decay of the energy of the system and the analyticity are analyzed. Moreover, an upper bound for the type of the generator shall be given. We also consider the beam with local damping, and for this case, the bending moment is controlled by the linear feedback of rotation angle and angular velocity while the shear force is controlled by the linear feedback of displacement and velocity. The exponential stability of the locally damped beam will be established. We use the operator semigroup technique, the multiplier technique, and the contradiction argument on the frequency domain method to establish the exponential stability of the semigroup associated with the beam equations. Representation theorems for sesquilinear forms were used to describe the spectrum of the generator of the semigroup.

Acknowledgments

I would like to express my deepest gratitude to the following, for without them the success of this research will not be possible.

First of all, I would like to thank my adviser Dr. Ricardo del Rosario for giving this topic for me and providing me the necessary papers and books I needed in this thesis. I am grateful that under your supervision, I have learned a lot of things in analysis.

To my co-adviser Dr. Priscilla Supnet-Macansantos for teaching me the fundamentals of functional and complex analysis, and I think that her supervision under these courses gave me maturity in these subjects.

I also thank my other professors in the graduate program, Dr. Teofina Rapanut and Prof. Wilfredo Alangui. I am indebted that you shared your knowledge in mathematics, showed its beauty and gave me inspiration in teaching mathematics.

The Department of Mathematics and Computer Science for granting me study load credits to finish my thesis and my last course work. Also, for letting me print drafts of my thesis and references.

To Kristine Hazel Batin for love and understanding, and to my sister Veronica for correcting some typographical errors. To Jensen Hahn and Johannes Drei for inspiration.

Finally, to God for the guidance and source of our knowledge.

Contents

Chapter 1

Introduction

The partial differential equations describing the longitudinal and transversal vibrations of the Euler-Bernoulli beam have been extensively studied for the past decades. There are numerous results establishing the exponential stability, strong asymptotic stability, analyticity and convergence of the approximations of the system with and without internal (or Kelvin-Voigt) damping and with different boundary conditions. Several methods have been developed to obtain these results. Also, harmonic vibrations $y(t,x) = Y(x)e^{i\omega t}$ of a homogeneous cantilever beam have been determined by transforming the system into a Volterra integral equation of the second kind and using a standard method in solving the obtained integral equation (see Tricomi [35]).

The transverse displacement $y(t,x)$ of an unforced beam at position x and time t having length ℓ, thickness h, width b, non-homogeneous density $\rho_0(x)$, non-homogeneous stiffness (or Young's modulus) $EI(x)$, non-homogeneous Kelvin-Voigt damping $c_DI(x)$ and (external) viscous air damping $c_a(x)$ is given by the Euler-Bernoulli equation

$$\begin{aligned}\rho_0(x)hb\frac{\partial^2 y(t,x)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y(t,x)}{\partial x^2}\right)\\ + \frac{\partial^2}{\partial x^2}\left(c_DI(x)\frac{\partial^3 y(t,x)}{\partial x^2 \partial t}\right) + c_a(x)\frac{\partial y(t,x)}{\partial t} = 0,\end{aligned} \tag{1.1}$$

where $(t,x) \in (0,\infty)\times(0,\ell)$. In this model, the axial direction coincides with the x-axis. For simplicity, we sometimes denote the derivative with respect to x by a prime and the derivative with respect to t by a dot, that is, $\partial y/\partial x = y'$ and $\partial y/\partial t = \dot{y}$, respectively. For a detailed discussion on the modeling aspect, we refer the readers to the works of Banks, Ito and Wang [2].

The damping is called global if it is distributed entirely in the domain and it is called local if it is distributed only in a subdomain. More precisely, we say that the Kelvin-Voigt damping c_DI is **globally distributed** if $c_DI(x) \geq c_0 > 0$ for all $x \in [0,\ell]$ and we say that it is **locally distributed** if $c_DI(x) \geq c_0 > 0$ for all x in some proper subinterval of $[0,\ell]$ and $c_DI = 0$ elsewhere.

Recent works in problems involving exponential stability of elastic materials involved with globally or locally distributed damping. Most of these were devoted either to viscous damping, i.e, damping is proportional to velocity (see Chen et. al. [6], Liu [24] and Liu and Zheng [28]) or to viscoelastic damping (see Zhao et. al. [39] and Liu and Zheng [28]) but not both. In this study, we shall consider both non-homogeneous viscoelastic Kelvin-Voigt and viscous damping. Numerical results using finite dimensional subspaces generated by cubic B-splines of the Euler-Bernoulli beam, where the density, Young's modulus, Kelvin-Voigt and viscous damping are all constants, are available in Cagnol and Zolésio [5].

In the absence of internal or Kelvin-Voigt damping and air damping, system (1.1) under the boundary conditions

$$\begin{aligned} EI(x)\frac{\partial^2 y(t,x)}{\partial x^2}\bigg|_{x=\ell} &= -\frac{\partial}{\partial x}\left(EI(x)\frac{\partial^2 y(t,x)}{\partial x^2}\right)\bigg|_{x=\ell} = 0, \quad t\in(0,\infty),\\ EI(x)\frac{\partial^2 y(t,x)}{\partial x^2}\bigg|_{x=0} &= k_1\frac{\partial y(t,0)}{\partial x} + k_2\frac{\partial^2 y(t,0)}{\partial t\partial x}, \qquad t\in(0,\infty),\\ -\frac{\partial}{\partial x}\left(EI(x)\frac{\partial^2 y(t,x)}{\partial x^2}\right)\bigg|_{x=0} &= k_3 y(t,0) + k_4\frac{\partial y(t,0)}{\partial t}, \qquad t\in(0,\infty), \end{aligned}$$

has been studied by Guo and Huang [17] and they gave necessary and sufficient conditions for the exponential stability of the system. The mechanical meaning of the above boundary conditions is that both ends are free, the bending moment at $x = 0$ is controlled by the linear feedback of rotation angle and angular velocity, while the shear force is controlled by the linear feedback of displacement and velocity.

For the system (1.1) with clamped edge boundary conditions

$$y(t,0) = y(t,\ell) = y'(t,0) = y'(t,\ell) = 0,$$

and without air damping, the exponential stability of the system has been established by K. Liu and Z. Liu [25]. For the beam equation without internal damping, exponential stability using the Riesz basis approach can be found in Guo [13, 14]. Existence and uniqueness of the class of equations which include the partial differential equation (1.1) can be found in Banks et. al. [2].

In this paper, we are interested in showing the exponential stability of the system where the bending moment M and shear force $-\partial M/\partial x$ is controlled via feedback, that is, we are interested in the boundary conditions of the form

$$y(t,0) = \frac{\partial y(t,0)}{\partial x} = 0, \quad M(t,\ell) = m(t), \quad -\frac{\partial M(t,\ell)}{\partial x} = h(t), \tag{1.2}$$

where

$$M(t,x) = -\left(EI(x)\frac{\partial^2 y(t,x)}{\partial x^2} + c_D I(x)\frac{\partial^3 y(t,x)}{\partial x^2\partial t}\right).$$

We consider the case where the functions $m(t)$ and $h(t)$ are given in feedback form, that is,

$$\begin{bmatrix} m(t) \\ h(t) \end{bmatrix} = F \begin{bmatrix} \dot{y}'(t,\ell) \\ \dot{y}(t,\ell) \end{bmatrix} + G \begin{bmatrix} y'(t,\ell) \\ y(t,\ell) \end{bmatrix} \tag{1.3}$$

where F and G are two diagonal matrices in $M_2(\mathbb{C})$. The matrices F and G allow a general description of cases where the bending moment is controlled via a linear combination of rotation angle and angular velocity and where the shear force is controlled via a linear combination of displacement and velocity. In the case where $m(t) = h(t) = 0$ and $c_a(x) = 0$, the exponential stability was already exhibited by Chen et. al. [7].

Let $F = \operatorname{diag}(k_2, k_4)$ and $G = \operatorname{diag}(k_1, k_3)$. In this study, we are interested in the following stabilization problem:

> *Under what conditions on the functions $\rho = hb\rho_0, EI, c_D I$, c_a and on the parameters $k_1, k_2, k_3,$ and k_4 do the semigroup associated with the PDE* (1.1) *under the initial conditions $y(0,x) = w_0(x)$, $\dot{y}(0,x) = v_0(x)$ and boundary conditions* (1.2) *satisfying* (1.3) *exponentially decays?*

A general scheme for analyzing Galerkin approximations of abstract second order (in time) variational systems of the form

$$\langle \ddot{w}(t), \psi \rangle_{V^*,V} + \sigma_2(\dot{w}(t), \psi) + \sigma_1(w(t), \psi) = \left\langle \tilde{\mathcal{B}}u(t), \psi \right\rangle_{V^*,V},$$

under the initial conditions $w(0) = w_0$ and $\dot{w}(0) = v_0$ can be found in Banks et. al. [2], and Banks and del Rosario [3]. Other approximation schemes in linear viscoelasticity are presented in Liu and Zheng [27]. In the above system, σ_1 and σ_2 are sesquilinear forms from $V \times V$ to $\mathbb{C}$ and $\tilde{\mathcal{B}}$ is the control operator on an input space to the dual space V^* of V. We remark that this variational system includes the partial differential equation (1.1) under the boundary conditions (1.2) with $m(t) = h(t) = 0$.

The system (1.1) under the boundary conditions (1.2) satisfying (1.3) with $G = 0$ and without Kelvin-Voigt and air damping was discussed in the book of Liu and Zheng [28] (see also del Rosario [9]). They have studied three feedback schemes and gave sufficient conditions for each scheme to obtain the exponential stability of the said system. Interestingly, these conditions are also sufficient for the exponential stability of the system that we consider provided that both the Kelvin-Voigt and air damping are continuous and the Kelvin-Voigt damping is globally distributed.

For the Euler-Bernoulli beam with nonlinear boundary feedback we refer the readers to the works of Guo and Song [15] and in the paper of Berrahmoune [4]. For angular velocity and torque controls of the undamped Euler-Bernoulli beam we refer the readers to Tucsnak and Weiss [36]. Inverse problems of an undamped beam were discussed in Gladwell [12]. The system we are investigating is dissipative, but a general method for exhibiting exponential stability of non-dissipative systems can be found in K. Liu, Z. Liu, and B. Rao [26].

A necessary assumption to verify the controllability of the system and the convergence of the resulting finite-dimensional approximating system when numerical methods are implemented is the stability of the semigroup associated from the system. Most of the theorems which guarantee controllability and convergence rely on the exponential stability of the associated semigroup, but strong convergence has been also considered for the controllability and convergence of strongly stable systems (see Oostven, Curtain and Ito [31], Oostven [30], and del Rosario [8].)

The partial differential equations describing flexible systems with actuators are abstractly formulated in the form

$$\dot{z}(t) = Az(t) + Bu(t) + F(t), \quad z(0) = z_0$$

where A is an infinitesimal generator of a semigroup on a Hilbert space. The Hilbert space is defined according to the smoothness requirements of the modeling equations. The stabilizability of the system will be established if we can find a gain operator K, which is linear and bounded, such that $A - BK$ generates an exponentially stable C_0-semigroup $S(t) = e^{(A-BK)t}$. Stabilizability is an important aspect in control theory. From control theory, it is known that if the system is stabilizable, then existence and uniqueness of a solution of the associated operator algebraic Ricatti equation is guaranteed. If the C_0-semigroup generated by A is already exponentially stable, then the stabilizability of the system can be established by taking K to be the zero operator.

In our results, we use the operator semigroup technique, multiplier technique and a contradiction argument on the frequency domain method. Loosely speaking, the frequency domain method is based on the boundedness of the imaginary axis of the resolvent of the associated semigroup generator in order to establish the exponential stability of the C_0-semigroup on a particular Hilbert space (see Gearhart [11], Huang [19], and Prüss [33]). The frequency domain method has been successfully applied by Liu and Zheng [28] to show exponential stability of various dissipative systems arising in mechanics such as linear thermoelastic systems, linear viscoelastic systems, linear thermoviscoelastic systems and as well as elastic systems with shear and boundary damping. Furthermore, we use some results on m-sectorial operators (see the classic work of Kato [22]) to prove that the C_0-semigroup is analytic in the case where the Kelvin-Voigt damping is distributed globally. For more details about the applications of exponential stability and analyticity in the theory of partial differential equations we refer the readers to Miklavčič [29], Renardy and Roberts [34], and Tucsnak and Weiss [36].

The theory of semigroups of bounded linear operators in a Banach space deals with the exponential functions in infinite dimensional spaces. It is concerned with the problem of finding a bounded linear operator valued-function $S(t)$, where $t \geq 0$, satisfying

$$S(t + s) = S(t)S(s), \quad S(0) = I.$$

This problem was independently investigated by E. Hille [18] and K. Yosida [37] in 1948.

To illustrate the semigroup technique, consider the wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \Delta u(x,t), \quad -\infty < t < \infty,$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and Δ is the Laplacian operator

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2},$$

with initial conditions

$$u(x,0) = f(x), \quad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = g(x)$$

where the functions f and g are given. If we let $v = \partial u/\partial t$ then the wave equation can be written in vector form as

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

with initial condition

$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} f(x) \\ g(x) \end{bmatrix}.$$

Notice that we have rewritten the wave equation in the form $\mathrm{d}y/\mathrm{d}t = \alpha y$. Since we know that the equation $\mathrm{d}y/\mathrm{d}t = \alpha y$ has the ordinary exponential function $y = Ce^{\alpha t}$ as a solution, it is suggested that the wave equation may be solved by properly defining the exponential function of the operator

$$\begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix}$$

in a suitable function space. This is the motivation of studying linear semigroup theory, its application to Cauchy's problem. For a more detailed discussion regarding the wave equation and its semigroup formulation the reader may consult Yosida [38]. In summary, the semigroup technique is to reformulate a time-dependent partial differential equation or a system of time-dependent partial differential equations as an ordinary differential equation on a suitable function space and study the properties of the associated operator using the results in semigroup theory.

As we have said earlier, there are several methods in showing the exponential stability of system (1.1). However, these methods are not readily extendable to the system we consider due to the non-homogeneity of the beam and the more general boundary condition at the end $x = \ell$. These additional constraints will give way in introducing some techniques to solve our problem and one approach is that we require the stiffness and internal damping to be sufficiently smooth. Also, difficulties arise when we consider Kelvin-Voigt and air damping.

To prove exponential stability, we will use the result that states that when a linear operator generates a strongly continuous semigroup of contractions on a Hilbert space then the semigroup is exponentially stable if and only if the imaginary axis lies in the resolvent of the operator and the imaginary axis of its resolvent is bounded (see Figure 1). Thus, the first step is to define the suitable Hilbert space on which our system will be studied. For us to show that the linear operator, which is associated with the abstract formulation of the system, generates a strongly continuous semigroup, we need to establish that it is densely defined, dissipative and the origin lies in its resolvent set. This result is a consequence of the well-known Lumer-Phillips Theorem. In this direction, the concept of m-accretive operators will be utilized to show that the operator is indeed densely defined.

From the above remarks, exponential stability can be achieved once we prove that the resolvent set of the generator contains the imaginary axis and that this part of the resolvent is bounded. The former will be shown to be true. This can be easily done if the infinitesimal generator has a compact inverse (see Le Gall et. al. [23], Guo [13, 14], Guo, Wang, and Yung [16], and Liu and Zheng [28]). But it is hard to decide whether the generator enjoys this property due to the presence of the structural damping. Knowing this limitation, we will use a method which is a hybrid of the methods presented by Banks et. al. [2], Chen et. al. [7], and Liu and Zheng [28].

To outline the said method, Riesz representation theorems for sesquilinear forms will be used to rewrite the infinitesimal generator in terms of two self-adjoint operators, one is bounded and the other has a compact inverse. Other characterizations of these self-adjoint operators shall be given, and from these we will obtain a relationship between the point spectrum and nonreal spectrum of the infinitesimal generator. We would like to point out that this approach is presented in Chen et. al. [7] and this method was successfully used in showing the stability of Euler-Bernoulli beam and Timoshenko beam models. For instance, see the papers of K. Liu and Z. Liu [25], Guo and Huang [17], and Zhao, Liu, and Zhang [39]. Afterwards, we will consider a unique continuation condition together with certain hypotheses of smoothness on the variable coefficient functions to finally prove that the imaginary axis lies in the resolvent of the infinitesimal generator. We then use a contradiction argument wherein we will be assuming that the system is not exponentially stable. The Resonance Theorem and multiplier technique will then give a certain contradiction.

This study is organized as follows. We will discuss some preliminaries on Chapter 2. Specifically, we will present some definitions and theorems regarding the resolvent of compact and self-adjoint operators, Sobolev spaces, interpolation inequalities, accretive and dissipative operators, linear semigroups and as well as analytic semigroups. In Chapter 3, we will establish the exponential stability of the non-homogeneous Euler-

Bernoulli beam (1.1) with boundary conditions of the form (1.2) and (1.3), where $G = 0$, and the Kelvin-Voigt damping is globally distributed. Also, we will give a necessary condition in order for the energy of the system to decay exponentially. Furthermore, we give an upper bound for the growth bound or type of the generator and show that the generator is also analytic. In Chapter 4, we also prove that, under some smoothness requirements on the density, stiffness, internal damping and viscous damping coefficient functions, if the Kelvin-Voigt damping is locally distributed on $[0, a]$ then the associated semigroup is also exponentially stable.

SEMIGROUP FORMULATION

m-accretive operators

$\downarrow$

THEOREM 2.21
If $0 \in \rho(\mathcal{A})$, $\overline{\mathcal{D}(\mathcal{A})} = \mathcal{H}$
and $\mathrm{Re}\,\langle \mathcal{A}z, z\rangle_{\mathcal{H}} \leq 0, \forall z \in \mathcal{H}$,
then $e^{\mathcal{A}t}$ is a C_0-semigroup
of contractions on $\mathcal{H}$.

$\longrightarrow$

THEOREM 2.22

HYPOTHESIS
$e^{\mathcal{A}t}$ is a C_0-semigroup
of contractions on a
Hilbert space $\mathcal{H}$.

$\longleftarrow$

STATE SPACE

Define the Hilbert
space $\mathcal{H}$.

$\downarrow$

EXPONENTIAL STABILITY

Resonance Theorem,
Contradiction Argument,
and Multiplier Technique

$\longrightarrow$

CONCLUSION
$e^{\mathcal{A}t}$ is exponentially stable iff
$i\mathbb{R} \subset \rho(\mathcal{A})$
and
$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$

$\uparrow$

Unique Continuation Condition

$\uparrow$

Relate the nonreal spectrum
and point spectrum of $\mathcal{A}$.

$\uparrow$

Express $\mathcal{A}$ in terms of
two self-adjoint operators.

$\uparrow$

Representation theorems
for sesquilinear forms

SPECTRAL PROPERTIES OF THE GENERATOR

Figure 1: Frequency Domain Method

Chapter 2

Preliminary Concepts

In this chapter we give some definitions and results in functional analysis which will be needed in the later chapters. Closed, compact, and self-adjoint operators and their spectral properties will be discussed in Section 1 and they can be found in the works of Miklavčič [29] and Yosida [38]. Sobolev spaces, Sobolev embedding theorem and interpolation inequalities were discussed in Section 2 and they were based on Adams [1], Liu and Zheng [28], and Miklavčič [29]. In Section 3, we state some results on absolutely continuous semigroups generated by dissipative systems and they were based in Engel and Nagel [10], Liu and Zheng [28] and Pazy [32]. Finally, Section 4 devotes to analytic semigroups and the results in this section can be also found in Miklavčič [29] and in the book of Kato [22].

2.1 Some Notes in Functional Analysis

Let X and Y be normed spaces with the same scalar field. A mapping T from a subspace of X, called the **domain** of T and denoted by $\mathcal{D}(T)$, into Y is called a **linear operator** from X to Y if

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty$$

for all $x, y \in \mathcal{D}(T)$ and for all scalars α and β. The set $\mathcal{R}(T) = \{Tx \,|\, x \in \mathcal{D}(T)\}$ is called the **range** of T. The **null space** or **kernel** of T is the set $\mathcal{N}(T) = \{x \in \mathcal{D}(T) \,|\, Tx = 0\}$. A linear operator T is one-to-one if and only if $\mathcal{N}(T) = \{0\}$. The operator T is said to be a **densely defined** operator if $\mathcal{D}(T)$ is dense in X, that is, $\overline{\mathcal{D}(T)} = X$. If S is another linear operator from X to Y such that $\mathcal{D}(T) \subset \mathcal{D}(S)$ and $Sx = Tx$ for all $x \in \mathcal{D}(T)$, then S is called an **extension** of T and T is called a **restriction** of S.

A mapping $T : \mathcal{D}(T) \subset X \to Y$ is said to be **bounded** if it maps bounded sets into bounded sets. In the case that T is a linear operator, T is bounded if and only if we can find a nonnegative real number m such that

$$\|Tx\|_Y \leq m\|x\|_X$$

for all $x \in \mathcal{D}(T)$. A linear operator is bounded if and only if it is continuous. The collection of all bounded linear operators from X to Y with $\mathcal{D}(T) = X$ will be denoted by $\mathcal{L}(X,Y)$. Further, we let $\mathcal{L}(X) = \mathcal{L}(X,X)$. If X is a Banach space and $\mathcal{L}(X)$ is equipped with the norm

$$\|T\|_{\mathcal{L}(X)} = \sup_{\|x\|\leq 1} \|Tx\|_X = \sup_{\|x\|=1} \|Tx\|_X,$$

then $\mathcal{L}(X)$ is a Banach space.

A linear operator T from X to Y is said to be **closed** if for every sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{D}(T)$ such that

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} Tx_n = y,$$

provided that these two limits exist, we have $x \in \mathcal{D}(T)$ and $Tx = y$.

Theorem 2.1. *If $T : \mathcal{D}(T) \subset X \to X$ is closed, then $T + \lambda I$ is also closed for any scalar λ.*

Proof. Let $\{x_n\}_{n=1}^{\infty} \subset \mathcal{D}(T)$ be such that

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} (T + \lambda I)x_n = y.$$

Since $Tx_n = (T + \lambda I)x_n - \lambda x_n$, it follows that

$$\lim_{n\to\infty} Tx_n = y - \lambda x.$$

Because T is closed, then $x \in \mathcal{D}(T)$ and $Tx = y - \lambda x$. Therefore $x \in \mathcal{D}(T + \lambda I)$ and $(T + \lambda I)x = y$ and so $T + \lambda I$ is closed. □

For the proof of the following theorem we refer the readers to Yosida [38, Proposition 3, p. 79].

Theorem 2.2. *Let T be a linear operator from X to Y. If T is one-to-one, then T is closed if and only if T^{-1} is closed.*

Theorem 2.3 (Closed Graph Theorem). *If X and Y are Banach spaces and $T : X \to Y$ is closed, then $T \in \mathcal{L}(X,Y)$.*

Let T be a linear operator in a normed linear space X with scalar field $\mathbb{K}$, where $\mathbb{K}$ is either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers. The **resolvent set** of T, denoted by $\rho(T)$, is defined to be the set of all scalars $\lambda \in \mathbb{K}$ for which there exists a bounded linear operator $R(\lambda) \in \mathcal{L}(X)$ such that

(1) for every $y \in X$ we have that $R(\lambda)y \in \mathcal{D}(T)$ and $(T - \lambda I)R(\lambda)y = y$,

(2) $R(\lambda)(T - \lambda I)x = x$ for all $x \in \mathcal{D}(T)$.

When $\lambda \in \rho(T)$, the bounded linear operator $R(\lambda)$ is called the **resolvent** of T at λ and will be denoted by $(T-\lambda I)^{-1}$. The set $\sigma(T) = \mathbb{K} \setminus \rho(T)$ is called the **spectrum** of T. The set of $\lambda \in \mathbb{K}$ for which there exists $x \in \mathcal{D}(T) \setminus \{0\}$ such that $Tx = \lambda x$ is called the **point spectrum** of T and it is denoted by $\sigma_p(T)$. Therefore the point spectrum of T is the set of all scalars $\lambda \in \mathbb{K}$ such that $T - \lambda I$ is not one-to-one. The elements of $\sigma_p(T)$ are called the **eigenvalues** of T and the elements of $\mathcal{N}(T-\lambda I) \setminus \{0\}$ are called the **eigenvectors** of T corresponding to the eigenvalue λ.

Note that if $\rho(T)$ is nonempty then T is closed. Indeed, let $\lambda \in \rho(T)$. Suppose $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty}(T-\lambda I)x_n = y$, where $\{x_n\}_{n=1}^{\infty} \subset \mathcal{D}(T)$. By the continuity of the resolvent $R(\lambda)$ we have

$$\lim_{n\to\infty} R(\lambda)(T-\lambda I)x_n = R(\lambda) \lim_{n\to\infty}(T-\lambda I)x_n = R(\lambda)y.$$

Using the second property of the resolvent we have $R(\lambda)(T-\lambda I)x_n = x_n$ for all $n \in \mathbb{N}$. From the uniqueness of limits and the first property of the resolvent we have $x = R(\lambda)y \in \mathcal{D}(T)$. Hence $(T-\lambda I)x = (T-\lambda I)R(\lambda)y = y$. Therefore $T - \lambda I$ is closed and this implies that $T = (T-\lambda I) + \lambda I$ is also a closed operator by Theorem 2.1.

Suppose that T is a closed operator in a Banach space X. Assume that $\lambda \in \rho(T)$. Suppose $x, y \in \mathcal{D}(T)$ satisfy $(T-\lambda I)x = (T-\lambda I)y$. Then applying $R(\lambda)$ both sides we get $x = R(\lambda)(T-\lambda I)x = R(\lambda)(T-\lambda I)y = y$, which shows that $T-\lambda I$ is one-to-one. Let $y \in X$. Since $x = R(\lambda)y \in \mathcal{D}(T)$ then $y = (T-\lambda I)R(\lambda)y = (T-\lambda I)x$. Thus $T-\lambda I$ is also onto. Therefore $T-\lambda I$ is one-to-one and onto. Conversely, assume that $T-\lambda I$ is one-to-one and onto, where T is a closed operator. Then $T-\lambda I$ is closed, and by Theorem 2.2 $(T-\lambda I)^{-1}$ is also closed. By the Closed Graph Theorem we have $(T-\lambda I)^{-1} \in \mathcal{L}(X)$ and it follows that $\lambda \in \rho(T)$. Thus, we have the following theorem.

Theorem 2.4. *If T is a closed operator in a Banach space X, then $\lambda \in \rho(T)$ if and only if $T-\lambda I$ is one-to-one and onto.*

Let X and Y be two Banach spaces. The operator $T \in \mathcal{L}(X,Y)$ is said to be **compact** if every bounded sequence $\{x_n\}_{n=1}^{\infty}$ in X has a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ such that the sequence $\{Tx_{n_j}\}_{j=1}^{\infty}$ converges to some element $y \in Y$.

Theorem 2.5 (Riesz-Schauder). *Let X be a Banach space and $T \in \mathcal{L}(X)$ be a compact operator. If $\lambda \neq 0$ is not an eigenvalue of T, then $\lambda \in \rho(T)$.*

The proof of this theorem was given in Yosida [38, Theorem 1, p. 283]. The following theorem states that the compact operators in $\mathcal{L}(X)$ constitute a two-sided ideal of the algebra $\mathcal{L}(X)$ (see [38, p. 278]).

Theorem 2.6. *Let X be a Banach space. The product of a compact operator $T \in \mathcal{L}(X)$ with a bounded linear operator $S \in \mathcal{L}(X)$ is compact.*

Let H be a Hilbert space and T be a linear operator in H. Define the **Hilbert space adjoint** $T^* : \mathcal{D}(T^*) \to H$ as follows. We have $y \in \mathcal{D}(T^*)$ if and only if there exists $z \in H$ such that $(Tx, y) = (x, z)$ for all $x \in \mathcal{D}(T)$. Because $\overline{\mathcal{D}(T)} = H$, it follows that for $y \in \mathcal{D}(T^*)$ we can find a unique element $z \in H$ such that $(Tx, y) = (x, z)$ for all $x \in \mathcal{D}(T)$ and in this case we define $T^*y = z$.

A linear operator T in a Hilbert space is called **symmetric** if

$$(Tx, y) = (x, Ty)$$

for all $x, y \in \mathcal{D}(T)$. A linear operator T is said to be **self-adjoint** if T is densely defined and $T = T^*$. If $T \in \mathcal{L}(H)$, then T is self-adjoint if and only if T is symmetric. Note that self-adjoint operators are closed. The following theorem characterizes the spectrum of self-adjoint operators (see [29, Theorem 2.6.2]).

Theorem 2.7. *If T is a self-adjoint operator in a Hilbert space H, then $\sigma(T) \subset \mathbb{R}$.*

2.2 Sobolev Spaces

Let Ω be a nonempty open subset of $\mathbb{R}^n$ where $n \in \mathbb{N}$. Let $C^m(\Omega)$, where $m \in \mathbb{N} \cup \{0\}$ or $m = \infty$, be the space of all m times continuously differentiable functions on Ω and $C_0^m(\Omega)$ be the space of $C^m(\Omega)$ functions with compact support in Ω. We define $C_B^m(\Omega)$ to be the set of those functions $f \in C^m(\Omega)$ for which

$$\|f\|_{m,\infty} = \max_{|\alpha| \le m} \sup_{x \in \Omega} |D^\alpha f(x)| < \infty.$$

Under the norm $\|\cdot\|_{m,\infty}$, $C_B^m(\Omega)$ is a Banach space.

For $p \in [1, \infty]$, we let

$$L^p_{\mathrm{loc}}(\Omega) = \{u \mid u \in L^p(K) \text{ for each compact } K \subset \Omega\}.$$

When $u \in L^1_{\mathrm{loc}}(\Omega)$ and α is a multi-index, we say that u has an **αth weak derivative** (or that $D^\alpha u$ exists) if there exists $v \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$(-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi = \int_\Omega v \varphi \tag{2.1}$$

for all $\varphi \in C_0^\infty(\Omega)$. The function φ is called a **test function**. If u has an αth weak derivative then there exists a unique $v \in L^1_{\mathrm{loc}}(\Omega)$ such that (2.1) is true for all $\varphi \in C_0^\infty(\Omega)$, and in this case we define $D^\alpha u = v$.

For $m \in \mathbb{N} \cup \{0\}$ and $p \in [1, \infty]$ define the **Sobolev space**

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \mid D^\alpha u \text{ exists and } D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \le m\}.$$

For $u \in W^{m,p}(\Omega)$, define the norm $\|\cdot\|_{W^{m,p}(\Omega)}$ by

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

if $1 \leq p < \infty$ and

$$\|u\|_{W^{m,\infty}(\Omega)} = \max_{|\alpha| \leq m} \|D^\alpha u\|_{L^\infty(\Omega)}$$

if $p = \infty$. When $p = 2$, we let $W^m(\Omega) = W^{m,2}(\Omega)$. For $u, v \in W^m(\Omega)$, define the inner product $\langle \cdot, \cdot \rangle_{W^m(\Omega)}$ by

$$\langle u, v \rangle_{W^m(\Omega)} = \sum_{|\alpha| \leq m} \int_\Omega (D^\alpha u)(\overline{D^\alpha v}).$$

The Sobolev space $W^{m,p}(\Omega)$ under the norm $\|\cdot\|_{W^{m,p}(\Omega)}$ is a Banach space and $W^m(\Omega)$ equipped with the inner product $\langle \cdot, \cdot \rangle_{W^m(\Omega)}$ is a Hilbert space.

We define $H^{m,p}(\Omega)$ to be the closure of $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ relative to the $W^{m,p}$-norm. If $p \in [1, \infty)$ and $m \in \mathbb{N} \cup \{0\}$ then (see [29, Theorem 3.5.5])

$$W^{m,p}(\Omega) = H^{m,p}(\Omega).$$

The following theorem is presented in Miklavčič [29, Theorem 3.5.3], but no proof was given there. Because we will use this theorem frequently in our discussions, we will give a proof of this statement.

Theorem 2.8. *Let Ω be a nonempty open subset of $\mathbb{R}^n$. If m is a nonnegative integer, $1 < p < \infty$, $u \in W^{m,p}(\Omega)$ and $v \in C^m_B(\Omega)$, then $uv \in W^{m,p}(\Omega)$ and*

$$\|uv\|_{W^{m,p}(\Omega)} \leq 2^m (1+m)^{\frac{n}{p}} \|u\|_{W^{m,p}(\Omega)} \|v\|_{m,\infty}.$$

Proof. It is clear that $uv \in W^{m,p}(\Omega)$ and so it remains for us to show the given estimate. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be two multi-indices such that $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq m$ and $\beta \leq \alpha$, that is, $\beta_i \leq \alpha_i$ for each $1 \leq i \leq n$. From the Binomial Theorem we have

$$2^k = \sum_{j=0}^{k} \binom{k}{j}$$

for each $k \in \mathbb{N} \cup \{0\}$. Hence it follows that

$$\binom{\alpha_i}{\beta_i} \leq 2^{\alpha_i}$$

for all $1 \leq i \leq n$, and consequently,

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n} \leq 2^{\alpha_1} \cdots 2^{\alpha_n} \leq 2^m. \tag{2.2}$$

Now, observe that $\{\alpha \,|\, |\alpha| \leq m\} \subset \{0, 1, \ldots, m\}^n$ and so we have

$$\sum_{|\alpha| \leq m} 1 = \mathrm{Card}(\{\alpha \,|\, |\alpha| \leq m\}) \leq (1+m)^n.$$

Using (2.2), Leibniz' formula and Hölder's inequality we get

$$\begin{aligned}
\|uv\|^p_{W^{m,p}(\Omega)} &= \sum_{|\alpha| \leq m} \int_\Omega |D^\alpha(uv)|^p \\
&\leq \sum_{|\alpha| \leq m} \int_\Omega \left(\sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |D^\beta u||D^{\alpha-\beta} v| \right)^p \\
&\leq 2^{mp} \sum_{|\alpha| \leq m} \int_\Omega \left(\sum_{|\beta| \leq m} |D^\beta u||D^{\alpha-\beta} v| \right)^p \\
&\leq 2^{mp} \sum_{|\alpha| \leq m} \int_\Omega \left(\sum_{|\beta| \leq m} |D^\beta u|^p \right) \left(\sum_{|\beta| \leq m} |D^{\alpha-\beta} v|^q \right)^{\frac{p}{q}}
\end{aligned}$$

where q satisfies $\frac{1}{p} + \frac{1}{q} = 1$. Note that

$$\sum_{|\beta| \leq m} |D^{\alpha-\beta} v|^q \leq (1+m)^n \|v\|^q_{m,\infty}$$

and so

$$\begin{aligned}
\|uv\|^p_{W^{m,p}(\Omega)} &\leq 2^{mp}(1+m)^{\frac{np}{q}} \|v\|^p_{m,\infty} \sum_{|\alpha| \leq m} \sum_{|\beta| \leq m} \int_\Omega |D^\beta u|^p \\
&\leq 2^{mp}(1+m)^{n(\frac{p}{q}+1)} \|u\|^p_{W^{m,p}(\Omega)} \|v\|^p_{m,\infty}.
\end{aligned}$$

Raising both sides to $\frac{1}{p}$ proves the theorem. □

Notice that $C_0^\infty(\Omega)$ is a subspace of $H^{m,p}(\Omega)$. For nonnegative integers m and $p \in [1, \infty)$, we define $H_0^{m,p}(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in $H^{m,p}(\Omega)$. Similarly, we let $H_0^m(\Omega) = H_0^{m,2}(\Omega)$. Then $H_0^{m,p}(\Omega)$ is a Banach space with norm $\|\cdot\|_{W^{m,p}(\Omega)}$ and $H_0^m(\Omega)$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{W^m(\Omega)}$. Further, we have $C_0^m(\Omega) \subset H_0^{m,p}(\Omega)$ and thus $H_0^{m,p}(\Omega)$ is the closure of $C_0^m(\Omega)$ in $W^{m,p}(\Omega)$.

Definition 2.9. For $a \in \mathbb{R}^n \setminus \{0\}$ and $\theta \in (0, \pi]$ define

$$\mathrm{cone}(a, \theta) = \{x \in \mathbb{R}^n \,|\, x \cdot a \geq |x||a| \cos\theta \text{ and } |x| \leq |a|\}.$$

A nonempty open set Ω in $\mathbb{R}^n, n \in \mathbb{N}$, is said to have the **cone property** if there exist $h \in (0, \infty)$, $\theta \in (0, \pi]$ such that for every $x \in \Omega$ we can find an element $a \in \mathbb{R}^n$ satisfying the properties $|a| = h$ and $x + \mathrm{cone}(a, \theta) \subset \Omega$.

The following proposition, together with the succeeding theorem will establish the compact embedding of the Sobolev spaces we are interested with (since our domain is in $\mathbb{R}$).

Proposition 2.10. *Let $b > 0$. Then $(0,b) \subset \mathbb{R}$ has the cone property.*

Proof. Let $\theta = \pi/2$ and $h = b/4$. Then $\operatorname{cone}(a,\pi/2) = \{x \in \mathbb{R} \,|\, xa \geq 0 \text{ and } |x| \leq |a|\}$ for each $a \in \mathbb{R} \setminus \{0\}$. If $a > 0$ then $\operatorname{cone}(a,\pi/2) = [0,a]$ and if $a < 0$ then $\operatorname{cone}(a,\pi/2) = [a,0]$. Let $x \in (0,b)$. First assume that $x \leq b/2$. If we take $a = b/4$ then $|a| = h$ and $x + \operatorname{cone}(a,\pi/2) = x + [0,b/4] = [x, x+b/4] \subset (0,b)$. Now suppose that $x > b/2$. If we take $a = -b/4$ then $x + \operatorname{cone}(a,\pi/2) = [x-b/4, x] \subset (0,b)$. Therefore $(0,b)$ has the cone property. □

Theorem 2.11. *Suppose Ω is bounded and satisfies the cone property. If $j \geq 0, m \geq 1, 1 \leq p < \infty, 1 \leq q < \infty, \frac{1}{q} > \frac{1}{p} - \frac{m}{n}$ and $\{u_i\}_{i=1}^{\infty}$ is a bounded sequence in $W^{j+m,p}(\Omega)$, then there exist integers $n_1 < n_2 < \cdots$ and $v \in W^{j,q}(\Omega)$ such that*

$$\lim_{i\to\infty} \|u_{n_i} - v\|_{W^{j,q}(\Omega)} = 0,$$

in other words, the embedding $W^{j+m,p}(\Omega) \hookrightarrow W^{j,q}(\Omega)$ is compact.

For the proof of this theorem see Miklavčič [29, Theorem 3.6.16]. In a particular case, the above theorem implies that the embedding $H^2(\Omega) \hookrightarrow L^2(\Omega)$ is compact whenever Ω is a bounded domain satisfying the cone property. A norm equivalent to the usual norm of Sobolev spaces is given in the following theorem (see Adams [1, Corollary 4.16]).

Theorem 2.12. *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain and has the cone property. Then the functional $(\!(u)\!)_{W^{m,p}(\Omega)}$ defined by*

$$(\!(u)\!)_{W^{m,p}(\Omega)} = \left(\|u\|_{L^p(\Omega)}^p + \sum_{|\alpha|=m} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}$$

is a norm, equivalent to the usual norm $\|\cdot\|_{W^{m,p}(\Omega)}$ in $W^{m,p}(\Omega)$.

We now state some interpolation inequalities (see Adams [1, Theorem 4.17] and Liu and Zheng [28, Theorem 1.4.5]).

Theorem 2.13. *If $\Omega \subset \mathbb{R}^n$ is bounded and has the cone property, and if $1 \leq p < \infty$, then there exists a constant $K = K(m,p,\Omega)$ such that for $0 \leq j \leq m$ and any $u \in W^{m,p}(\Omega)$,*

$$\|u\|_{W^{j,p}(\Omega)} \leq K \|u\|_{W^{m,p}(\Omega)}^{j/m} \|u\|_{L^p(\Omega)}^{(m-j)/m}.$$

Before stating the Gagliardo-Nirenberg Inequality we introduce some notation. For $p > 0$, let $|u|_{p,\Omega} = \|u\|_{L^p(\Omega)}$. For $p < 0$, set $h = \lfloor -n/p \rfloor, -\alpha = h + n/p$ and define

$$|u|_{p,\Omega} = \begin{cases} \displaystyle\sum_{|\beta|=h} \sup_{x\in\Omega} |D^\beta u(x)|, & \text{if } \alpha = 0, \\ \displaystyle\sum_{|\beta|=h} \sup_{x,y\in\Omega} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}, & \text{if } \alpha > 0. \end{cases}$$

Theorem 2.14 (Gagliardo-Nirenberg Inequality). *Let j and m be any integers satisfying $0 \le j < m$, and let $1 \le q \le \infty$, $1 \le r \le \infty$, $p \in \mathbb{R}$, $j/m \le a \le 1$ such that*

$$\frac{1}{p} - \frac{j}{n} = a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}.$$

For any $u \in W^{m,r}(\Omega) \cap L^q(\Omega)$, where Ω is a bounded domain with smooth boundary, there exist constants $K_1, K_2 > 0$ such that

$$|D^j u|_{p,\Omega} \le K_1 |D^m u|^a_{r,\Omega} |u|^{1-a}_{q,\Omega} + K_2 |u|_{q,\Omega} \tag{2.3}$$

with the following exception: if $1 < r < \infty$ and $m - j - n/r$ is a nonnegative integer, then (2.3) holds only for a satisfying $j/m \le a < 1$.

2.3 Semigroup Theory

In this section, we present some definitions and results in the theory of operator semigroups.

Definition 2.15. If A is an operator on $\mathcal{D}(A) \subset H$ into a Hilbert space H, we say that A is **dissipative** if $\mathrm{Re}\,\langle Ax, x\rangle_H \le 0$ for all $x \in \mathcal{D}(A)$.

Definition 2.16. A family $\{S(t)\}_{t\ge0}$ of bounded linear operators in a Banach space X is called a **strongly continuous semigroup** or a C_0-**semigroup** if

(1) $S(0) = I$,

(2) $S(t_1 + t_2) = S(t_1)S(t_2)$ for all $t_1, t_2 \ge 0$ (semigroup property),

(3) $S(\cdot)x : [0, \infty) \to X$ is continuous for each $x \in X$ (strong continuity).

We associate an operator $A : \mathcal{D}(A) \subset X \to X$ with the C_0-semigroup $\{S(t)\}_{t\ge0}$. The domain of A is given by

$$\mathcal{D}(A) = \left\{ x \in X \;\middle|\; \lim_{h\downarrow 0} \frac{S(h)x - x}{h} \text{ exists} \right\}$$

and for $x \in \mathcal{D}(A)$ we define A as

$$Ax = \lim_{h \downarrow 0} \frac{S(h)x - x}{h}.$$

We call A the **infinitesimal generator** of the C_0-semigroup $\{S(t)\}_{t \geq 0}$. Given any operator A in X, if A coincides with the infinitesimal generator of the C_0-semigroup $\{S(t)\}_{t \geq 0}$, then we say that A **generates** a strongly continuous semigroup $\{S(t)\}_{t \geq 0}$. We also denote $S(t)$ by e^{tA}.

For every strongly continuous semigroup $\{S(t)\}_{t \geq 0}$ we can find constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that $\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$ for all $t \geq 0$. The C_0-semigroup $\{S(t)\}_{t \geq 0}$ is said to be **exponentially stable** if there exist constants $\alpha > 0$ and $M \geq 1$ such that

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{-\alpha t}$$

for all $t \geq 0$. If $\|S(t)\|_{\mathcal{L}(X)} \leq 1$ for all $t \geq 0$, then the C_0-semigroup $\{S(t)\}_{t \geq 0}$ is called a C_0-**semigroup of contractions**.

The abstract Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} &= Au(t), \qquad \text{for } t \geq 0, \\ u(0) &= x_0 \end{cases}$$

associated with a closed operator $A : \mathcal{D}(A) \subset X \to X$ in a Banach space X is called **well-posed** if

(1) for every $x \in \mathcal{D}(A)$ there exists a unique solution $u(\cdot, x)$ of the abstract Cauchy problem,

(2) $\mathcal{D}(A)$ is dense in X, and

(3) for every sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{D}(A)$ satisfying $\lim_{n \to \infty} x_n = 0$, we have

$$\lim_{n \to \infty} u(t, x_n) = 0$$

uniformly on compact intervals $[0, t_0]$.

This definition of well-posedness of the abstract Cauchy problem is based on Engel and Nagel [10]. Intuitively, this definition expresses what we want for the solutions of problems: the existence, uniqueness and continuous dependence on the data. The following result is based in [10, Corollary 6.9].

Theorem 2.17. *For a closed operator $A : \mathcal{D}(A) \subset X \to X$ in a Banach space X, the associated abstract Cauchy problem is well-posed if and only if A generates a strongly continuous semigroup on X.*

Next, we shall state some generation theorems regarding strongly continuous semigroup of contractions.

Theorem 2.18 (Hille-Yosida). *A linear unbounded operator $A : \mathcal{D}(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup of contractions $\{S(t)\}_{t\geq 0}$ if and only if*

(1) *A is closed and $\overline{\mathcal{D}(A)} = H$,*

(2) *$(0, \infty) \subset \rho(A)$ and for every $\lambda > 0$,*

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(H)} \leq \frac{1}{\lambda}.$$

Theorem 2.19. *Let A be a densely defined linear operator in a Hilbert space H. Then A generates a C_0-semigroup of contractions on H if and only if A is dissipative and $\mathcal{R}(I - A) = H$.*

The following theorem states that the condition $\mathcal{R}(I - A) = H$ can be replaced by $\mathcal{R}(\lambda_0 I - A) = H$, where $\lambda_0 > 0$.

Theorem 2.20 (Lumer-Phillips). *Let A be a linear operator with a dense domain $\mathcal{D}(A)$ in a Hilbert space H. If A is dissipative and there is a $\lambda_0 > 0$ such that $\mathcal{R}(\lambda_0 I - A) = H$, then A is the infinitesimal generator of a C_0-semigroup of contractions on H.*

To show that an operator in a Hilbert space generates a C_0-semigroup of contractions we will use the following consequence (see Liu and Zheng [28, Theorem 1.2.4]) of the Lumer-Phillips Theorem.

Theorem 2.21. *Let A be a linear operator with dense domain $\mathcal{D}(A)$ in a Hilbert space H. If A is dissipative and $0 \in \rho(A)$, then A is the infinitesimal generator of a C_0-semigroup of contractions on H.*

To establish exponential stability of a C_0-semigroup of contractions we will use the following theorem (see Liu and Zheng [28, Theorem 1.3.2]).

Theorem 2.22. *Let $\{S(t)\}_{t\geq 0}$ be a C_0-semigroup of contractions on a Hilbert space H with infinitesimal generator A. Then $\{S(t)\}_{t\geq 0}$ is exponentially stable if and only if*

$$i\mathbb{R} = \{i\beta \mid \beta \in \mathbb{R}\} \subset \rho(A)$$

and

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - A)^{-1}\|_{\mathcal{L}(H)} < \infty.$$

2.4 Analytic Semigroups

Let H be a Hilbert space such that dim $H > 0$. Suppose that A is a linear operator in H. The **numerical range** or **field of values** of A is the set

$$\Theta(A) = \{\langle Au, u\rangle_H \in \mathbb{C} \,|\, u \in \mathcal{D}(A) \text{ and } \|u\|_H = 1\}.$$

The operator A is said to be **accretive** if the numerical range of A lies in the right half-plane, that is,

$$\Theta(A) \subset \{z \in \mathbb{C} \,|\, \mathrm{Re}\, z \geq 0\}.$$

Thus, A is accretive if and only if $\mathrm{Re}\,\langle Au, u\rangle_H \geq 0$ for all $u \in \mathcal{D}(A)$. From this, it follows that A is dissipative if and only if $-A$ is accretive.

An **m-accretive** operator is an accretive operator such that $\mathcal{R}(A - \lambda I) = H$ for some $\lambda \in \mathbb{K}$ with $\mathrm{Re}\,\lambda < 0$. Further, an m-accretive operator is a maximal accretive operator in the sense that it is accretive and it has no proper accretive extensions. The following theorem (see Miklavčič [29, Theorem 2.3.2]) implies that m-accretive operators are closed.

Theorem 2.23. *Let A be a linear operator in a Hilbert space H. Then A is m-accretive if and only if every scalar ζ with* $\mathrm{Re}\,\zeta < 0$ *belongs to $\rho(A)$ and*

$$\|(A - \zeta I)^{-1}\|_{\mathcal{L}(H)} \leq \frac{1}{|\mathrm{Re}\,\zeta|}.$$

An m-accretive operator is necessarily densely defined, that is, if A is m-accretive then $\overline{\mathcal{D}(A)} = H$ (see Kato [22, p. 279] or Miklavčič [29, p. 58]). We shall say that A is **quasi-accretive** if $A + \alpha I$ is accretive for some $\alpha \in \mathbb{C}$. The operator A is said to be a **quasi-m-accretive** operator if $A + \alpha I$ is m-accretive for some complex number α. If we take $\alpha = 0$, it follows that an m-accretive operator is a quasi-m-accretive operator.

A **sectorially-valued** or simply **sectorial** operator A in a Hilbert space H is an operator such that its numerical range lies in a sector of the complex plane, that is,

$$\Theta(A) \subset \left\{\zeta \in \mathbb{C} \,\middle|\, |\arg(\zeta - \gamma)| \leq \theta < \frac{\pi}{2}\right\}.$$

The complex number γ is called a **vertex** and the angle θ, which is in radian measure, is called a **semi-angle** of the sectorial operator A. We note that the vertex and the semi-angle of a sectorial operator are not uniquely determined. An operator is called **m-sectorial** if it is sectorial and quasi-m-accretive. Hence, an m-accretive sectorial operator is m-sectorial.

If A is m-sectorial with vertex γ and semi-angle θ, then

$$\sigma(A) \subset \{\zeta \in \mathbb{C} \,|\, |\arg(\zeta - \gamma)| \leq \theta\},$$

in other words,
$$\{\zeta \in \mathbb{C} \,|\, |\arg(\zeta - \gamma)| > \theta\} \subset \rho(A).$$

We let $\mathscr{H}(\omega, 0)$ be the set of all densely defined closed operators A on a Hilbert space H such that

$$\left\{\zeta \in \mathbb{C} \,\middle|\, |\arg \zeta| < \frac{\pi}{2} + \omega,\ \omega \in (0, \pi/2]\right\} \subset \rho(-A)$$

and for every $\epsilon > 0$,
$$\|(A + \zeta I)^{-1}\|_{\mathcal{L}(H)} \leq \frac{M_\epsilon}{|\zeta|}$$
for $|\arg \zeta| \leq \pi/2 + \omega - \epsilon$, where $M_\epsilon > 0$ is independent of ζ. Furthermore, for $\beta \in \mathbb{R}$ we let

$$\mathscr{H}(\omega, \beta) = \{A : \mathcal{D}(A) \subset H \to H \,|\, A = A_0 - \beta I \ \text{ for some } A_0 \in \mathscr{H}(\omega, 0)\}.$$

Definition 2.24. A C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ is said to be **analytic** (or **holomorphic**) on a Hilbert space H if e^{tA} admits an extension $T(\lambda)$ for $\lambda \in \Delta_\theta = \{\lambda \in \mathbb{C} \,|\, |\arg \lambda| < \theta\}$ for some $\theta > 0$ such that

(1) $\lambda \mapsto T(\lambda)$ is analytic,

(2) $\lim\limits_{\Delta_\theta \ni \lambda \to 0} \|T(\lambda)z - z\|_X = 0$ for all $z \in H$,

(3) $T(\lambda + \mu) = T(\lambda)T(\mu)$ for all $\lambda, \mu \in \Delta_\theta$.

Analytic semigroups are used in analyzing a partial differential equation. In comparison to C_0-semigroups, analytic semigroups have better regularity in relation to the solutions of initial-value problems and they have better results concerning perturbations of the infinitesimal generator. For a more rigorous discussions regarding these properties of analytic semigroups we refer the readers to Renardy and Roberts [34].

Theorem 2.25. *If $A \in \mathscr{H}(\omega, \beta)$, where $\beta \in \mathbb{R}$, then e^{-tA} is a semigroup holomorphic or analytic for $|\arg t| < \omega$.*

The above theorem can be seen in Kato [22, p. 490]. The following theorem, which is Theorem 1.24 in Kato [22], gives another criterion for $-A$ to be the generator of an analytic semigroup.

Theorem 2.26. *Let A be an m-sectorial operator in a Hilbert space H with a vertex 0 so that*
$$\Theta(A) \subset \left\{\zeta \in \mathbb{C} \,\middle|\, |\arg \zeta| \leq \frac{\pi}{2} - \omega,\ \omega \in (0, \pi/2]\right\}.$$
Then $A \in \mathscr{H}(\omega, 0)$ and e^{-tA} is analytic for $|\arg t| < \omega$.

Chapter 3

Globally Damped Euler-Bernoulli Beam

In this chapter, we consider the following initial boundary value problem for the non-homogeneous Euler-Bernoulli beam equation with Kelvin-Voigt and air damping

$$\rho(x)\frac{\partial^2 y(t,x)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y(t,x)}{\partial x^2}\right) + \frac{\partial^2}{\partial x^2}\left(c_D I(x)\frac{\partial^3 y(t,x)}{\partial x^2 \partial t}\right) + c_a(x)\frac{\partial y(t,x)}{\partial t} = 0,$$

where $(t,x) \in (0,\infty) \times (0,\ell)$ and $\rho(x) = hb\rho_0(x)$, with initial conditions

$$y(0,x) \;=\; w_0(x), \qquad \left.\frac{\partial y(t,x)}{\partial t}\right|_{t=0} = v_0(x),$$

for $x \in (0,\ell)$. The functions w_0 and v_0 are the initial deflection and initial velocity, respectively. We are interested in the following boundary conditions

$$\begin{cases} y(t,0) \;=\; \left.\dfrac{\partial y(t,x)}{\partial x}\right|_{x=0} &= \;0, \qquad t \in (0,\infty), \\ -\left.\left(EI(x)\dfrac{\partial^2 y(t,x)}{\partial x^2} + c_D I(x)\dfrac{\partial^3 y(t,x)}{\partial x^2 \partial t}\right)\right|_{x=\ell} &= \;m(t), \quad t \in (0,\infty), \\ \dfrac{\partial}{\partial x}\left.\left(EI(x)\dfrac{\partial^2 y(t,x)}{\partial x^2} + c_D I(x)\dfrac{\partial^3 y(t,x)}{\partial x^2 \partial t}\right)\right|_{x=\ell} &= \;h(t), \quad t \in (0,\infty). \end{cases}$$

In this case, cantilever end conditions are assumed in the beam with clamped edge at $x = 0$ and free edge at $x = \ell$.

The mechanical meaning of the boundary conditions at one end of the beam $x = \ell$ is that the bending moment and shear force applied to that end is $m(t)$ and $h(t)$, respectively. We consider the case where the functions $m(t)$ and $h(t)$ satisfy the feedback form

$$\begin{bmatrix} m(t) \\ h(t) \end{bmatrix} = F \begin{bmatrix} \dot{y}'(t,\ell) \\ \dot{y}(t,\ell) \end{bmatrix}$$

where

$$F = \begin{bmatrix} k_2 & 0 \\ 0 & k_4 \end{bmatrix} \tag{3.1}$$

and $k_2, k_4 \geq 0$ are called the **feedback coefficients**. In this form of $[m(t)\ h(t)]^\mathsf{T}$, the bending moment $-(EI(\ell)y''(t,\ell) + c_D I(\ell)\dot{y}''(t,\ell))$ is controlled by the linear feedback of the angular velocity $\dot{y}'(t,\ell)$, and the shear force $(EI(\ell)y''(t,\ell) + c_D I(\ell)\dot{y}''(t,\ell))'$ is controlled by the linear feedback of the velocity $y'(t,\ell)$.

If the resulting system (a closed-loop system) is exponentially stable, then we say that the original system is **exponentially stabilizable**. All throughout this chapter, we will assume that

(H1) $c_a \in C[0,\ell]$, $\rho \in C^1[0,\ell]$, $EI \in C^2[0,\ell]$, $c_D I \in C[0,\ell]$, $\rho(x), EI(x), c_D I(x) \geq c_0 > 0$ and $c_a(x) \geq 0$ for all $x \in [0,\ell]$.

Under the hypothesis (H1), we assume that the Kelvin-Voigt damping is globally distributed. Also, this hypothesis together with the Extreme Value Theorem imply that we can find a constant $C_0 > 0$ such that

$$\rho(x), EI(x), c_D I(x), c_a(x) \leq C_0 \text{ for all } x \in [0,\ell].$$

The **energy** of the beam at time $t \geq 0$ is defined by

$$E(t) = \frac{1}{2}\int_0^\ell \left(EI(x)\left|\frac{\partial^2 y(t,x)}{\partial x^2}\right|^2 + \rho(x)\left|\frac{\partial y(t,x)}{\partial t}\right|^2 \right) \mathrm{d}x. \tag{3.2}$$

The first term in (3.2) represents the negative of the potential energy due to internal forces while the second term in (3.2) represents the kinetic energy of the beam.

In Section 1, we will define the state space, in other words, the Hilbert space on which we shall study the partial differential equation (1.1). Then we will reformulate this partial differential equation as an abstract Cauchy problem on the state space and prove that the associated operator generates a contractive C_0-semigroup in Section 2. Using the representation theorems for sesquilinear forms we will characterize the nonreal spectrum of the infinitesimal generator in Section 3. Section 4 deals with the exponential stability of the corresponding C_0-semigroup. We will also give an upper bound for the type of the infinitesimal generator in Section 5. In Section 6, we will give a complete proof showing that the exponential stability of the associated C_0-semigroup is equivalent to the **exponential decay** of the energy, that is,

$$E(t) \leq Me^{-\alpha t}E(0), \quad \text{for all } t \geq 0,$$

for some constants $\alpha > 0$ and $M \geq 1$. Finally, the analyticity of the C_0-semigroup is established in Section 7.

3.1 Finite Energy State Space

Let H (also denoted as $L^2_\rho(0,\ell)$) be the space of all equivalence classes of measurable functions v on $(0,\ell)$ such that

$$\int_0^\ell \rho(x)|v(x)|^2\,\mathrm{d}x < \infty,$$

where two functions belong to the same equivalence class if they differ only on a set of measure 0. For simplicity, we also consider H as a set of functions. Define the inner product on H by

$$\langle v_1, v_2\rangle_H = \int_0^\ell \rho(x)v_1(x)\overline{v_2(x)}\,\mathrm{d}x.$$

Let $V = \{w \in H^2(0,\ell)\,|\,w(0) = w'(0) = 0\}$ and define the inner product on V by

$$\langle w_1, w_2\rangle_V = \int_0^\ell EI(x)w_1''(x)\overline{w_2''(x)}\,\mathrm{d}x.$$

The corresponding norms are given by

$$\|v\|_H = \left(\int_0^\ell \rho(x)|v(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}$$

and

$$\|w\|_V = \left(\int_0^\ell EI(x)|w''(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}},$$

respectively. In the following theorems, we characterize the normed spaces $(H, \|\cdot\|_H)$ and $(V, \|\cdot\|_V)$.

Theorem 3.1. *The L^2-norm is equivalent to the H-norm. Furthermore, the space H is a Hilbert space and the mapping $U : H \to L^2(0,\ell)$ defined by $Uv = \rho^{\frac{1}{2}}v$ is a unitary isomorphism between H and $L^2(0,\ell)$.*

Proof. Since

$$\sqrt{c_0}\|v\|_{L^2(0,\ell)} \le \|v\|_H \le \sqrt{C_0}\|v\|_{L^2(0,\ell)} \tag{3.3}$$

it follows that in H, the L^2-norm $\|\cdot\|_{L^2(0,\ell)}$ is equivalent to the H-norm $\|\cdot\|_H$. Next, let $\{v_n\}_{n=1}^\infty$ be a Cauchy sequence in H. Then it follows that the sequence $\{\rho^{\frac{1}{2}}v_n\}_{n=1}^\infty$ is a Cauchy sequence in $L^2(0,\ell)$, and by the completeness of $L^2(0,\ell)$ we can find an element

$v \in L^2(0,\ell)$ such that $\|\rho^{\frac{1}{2}} v_n - v\|_{L^2(0,\ell)} \to 0$ as $n \to \infty$. Thus, $\|v_n - \rho^{-\frac{1}{2}} v\|_H \to 0$ as $n \to \infty$. Because $\|\rho^{-\frac{1}{2}} v\|_H = \|v\|_{L^2(0,\ell)} < \infty$ we have $\rho^{-\frac{1}{2}} v \in H$ and this shows that H is a Hilbert space. Let $v_1, v_2 \in H$ and α_1, α_2 be scalars. Then

$$\begin{aligned} U(\alpha_1 v_1 + \alpha_2 v_1) &= \rho^{\frac{1}{2}}(\alpha_1 v_1 + \alpha_2 v_2) \\ &= \alpha_1(\rho^{\frac{1}{2}} v_1) + \alpha_2(\rho^{\frac{1}{2}} v_2) \\ &= \alpha_1 U v_1 + \alpha_2 U v_2, \end{aligned}$$

which means that U is linear.

If $Uv_1 = Uv_2$ then $\rho^{\frac{1}{2}} v_1 = \rho^{\frac{1}{2}} v_2$ and dividing both sides by the positive $\rho^{\frac{1}{2}}$ we get $v_1 = v_2$. Given $v \in L^2(0,\ell)$ we have $U(\rho^{-\frac{1}{2}} v) = v$ and $\|\rho^{-\frac{1}{2}} v\|_H = \|v\|_{L^2(0,\ell)} < \infty$ so that $\rho^{-\frac{1}{2}} v \in H$. Therefore U is both one-to-one and onto. Finally, U preserves norm since $\|Uv\|_{L^2(0,\ell)} = \|\rho^{\frac{1}{2}} v\|_{L^2(0,\ell)} = \|v\|_H$ for all $v \in H$, and we conclude that U is indeed a unitary isomorphism between H and $L^2(0,\ell)$. □

Theorem 3.2. *The space V equipped with the inner product $\langle\cdot,\cdot\rangle_V$ is a Hilbert space. Moreover, the Sobolev norm $\|\cdot\|_{H^2(0,\ell)}$ is equivalent to the V-norm $\|\cdot\|_V$.*

Proof. First, let us show that the Sobolev norm is equivalent to the V-norm. Let $w \in V$. Observe that

$$\sqrt{c_0}\|w''\|_{L^2(0,\ell)} \le \|w\|_V \le \sqrt{C_0}\|w''\|_{L^2(0,\ell)}.$$

For each $x \in [0,\ell]$

$$\int_0^x w'(s)\,\mathrm{d}s = w(x) - w(0) = w(x).$$

Hence

$$\begin{aligned} |w(x)|^2 &= \left|\int_0^x w'(s)\,\mathrm{d}s\right|^2 \\ &\le \left(\int_0^x |w'(s)|\,\mathrm{d}s\right)^2 \\ &\le \left(\int_0^\ell |w'(s)|\,\mathrm{d}s\right)^2. \end{aligned} \tag{3.4}$$

By the Cauchy-Schwartz inequality

$$\begin{aligned} \left(\int_0^\ell |w'(s)|\,\mathrm{d}s\right)^2 &\le \left(\int_0^\ell \mathrm{d}s\right)\left(\int_0^\ell |w'(s)|^2\,\mathrm{d}s\right) \\ &= \ell\|w'\|^2_{L^2(0,\ell)}. \end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5), we obtain that for all $x \in [0,\ell]$,

$$|w(x)| \le \sqrt{\ell}\|w'\|_{L^2(0,\ell)} \tag{3.6}$$

and so $\|w\|_{L^2(0,\ell)} \le \ell \|w'\|_{L^2(0,\ell)}$.

In a similar way, using the fact that $w'(0) = 0$ we get

$$|w'(x)| \le \sqrt{\ell}\|w''\|_{L^2(0,\ell)}, \tag{3.7}$$

for all $x \in [0,\ell]$ and $\|w'\|_{L^2(0,\ell)} \le \ell\|w''\|_{L^2(0,\ell)}$. Hence

$$\begin{aligned}
\|w\|_{H^2(0,\ell)} &= \left(\sum_{i=0}^{2} \|D^i w\|^2_{L^2(0,\ell)}\right)^{\frac{1}{2}} \\
&\le \left(\ell^2\|w'\|^2_{L^2(0,\ell)} + \ell^2\|w''\|^2_{L^2(0,\ell)} + \|w''\|^2_{L^2(0,\ell)}\right)^{\frac{1}{2}} \\
&\le (\ell^4 + \ell^2 + 1)^{\frac{1}{2}}\|w''\|_{L^2(0,\ell)} \\
&\le \left(\frac{\ell^4+\ell^2+1}{c_0}\right)^{\frac{1}{2}} \|w\|_V.
\end{aligned}$$

For the other inequality, observe that $\|w''\|_{L^2(0,\ell)} \le \|w\|_{H^2(0,\ell)}$ and so

$$\frac{1}{\sqrt{C_0}}\|w\|_V \le \|w\|_{H^2(0,\ell)}.$$

Therefore

$$\frac{1}{\sqrt{C_0}}\|w\|_V \le \|w\|_{H^2(0,\ell)} \le \sqrt{C_\ell}\|w\|_V \tag{3.8}$$

where $C_\ell = (\ell^4 + \ell^2 + 1)/c_0$, which proves that the Sobolev norm is equivalent to the V-norm.

Let $\{w_n\}_{n=1}^\infty$ be a Cauchy sequence in V. Then given $\epsilon > 0$, there exists a positive integer N such that

$$\left(\int_0^\ell |EI^{\frac{1}{2}}(x)w''_m(x) - EI^{\frac{1}{2}}(x)w''_n(x)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} < \epsilon$$

whenever $m, n \ge N$. Thus $\{EI^{\frac{1}{2}}w''_n\}_{n=1}^\infty$ is a Cauchy sequence in $L^2(0,\ell)$ and because $L^2(0,\ell)$ is a Banach space we can find an element $v \in L^2(0,\ell)$ such that $EI^{\frac{1}{2}}w''_n \to v$ as $n \to \infty$ with respect to the L^2-norm. Let

$$w(x) = \int_0^x \int_0^s EI^{-\frac{1}{2}}(\tau)v(\tau)\,\mathrm{d}\tau\,\mathrm{d}s.$$

We show that $w \in V$. The first and second derivatives of w are given by

$$w'(x) = \int_0^x EI^{-\frac{1}{2}}(\tau)v(\tau)\,\mathrm{d}\tau$$

and $w''(x) = EI^{-\frac{1}{2}}(x)v(x)$, respectively, and so $w(0) = w'(0) = 0$. Further, using the properties of the integral and the Cauchy-Schwartz inequality we obtain

$$\begin{aligned}
\int_0^\ell |w(x)|^2 \,\mathrm{d}x &= \int_0^\ell \left| \int_0^x \int_0^s EI^{-\frac{1}{2}}(\tau)v(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \right|^2 \mathrm{d}x \\
&\le \int_0^\ell \left(\int_0^x \int_0^s |EI^{-\frac{1}{2}}(\tau)v(\tau)|\,\mathrm{d}\tau\,\mathrm{d}s \right)^2 \mathrm{d}x \\
&\le \int_0^\ell \left(\int_0^\ell \int_0^\ell |EI^{-\frac{1}{2}}(\tau)v(\tau)|\,\mathrm{d}\tau\,\mathrm{d}s \right)^2 \mathrm{d}x \\
&= \ell^3 \left(\int_0^\ell |EI^{-\frac{1}{2}}(\tau)v(\tau)|\,\mathrm{d}\tau \right)^2 \\
&\le \ell^4 \int_0^\ell |EI^{-\frac{1}{2}}(\tau)v(\tau)|^2 \,\mathrm{d}\tau \\
&\le \ell^4 c_0^{-1} \|v\|^2_{L^2(0,\ell)}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\int_0^\ell |w'(x)|^2 \,\mathrm{d}x &= \int_0^\ell \left| \int_0^x EI^{-\frac{1}{2}}(\tau)v(\tau)\,\mathrm{d}\tau \right|^2 \mathrm{d}x \\
&\le \int_0^\ell \left(\int_0^\ell |EI^{-\frac{1}{2}}(\tau)v(\tau)|\,\mathrm{d}\tau \right)^2 \mathrm{d}x \\
&\le \ell^2 \int_0^\ell |EI^{-\frac{1}{2}}(\tau)v(\tau)|^2 \,\mathrm{d}\tau \\
&= \ell^2 c_0^{-1} \|v\|^2_{L^2(0,\ell)}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\ell |w''(x)|^2 \,\mathrm{d}x &= \int_0^\ell |E^{-\frac{1}{2}}(x)v(x)|^2 \,\mathrm{d}x \\
&\le c_0^{-1} \|v\|^2_{L^2(0,\ell)}.
\end{aligned}$$

Using these estimates we have $\|w\|_{H^2(0,\ell)} \le (\ell^4 + \ell^2 + 1)^{\frac{1}{2}} c_0^{-\frac{1}{2}} \|v\|_{L^2(0,\ell)}$ and $w \in H^2(0,\ell)$. Therefore $w \in V$ and

$$\begin{aligned}
\|w_n - w\|_V &= \left(\int_0^\ell EI(x)|w_n''(x) - w''(x)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \\
&= \left(\int_0^\ell EI(x)|w_n''(x) - EI^{-\frac{1}{2}}(x)v(x)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \\
&= \left(\int_0^\ell |EI^{\frac{1}{2}}(x)w_n''(x) - v(x)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}.
\end{aligned}$$

Since $EI^{\frac{1}{2}} w_n'' \to v$ as $n \to \infty$ with respect to the L^2-norm, it now follows that $w_n \to w$ as $n \to \infty$ in V and thus V is a Hilbert space. $\square$

Theorem 3.3. *The embedding $V \hookrightarrow H$ is compact, continuous and dense.*

Proof. Let $\{v_n\}_{n=1}^{\infty}$ be a bounded sequence in V. By the equivalence of the Sobolev norm and the V-norm it follows that $\{v_n\}_{n=1}^{\infty}$ is also a bounded sequence in $H^2(0,\ell)$ and since the embedding $H^2(0,\ell) \hookrightarrow L^2(0,\ell)$ is compact, there exists a subsequence $\{v_{n_j}\}_{j=1}^{\infty}$ of $\{v_n\}_{n=1}^{\infty}$ such that $\|v_{n_j} - v\|_{L^2(0,\ell)} \to 0$ as $j \to \infty$ for some $v \in L^2(0,\ell)$. Since the L^2-norm is equivalent to the H-norm we also have $v \in H$ and $\|v_{n_j} - v\|_H \to 0$ as $j \to \infty$. Thus $V \hookrightarrow H$ is a compact embedding.

Suppose $v \in V$. Then $\|v\|_H \leq \sqrt{C_0}\|v\|_{L^2(0,\ell)} \leq \sqrt{C_0}\|v\|_{H^2(0,\ell)} \leq \sqrt{C_0 C_\ell}\|v\|_V$. This shows that the embedding $V \hookrightarrow H$ is bounded and so it is continuous. Finally, to show that the embedding is dense, let $w \in H$ and $\epsilon > 0$. Then $w \in L^2(0,\ell)$ and since $C_0^\infty(0,\ell)$ is dense in $L^2(0,\ell)$ (see Jurgen [21, Corollary 19.20]) we can find an element $v_\epsilon \in C_0^\infty(0,\ell)$ such that $\|w - v_\epsilon\|_{L^2(0,\ell)} < \epsilon/\sqrt{C_0}$. Since $C_0^\infty(0,\ell) \subset V$ we have $v_\epsilon \in V$ and $\|w - v_\epsilon\|_H < \epsilon$. $\square$

Define $\mathcal{H}$ to be the product $V \times H$ of the two Hilbert spaces H and V equipped with the inner product

$$\left\langle \begin{bmatrix} w_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} w_2 \\ v_2 \end{bmatrix} \right\rangle_{\mathcal{H}} = \langle w_1, w_2 \rangle_V + \langle v_1, v_2 \rangle_H$$

and the corresponding energy-induced norm is given by

$$\left\| \begin{bmatrix} w \\ v \end{bmatrix} \right\|_{\mathcal{H}} = (\|w\|_V^2 + \|v\|_H^2)^{\frac{1}{2}} = \left(\int_0^\ell [EI(x)|w''(x)|^2 + \rho(x)|v(x)|^2] \, \mathrm{d}x \right)^{\frac{1}{2}}.$$

Using Theorems 3.1 and 3.2 we can see that $\mathcal{H}$ is a Hilbert space which is called the **finite energy state space**.

3.2 Semigroup Formulation

In this section, we reformulate the partial differential equation

$$\begin{aligned} \rho(x)\frac{\partial^2 y(t,x)}{\partial t^2} &+ \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y(t,x)}{\partial x^2} \right) \\ &+ \frac{\partial^2}{\partial x^2}\left(c_D I(x)\frac{\partial^3 y(t,x)}{\partial x^2 \partial t} \right) + c_a(x)\frac{\partial y(t,x)}{\partial t} = 0 \end{aligned} \tag{3.9}$$

with initial conditions

$$y(0,x) = w_0(x), \quad \dot{y}(0,x) = v_0(x), \tag{3.10}$$

and boundary conditions

$$\begin{aligned} y(t,0) \;=\; y'(t,0) &= 0, \\ -(EI(\ell)y''(t,\ell)+c_D I(\ell)\dot{y}''(t,\ell)) &= k_2\dot{y}'(t,\ell), \\ (EI(\ell)y''(t,\ell)+c_D I(\ell)\dot{y}''(t,\ell))' &= k_4\dot{y}(t,\ell), \end{aligned} \tag{3.11}$$

as an abstract Cauchy problem on the Hilbert space $\mathcal{H}$. Note that we can re-write Equation (3.9) as

$$\frac{\partial^2 y}{\partial t^2} = -\frac{1}{\rho(x)}\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2} + c_D I(x)\frac{\partial^2}{\partial x^2}\left(\frac{\partial y}{\partial t}\right)\right] - \frac{c_a(x)}{\rho(x)}\frac{\partial y}{\partial t}. \tag{3.12}$$

Consider the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$, where its domain is given by

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{bmatrix} w \\ v \end{bmatrix} \in \mathcal{H} \;\middle|\; \begin{array}{c} w, v \in V,\ EIw'' + c_D I v'' \in H^2(0,\ell), \\ \begin{bmatrix} -(EI(\ell)w''(\ell) + c_D I(\ell)v''(\ell)) \\ (EI(\ell)w''(\ell) + c_D I(\ell)v''(\ell))' \end{bmatrix} = F \begin{bmatrix} v'(\ell) \\ v(\ell) \end{bmatrix} \end{array} \right\}$$

and F is the diagonal matrix given by Equation (3.1), defined by

$$\mathcal{A}z = \mathcal{A}\begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} v \\ -\frac{1}{\rho}(EIw'' + c_D I v'')'' - \frac{1}{\rho}c_a v \end{bmatrix}$$

for all $z = [w\ v]^{\mathsf{T}} \in \mathcal{D}(\mathcal{A})$.

Let $w = y$ and $v = \dot{w}$. Then $\ddot{y} = \ddot{w} = \dot{v}$ and $\dot{y}'' = \dot{w}'' = v''$. From Equation (3.12) we have

$$\ddot{y} = -\frac{1}{\rho}(EIy'' + c_D I\dot{y}'')'' - \frac{c_a}{\rho}\dot{y}$$

and in terms of w and v we have

$$\dot{v} = -\frac{1}{\rho}(EIw'' + c_D I v'')'' - \frac{c_a}{\rho}v.$$

Therefore (3.9), (3.10) and (3.11) can be phrased as an abstract Cauchy problem on $\mathcal{H}$ as

$$\begin{cases} \dfrac{\mathrm{d}z(t)}{\mathrm{d}t} = \mathcal{A}z(t), \quad \text{for all } t > 0, \\ z(0) = z_0 = [w_0, v_0]^{\mathsf{T}}. \end{cases} \tag{3.13}$$

3.2.1 Well-Posedness

If $k_2, k_4 \geq 0$, then F is symmetric and nonnegative definite, that is, $x^{\mathsf{T}}Fx \geq 0$ for any $x \in \mathbb{R}^2$. We will use this fact, together with hypothesis (H1) to show that if the feedback coefficients are nonnegative then the operator $\mathcal{A}$ is a densely defined dissipative operator having a bounded inverse. From Theorem 2.21 this will imply that the operator $\mathcal{A}$ will generate a strongly continuous semigroup of contractions. Furthermore, this will lead us to the well-posedness of the abstract Cauchy problem (3.13) associated with $\mathcal{A}$ by Theorem 2.17. This is the content of the following theorem.

Theorem 3.4. *If $k_2, k_4 \geq 0$, then $\mathcal{A}$ generates a C_0-semigroup $S(t) = e^{\mathcal{A}t}$ of contractions on $\mathcal{H}$. Furthermore, $\mathcal{A}^{-1} \in \mathcal{L}(\mathcal{H})$ is a closed operator.*

Before proving this theorem, we have the following three lemmas. These lemmas will be used in proving the above theorem.

Lemma 3.5. *We have*

$$\begin{aligned} \langle \mathcal{A}z, z\rangle_{\mathcal{H}} &= -\,[|v'(\ell)|\ |v(\ell)|]F\begin{bmatrix} |v'(\ell)| \\ |v(\ell)| \end{bmatrix} - \int_0^\ell (c_a|v|^2 + c_D I|v''|^2)\,\mathrm{d}x \\ &\quad + 2i\mathrm{Im}\,\langle v, w\rangle_V\,, \end{aligned}$$

for $z = [w\ v]^\mathsf{T} \in \mathcal{D}(\mathcal{A})$. In particular, if $k_2, k_4 \geq 0$ then $\mathcal{A}$ is dissipative.

Proof. Let $z = [w\ v]^\mathsf{T} \in \mathcal{D}(\mathcal{A})$. Using the definition of $\mathcal{A}$ and the definition of the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ on $\mathcal{H}$ we have

$$\begin{aligned} \left\langle \mathcal{A}\begin{bmatrix} w \\ v \end{bmatrix}, \begin{bmatrix} w \\ v \end{bmatrix} \right\rangle_{\mathcal{H}} &= \left\langle \begin{bmatrix} v \\ -\frac{1}{\rho}(EIw'' + c_D Iv'')'' - \frac{c_a}{\rho}v \end{bmatrix}, \begin{bmatrix} w \\ v \end{bmatrix} \right\rangle_{\mathcal{H}} \\ &= \langle v, w\rangle_V + \left\langle -\frac{1}{\rho}(EIw'' + c_D Iv'')'' - \frac{c_a}{\rho}v, v \right\rangle_H \\ &= \langle v, w\rangle_V + \int_0^\ell \rho\left(-\frac{1}{\rho}(EIw'' + c_D Iv'')''\overline{v} - \frac{c_a}{\rho}|v|^2\right)\mathrm{d}x \\ &= \langle v, w\rangle_V - \int_0^\ell (EIw'' + c_D Iv'')''\overline{v}\,\mathrm{d}x - \int_0^\ell c_a|v|^2\,\mathrm{d}x. \end{aligned}$$

Integrating the second term of the above last expression by parts twice and using the boundary conditions at $x = \ell$ yield

$$\begin{aligned} \int_0^\ell (EIw'' + c_D Iv'')''\overline{v}\,\mathrm{d}x &= (EIw'' + c_D Iv'')'\overline{v}\Big|_0^\ell - \int_0^\ell (EIw'' + c_D Iv'')'\overline{v'}\,\mathrm{d}x \\ &= (EIw'' + c_D Iv'')'(\ell)\overline{v(\ell)} - \int_0^\ell (EIw'' + c_D Iv'')'\overline{v'}\,\mathrm{d}x \\ &= (EIw'' + c_D Iv'')'(\ell)\overline{v(\ell)} - (EIw'' + c_D Iv'')(\ell)\overline{v'(\ell)} \\ &\quad + \int_0^\ell (EIw'' + c_D Iv'')\overline{v''}\,\mathrm{d}x \\ &= \left[\overline{v'(\ell)}\ \ \overline{v(\ell)}\right] F\begin{bmatrix} v'(\ell) \\ v(\ell) \end{bmatrix} + \langle w, v\rangle_V + \int_0^\ell c_D I|v''|^2\,\mathrm{d}x. \end{aligned}$$

Note that $\langle v,w\rangle_V - \langle w,v\rangle_V = \langle v,w\rangle_V - \overline{\langle v,w\rangle_V} = 2i\mathrm{Im}\,\langle v,w\rangle_V$. Combining these we get

$$\begin{aligned}\left\langle \mathcal{A}\begin{bmatrix} w \\ v \end{bmatrix}, \begin{bmatrix} w \\ v \end{bmatrix}\right\rangle_{\mathcal{H}} &= \langle v,w\rangle_V - \begin{bmatrix}\overline{v'(\ell)} & \overline{v(\ell)}\end{bmatrix} F \begin{bmatrix} v'(\ell) \\ v(\ell) \end{bmatrix} - \langle w,v\rangle_V \\ &\quad - \int_0^\ell (c_a|v|^2 + c_D I|v''|^2)\,\mathrm{d}x \\ &= -\begin{bmatrix}\overline{v'(\ell)} & \overline{v(\ell)}\end{bmatrix} F \begin{bmatrix} v'(\ell) \\ v(\ell) \end{bmatrix} - \int_0^\ell (c_a|v|^2 + c_D I|v''|^2)\,\mathrm{d}x \\ &\quad + 2i\mathrm{Im}\,\langle v,w\rangle_V .\end{aligned}$$

Since

$$\begin{bmatrix}\overline{v'(\ell)} & \overline{v(\ell)}\end{bmatrix} F \begin{bmatrix} v'(\ell) \\ v(\ell) \end{bmatrix} = [|v'(\ell)|\ |v(\ell)|] F \begin{bmatrix} |v'(\ell)| \\ |v(\ell)| \end{bmatrix}$$

we have the following formula for $\langle \mathcal{A}[w\ v]^\mathsf{T}, [w\ v]^\mathsf{T}\rangle_{\mathcal{H}}$

$$\begin{aligned}\left\langle \mathcal{A}\begin{bmatrix} w \\ v \end{bmatrix}, \begin{bmatrix} w \\ v \end{bmatrix}\right\rangle_{\mathcal{H}} &= -\,[|v'(\ell)|\ |v(\ell)|] F \begin{bmatrix} |v'(\ell)| \\ |v(\ell)| \end{bmatrix} - \int_0^\ell (c_a|v|^2 + c_D I|v''|^2)\,\mathrm{d}x \\ &\quad + 2i\mathrm{Im}\,\langle v,w\rangle_V .\end{aligned}$$

Taking the real part gives us

$$\begin{aligned}\mathrm{Re}\left\langle \mathcal{A}\begin{bmatrix} w \\ v \end{bmatrix}, \begin{bmatrix} w \\ v \end{bmatrix}\right\rangle_{\mathcal{H}} &= -\,[|v'(\ell)|\ |v(\ell)|] F \begin{bmatrix} |v'(\ell)| \\ |v(\ell)| \end{bmatrix} - \int_0^\ell c_a|v|^2\,\mathrm{d}x \\ &\quad - \int_0^\ell c_D I|v''|^2\,\mathrm{d}x. \end{aligned} \tag{3.14}$$

Because $k_2, k_4 \geq 0$, then F is a diagonal matrix with nonnegative entries and so the first term of the right hand side of the above equality is nonpositive. Moreover, the inequalities $c_a \geq 0$ and $c_D I \geq c_0 > 0$ imply that the integrals in Equation (3.14) are nonnegative. Thus, $\mathrm{Re}\,\langle \mathcal{A}z, z\rangle_{\mathcal{H}} \leq 0$ for all $z \in \mathcal{H}$ and hence $\mathcal{A}$ is dissipative. □

Lemma 3.6. *The linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is one-to-one and onto.*

Proof. Given $[f\ g]^\mathsf{T} \in \mathcal{H}$, we claim that there exists a unique $[w\ v]^\mathsf{T} \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}[w\ v]^\mathsf{T} = [f\ g]^\mathsf{T}$. The equation $\mathcal{A}[w\ v]^\mathsf{T} = [f\ g]^\mathsf{T}$ means that we have the system

$$\begin{aligned} v &= f \\ (EIw'' + c_D I v'')'' + c_a v &= -\rho g. \end{aligned} \tag{3.15}$$

Now, we will solve the differential equation in (3.15). Integrating twice the second equality in (3.15), we obtain

$$\int_x^\ell \int_\xi^\ell (EI(\tau)w''(\tau) + c_D I(\tau)v''(\tau))''\,\mathrm{d}\tau\,\mathrm{d}\xi = -\int_x^\ell \int_\xi^\ell (\rho(\tau)g(\tau) + c_a(\tau)v(\tau))\,\mathrm{d}\tau\,\mathrm{d}\xi.$$

Evaluating the first integral on the left hand side yields

$$\begin{aligned}
&\int_x^\ell \int_\xi^\ell (EI(\tau)w''(\tau) + c_D I(\tau)v''(\tau))''\,\mathrm{d}\tau\,\mathrm{d}\xi \\
&= \int_x^\ell [(EI(\ell)w''(\ell) + c_D I(\ell)v''(\ell))' - (EI(\xi)w''(\xi) + c_D I(\xi)f''(\xi))']\,\mathrm{d}\xi \\
&= (EI(\ell)w''(\ell) + c_D I(\ell)v''(\ell))'(\ell - x) - \int_x^\ell (EI(\xi)w''(\xi) + c_D I(\xi)f''(\xi))'\,\mathrm{d}\xi \\
&= (\ell - x)(EI(\ell)w''(\ell) + c_D I(\ell)v''(\ell))' - (EI(\ell)w''(\ell) + c_D I(\ell)v''(\ell)) \\
&\quad + EI(x)w''(x) + c_D I(x)v''(x) \\
&= [1\ (\ell - x)]F \begin{bmatrix} v'(\ell) \\ v(\ell) \end{bmatrix} + EI(x)w''(x) + c_D I(x)v''(x).
\end{aligned}$$

Thus, solving for EIw'' gives us

$$\begin{aligned}
EI(x)w''(x) &= -[1\ (\ell - x)]F \begin{bmatrix} v'(\ell) \\ v(\ell) \end{bmatrix} - c_D I(x)v''(x) - \int_x^\ell \int_\xi^\ell \rho(\tau)g(\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi \\
&\quad - \int_x^\ell \int_\xi^\ell c_a(\tau)v(\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi. \qquad (3.16)
\end{aligned}$$

Using the condition $w(0) = w'(0) = 0$ we have

$$\int_0^x \int_0^t w''(s)\,\mathrm{d}s\,\mathrm{d}t = \int_0^x w'(t)\,\mathrm{d}t = w(x). \qquad (3.17)$$

Dividing both sides of Equation (3.16) by the positive EI, integrating from 0 to x and noting that $v = f$ we have

$$\begin{aligned}
w'(x) &= -\int_0^x (EI(s))^{-1}[1\ (\ell - s)]F \begin{bmatrix} f'(\ell) \\ f(\ell) \end{bmatrix} \mathrm{d}s - \int_0^x \frac{c_D I(s)}{EI(s)} f''(s)\,\mathrm{d}s \\
&\quad - \int_0^x \int_s^\ell \int_\xi^\ell \frac{\rho(\tau)g(\tau) + c_a(\tau)f(\tau)}{EI(s)}\,\mathrm{d}\tau\,\mathrm{d}\xi\,\mathrm{d}s.
\end{aligned}$$

Again, if we integrate both sides of the above equality from 0 to x we obtain

$$\begin{aligned}
w(x) &= -\int_0^x \int_0^t (EI(s))^{-1}[1\ (\ell - s)]F \begin{bmatrix} f'(\ell) \\ f(\ell) \end{bmatrix} \mathrm{d}s\,\mathrm{d}t - \int_0^x \int_0^t \frac{c_D I(s)}{EI(s)} f''(s)\,\mathrm{d}s\,\mathrm{d}t \\
&\quad - \int_0^x \int_0^t \int_s^\ell \int_\xi^\ell \frac{\rho(\tau)g(\tau) + c_a(\tau)f(\tau)}{EI(s)}\,\mathrm{d}\tau\,\mathrm{d}\xi\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

Observe that $v = f \in V$. From (3.16) and using the fact that $w(0) = w'(0) = 0$ we have $w \in V$. Further, since $(EIw'' + c_D Iv'')'' = -\rho g - c_a f \in L^2(0, \ell)$ and

$$\begin{bmatrix} -(EI(\ell)w''(\ell) + c_D I(\ell)v''(\ell)) \\ (EI(\ell)w''(\ell) + c_D I(\ell)v''(\ell))' \end{bmatrix} = F \begin{bmatrix} v'(\ell) \\ v(\ell) \end{bmatrix}$$

from (3.16), it follows that $[w\ v]^\mathsf{T} \in \mathcal{D}(\mathcal{A})$. Therefore, for all $[f\ g]^\mathsf{T} \in \mathcal{H}$ we can find a vector $[w\ v]^\mathsf{T} \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}[w\ v]^\mathsf{T} = [f\ g]^\mathsf{T}$. Let us show that such vector is unique. Let $[w_1\ v_1]^\mathsf{T}, [w_2\ v_2]^\mathsf{T} \in \mathcal{D}(\mathcal{A})$ be such that $\mathcal{A}[w_1\ v_1]^\mathsf{T} = [f\ g]^\mathsf{T} = \mathcal{A}[w_2\ v_2]^\mathsf{T}$. Then $v_1 = f = v_2$ and

$$(EIw_1'' + c_D v_1'')'' = (EIw_2'' + c_D v_2'')''.$$

Integrating twice and using the boundary conditions at $x = \ell$ we have

$$\begin{aligned}
&[1\ (\ell - x)]F \begin{bmatrix} v_1'(\ell) \\ v_1(\ell) \end{bmatrix} + EI(x)w_1''(x) + c_D I(x)v_1''(x) \\
&\quad = [1\ (\ell - x)]F \begin{bmatrix} v_2'(\ell) \\ v_2(\ell) \end{bmatrix} + EI(x)w_2''(x) + c_D I(x)v_2''(x)
\end{aligned}$$

and since $v_1 = f = v_2$ it follows that $EIw_1'' = EIw_2''$. Dividing both sides by $EI > 0$ we get $w_1'' = w_2''$. Integrating twice from 0 to x and using the conditions $w_1(0) = w_1'(0) = w_2(0) = w_2'(0) = 0$ we obtain that $w_1 = w_2$. Hence $[w_1\ v_1]^\mathsf{T} = [w_2\ v_2]^\mathsf{T}$. This completes the proof that $\mathcal{A}$ is both one-to-one and onto. □

Lemma 3.7. *Assume that $k_2, k_4 \geq 0$. The inverse $\mathcal{A}^{-1} : \mathcal{H} \to \mathcal{H}$ of $\mathcal{A}$ is bounded.*

Proof. It follows from Lemma 3.6 that $\mathcal{A}^{-1}$ exists. To show boundedness, let $[f\ g]^\mathsf{T} \in \mathcal{H}$ and so $\mathcal{A}[w\ v]^\mathsf{T} = [f\ g]^\mathsf{T}$ for some $[w\ v]^\mathsf{T} \in \mathcal{D}(\mathcal{A})$. First, we have

$$\|v\|_H^2 = \|f\|_H^2 \leq C_0 \|f\|_{H^2(0,\ell)}^2 \leq C_0 C_\ell \|f\|_V^2$$

and

$$\|f\|_V^2 \leq \|f\|_V^2 + \|g\|_H^2 = \left\| \begin{bmatrix} f \\ g \end{bmatrix} \right\|_{\mathcal{H}}^2.$$

Thus

$$\|v\|_H^2 \leq C_0 C_\ell \left\| \begin{bmatrix} f \\ g \end{bmatrix} \right\|_{\mathcal{H}}^2.$$

Next, we also show that the V-norm of w is bounded above by a constant multiple of the $\mathcal{H}$-norm of $[f\ g]^\mathsf{T}$. From (3.16) we get

$$\begin{aligned}
EI(x)w''(x) + c_D I(x)v''(x) &= -\int_x^\ell \int_\xi^\ell \rho(\tau)g(\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi - \int_x^\ell \int_\xi^\ell c_a(\tau)v(\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi \\
&\quad - [1\ (\ell - x)]F \begin{bmatrix} v'(\ell) \\ v(\ell) \end{bmatrix}.
\end{aligned}$$

Therefore, taking the norm both sides in $L^2(0,\ell)$ and using the triangle inequality we get

$$\begin{aligned}
&\|EIw'' + c_D I v''\|_{L^2(0,\ell)} \\
&\quad \leq \left\| \int_x^\ell \int_\xi^\ell \rho(\tau) g(\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi \right\|_{L^2(0,\ell)} + \left\| \int_x^\ell \int_\xi^\ell c_a(\tau) v(\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi \right\|_{L^2(0,\ell)} \\
&\qquad + \left\| [1\ (\ell - x)] F \begin{bmatrix} v'(\ell) \\ v(\ell) \end{bmatrix} \right\|_{L^2(0,\ell)}.
\end{aligned}$$

Now let us find an estimate in terms of the V-norm of f and H-norm of g for each term of the right hand side of this inequality. Using the Cauchy-Schwartz Inequality and properties of the integral, we have the following estimate for the first term

$$\begin{aligned}
\left\| \int_x^\ell \int_\xi^\ell \rho(\tau) g(\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi \right\|^2_{L^2(0,\ell)} &= \int_0^\ell \left| \int_x^\ell \int_\xi^\ell \rho(\tau) g(\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi \right|^2 \mathrm{d}x \\
&\leq \int_0^\ell \left(\int_x^\ell \int_\xi^\ell \rho(\tau) |g(\tau)|\,\mathrm{d}\tau\,\mathrm{d}\xi \right)^2 \mathrm{d}x \\
&\leq \int_0^\ell \left(\int_0^\ell \int_0^\ell \rho(\tau) |g(\tau)|\,\mathrm{d}\tau\,\mathrm{d}\xi \right)^2 \mathrm{d}x \\
&= \ell^3 \left(\int_0^\ell \rho(\tau) |g(\tau)|\,\mathrm{d}\tau \right)^2 \\
&\leq \ell^3 \|\rho\|^2_{L^2(0,\ell)} \|g\|^2_{L^2(0,\ell)}.
\end{aligned}$$

Similarly, an estimate for the second term is given by

$$\left\| \int_x^\ell \int_\xi^\ell c_a(\tau) v(\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi \right\|_{L^2(0,\ell)} \leq \ell^{\frac{3}{2}} \|c_a\|_{L^2(0,\ell)} \|v\|_{L^2(0,\ell)}.$$

Also, for the third term we have

$$\begin{aligned}
\left\| [1\ (\ell - x)] F \begin{bmatrix} v'(\ell) \\ v(\ell) \end{bmatrix} \right\|_{L^2(0,\ell)} &= \|k_2 v'(\ell) + k_4(\ell - x) v(\ell)\|_{L^2(0,\ell)} \\
&\leq k_2 \ell^{\frac{1}{2}} |v'(\ell)| + k_4 |v(\ell)| \|x - \ell\|_{L^2(0,\ell)}.
\end{aligned}$$

Note that

$$\|x - \ell\|_{L^2(0,\ell)} = \left(\int_0^\ell (x - \ell)^2\,\mathrm{d}x \right)^{\frac{1}{2}} = \frac{\ell^{\frac{3}{2}}}{\sqrt{3}}.$$

Using this together with (3.6) and (3.7) we have

$$\begin{aligned}\left\|[1\ (\ell-x)]F\begin{bmatrix} v'(\ell)\\ v(\ell)\end{bmatrix}\right\|_{L^2(0,\ell)} &\leq k_2\ell^{\frac{1}{2}}|v'(\ell)| + \frac{k_4\ell^{\frac{3}{2}}}{\sqrt{3}}|v(\ell)|\\ &\leq k_2\ell\|v''\|_{L^2(0,\ell)} + \frac{k_4\ell^2}{\sqrt{3}}\|v'\|_{L^2(0,\ell)}.\end{aligned}$$

Combining all the estimates we obtain

$$\begin{aligned}\|EIw''+c_DIv''\|_{L^2(0,\ell)} &\leq \ell^{\frac{3}{2}}\|\rho\|_{L^2(0,\ell)}\|g\|_{L^2(0,\ell)} + \ell^{\frac{3}{2}}\|c_a\|_{L^2(0,\ell)}\|v\|_{L^2(0,\ell)}\\ &\quad + k_2\ell\|v''\|_{L^2(0,\ell)} + \frac{k_4\ell^2}{\sqrt{3}}\|v'\|_{L^2(0,\ell)}\\ &\leq C_1\|g\|_H + C_2\|v\|_{H^2(0,\ell)},\end{aligned}$$

where $C_1 = \ell^{\frac{3}{2}}c_0^{-\frac{1}{2}}\|\rho\|_{L^2(0,\ell)}$ and $C_2 = 3\max\{\ell^{\frac{3}{2}}\|c_a\|_{L^2(0,\ell)}, k_2\ell, k_4\ell^2/\sqrt{3}\}$. Thus, the triangle inequality gives us

$$\begin{aligned}\|EIw''\|_{L^2(0,\ell)} &\leq \|EIw''+c_DIv''\|_{L^2(0,\ell)} + \|-c_DIv''\|_{L^2(0,\ell)}\\ &\leq C_1\|g\|_H + C_2\|v\|_{H^2(0,\ell)} + \|c_DIv''\|_{L^2(0,\ell)}\\ &\leq C_1\|g\|_H + C_2\|v\|_{H^2(0,\ell)} + C_0\|v''\|_{L^2(0,\ell)}\\ &\leq C_1\|g\|_H + (C_0+C_2)\|v\|_{H^2(0,\ell)}\\ &\leq C_1\|g\|_H + (C_0+C_2)\sqrt{C_\ell}\|v\|_V\end{aligned}$$

Since $\|EIw''\|_{L^2(0,\ell)} \geq \sqrt{c_0}\|w\|_V$ and $v=f$, we have

$$\|w\|_V \leq (C_0+C_2)\sqrt{\frac{C_\ell}{c_0}}\|f\|_V + \frac{C_1}{\sqrt{c_0}}\|g\|_H.$$

To simplify notation, let $C_3 = \max\{(C_0+C_2)\sqrt{C_\ell/c_0}, C_1/\sqrt{c_0}\}$. We claim that

$$\|w\|_V^2 \leq 2C_3^2\left\|\begin{bmatrix} f\\ g\end{bmatrix}\right\|_{\mathcal{H}}^2.$$

Indeed, squaring both sides of $\|w\|_V \leq C_3(\|f\|_V + \|g\|_H)$ gives

$$\|w\|_V^2 \leq C_3^2(\|f\|_V^2 + 2\|f\|_V\|g\|_H + \|g\|_H^2).$$

Using Young's inequality we obtain

$$\|f\|_V\|g\|_H \leq \frac{\|f\|_V^2}{2} + \frac{\|g\|_H^2}{2}$$

so that

$$\|w\|_V^2 \le 2C_3^2(\|f\|_V^2 + \|g\|_H^2)$$

and our claim is proved. Therefore

$$\left\| \mathcal{A}^{-1} \begin{bmatrix} f \\ g \end{bmatrix} \right\|_{\mathcal{H}} = \left\| \begin{bmatrix} w \\ v \end{bmatrix} \right\|_{\mathcal{H}} \le \sqrt{2C_3^2 + C_0 C_\ell} \left\| \begin{bmatrix} f \\ g \end{bmatrix} \right\|_{\mathcal{H}},$$

where C_0, C_ℓ and C_3 are constants independent of f and g. This shows that $\mathcal{A}^{-1} \in \mathcal{L}(\mathcal{H})$. □

Proof of Theorem 3.4. From Theorem 2.21, we only need to show that the operator $\mathcal{A}$ is a densely defined dissipative operator and that 0 lies in the resolvent set of $\mathcal{A}$. We have already shown in Lemma 3.5 that $\mathcal{A}$ is dissipative. We can see that the membership $0 \in \rho(\mathcal{A})$ is a consequence of Lemma 3.7. Thus, $\mathcal{A}$ is closed and by Theorem 2.2 $\mathcal{A}^{-1}$ is also closed. Because $\mathcal{A}$ is dissipative then $-\mathcal{A}$ is accretive. Furthermore, it is clear that 0 lies also in the resolvent of $-\mathcal{A}$ and since the resolvent set is an open set, we can find an $\epsilon > 0$ such that the ϵ-neighborhood of the origin lies in the resolvent set of $-\mathcal{A}$, that is, $\lambda \in \rho(-\mathcal{A})$ whenever $|\lambda| < \epsilon$. From this we can see that $-\epsilon/2 \in \rho(-\mathcal{A})$. Since $-\mathcal{A}$ is closed, $\mathcal{R}(-\mathcal{A} - (-\epsilon/2)I) = \mathcal{H}$ by Theorem 2.4. Hence $-\mathcal{A}$ is m-accretive and because m-accretive operators have dense domain, it follows that $\mathcal{D}(-\mathcal{A}) = \mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. Therefore $\mathcal{A}$ generates a C_0-semigroup of contractions in $\mathcal{H}$. □

3.2.2 Existence and Uniqueness

From the proof of Theorem 3.4, the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is closed. It follows that $\mathcal{D}(\mathcal{A})$ is a Hilbert space with the **graph norm** $\|\cdot\|_{gr}$ given by

$$\|z\|_{gr}^2 = \|z\|_{\mathcal{H}}^2 + \|\mathcal{A}z\|_{\mathcal{H}}^2.$$

Before interpreting our results in this section, we first state a theorem in Tucsnak [36, Proposition 2.3.5] necessary in this interpretation.

Theorem 3.8. *Let $S(t) = e^{At}$ be a C_0-semigroup on X where A is the infinitesimal generator. Let $z_0 \in \mathcal{D}(A)$ and define the function $z : [0, \infty) \to \mathcal{D}(A)$ by $z(t) = e^{At} z_0$. Then z is continuous, if we consider on $\mathcal{D}(A)$ the graph norm, and we also have $z \in C^1([0, \infty); X)$. Moreover, z is the unique function with the above properties satisfying the initial value problem*

$$\frac{\mathrm{d}z}{\mathrm{d}t} = Az, \quad z(0) = z_0.$$

Now let us interpret Theorem 3.4 in terms of the partial differential equation (3.9) with initial and boundary conditions (3.10) and (3.11), respectively. For $z_0 = [w_0\ v_0]^\mathsf{T} \in \mathcal{D}(\mathcal{A})$, the system (3.9)−(3.11) admits a unique solution y such that

$$[y\ \partial y/\partial t]^\mathsf{T} \in C([0,\infty);\mathcal{D}(\mathcal{A})) \cap C^1([0,\infty);\mathcal{H}).$$

Indeed, by setting

$$z(t) = \begin{bmatrix} y(t,\cdot) \\ \partial y(t,\cdot)/\partial t \end{bmatrix}, \tag{3.18}$$

we can see that y satisfies the above conditions if and only if z is a continuous $\mathcal{D}(\mathcal{A})$-valued function (where $\mathcal{D}(\mathcal{A})$ is endowed with the graph norm), continuously differentiable $\mathcal{H}$-valued function and it satisfies the abstract Cauchy problem

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = \mathcal{A}z(t), \quad z(0) = [w_0\ v_0]^\mathsf{T}. \tag{3.19}$$

Since we have established in Theorem 3.4 that $\mathcal{A}$ generates a C_0-semigroup, then Theorem 3.8 implies the existence and uniqueness of z (and consequently of y) with the above properties. Furthermore, z is given by $z(t) = e^{\mathcal{A}t}z_0$. Let $[y\ y_0]^\mathsf{T}$ be the vector form of z. Then Equation (3.19) implies that $y_0 = \partial y/\partial t$. Thus

$$z(t)(x) = \begin{bmatrix} y(t,x) \\ \partial y(t,x)/\partial t \end{bmatrix} = e^{\mathcal{A}t}\begin{bmatrix} w_0(x) \\ v_0(x) \end{bmatrix}.$$

If $\pi_1 : \mathcal{H} \to V$ is the projection map onto V then $y(t,x) = \pi_1(e^{\mathcal{A}t}[w_0(x)\ v_0(x)]^\mathsf{T})$.

3.3 Spectral Properties of the Generator

3.3.1 Sesquilinear Forms and Represenation Theorems

Let H and H' be two Hilbert spaces. A function $f : H \times H' \to \mathbb{C}$ is called a **sesquilinear form** on $H \times H'$ if

$$f[\alpha u_1 + \beta u_2, v] = \alpha f[u_1, v] + \beta f[u_2, v]$$

and

$$f[u, \alpha v_1 + \beta v_2] = \overline{\alpha} f[u, v_1] + \overline{\beta} f[u, v_2]$$

for all $u, u_1, u_2 \in H$, $v, v_1, v_2 \in H'$ and $\alpha, \beta \in \mathbb{C}$. In other words, f is sesquilinear if it is linear in the first component and semilinear in the second component. In particular, if $H = H'$ we say that f is a sesquilinear form on H. The inner product on H is a special case of a sesquilinear form on H.

Now we will consider forms on a subspace D of a Hilbert space H. If $f : D \times D \to \mathbb{C}$ is a sesquilinear form, the space D is called the **domain** of f and will be denoted by

$\mathcal{D}(f)$. The form f is said to be **densely defined** if $\overline{\mathcal{D}(f)} = H$. A form f is called **symmetric** if

$$f[u,v] = \overline{f[v,u]}$$

for all $u, v \in \mathcal{D}(f)$. For each $u \in \mathcal{D}(f)$ we let $f[u] = f[u,u]$.

A symmetric form is said to be **bounded from below** if

$$f[u] \geq \gamma \|u\|_H^2 \tag{3.20}$$

for all $u \in \mathcal{D}(f)$ and in this case we write $f \geq \gamma$. If $f \geq 0$ then f is said to be **nonnegative**. The largest number γ satisfying inequality (3.20) is called the **lower bound** of f and in the event that f has a positive lower bound it is called **positive definite**. Similarly, f is said to be **bounded from above** if

$$f[u] \leq \mu \|u\|_H^2$$

for all $u \in \mathcal{D}(f)$ and the smallest number μ satisfying this inequality is called the **upper bound** of f. Moreover, f is called **bounded** if it is both bounded from below and from above.

The set $\Theta(f) = \{f[u] \,|\, u \in \mathcal{D}(f) \text{ and } \|u\|_H = 1\}$ is called the **numerical range** of the form f. A form f is said to be **sectorially bounded from the left** (or simply **sectorial**) if

$$\Theta(f) \subset \{\zeta \in \mathbb{C} \,|\, |\arg(\zeta - \gamma)| \leq \theta\}$$

for some $\theta \in [0, \pi/2)$ and $\gamma \in \mathbb{R}$. A sectorial form f is called **closed** if $\|u_n - u\|_H \to 0$ and $f[u_n - u_m] \to 0$ as $n, m \to \infty$, where $u_n \in \mathcal{D}(f)$ and $u \in H$, imply that $u \in \mathcal{D}(f)$ and $f[u_n - u] \to 0$ as $n \to \infty$.

Before proving our main result in this section, we will first state the Riesz Representation Theorems. For the proofs of these theorems, we refer the readers to the work of Kato [22, pp. 322-333].

Theorem 3.9 (First Representation Theorem). *Let $f[w,v]$ be a densely defined, closed, sectorial sesquilinear form in a Hilbert space H. There exists an m-sectorial operator T such that $\mathcal{D}(T) \subset \mathcal{D}(f)$ and $f[w,v] = \langle Tw, v\rangle_H$ for all $w \in \mathcal{D}(T)$ and for all $v \in \mathcal{D}(f)$ and the m-sectorial operator T is uniquely determined by these properties. The form f is bounded if and only if T is bounded. Furthermore, f is symmetric if and only if T is self-adjoint.*

Theorem 3.10 (Second Representation Theorem). *Let $f[w,v]$ be a densely defined, closed, symmetric sesquilinear form in a Hilbert space H and $f \geq 0$. Let T be the associated self-adjoint operator obtained from Theorem 3.9. Then we have $\mathcal{D}(T^{\frac{1}{2}}) = \mathcal{D}(f)$ and*

$$f[w,v] = \langle T^{\frac{1}{2}} w, T^{\frac{1}{2}} v\rangle_H$$

for all $w, v \in \mathcal{D}(f)$. Thus $T^{\frac{1}{2}}$ is self-adjoint and $(T^{\frac{1}{2}})^2 = T$. Moreover T and f have the same lower bound.

Consider the function $b[\cdot,\cdot] : V \times V \to \mathbb{C}$ defined by

$$b[w,v] = k_2 w'(\ell)\overline{v'(\ell)} + k_4 w(\ell)\overline{v(\ell)} + \int_0^\ell (c_D I w''\overline{v''} + c_a w\overline{v})\,\mathrm{d}x.$$

Further we let $b[w] = b[w,w]$ for all $w \in V$.

Theorem 3.11. *The form b is a nonnegative, closed, symmetric, sectorial form on V. Furthermore, there exists a constant $K > 0$ such that $b[w] \leq K\|w\|_V^2$ for all $w \in V$.*

Proof. If $w_1, w_2, v \in V$ and $\alpha, \beta \in \mathbb{C}$ then

$$\begin{aligned} b[\alpha w_1 + \beta w_2, v] &= k_2(\alpha w_1 + \beta w_2)'(\ell)\overline{v'(\ell)} + k_4(\alpha w_1 + \beta w_2)(\ell)\overline{v(\ell)} \\ &\quad + \int_0^\ell [c_D I(\alpha w_1 + \beta w_2)''\overline{v''} + c_a(\alpha w_1 + \beta w_2)\overline{v}]\,\mathrm{d}x \\ &= \alpha\left(k_2 w_1'(\ell)\overline{v'(\ell)} + k_4 w_1(\ell)\overline{v(\ell)} + \int_0^\ell (c_D I w_1''\overline{v''} + c_a w_1\overline{v})\,\mathrm{d}x\right) \\ &\quad + \beta\left(k_2 w_2'(\ell)\overline{v'(\ell)} + k_4 w_2(\ell)\overline{v(\ell)} + \int_0^\ell (c_D I w_2''\overline{v''} + c_a w_2\overline{v})\,\mathrm{d}x\right) \\ &= \alpha b[w_1, v] + \beta b[w_2, v]. \end{aligned}$$

Therefore b is linear in the first component. Also, b is semilinear in the second component since

$$\begin{aligned} b[w, \alpha v_1 + \beta v_2] &= k_2 w'(\ell)[\overline{\alpha v_1'(\ell) + \beta v_2'(\ell)}] + k_4 w(\ell)[\overline{\alpha v_1(\ell) + \beta v_2(\ell)}] \\ &\quad + \int_0^\ell [c_D I w''(\overline{\alpha v_1'' + \beta v_2''}) + c_a w(\overline{\alpha v_1 + \beta v_2})]\,\mathrm{d}x \\ &= \overline{\alpha}\left(k_2 w'(\ell)\overline{v_1'(\ell)} + k_4 w(\ell)\overline{v_1(\ell)} + \int_0^\ell (c_D I w''\overline{v_1''} + c_a w\overline{v_1})\,\mathrm{d}x\right) \\ &\quad + \overline{\beta}\left(k_2 w'(\ell)\overline{v_2'(\ell)} + k_4 w(\ell)\overline{v_2(\ell)} + \int_0^\ell (c_D I w''\overline{v_2''} + c_a w\overline{v_2})\,\mathrm{d}x\right) \\ &= \overline{\alpha} b[w, v_1] + \overline{\beta} b[w, v_2] \end{aligned}$$

for all $w, v_1, v_2 \in V$.

Further we can see that $b[w,v] = \overline{b[v,w]}$ for all pairs $w, v \in V$ and hence b is symmetric. Since

$$b[w] = k_2|w'(\ell)|^2 + k_4|w(\ell)|^2 + \int_0^\ell (c_D I|w''|^2 + c_a|w|^2)\,\mathrm{d}x$$

we have $b[w] \geq 0$. Also, from (3.6) and (3.7) we have

$$\begin{aligned} b[w] &\leq (C_0 + k_2\ell + k_4\ell^3 + C_0\ell^4)\|w''\|_{L^2(0,\ell)}^2 \\ &\leq \frac{C_0 + k_2\ell + k_4\ell^3 + C_0\ell^4}{c_0}\|w\|_V^2 \end{aligned}$$

for all $w \in V$. Because b is nonnegative then its numerical range is a subset of the set of nonnegative real numbers and thus b is sectorial. Let $w_n \in \mathcal{D}(b) = V$ for all $n \in \mathbb{N}$, $\|w_n - w\|_V \to 0$ and $b[w_m - w_n] \to 0$ as $n, m \to \infty$. The completeness of V implies that $w \in V$ and the inequality

$$b[w_n - w] \le \frac{C_0 + k_2\ell + k_4\ell^3 + C_0\ell^4}{c_0}\|w_n - w\|_V^2$$

implies that $b[w_n - w] \to 0$. Therefore b is closed. □

We can reformulate the partial differential equation (3.9) as a variational evolution equation in terms of the inner products in V and H and the sesquilinear form b.

Theorem 3.12. *Suppose that y is a solution of the partial differential equation* (3.9) *with initial conditions* (3.10) *and boundary conditions* (3.11)*. Then y satisfies the variational evolution equation*

$$\langle \ddot{y}(t), \xi\rangle_H + b[\dot{y}(t), \xi] + \langle y(t), \xi\rangle_V = 0 \tag{3.21}$$

for all $\xi \in V$ and for all $t > 0$ with initial conditions

$$y(0) = w_0, \quad \dot{y}(0) = v_0. \tag{3.22}$$

Proof. For every $\xi \in V$ we have

$$\langle \ddot{y}, \xi\rangle_H = \left\langle -\frac{1}{\rho}(EIy'' + c_D I\dot{y}'')'' - \frac{c_a\dot{y}}{\rho}, \xi\right\rangle_H .$$

The definitions of the inner products in H and V and the sesquilinear form b on V together with integration by parts give us

$$\begin{aligned}
\langle y, \xi\rangle_V + b[\dot{y}, \xi] &= \int_0^\ell (EIy'' + c_D I\dot{y}'')\overline{\xi''}\,\mathrm{d}x + k_2\dot{y}'(\ell)\overline{\xi'(\ell)} + k_4\dot{y}(\ell)\overline{\xi(\ell)} + \int_0^\ell c_a\dot{y}\overline{\xi}\,\mathrm{d}x \\
&= (EI(\ell)y''(\ell) + c_D I(\ell)\dot{y}''(\ell))\overline{\xi'(\ell)} - (EI(\ell)y''(\ell) + c_D I(\ell)\dot{y}''(\ell))'\overline{\xi(\ell)} \\
&\quad + \int_0^\ell (EIy'' + c_D I\dot{y}'')''\overline{\xi}\,\mathrm{d}x + k_2\dot{y}'(\ell)\overline{\xi'(\ell)} + k_4\dot{y}(\ell)\overline{\xi(\ell)} + \int_0^\ell c_a\dot{y}\overline{\xi}\,\mathrm{d}x \\
&= \int_0^\ell ((EIy'' + c_D I\dot{y}'')'' + c_a\dot{y})\overline{\xi}\,\mathrm{d}x \\
&= \left\langle \frac{1}{\rho}(EIy'' + c_D I\dot{y}'')'' + \frac{c_a\dot{y}}{\rho}, \xi\right\rangle_H \\
&= -\langle \ddot{y}, \xi\rangle_H .
\end{aligned}$$

Therefore y satisfies the variational evolution equation (3.21) with initial conditions (3.22). □

The domain of the symmetric sesquilinear form $\langle\cdot,\cdot\rangle_V$ is V and hence it is a densely defined form in H by Theorem 3.3. Further $\langle\cdot,\cdot\rangle_V$ is positive definite since $\|w\|_V \geq (C_0C_\ell)^{-\frac{1}{2}}\|w\|_H$ for all $w \in V$, where $(C_0C_\ell)^{-\frac{1}{2}}$ is a positive constant independent of w. This implies that $\langle\cdot,\cdot\rangle_V$ is sectorial. Let $w_n \in V$ for each $n \in \mathbb{N}$, $\|w_n - w\|_H \to 0$ and $\|w_n - w_m\|_V \to 0$ as $n, m \to \infty$. Again, the completeness of V implies that $\|w_n - w_0\|_V \to 0$ as $n \to \infty$ for some $w_0 \in V$. But $\|w_n - w_0\|_H \leq (C_0C_\ell)^{\frac{1}{2}}\|w_n - w_0\|_V$ and so $\|w_n - w_0\|_H \to 0$. The uniqueness of limits implies that $w = w_0$ and therefore $\langle\cdot,\cdot\rangle_V$ is closed. In summary, $\langle\cdot,\cdot\rangle_V$ is a positive definite, densely defined, closed, symmetric sesquilinear form in H.

The Riesz representation theorems and the preceding paragraph imply that there exists a positive self-adjoint operator $A : \mathcal{D}(A) \subset H \to H$ such that

$$\begin{aligned} \mathcal{D}(A^{\frac{1}{2}}) &= V, \\ \langle w, v\rangle_V &= \langle A^{\frac{1}{2}}w, A^{\frac{1}{2}}v\rangle_H, \end{aligned} \tag{3.23}$$

for all $w, v \in V$. Furthermore $A^{\frac{1}{2}}$ is a self-adjoint operator and if $w \in \mathcal{D}(A)$ and $v \in V$ then

$$\langle w, v\rangle_V = \langle A^{\frac{1}{2}}(A^{\frac{1}{2}}w), v\rangle_H = \langle Aw, v\rangle_H\,. \tag{3.24}$$

Theorem 3.11 and the boundedness of the sesquilinear form b implies that there exists a nonnegative self-adjoint operator $S \in \mathcal{L}(V)$ such that

$$b[w, v] = \langle S^{\frac{1}{2}}w, S^{\frac{1}{2}}v\rangle_V$$

for all $w, v \in V$. Similarly, we obtain that

$$b[w, v] = \langle Sw, v\rangle_V \tag{3.25}$$

for all $w, v \in V$.

Theorem 3.13. *Let $w, v \in V$. The following conditions are equivalent.*

(1) $w + Sv \in \mathcal{D}(A)$.

(2) *There exists $u \in H$ such that $\langle w + Sv, \xi\rangle_V = \langle u, \xi\rangle_H$ for all $\xi \in V$.*

(3) $A(w + Sv) = u$.

Proof. It is obvious that (3) implies (1). If $w + Sv \in \mathcal{D}(A)$ then for all $\xi \in V$ we have $\langle w + Sv, \xi\rangle_V = \langle A(w + Sv), \xi\rangle_H$ by Equation (3.24) and we may take $u = A(w + Sv)$. Thus (1) implies (2). Finally, assume that condition (2) is satisfied. Then $\langle w + Sv, \xi\rangle_V = \langle Az, \xi\rangle_H$ where $z = A^{-1}u \in \mathcal{D}(A)$ and hence $\langle w + Sv, \xi\rangle_V = \langle z, \xi\rangle_V$ for all $\xi \in V$ by (3.24). Further this implies that $\langle w + Sv - z, \xi\rangle_V = 0$ for all $\xi \in V$ and it follows that $w + Sv = z$ by the density of V in H. Hence condition (2) implies (3). $\square$

3.3.2 A Decomposition of the Generator

Define the operator $\mathcal{A}_1 : \mathcal{D}(\mathcal{A}_1) \subset \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{A}_1 \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} v \\ -A(w + Sv) \end{bmatrix}$$

where $\mathcal{D}(\mathcal{A}_1) = \{[w\ v]^\mathsf{T} \in \mathcal{H} \,|\, v \in V, w + Sv \in \mathcal{D}(A)\}$. Suppose $[w\ v]^\mathsf{T} \in \mathcal{D}(\mathcal{A})$. Then $w, v \in V$, $EIw'' + c_D Iv'' \in H^2(0,\ell)$ and from (3.25)

$$\begin{aligned} \langle w + Sv, \xi \rangle_V &= \langle w, \xi \rangle_V + \langle Sv, \xi \rangle_V \\ &= \langle w, \xi \rangle_V + b[v, \xi] \\ &= \left\langle \frac{1}{\rho}(EIw'' + c_D Iv'')'' + \frac{c_a v}{\rho}, \xi \right\rangle_H \end{aligned}$$

for all $\xi \in V$, where the last term is obtained from the boundary conditions at $x = \ell$. Since $\frac{1}{\rho}(EIw'' + c_D Iv'')'' + \frac{1}{\rho} c_a v \in H$, Theorem 3.13 implies that we must have $w + Sv \in \mathcal{D}(A)$ and

$$A(w + Sv) = \frac{1}{\rho}(EIw'' + c_D Iv'')'' + \frac{c_a v}{\rho}.$$

Therefore $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{A}_1)$ and $\mathcal{A}_1$ is an extension of $\mathcal{A}$.

Since A is positive definite then $A^{-\frac{1}{2}} \in \mathcal{L}(H)$ and $A^{-1} = (A^{-\frac{1}{2}})^2 \in \mathcal{L}(H)$ (see Kato [22, p. 283]). Consider the operator $A^{-\frac{1}{2}} : H \to H$. Let $\{v_n\}_{n=1}^\infty$ be a bounded sequence in H. Since $\mathcal{R}(A^{-\frac{1}{2}}) = \mathcal{D}(A^{\frac{1}{2}}) = V$ we have $A^{-\frac{1}{2}} v_n \in V$ for all $n \in \mathbb{N}$. This implies that

$$\|A^{-\frac{1}{2}} v_n\|_V = \|v_n\|_H$$

for all $n \in \mathbb{N}$. The above equality shows that $\{A^{-\frac{1}{2}} v_n\}_{n=1}^\infty$ is a bounded sequence in V, so that by the compactness of the embedding $V \hookrightarrow H$,

$$\lim_{j \to \infty} \|A^{-\frac{1}{2}} v_{n_j} - v\|_H = 0$$

for some subsequence $\{A^{-\frac{1}{2}} v_{n_j}\}_{j=1}^\infty$ of $\{A^{-\frac{1}{2}} v_n\}_{n=1}^\infty$ and for some $v \in H$. Hence $A^{-\frac{1}{2}}$ is compact. Moreover, the restriction $A^{-\frac{1}{2}}|_V : V \to V$ of $A^{-\frac{1}{2}}$ in V is also compact. Indeed, let $\{w_n\}_{n=1}^\infty$ be a bounded sequence in V. Then for some subsequence $\{w_{n_j}\}_{j=1}^\infty$ of $\{w_n\}_{n=1}^\infty$ and some element $w_0 \in H$ we have

$$\lim_{j \to \infty} \|w_{n_j} - w_0\|_H = 0.$$

Let $w = A^{-\frac{1}{2}} w_0 \in V$. Then

$$\begin{aligned} \|A^{-\frac{1}{2}}|_V w_{n_j} - w\|_V &= \|A^{-\frac{1}{2}} w_{n_j} - w\|_V \\ &= \|A^{-\frac{1}{2}}(w_{n_j} - w_0)\|_V \\ &= \|w_{n_j} - w_0\|_H \end{aligned}$$

and so

$$\lim_{j\to\infty} \|A^{-\frac{1}{2}}|_V w_{n_j} - w_0\|_V = 0,$$

showing that $A^{-\frac{1}{2}}|_V$ is compact.

In the following discussion, we are going to show that the operator $A^{-1}: H \to H$ is also compact. First note that $\mathcal{R}(A^{-1}) = \mathcal{D}(A) \subset V$. Let $\{v_n\}_{n=1}^{\infty}$ be a bounded sequence of functions in H. Because $A^{-1}v_n \in V$ we have

$$\begin{aligned} \|A^{-1}v_n\|_V^2 &= \left\langle A^{-1}v_n, A^{-1}v_n \right\rangle_V \\ &= \langle A^{\frac{1}{2}}(A^{-1}v_n), A^{\frac{1}{2}}(A^{-1}v_n)\rangle_H \\ &= \langle A^{-\frac{1}{2}}v_n, A^{-\frac{1}{2}}v_n\rangle_H \\ &= \|A^{-\frac{1}{2}}v_n\|_H^2 \end{aligned}$$

and since $\|A^{-\frac{1}{2}}v_n\|_H \le \|A^{-\frac{1}{2}}\|_{\mathcal{L}(H)}\|v_n\|_H$ we arrive at

$$\|A^{-1}v_n\|_V \le \|A^{-\frac{1}{2}}\|_{\mathcal{L}(H)}\|v_n\|_H$$

which implies that the sequence $\{A^{-1}v_n\}_{n=1}^{\infty}$ is a bounded sequence in V. The compactness of the embedding $V \hookrightarrow H$ implies that we can find a subsequence $\{A^{-1}v_{n_j}\}_{j=1}^{\infty}$ of $\{A^{-1}v_n\}_{n=1}^{\infty}$ such that $\|A^{-1}v_{n_j} - w\|_H \to 0$ as $j \to \infty$ for some $w \in H$. Therefore the operator A^{-1} is compact.

Next we consider the restriction $A^{-1}|_V : V \to V$ of A^{-1} in V. For convenience we let $A_V^{-1} = A^{-1}|_V$. Suppose that $\{w_n\}_{n=1}^{\infty}$ is a bounded sequence in V. The compactness of $V \hookrightarrow H$ implies the existence of a function $w_0 \in H$ such that

$$\lim_{j\to\infty} \|w_{n_j} - w_0\|_H = 0$$

for some subsequence $\{w_{n_j}\}_{j=1}^{\infty}$ of $\{w_n\}_{n=1}^{\infty}$. Let $w = A^{-1}w_0 \in V$. Using the inequality

$$\begin{aligned} \|A_V^{-1}w_{n_j} - w\|_V &= \|A^{-1}w_{n_j} - w\|_V \\ &= \|A^{-1}(w_{n_j} - w_0)\|_V \\ &\le \|A^{-\frac{1}{2}}\|_{\mathcal{L}(H)}\|w_{n_j} - w_0\|_H \end{aligned}$$

we have

$$\lim_{j\to\infty} \|A_V^{-1}w_{n_j} - w\|_V = 0.$$

Thus, A_V^{-1} is compact. These prove the following theorem.

Theorem 3.14. *The inverse $A^{-1}: H \to H$ of the operator A and its restriction $A_V^{-1} = A^{-1}|_V : V \to V$ in V are compact operators. Furthermore*

$$\|A^{-1}w\|_V \le \|A^{-\frac{1}{2}}\|_{\mathcal{L}(H)}\|w\|_H \tag{3.26}$$

for all $w \in H$. Also, $A^{-\frac{1}{2}}: H \to H$ and $A^{-\frac{1}{2}}|_V : V \to V$ are compact operators.

From Equation (3.26) it follows that

$$\|A^{-1}w\|_V \leq \sqrt{C_0 C_\ell}\|A^{-\frac{1}{2}}\|_{\mathcal{L}(H)}\|w\|_V$$

for all $v \in V$. Thus $A_V^{-1} \in \mathcal{L}(V)$.

Let us show that the resolvent of $\mathcal{A}_1$ contains the scalar 0. If $\mathcal{A}_1[w\ v]^\mathsf{T} = 0$ then $v = 0$ and so $Aw = 0$. Thus $w = A^{-1}0 = 0$ and so $\mathcal{N}(\mathcal{A}_1) = \{0\}$. Now assume that $[f\ g]^\mathsf{T} \in \mathcal{H}$. Let $v = f$ and $w = -A^{-1}g - Sf$. Then $v \in V$ and $w + Sv = -A^{-1}g \in \mathcal{D}(A)$ so that $[w\ v]^\mathsf{T} \in \mathcal{D}(\mathcal{A}_1)$. Also $\mathcal{A}_1[w\ v]^\mathsf{T} = [f\ g]^\mathsf{T}$ and therefore $\mathcal{A}_1^{-1}$ exists and

$$\begin{aligned}
\left\|\mathcal{A}_1^{-1}\begin{bmatrix} f \\ g \end{bmatrix}\right\|_{\mathcal{H}}^2 &= \left\|\begin{bmatrix} -A^{-1}g - Sf \\ f \end{bmatrix}\right\|_{\mathcal{H}}^2 \\
&= \|A^{-1}g + Sf\|_V^2 + \|f\|_H^2.
\end{aligned}$$

Recall that $\|f\|_H^2 \leq C_0 C_\ell \|f\|_V^2$. Also, by Theorem 3.14 and Young's inequality

$$\begin{aligned}
\|A^{-1}g + Sf\|_V^2 &\leq (\|A^{-1}g\|_V + \|Sf\|_V)^2 \\
&\leq (\|A^{-\frac{1}{2}}\|_{\mathcal{L}(H)}\|g\|_H + \|S\|_{\mathcal{L}(V)}\|f\|_V)^2 \\
&= \|A^{-\frac{1}{2}}\|_{\mathcal{L}(H)}^2\|g\|_H^2 + 2\|A^{-\frac{1}{2}}\|_{\mathcal{L}(H)}\|S\|_{\mathcal{L}(V)}\|g\|_H\|f\|_V \\
&\quad + \|S\|_{\mathcal{L}(V)}^2\|f\|_V^2 \\
&= \|A^{-\frac{1}{2}}\|_{\mathcal{L}(H)}^2\|g\|_H^2 + \|A^{-\frac{1}{2}}\|_{\mathcal{L}(H)}\|S\|_{\mathcal{L}(V)}(\|g\|_H^2 + \|f\|_V^2) \\
&\quad + \|S\|_{\mathcal{L}(V)}^2\|f\|_V^2 \\
&\leq 2K(\|g\|_H^2 + \|f\|_V^2).
\end{aligned}$$

where $K = \max\{\|A^{-\frac{1}{2}}\|_{\mathcal{L}(H)}^2, \|S\|_{\mathcal{L}(V)}^2\}$. Therefore

$$\begin{aligned}
\left\|\mathcal{A}_1^{-1}\begin{bmatrix} f \\ g \end{bmatrix}\right\|_{\mathcal{H}}^2 &\leq (2K + C_0C_\ell)\|f\|_V^2 + 2K\|g\|_H^2 \\
&\leq (2K + C_0C_\ell)\left\|\begin{bmatrix} f \\ g \end{bmatrix}\right\|_{\mathcal{H}}^2.
\end{aligned}$$

Hence $\mathcal{A}_1^{-1} \in \mathcal{L}(\mathcal{H})$ and therefore $0 \in \rho(\mathcal{A}_1)$. Because $0 \in \rho(\mathcal{A}) \cap \rho(\mathcal{A}_1)$ and $\mathcal{A}_1$ is an extension of $\mathcal{A}$ it follows from the succeeding theorem that $\mathcal{A}_1 = \mathcal{A}$ (see Miklavčič [29, Lemma 1.6.14]).

Theorem 3.15. *If A and B are linear operators in a Banach space X such that A is an extension of B and $\rho(A) \cap \rho(B)$ is nonempty, then $A = B$.*

Proof. We note that it is enough to show that $\mathcal{D}(A) \subset \mathcal{D}(B)$. Let $x \in \mathcal{D}(A)$ and $\lambda \in \rho(A) \cap \rho(B)$. Then $(A - \lambda I)x \in X$. The operators A and B are closed since they have nonempty spectrum, and by Theorem 2.4 $A - \lambda I$ and $B - \lambda I$ are both one-to-one and onto. Specifically, since B is onto this means that we can find an element $y \in \mathcal{D}(B)$ such that $(B - \lambda I)y = (A - \lambda I)x \in X$. Since A is an extension of B, then $(A - \lambda I)y = (B - \lambda I)y$. Hence $(A - \lambda I)(x - y) = 0$ so that $x = y$. This shows that $x \in \mathcal{D}(B)$. Therefore $A = B$. □

We summarize the results of the above discussion in the following theorem.

Theorem 3.16. *The infinitesimal generator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ can be also defined as*

$$\mathcal{A} \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} v \\ -A(w + Sv) \end{bmatrix}$$

where $\mathcal{D}(\mathcal{A}) = \{[w \ v]^\mathsf{T} \in \mathcal{H} \,|\, v \in V \text{ and } w + Sv \in \mathcal{D}(A)\}$, for some positive definite self-adjoint operator $A : \mathcal{D}(A) \subset H \to H$ and for some nonnegative self-adjoint operator $S \in \mathcal{L}(V)$.

3.3.3 Nonreal and Point Spectra

The following theorem, which is the main result of this section and plays a crucial role in the proof of the exponential stability of the system that we consider, implies that the nonreal spectrum of the infinitesimal generator $\mathcal{A}$ lies in its point spectrum. We remark that this result is due to Chen, Liu and Liu [7, Lemma 4.1] and its proof follows the proof in their paper.

Theorem 3.17. *Let $\Lambda = \{z \in \mathbb{C} \,|\, \mathrm{Im}\ z \neq 0\}$ and $\Lambda_0 = \Lambda \cap \sigma(\mathcal{A})$. Then $\Lambda_0 \subset \sigma_p(\mathcal{A})$.*

Proof. We claim that $\Lambda \cap (\mathbb{C} \setminus \sigma_p(\mathcal{A})) \subset \rho(\mathcal{A})$. Let $\lambda \in \Lambda \cap (\mathbb{C} \setminus \sigma_p(\mathcal{A}))$. Then $\mathrm{Im}\ \lambda \neq 0$ and $\lambda \notin \sigma_p(\mathcal{A})$. Hence $\mathcal{A} - \lambda I$ is one-to-one and so $\mathcal{N}(\mathcal{A} - \lambda I) = \{0\}$. If we can show that $\mathcal{R}(\mathcal{A} - \lambda I) = \mathcal{H}$ then $\lambda \in \rho(\mathcal{A})$. Indeed, since $\mathcal{A}$ is closed, $\mathcal{A} - \lambda I$ is one-to-one and $\mathcal{R}(\mathcal{A} - \lambda I) = \mathcal{H}$, so that $\mathcal{A} - \lambda I$ is one-to-one and onto, it follows from Theorem 2.4 that $\lambda \in \rho(\mathcal{A})$.

Define the operator $\Delta_\lambda : V \to V$ by

$$\Delta_\lambda = I + \lambda S + \lambda^2 A_V^{-1}. \tag{3.27}$$

Since $S, A_V^{-1} \in \mathcal{L}(V)$ and $\|\Delta_\lambda\|_{\mathcal{L}(V)} \leq 1 + |\lambda| \|S\|_{\mathcal{L}(V)} + |\lambda|^2 \|A_V^{-1}\|_{\mathcal{L}(V)}$ we have $\Delta_\lambda \in \mathcal{L}(V)$. We claim that $\mathcal{N}(\Delta_\lambda) = \{0\}$. Indeed, assume the contrary and let v be a nonzero element of $\mathcal{N}(\Delta_\lambda)$. Then v satisfies the equation

$$v + \lambda S v + \lambda^2 A_V^{-1} v = v + \lambda S v + \lambda^2 A^{-1} v = 0.$$

Dividing by the nonzero λ and using the linearity of A^{-1} we get

$$v/\lambda + Sv = -A^{-1}(\lambda v)$$

and so

$$A(v/\lambda + Sv) = -\lambda v.$$

Using these we get

$$(\mathcal{A} - \lambda I)\begin{bmatrix} v/\lambda \\ v \end{bmatrix} = \begin{bmatrix} v \\ -A(v/\lambda + Sv) \end{bmatrix} - \begin{bmatrix} v \\ \lambda v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus $[v/\lambda \ v]^{\mathsf{T}} \in \mathcal{N}(\mathcal{A} - \lambda I)$, which is a contradiction. Therefore we must have $\mathcal{N}(\mathcal{A} - \lambda I) = \{0\}$.

It follows from Theorem 2.7 and the self-adjointness of S that $\lambda_1 \in \rho(S)$ for any nonreal λ_1. Thus, taking $\lambda_1 = -\lambda^{-1}$, the operator $S - (-\lambda^{-1})I$ has a continuous inverse. Since $\lambda S + I = \lambda(S - (-\lambda^{-1})I)$ and $-\lambda^{-1} \in \rho(S)$ then $0 \in \rho(\lambda S + I)$. Notice that we can re-write Δ_λ as

$$\begin{aligned} \Delta_\lambda &= (\lambda S + I)(I + \lambda^2(\lambda S + I)^{-1}A_V^{-1}) \\ &= \lambda^2(\lambda S + I)((\lambda S + I)^{-1}A_V^{-1} - (-\lambda^{-2})I). \end{aligned} \tag{3.28}$$

Since $(\lambda S+I)^{-1} \in \mathcal{L}(V)$ and A_V^{-1} is compact, by Theorem 2.6, the product $(\lambda S+I)^{-1}A_V^{-1}$ is also compact. We claim that $-\lambda^{-2} \in \rho((\lambda S + I)^{-1}A_V^{-1})$. Indeed, assume that $-\lambda^{-2}$ is an eigenvalue of $(\lambda S + I)^{-1}A_V^{-1}$. Then

$$(\lambda S + I)^{-1}A_V^{-1}v = -\lambda^{-2}v$$

for some nonzero v. Thus

$$\begin{aligned} \Delta_\lambda v &= \lambda^2(\lambda S + I)((\lambda S + I)^{-1}A_V^{-1}v - (-\lambda^{-2})v) \\ &= \lambda^2(\lambda S + I)0 \\ &= 0. \end{aligned}$$

Therefore $v \in \mathcal{N}(\Delta_\lambda)$, which is a contradiction. Thus $-\lambda^{-2} \notin \sigma_p((\lambda S+I)^{-1}A_V^{-1})$ and by Theorem 2.5 we have $-\lambda^{-2} \in \rho((\lambda S+I)^{-1}A_V^{-1})$. Using the facts that $0 \in \rho(\lambda S+I)$ and $-\lambda^{-2} \in \rho((\lambda S + I)^{-1}A_V^{-1})$ we obtain from (3.28) that $0 \in \rho(\Delta_\lambda)$. Therefore $\rho(\Delta_\lambda) \neq \emptyset$ and so Δ_λ is closed. Further, $0 \in \rho(\Delta_\lambda)$ implies that Δ_λ is one-to-one and onto by Theorem 2.4. Hence $\mathcal{R}(\Delta_\lambda) = V$.

Let $[f \ g]^{\mathsf{T}} \in \mathcal{H}$. Since $f - \lambda A^{-1}g \in V$ we can find an element $v \in V$ such that

$$\Delta_\lambda v = f - \lambda A^{-1}g.$$

Thus $v = \Delta_\lambda^{-1}(f - \lambda A^{-1}g)$. Let $w = \lambda^{-1}(v - f)$. Because $v, f \in V$ it follows that $w \in V$. Also,

$$\begin{aligned} Sv &= \lambda^{-1}(\Delta_\lambda v - v - \lambda^2 A_V^{-1}v) \\ &= \lambda^{-1}(\Delta_\lambda v - v - \lambda^2 A^{-1}v) \end{aligned}$$

and

$$\begin{aligned} w + Sv &= \lambda^{-1}(v - f) + \lambda^{-1}(\Delta_\lambda v - v - \lambda^2 A^{-1} v) \\ &= \lambda^{-1}(v - f) + \lambda^{-1}(f - \lambda A^{-1} g - v - \lambda^2 A^{-1} v) \\ &= -A^{-1} g - \lambda A^{-1} v \\ &= A^{-1}(-g - \lambda v) \in \mathcal{D}(A). \end{aligned}$$

Therefore $[w\ v]^\mathsf{T} \in \mathcal{D}(\mathcal{A})$ by Theorem 3.16 and

$$\begin{aligned} (\mathcal{A} - \lambda I)\begin{bmatrix} w \\ v \end{bmatrix} &= \begin{bmatrix} v \\ -A(w + Sv) \end{bmatrix} - \begin{bmatrix} \lambda w \\ \lambda v \end{bmatrix} \\ &= \begin{bmatrix} v \\ g + \lambda v \end{bmatrix} - \begin{bmatrix} \lambda w \\ \lambda v \end{bmatrix} \\ &= \begin{bmatrix} f \\ g \end{bmatrix}. \end{aligned}$$

This shows that $\mathcal{R}(\mathcal{A} - \lambda I) = \mathcal{H}$ and this completes the proof of our claim.

To finish our proof, let $\lambda \notin \sigma_p(\mathcal{A})$. If $\operatorname{Im}\lambda = 0$ then $\lambda \notin \Lambda$ so that in particular $\lambda \notin \Lambda_0$. Suppose $\operatorname{Im}\lambda \neq 0$. Then $\lambda \in \Lambda \cap (\mathbb{C} \setminus \sigma_p(\mathcal{A}))$ and hence we have $\lambda \in \rho(\mathcal{A})$. This means that λ does not lie in the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$. Thus $\lambda \notin \Lambda_0$. Therefore we have shown that if $\lambda \notin \sigma_p(\mathcal{A})$ then $\lambda \notin \Lambda_0$. Hence $\Lambda_0 \subset \sigma_p(\mathcal{A})$. This completes the proof of the theorem. □

3.4 Exponential Stability

In this section, we will consider three feedback schemes in our exponential stabilization problem. We remark that these feedback schemes are the same as those with Liu and Zheng [28]. The following theorem implies that the presence of the Kelvin-Voigt and air damping do not affect the exponential stability of the system provided that the stiffness coefficient function is continuously twice differentiable and the internal and viscous damping coefficient functions are continuous on the whole interval $[0, \ell]$.

Theorem 3.18. *The C_0-semigroup generated by $\mathcal{A}$ is exponentially stable, that is, there exist constants $M \geq 1$ and $\alpha > 0$ such that*

$$\|e^{t\mathcal{A}}\|_{\mathcal{L}(\mathcal{H})} \leq M e^{-\alpha t}$$

for all $t \geq 0$, if hypothesis (H1) *and one of the following conditions is satisfied:*

(1) *the feedback coefficients k_2 and k_4 are positive* (*i.e., the bending moment and shear force are given simultaneously in a feedback form*);

(2) $k_2 = 0$, $k_4 > 0$, $\rho(x) + x\rho'(x) \geq 0$ *and* $3EI(x) - xEI'(x) \geq k_0 > 0$ *for all* $x \in [0, \ell]$ (*i.e., shear force is given in feedback form and bending moment is free*);

(3) $k_2 > 0$, $k_4 = 0$ *and there exist constants* $k_0 > 0$ *and* $\kappa \geq 0$ *such that*

$$\rho'(x)\left(\kappa + \int_0^x \frac{\mathrm{d}s}{EI(s)}\right) \leq \frac{\rho(x)}{EI(x)} \tag{3.29}$$

and

$$-EI'(x)\left(\kappa + \int_0^x \frac{\mathrm{d}s}{EI(s)}\right) \leq 3 - k_0 \tag{3.30}$$

for all $x \in [0, \ell]$ (*i.e., bending moment is given in feedback form and shear force is free*).

Before proving the theorem, we note that if ρ and EI are constants then the conditions in the second and third feedback schemes in the above theorem are satisfied.

3.4.1 A Unique Continuation Condition

We divide the proof of Theorem 3.18 into several steps which consists of the following lemmas. First, we will consider the following unique continuation condition.

(UCC) If $w \in V$, $\langle w, \xi \rangle_V - \beta^2 \langle w, \xi \rangle_H = 0$ for all $\xi \in V$, $b[w] = 0$ and

$$k_4 w(\ell) = k_2 w'(\ell) = w''(\ell) = w'''(\ell) = 0$$

then $w = 0$ in V, for all $\beta \in \mathbb{R}$.

The following lemma says that hypothesis (H1) together with (UCC) implies that the imaginary axis lies in the resolvent set of $\mathcal{A}$.

Lemma 3.19. *If* $k_2, k_4 \geq 0$, (H1) *and* (UCC) *are satisfied then* $i\mathbb{R} \subset \rho(\mathcal{A})$.

Proof. We claim that $i\mathbb{R} \cap \sigma_p(\mathcal{A}) = \emptyset$. Let $\beta \in \mathbb{R}$. Since $0 \in \rho(\mathcal{A})$ it follows that $0 \notin \sigma_p(A)$. Therefore, we may assume that $\beta \neq 0$. Let $[w \ v]^\mathsf{T} \in \mathcal{N}(\mathcal{A} - i\beta I)$. By definition

$$\begin{bmatrix} v - i\beta w \\ -\frac{1}{\rho}(EIw'' + c_D I v'')'' - \frac{1}{\rho} c_a v - i\beta v \end{bmatrix} = (\mathcal{A} - i\beta I) \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus,

$$-\frac{1}{\rho}(EIw'' + c_D I v'')'' - \frac{c_a v}{\rho} - i\beta v = 0.$$

Multiplying both sides by $-\rho$ and noting that $v = i\beta w$ we get

$$(EIw'' + c_D I v'')'' + c_a v - \beta^2 \rho w = 0.$$

Taking the inner product of $(EIw'' + c_D I v'')'' + c_a v - \beta^2 \rho w$ and $\xi \in V$ in $L^2(0,\ell)$ gives

$$\langle (EIw'' + c_D I v'')'', \xi \rangle_{L^2(0,\ell)} + \langle c_a v, \xi \rangle_{L^2(0,\ell)} - \beta^2 \langle w, \xi \rangle_H = 0. \tag{3.31}$$

Now, notice that the first term of the left hand side of (3.31) is equal to

$$\begin{aligned}
\langle (EIw'' + c_D I v'')'', \xi \rangle_{L^2(0,\ell)} &= \int_0^\ell (EIw'' + c_D I v'')'' \overline{\xi} \, \mathrm{d}x \\
&= (EIw'' + c_D I v'')'(\ell)\overline{\xi(\ell)} - \int_0^\ell (EIw'' + c_D I v'')' \overline{\xi'} \, \mathrm{d}x \\
&= (EIw'' + c_D I v'')'(\ell)\overline{\xi(\ell)} - (EIw'' + c_D I v'')(\ell)\overline{\xi'(\ell)} \\
&\quad + \int_0^\ell (EIw'' + c_D I v'')\overline{\xi''} \, \mathrm{d}x \\
&= \begin{bmatrix} \overline{\xi'(\ell)} & \overline{\xi(\ell)} \end{bmatrix} F \begin{bmatrix} v'(\ell) \\ v(\ell) \end{bmatrix} + \langle EIw'' + c_D I v'', \xi'' \rangle_{L^2(0,\ell)}.
\end{aligned}$$

Since the second term of the above last expression can be written as

$$\begin{aligned}
\langle EIw'' + c_D I v'', \xi'' \rangle_{L^2(0,\ell)} &= \langle EIw'', \xi'' \rangle_{L^2(0,\ell)} + \langle c_D I v'', \xi'' \rangle_{L^2(0,\ell)} \\
&= \langle w, \xi \rangle_V + b[v, \xi] - \begin{bmatrix} \overline{\xi'(\ell)} & \overline{\xi(\ell)} \end{bmatrix} F \begin{bmatrix} v'(\ell) \\ v(\ell) \end{bmatrix} \\
&\quad - \langle c_a v, \xi \rangle_{L^2(0,\ell)}
\end{aligned}$$

we have

$$\langle (EIw'' + c_D I v'')'', \xi \rangle_{L^2(0,\ell)} = \langle w, \xi \rangle_V + i\beta b[w, \xi] - \langle c_a v, \xi \rangle_{L^2(0,\ell)}. \tag{3.32}$$

Substituting (3.32) in (3.31) we obtain

$$\langle w, \xi \rangle_V + i\beta b[w, \xi] - \beta^2 \langle w, \xi \rangle_H = 0, \tag{3.33}$$

for all $\xi \in V$. If we let $\xi = w$ in (3.33) we get

$$\|w\|_V^2 - \beta^2 \|w\|_H^2 + i\beta b[w] = 0.$$

Taking the imaginary part and noting that β is nonzero we conclude that $b[w] = 0$. Using (3.25) we get

$$\|S^{\frac{1}{2}} w\|_V^2 = \langle Sw, w \rangle_V = b[w]$$

for all $\xi \in V$ and so we have $S^{\frac{1}{2}}w = 0$. Thus $Sw = S^{\frac{1}{2}}(S^{\frac{1}{2}}w) = 0$ and $b[w,\xi] = \langle Sw, \xi \rangle_V = 0$ for any $\xi \in V$. Therefore Equation (3.33) is reduced to

$$\langle w, \xi \rangle_V - \beta^2 \langle w, \xi \rangle_H = 0,$$

for all $\xi \in V$.

Getting the inner product of $(\mathcal{A} - i\beta I)[w\ v]^\mathsf{T}$ and $[w\ v]^\mathsf{T}$ in $\mathcal{H}$ gives

$$\begin{aligned}
0 &= \left\langle (\mathcal{A} - i\beta I) \begin{bmatrix} w \\ v \end{bmatrix}, \begin{bmatrix} w \\ v \end{bmatrix} \right\rangle_{\mathcal{H}} \\
&= \left\langle \mathcal{A} \begin{bmatrix} w \\ v \end{bmatrix}, \begin{bmatrix} w \\ v \end{bmatrix} \right\rangle_{\mathcal{H}} - i\beta \left\langle \begin{bmatrix} w \\ v \end{bmatrix}, \begin{bmatrix} w \\ v \end{bmatrix} \right\rangle_{\mathcal{H}} \\
&= -\left[|v'(\ell)|\ |v(\ell)|\right] F \begin{bmatrix} |v'(\ell)| \\ |v(\ell)| \end{bmatrix} - \int_0^\ell (c_D I |v''|^2 + c_a |v|^2)\, \mathrm{d}x \\
&\quad + i \left(2\mathrm{Im}\, \langle v, w \rangle_V - \beta \left\| \begin{bmatrix} w \\ v \end{bmatrix} \right\|_{\mathcal{H}}^2 \right).
\end{aligned}$$

Because F is nonnegative definite, $c_a \geq 0$ and $c_D I > 0$, it follows by taking the real part of the above equation that

$$\int_0^\ell (c_D I |v''|^2 + c_a |v|^2)\, \mathrm{d}x = 0$$

and

$$k_2 |v'(\ell)|^2 + k_4 |v(\ell)|^2 = \left[|v'(\ell)|\ |v(\ell)|\right] F \begin{bmatrix} |v'(\ell)| \\ |v(\ell)| \end{bmatrix} = 0.$$

Therefore we have $k_2 |v'(\ell)|^2 = k_4 |v(\ell)|^2 = 0$, or equivalently, $k_2 v'(\ell) = k_4 v(\ell) = 0$. Since $w = -iv/\beta$ then $k_2 w'(\ell) = k_4 w(\ell) = 0$. Using the identity $v = i\beta w$ we have $EIw'' + c_D I v'' = EIw'' + c_D I (i\beta w)'' = (EI + i\beta c_D I) w''$. Thus

$$w''(\ell) = \frac{EI(\ell) w''(\ell) + c_D(\ell) v''(\ell)}{EI(\ell) + i\beta c_D I(\ell)} = -\frac{k_2 v'(\ell)}{EI(\ell) + i\beta c_D I(\ell)} = 0.$$

Further, the equality

$$(EIw'' + c_D I v'')' = (EI + i\beta c_D I)' w'' + (EI + i\beta c_D I) w'''$$

implies that

$$w'''(\ell) = \frac{(EI(\ell) w''(\ell) + c_D(\ell) v''(\ell))'}{EI(\ell) + i\beta c_D I(\ell)} = \frac{k_4 v(\ell)}{EI(\ell) + i\beta c_D I(\ell)} = 0.$$

Therefore we have shown that $\langle w, \xi \rangle_V - \beta^2 \langle w, \xi \rangle_H = 0$ for all $\xi \in V$, $b[w] = 0$, and $k_4 w(\ell) = k_2 w'(\ell) = w''(\ell) = w'''(\ell) = 0$, and by (UCC) we must have $w = 0$ and also

$v = i\beta w = 0$. This shows that $\mathcal{N}(\mathcal{A} - i\beta I) = \{0\}$. Hence $i\beta$ is not an eigenvalue of $\mathcal{A}$ for all $\beta \in \mathbb{R}$ and this proves that $i\mathbb{R} \cap \sigma_p(\mathcal{A}) = \emptyset$. With the aid of Theorem 3.17 we finally have $i\mathbb{R} \subset \rho(\mathcal{A})$. □

The succeeding three lemmas states that (H1) together with any one of the three feedback schemes stated in Theorem 3.18 guarantee that the unique continuation condition (UCC) holds.

Lemma 3.20. *If* (H1) *and condition* (1) *in Theorem 3.18 are satisfied then* (UCC) *holds.*

Proof. Suppose that the hypotheses in (UCC) holds. Then $\|c_D I^{\frac{1}{2}} w''\|_{L^2(0,\ell)} = 0$. Using integration by parts and noting that $\xi(0) = \xi'(0) = w''(\ell) = w'''(\ell) = 0$, we get

$$\begin{aligned}
\langle w, \xi \rangle_V &= \int_0^\ell EIw''\overline{\xi''}\,\mathrm{d}x \\
&= EIw''\overline{\xi'}\Big|_0^\ell - \int_0^\ell (EIw'')'\overline{\xi'}\,\mathrm{d}x \\
&= -\int_0^\ell (EIw'')'\overline{\xi'}\,\mathrm{d}x.
\end{aligned}$$

Using the same boundary conditions stated above we further have

$$\begin{aligned}
\int_0^\ell (EIw'')'\overline{\xi'}\,\mathrm{d}x &= (EIw'')'\overline{\xi}\Big|_0^\ell - \int_0^\ell (EIw'')''\overline{\xi}\,\mathrm{d}x \\
&= -\int_0^\ell (EIw'')''\overline{\xi}\,\mathrm{d}x
\end{aligned}$$

and so

$$\langle w, \xi \rangle_V = \int_0^\ell (EIw'')''\overline{\xi}\,\mathrm{d}x = \langle (EIw'')'', \xi \rangle_{L^2(0,\ell)}.$$

Therefore the condition $\langle w, \xi \rangle_V - \beta^2 \langle w, \xi \rangle_H = 0$ for all $\xi \in V$ in terms of the L^2-norm is

$$\left\langle (EIw'')'' - \beta^2 \rho w, \xi \right\rangle_{L^2(0,\ell)} = 0 \tag{3.34}$$

for all $\xi \in V$. Because V is dense in H, then V is also dense in $L^2(0,\ell)$. Since $(EIw'')'' - \beta^2 \rho w \in L^2(0,\ell)$, we can find a sequence $\{\xi_n\}_{n=1}^\infty \subset V$ such that

$$\lim_{n\to\infty} \|\xi_n - ((EIw'')'' - \beta^2\rho w)\|_{L^2(0,\ell)} = 0.$$

By Cauchy-Schwartz inequality, and if we let $u = (EIw'')'' - \beta^2 \rho w$ then

$$\begin{aligned}
|\langle u, \xi_n \rangle_{L^2(0,\ell)} - \langle u, u \rangle_{L^2(0,\ell)}| &= |\langle u, \xi_n - u \rangle_{L^2(0,\ell)}| \\
&\leq \|u\|_{L^2(0,\ell)} \|\xi_n - u\|_{L^2(0,\ell)}
\end{aligned}$$

and this implies that

$$\lim_{n\to\infty} \left\langle (EIw'')'' - \beta^2 \rho w, \xi_n \right\rangle_{L^2(0,\ell)} = \|(EIw'')'' - \beta^2 \rho w\|^2_{L^2(0,\ell)}.$$

Since $\langle (EIw'')'' - \beta^2\rho w, \xi_n\rangle_{L^2(0,\ell)} = 0$ for all $n \in \mathbb{N}$ from Equation (3.34), the uniqueness of limits implies that $\|(EIw'')'' - \beta^2\rho w\|^2_{L^2(0,\ell)} = 0$. Therefore, we have the following ordinary differential equation

$$(EIw'')'' - \beta^2 \rho w = 0$$

with initial conditions

$$w(\ell) = w'(\ell) = w''(\ell) = w'''(\ell) = 0.$$

Since $EI \in C^2[0,\ell]$ and $\rho \in C[0,\ell]$, the solution of the above initial value problem is unique from the theory of Ordinary Differential Equations and the zero conditions at the boundary $x = \ell$ guarantee that the solution is $w = 0$. Therefore the unique continuation condition (UCC) is satisfied. □

Lemma 3.21. *If* (H1) *and condition* (2) *in Theorem 3.18 are satisfied then* (UCC) *holds.*

Proof. Using a similar argument in the previous lemma, we have the ordinary differential equation

$$(EIw'')'' - \beta^2 \rho w = 0 \tag{3.35}$$

with initial conditions

$$w(0) = w(\ell) = w'(0) = w''(\ell) = w'''(\ell) = 0. \tag{3.36}$$

Taking the real part of the inner product of the above ordinary differential equation with xw' in $L^2(0,\ell)$ we have

$$\mathrm{Re}\int_0^\ell ((EIw'')'' - \beta^2\rho w)x\overline{w'}\,\mathrm{d}x = \mathrm{Re}\int_0^\ell \left[(EIw'')''x\overline{w'} - \beta^2 x\rho w\overline{w'}\right]\,\mathrm{d}x = 0.$$

We claim that the second term in the above equalities is nonnegative. Indeed, notice that

$$\begin{aligned}\mathrm{Re}\int_0^\ell \beta^2 x\rho w\overline{w'}\,\mathrm{d}x &= \frac{1}{2}x\rho|\beta w|^2\Big|_0^\ell - \frac{1}{2}\int_0^\ell (x\rho)'|\beta w|^2\,\mathrm{d}x \\ &= -\frac{1}{2}\int_0^\ell (\rho + x\rho')|\beta w|^2\,\mathrm{d}x.\end{aligned}$$

Also observe that

$$\begin{aligned}\operatorname{Re}\int_0^\ell (EIw'')''x\overline{w'}\,\mathrm{d}x &= \operatorname{Re}\left((EIw'')'x\overline{w'}\Big|_0^\ell - \int_0^\ell (EIw'')'(x\overline{w'})'\,\mathrm{d}x\right)\\ &= -\operatorname{Re}\left(\int_0^\ell (EIw'')'\overline{w'}\,\mathrm{d}x + \int_0^\ell (EIw'')'x\overline{w''}\,\mathrm{d}x\right).\end{aligned}$$

Let us rewrite the last two integrals. Note that

$$\begin{aligned}\operatorname{Re}\int_0^\ell (EIw'')'\overline{w'}\,\mathrm{d}x &= \operatorname{Re}\left((EIw'')\overline{w'}|_0^\ell - \int_0^\ell (EIw'')\overline{w''}\,\mathrm{d}x\right)\\ &= -\int_0^\ell EI|w''|^2\,\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\operatorname{Re}\int_0^\ell (EIw'')'x\overline{w''}\,\mathrm{d}x &= \int_0^\ell xEI'|w''|^2\,\mathrm{d}x + \operatorname{Re}\int_0^\ell xEIw'''\overline{w''}\,\mathrm{d}x\\ &= \int_0^\ell xEI'|w''|^2\,\mathrm{d}x + \frac{1}{2}(xEI)|w''|^2\Big|_0^\ell - \frac{1}{2}\int_0^\ell (xEI)'|w''|^2\,\mathrm{d}x\\ &= \int_0^\ell xEI'|w''|^2\,\mathrm{d}x - \frac{1}{2}\int_0^\ell (EI + xEI')|w''|^2\,\mathrm{d}x\\ &= \frac{1}{2}\int_0^\ell xEI'|w''|^2\,\mathrm{d}x - \frac{1}{2}\int_0^\ell EI|w''|^2\,\mathrm{d}x.\end{aligned}$$

Hence

$$\operatorname{Re}\int_0^\ell (EIw'')''x\overline{w'}\,\mathrm{d}x = \frac{1}{2}\int_0^\ell (3EI - xEI')|w''|^2\,\mathrm{d}x.$$

Therefore

$$\operatorname{Re}\int_0^\ell ((EIw'')' - \beta^2\rho w)x\overline{w'}\,\mathrm{d}x = \frac{1}{2}\int_0^\ell \left[(3EI - xEI')|w''|^2 + (\rho + x\rho')|\beta w|^2\right]\,\mathrm{d}x.$$

Since $\rho(x) + x\rho'(x) \geq 0$ and $3EI(x) - xEI'(x) \geq k_0 > 0$ for all $x \in [0,\ell]$ it follows that $\|w''\|_{L^2(0,\ell)} = 0$. As a consequence we must have $w = 0$ and the unique continuation condition (UCC) is satisfied. □

Lemma 3.22. *If* (H1) *and condition* (3) *in Theorem 3.18 are satisfied then* (UCC) *holds.*

Proof. Again, a similar method as in the previous lemmas gives us the ordinary differential equation

$$(EIw'')'' - \beta^2\rho w = 0 \tag{3.37}$$

with initial conditions

$$w(0) = w'(0) = w'(\ell) = w''(\ell) = w'''(\ell) = 0, \tag{3.38}$$

which we are going to find a solution. Let us introduce another multiplier. Define a function $q : [0, \ell] \to \mathbb{R}$ by

$$q(x) = \frac{1}{\rho(x)} \left(\kappa + \int_0^x \frac{\mathrm{d}s}{EI(s)} \right) \tag{3.39}$$

where κ is the nonnegative constant in condition (3) of Theorem 3.18. If we get the real part of the inner product of the above ordinary differential equation with $q(EIw'')'$ we have

$$\mathrm{Re} \int_0^\ell q(EIw'')''\overline{(EIw'')'}\,\mathrm{d}x - \mathrm{Re} \int_0^\ell \beta^2 \rho q w \overline{(EIw'')'}\,\mathrm{d}x = 0. \tag{3.40}$$

Using integration by parts and the initial conditions $w''(\ell) = w'''(\ell) = 0$, the first term on the left hand side of Equation (3.40) can be rewritten as

$$\begin{aligned} \mathrm{Re} \int_0^\ell q(EIw'')''\overline{(EIw'')'}\,\mathrm{d}x &= -q(0)|(EIw'')'(0)|^2 - \int_0^\ell q'|(EIw'')'|^2\,\mathrm{d}x \\ &\quad - \mathrm{Re} \int_0^\ell q(EIw'')'\overline{(EIw'')''}\,\mathrm{d}x. \end{aligned}$$

Let

$$I_0 = \int_0^\ell q(EIw'')''\overline{(EIw'')'}\,\mathrm{d}x$$

so that we have

$$\mathrm{Re}(I_0 + \overline{I_0}) = -q(0)|(EIw'')'(0)|^2 - \int_0^\ell q'|(EIw'')'|^2\,\mathrm{d}x$$

But notice that $\mathrm{Re}(I_0 + \overline{I_0}) = 2\mathrm{Re}\, I_0$ and we therefore obtain

$$\int_0^\ell q(EIw'')''\overline{(EIw'')'}\,\mathrm{d}x = -\frac{1}{2}q(0)|(EIw'')'(0)|^2 - \frac{1}{2}\int_0^\ell q'|(EIw'')'|^2\,\mathrm{d}x \tag{3.41}$$

Now, we get a similar form for the second term of the left hand side of the (3.40). Integration by parts and the initial conditions (3.38) give us

$$\begin{aligned} \mathrm{Re} \int_0^\ell \beta^2 \rho q w \overline{(EIw'')'}\,\mathrm{d}x &= \beta^2 \mathrm{Re} \Big((\rho q w)(\ell)\overline{(EIw'')(\ell)} - (\rho q w)(0)\overline{(EIw'')(0)} \Big) \\ &\quad - \mathrm{Re} \int_0^\ell \beta^2 (\rho q w)' EI\overline{w''}\,\mathrm{d}x \\ &= -\mathrm{Re} \int_0^\ell \beta^2 (\rho q)' EIw\overline{w''}\,\mathrm{d}x - \mathrm{Re} \int_0^\ell \beta^2 \rho q EIw'\overline{w''}\,\mathrm{d}x. \end{aligned}$$

From the definition of the function q we have $(\rho(x)q(x))'EI(x) = 1$ for all $x \in [0,\ell]$. Thus

$$\begin{aligned}
\mathrm{Re}\int_0^\ell \beta^2(\rho q)'EIw\overline{w''}\,\mathrm{d}x &= \mathrm{Re}\int_0^\ell \beta^2 w\overline{w''}\,\mathrm{d}x \\
&= \beta^2\mathrm{Re}\left(w(\ell)\overline{w'(\ell)} - w(0)\overline{w'(0)}\right) - \int_0^\ell |\beta w'|^2\,\mathrm{d}x \\
&= -\int_0^\ell |\beta w'|^2\,\mathrm{d}x.
\end{aligned}$$

Also,

$$\begin{aligned}
\mathrm{Re}\int_0^\ell \beta^2\rho qEIw'\overline{w''}\,\mathrm{d}x &= \frac{1}{2}\left(\rho(\ell)q(\ell)EI(\ell)|\beta w'(\ell)|^2 - \rho(0)q(0)EI(0)|\beta w'(0)|^2\right) \\
&\quad - \frac{1}{2}\int_0^\ell (\rho qEI)'|\beta w'|^2\,\mathrm{d}x \\
&= -\frac{1}{2}\int_0^\ell ((\rho q)'EI + \rho qEI')|\beta w'|^2\,\mathrm{d}x \\
&= -\frac{1}{2}\int_0^\ell (1 + \rho qEI')|\beta w'|^2\,\mathrm{d}x.
\end{aligned}$$

Therefore the second term of Equation (3.40) can be written as

$$\mathrm{Re}\int_0^\ell \beta^2\rho qw\overline{(EIw'')'}\,\mathrm{d}x = \frac{1}{2}\int_0^\ell (3 + \rho qEI')|\beta w'|^2\,\mathrm{d}x. \tag{3.42}$$

Combining our results (3.41) and (3.42) yield

$$\frac{1}{2}q(0)|(EIw'')'(0)|^2 + \frac{1}{2}\int_0^\ell \left[q'|(EIw'')'|^2 + (3 + \rho qEI')|\beta w'|^2\right]\,\mathrm{d}x = 0. \tag{3.43}$$

From condition (3) in Theorem 3.18 we have $q(0) = \kappa/\rho(0) \geq 0$,

$$q'(x) = \frac{1}{(\rho(x))^2}\left[\frac{\rho(x)}{EI(x)} - \left(\kappa + \int_0^x \frac{\mathrm{d}s}{EI(x)}\right)\rho'(x)\right] \geq 0$$

and

$$3 + \rho(x)q(x)EI'(x) = 3 + \left(\kappa + \int_0^x \frac{\mathrm{d}s}{EI(x)}\right)EI'(x) \geq k_0 > 0$$

for all $x \in [0,\ell]$. Observe that $\|(\beta w)'\|^2_{L^2(0,\ell)} \leq k_0^{-1}\|(3 + \rho qEI')^{\frac{1}{2}}\beta w'\|^2_{L^2(0,\ell)}$. Therefore, by Equation (3.43) and the above remarks we have $\|(\beta w)'\|_{L^2(0,\ell)} = 0$. Hence $\beta w = 0$ in $H_0^1(0,\ell)$ so that $w = 0$ whenever $\beta \neq 0$. Thus, (UCC) is satisfied and this completes the proof the lemma. $\square$

Combining the above four lemmas, it follows that hypothesis (H1) together with any one of the feedback schemes stated in Theorem 3.18 show that $i\mathbb{R} \subset \rho(\mathcal{A})$.

3.4.2 Estimates by the Multiplier Technique

To complete the proof of Theorem 3.18, we need the following theorem in functional analysis which is taken in the book of Yosida [38, p. 69].

Theorem 3.23 (Resonance Theorem). *Let $\{T_a \,|\, a \in A\}$ be a family of bounded operators on a Banach space X into a normed linear space Y. Then the boundedness of $\{\|T_a x\|_X \,|\, a \in A\}$ at each $x \in X$ implies the boundedness of $\{\|T_a\|_{\mathcal{L}(X,Y)} \,|\, a \in A\}$, i.e., if $\sup_{a\in A} \|T_a x\|_X < \infty$ for all $x \in X$ then $\sup_{a\in A} \|T_a\|_{\mathcal{L}(X,Y)} < \infty$.*

Lemma 3.24. *If hypothesis* (H1) *and one of the three conditions in the statement of Theorem 3.18 are satisfied, then the C_0-semigroup generated by $\mathcal{A}$ is exponentially stable.*

Proof. We proceed by contradiction. Suppose that the semigroup generated by $\mathcal{A}$ is not exponentially stable. Then the family $\{(i\beta I - \mathcal{A})^{-1} \,|\, \beta \in \mathbb{R}\}$ is not bounded by Theorem 2.22. The resonance theorem implies that there exists a nonzero vector $\zeta \in \mathcal{H}$ satisfying

$$\sup_{\beta\in\mathbb{R}} \|(i\beta I - \mathcal{A})^{-1}\zeta\|_{\mathcal{H}} = \infty.$$

Thus, we can find a sequence of real numbers $\{\beta_n\}_{n=1}^{\infty}$ such that $|\beta_n| \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \|(i\beta_n I - \mathcal{A})^{-1}\zeta\|_{\mathcal{H}} = \infty.$$

Let $\epsilon > 0$. Then we can find a positive integer N such that

$$\|(i\beta_n I - \mathcal{A})^{-1}\zeta\|_{\mathcal{H}} > \frac{\|\zeta\|_{\mathcal{H}}}{\epsilon}$$

for all $n \geq N$.

Consider a new sequence of vectors $\{z_n\}_{n=1}^{\infty}$ defined by

$$z_n = \frac{(i\beta_n I - \mathcal{A})^{-1}\zeta}{\|(i\beta_n I - \mathcal{A})^{-1}\zeta\|_{\mathcal{H}}}.$$

Thus $\|(i\beta_n I - \mathcal{A})^{-1}\zeta\|_{\mathcal{H}}(i\beta_n I - \mathcal{A})z_n = \zeta$ and $\{z_n\}_{n=1}^{\infty}$ is a sequence of unit vectors in $\mathcal{D}(\mathcal{A})$ such that

$$\|(i\beta_n I - \mathcal{A})z_n\|_{\mathcal{H}} = \frac{\|\zeta\|_{\mathcal{H}}}{\|(i\beta_n I - \mathcal{A})^{-1}\zeta\|_{\mathcal{H}}}$$

for all $n \in \mathbb{N}$ and so

$$\|(i\beta_n I - \mathcal{A})z_n\|_{\mathcal{H}} < \epsilon$$

for all $n \geq N$. Therefore

$$\lim_{n\to\infty} \|(i\beta_n I - \mathcal{A})z_n\|_{\mathcal{H}} = 0. \tag{3.44}$$

For each $n \in \mathbb{N}$ let

$$z_n = \begin{bmatrix} w_n \\ v_n \end{bmatrix} \quad \text{and} \quad h_n = \begin{bmatrix} f_n \\ g_n \end{bmatrix} = (i\beta_n I - \mathcal{A}) \begin{bmatrix} w_n \\ v_n \end{bmatrix}.$$

Using these representations in (3.44) we get

$$\lim_{n\to\infty} \left\| \begin{bmatrix} f_n \\ g_n \end{bmatrix} \right\|_{\mathcal{H}} = \lim_{n\to\infty} \left\| (i\beta_n I - \mathcal{A}) \begin{bmatrix} w_n \\ v_n \end{bmatrix} \right\|_{\mathcal{H}} = 0.$$

In terms of the V-norm and H-norm this is equivalent to $\|f_n\|_V \to 0$ and $\|g_n\|_H \to 0$ as $n \to \infty$, that is, $f_n \to 0$ in V and $g_n \to 0$ in H.

Using the definition of $\mathcal{A}$ we have the system of differential equations

$$\begin{aligned} f_n &= i\beta_n w_n - v_n, \\ g_n &= i\beta_n v_n + \frac{1}{\rho}(EIw_n'' + c_D I v_n'')'' + \frac{c_a v_n}{\rho}\ . \end{aligned} \tag{3.45}$$

Since $\|w_n\|_V \le \|z_n\|_{\mathcal{H}} = 1$ and $\|v_n\|_H \le \|z_n\|_{\mathcal{H}} = 1$ it follows that the sequence $\{w_n\}_{n=1}^{\infty}$ is bounded in V and the sequence $\{v_n\}_{n=1}^{\infty}$ is bounded in H. By the Cauchy-Schwartz inequality

$$|\langle f_n, w_n\rangle_V| \le \|f_n\|_V \|w_n\|_V \to 0 \tag{3.46}$$

and

$$|\langle g_n, v_n\rangle_H| \le \|g_n\|_H \|v_n\|_H \to 0 \tag{3.47}$$

as $n \to \infty$. Getting the inner product of f_n and w_n in V and using the first equation of system (3.45) give

$$\begin{aligned} \langle f_n, w_n\rangle_V &= \langle i\beta_n w_n - v_n, w_n\rangle_V \\ &= i\beta_n \langle w_n, w_n\rangle_V - \langle v_n, w_n\rangle_V \\ &= i\beta_n \|w_n\|_V^2 - \langle v_n, w_n\rangle_V\ . \end{aligned}$$

Similarly, getting the inner product of g_n and v_n in H gives

$$\begin{aligned} \langle g_n, v_n\rangle_H &= \left\langle i\beta_n v_n + \frac{1}{\rho}(EIw_n'' + c_D I v_n'')'' + \frac{c_a v_n}{\rho}, v_n \right\rangle_H \\ &= i\beta_n \langle v_n, v_v\rangle_H + \left\langle \frac{1}{\rho}(EIw_n'' + c_D I v_n'')'' + \frac{c_a v_n}{\rho}, v_n \right\rangle_H \\ &= i\beta_n \|v_n\|_H^2 + [|v_n'(\ell)|\ |v_n(\ell)|] F \begin{bmatrix} |v_n'(\ell)| \\ |v_n(\ell)| \end{bmatrix} \\ &\quad + \int_0^\ell (c_D I|v_n''|^2 + c_a|v_n|^2)\, \mathrm{d}x + \langle w_n, v_n\rangle_V\ . \end{aligned}$$

Note that $\mathcal{A}z_n = i\beta_n z_n - h_n$ and so $\langle \mathcal{A}z_n, z_n\rangle_{\mathcal{H}} = i\beta_n\|z_n\|^2_{\mathcal{H}} - \langle h_n, z_n\rangle_{\mathcal{H}} = -\mathrm{Re}\,\langle h_n, z_n\rangle_{\mathcal{H}} + i(\beta_n - \mathrm{Im}\,\langle h_n, z_n\rangle_{\mathcal{H}})$. Using the Cauchy-Schwartz inequality,

$$|\langle h_n, z_n\rangle_{\mathcal{H}}| \le \|h_n\|_{\mathcal{H}}\|z_n\|_{\mathcal{H}} = \|h_n\|_{\mathcal{H}}.$$

Therefore, using the fact that $\|h_n\|_{\mathcal{H}} \to 0$ as $n \to \infty$, we have $\langle h_n, z_n\rangle_{\mathcal{H}} \to 0$ as $n \to \infty$. This implies that $\mathrm{Re}\,\langle \mathcal{A}z_n, z_n\rangle_{\mathcal{H}} = -\mathrm{Re}\,\langle h_n, z_n\rangle_{\mathcal{H}} \to 0$ as $n \to \infty$. According to the definition of $\mathcal{A}$,

$$\langle \mathcal{A}z_n, z_n\rangle_{\mathcal{H}} = -[|v_n'(\ell)|\ |v_n(\ell)|]F\begin{bmatrix} |v_n'(\ell)| \\ |v_n(\ell)| \end{bmatrix} - \int_0^\ell (c_D I|v_n''|^2 + c_a|v_n|^2)\,\mathrm{d}x + 2i\mathrm{Im}\,\langle v_n, w_n\rangle_V$$

and we have

$$\lim_{n\to\infty}\left([|v_n'(\ell)|\ |v_n(\ell)|]F\begin{bmatrix} |v_n'(\ell)| \\ |v_n(\ell)| \end{bmatrix} + \int_0^\ell (c_D I|v_n''|^2 + c_a|v_n|^2)\,\mathrm{d}x\right) = 0. \tag{3.48}$$

Since F is nonnegative definite, $c_a \ge 0$, and $c_D I \ge c_0 > 0$ it follows that

$$[|v_n'(\ell)|\ |v_n(\ell)|]F\begin{bmatrix} |v_n'(\ell)| \\ |v_n(\ell)| \end{bmatrix}, \int_0^\ell c_D I|v_n''|^2\,\mathrm{d}x, \int_0^\ell c_a|v_n|^2\,\mathrm{d}x \ge 0.$$

Using these observations, it follows that

$$\lim_{n\to\infty}\left(\int_0^\ell c_D I|v_n''|^2\mathrm{d}x\right) = 0. \tag{3.49}$$

Also, from (3.47) and (3.48) we get

$$\begin{aligned}
&\lim_{n\to\infty}(i\beta_n\|v_n\|^2_H + \langle w_n, v_n\rangle_V) \\
&= \lim_{n\to\infty}\left(\langle g_n, v_n\rangle_H - [|v_n'(\ell)|\ |v_n(\ell)|]F\begin{bmatrix} |v_n'(\ell)| \\ |v_n(\ell)| \end{bmatrix} - \int_0^\ell (c_D I|v_n''|^2 + c_a|v_n|^2)\,\mathrm{d}x\right) \\
&= \lim_{n\to\infty}\langle g_n, v_n\rangle_H - \lim_{n\to\infty}\left([|v_n'(\ell)|\ |v_n(\ell)|]F\begin{bmatrix} |v_n'(\ell)| \\ |v_n(\ell)| \end{bmatrix} + \int_0^\ell (c_D I|v_n''|^2 + c_a|v_n|^2)\,\mathrm{d}x\right) \\
&= 0.
\end{aligned}$$

Therefore

$$\lim_{n\to\infty}(i\beta_n\|w_n\|^2_V - \langle v_n, w_n\rangle_V - i\beta_n\|v_n\|^2_H - \langle w_n, v_n\rangle_V) = 0.$$

Let

$$\begin{aligned}
\gamma_n &= -(\langle v_n, w_n\rangle_V + \langle w_n, v_n\rangle_V) + i\beta_n(\|w_n\|^2_V - \|v_n\|^2_H) \\
&= -2\mathrm{Re}\,\langle v_n, w_n\rangle_V + i\beta_n(\|w_n\|^2_V - \|v_n\|^2_H).
\end{aligned}$$

Because $\gamma_n \to 0$ as $n \to \infty$ we also have Im $\gamma_n \to 0$ as $n \to \infty$. Thus

$$\lim_{n\to\infty}(\|w_n\|_V^2 - \|v_n\|_H^2) = \lim_{n\to\infty}\frac{\text{Im }\gamma_n}{\beta_n} = 0.$$

Since $\|w_n\|_V^2 + \|v_n\|_H^2 = \|z_n\|_{\mathcal{H}}^2 = 1$ we have $\|w_n\|_V^2 - \|v_n\|_H^2 = 1 - 2\|v_n\|_H^2$. This further leads to $1 - 2\|v_n\|_H^2 \to 0$ as $n \to \infty$. Thus

$$\lim_{n\to\infty}\|v_n\|_H^2 = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{2}(1 - 2\|v_n\|_H^2)\right) = \frac{1}{2}. \tag{3.50}$$

Note that $c_D I$ is positive on the whole interval $[0,\ell]$ so that

$$\begin{aligned}\|v_n\|_V^2 &= \int_0^\ell EI|v_n''|^2 \mathrm{d}x \\ &= \int_0^\ell \left(\frac{EI}{c_D I}\right) c_D I|v_n''|^2 \mathrm{d}x \\ &\le \frac{C_0}{c_0}\int_0^\ell c_D I|v_n''|^2 \mathrm{d}x.\end{aligned}$$

Consequently using (3.49), $\|v_n\|_V^2 \to 0$ as $n \to \infty$. But $\|v_n\|_H^2 \le C_0\|v_n\|_{L^2(0,\ell)}^2 \le C_0\|v_n\|_{H^2(0,\ell)}^2$ and $\|v_n\|_{H^2(0,\ell)}^2 \le C_\ell\|v_n\|_V^2$, so that we have $\|v_n\|_H^2 \le C_0 C_\ell \|v_n\|_V^2$. Therefore

$$\lim_{n\to\infty}\|v_n\|_H^2 = 0,$$

which is a contradiction to (3.50). This contradiction proves the lemma. □

The proof of Theorem 3.18 is now completed by combining the above five lemmas.

3.5 Growth Bound of the Generator

If $A : \mathcal{D}(A) \subset X \to X$ is a closed operator then

$$\sigma_0(A) = \sup\{\mathrm{Re}\,\lambda \,|\, \lambda \in \sigma(A)\}$$

is called the **spectral bound** of A. Let A be the generator of a strongly continuous semigroup $\{S(t)\}_{t\ge 0}$ in a Banach space X. The **growth bound** or **type** of A is denoted and defined by

$$\omega_0(A) = \inf\{w \in \mathbb{R} \,|\, \text{there exists } M_w \ge 1 \text{ such that } \|S(t)\|_{\mathcal{L}(X)} \le M_w e^{wt} \text{ for all } t \ge 0\}.$$

The following theorem relates the spectral radius of the operators $S(t)$ to the growth bound of their infinitesimal generator. For the proof of this theorem one may consult Engel and Nagel [10, Proposition 2.2].

Theorem 3.25. *Let A be the generator of a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ in a Banach space X. Then*

$$\sigma_0(A) \leq \omega_0(A) = \lim_{t\to\infty} \frac{\ln \|S(t)\|_{\mathcal{L}(X)}}{t} = \inf_{t>0} \frac{\ln \|S(t)\|_{\mathcal{L}(X)}}{t}.$$

Moreover, the spectral radius $r(S(t))$ of the semigroup operator $S(t)$ is given by

$$r(S(t)) = e^{\omega_0(A)t}$$

for all $t \geq 0$.

From the exponential stability of $e^{\mathcal{A}t}$ it follows that $\omega_0(\mathcal{A}) \leq -\alpha$ where α is the positive constant appearing in Theorem 3.18. Also, Theorem 3.25 implies that $\sigma_0(\mathcal{A}) \leq -\alpha$ and we have

$$r(e^{\mathcal{A}t}) \leq e^{-\alpha t}$$

for all $t \geq 0$.

The following theorem gives us an upper bound for the type of the operator $\mathcal{A}$. This result is due to Chen, Liu and Liu [7, Theorem 3.1]. We remark that this result incorporates the maximum and minimum of the coefficient functions and as well as the length of the beam.

Theorem 3.26. *If* (H1) *and one of the three feedback schemes in Theorem 3.18 hold then*

$$\omega_0(\mathcal{A}) \leq \max\left\{-\frac{c_0^2}{2C_0^2(\ell^4+\ell^2+1)}, \sigma_0(\mathcal{A})\right\}. \tag{3.51}$$

Proof. We will use the formula (see Huang [19] and Prüss [33])

$$\omega_0(\mathcal{A}) = \inf\left\{\sigma > \sigma_0(\mathcal{A}) \,\middle|\, \sup_{\operatorname{Re}\lambda\geq\sigma} \|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty\right\}. \tag{3.52}$$

Notice that

$$\frac{c_0}{2C_0^2 C_\ell} = \frac{c_0^2}{2C_0^2(\ell^4+\ell^2+1)}.$$

Suppose

$$\max\left\{-\frac{c_0}{2C_0^2 C_\ell}, \sigma_0(\mathcal{A})\right\} < \sigma. \tag{3.53}$$

Hence $\sigma > \sigma_0(\mathcal{A})$. If λ is a complex number such that $\operatorname{Re}\lambda \geq \sigma$ then $\operatorname{Re}\lambda > \sigma_0(\mathcal{A})$ so that $\lambda \in \rho(\mathcal{A})$.

If μ is a complex number such that Re $\mu > \omega_0(\mathcal{A})$, then using the integral representation of the resolvent of $\mathcal{A}$ at μ and the exponential stability of the semigroup $e^{\mathcal{A}t}$ (see Tucsnak [36, Proposition 2.3.1]) we obtain

$$\begin{aligned}\|(\mu I - \mathcal{A})^{-1} z\|_{\mathcal{H}} &= \left\| \int_0^\infty e^{-\mu t} e^{\mathcal{A}t} z \,\mathrm{d}t \right\|_{\mathcal{H}} \\ &\leq \int_0^\infty |e^{-\mu t}| \|e^{\mathcal{A}t} z\|_{\mathcal{H}} \,\mathrm{d}t \\ &\leq M\|z\|_{\mathcal{H}} \int_0^\infty e^{-(\mathrm{Re}\ \mu)t} e^{-\alpha t} \,\mathrm{d}t \\ &= \frac{M}{\mathrm{Re}\ \mu + \alpha} \|z\|_{\mathcal{H}}.\end{aligned}$$

Therefore

$$\|(\mu I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{M}{\mathrm{Re}\ \mu + \alpha}$$

whenever Re $\mu > \omega_0(\mathcal{A})$.

If $\sigma \geq 1$ then $\mathrm{Re}\,\lambda \geq 1 > \omega_0(\mathcal{A})$. Using this and the estimate

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{M}{\mathrm{Re}\,\lambda + \alpha} \tag{3.54}$$

we have

$$\sup_{\mathrm{Re}\,\lambda \geq \sigma} \|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{M}{1 + \alpha} < \infty$$

whenever $\sigma \geq 1$. Now let us assume that $\sigma < 1$. If $\mathrm{Re}\,\lambda \geq 1 > \omega_0(\mathcal{A})$ we again satisfy Equation (3.54) and so

$$\sup_{\mathrm{Re}\,\lambda \in [1,\infty)} \|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{M}{1 + \alpha} < \infty.$$

Thus, for the case $\sigma < 1$, it is sufficient to show that

$$\sup_{\mathrm{Re}\,\lambda \in [\sigma,1]} \|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{3.55}$$

We argue by contradiction. Suppose that the inequality (3.55) is not true. Since $\sigma > \sigma_0(\mathcal{A})$ it follows that the strip $\{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\,\lambda \in [\sigma, 1]\}$ lies in $\rho(\mathcal{A})$. Because $\rho(\mathcal{A})$ is nonempty then the resolvent $(\lambda I - \mathcal{A})^{-1}$ is an $\mathcal{H}$-valued analytic function on $\rho(\mathcal{A})$. Hence, $\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}$ is bounded on the compact set

$$\{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\,\lambda \in [\sigma, 1] \text{ and } |\mathrm{Im}\,\lambda| \leq 1\}.$$

Thus, we must have

$$\sup\{\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \,|\, \operatorname{Re}\lambda \in [\sigma, 1] \text{ and } |\operatorname{Im}\lambda| \geq 1\} = \infty.$$

By the Resonance Theorem, we can find a sequence λ_n of complex numbers such that $|\operatorname{Im}\lambda_n| \geq 1$ for all $n \geq 1$ and

$$\lim_{n\to\infty} \operatorname{Re}\lambda_n = \lambda_0 \in [\sigma, 1]$$

and a sequence $z_n = [w_n \; v_n]^{\mathsf{T}} \in \mathcal{D}(\mathcal{A})$ of unit vectors such that

$$\lim_{n\to\infty} \|(\lambda_n I - \mathcal{A})z_n\|_{\mathcal{H}} = 0.$$

Therefore we have

$$\lambda_n w_n - v_n \to 0 \quad \text{in } V \tag{3.56}$$

and

$$\lambda_n v_n + \frac{1}{\rho}(EIw_n'' + c_D I v_n'')'' + \frac{c_a}{\rho} v_n \to 0 \quad \text{in } H. \tag{3.57}$$

Getting the inner product of the left hand side of (3.56) with w_n in V and using the Cauchy-Schwartz Inequality we obtain

$$\lim_{n\to\infty} \left(\lambda_n \|w_n\|_V^2 - \langle v_n, w_n\rangle_V\right) = 0. \tag{3.58}$$

Similarly, taking the inner product of the left hand side of (3.57) with v_n in H gives us

$$\lim_{n\to\infty} \left(\lambda_n \|v_n\|_H^2 + b[v_n] + \langle w_n, v_n\rangle_V\right) = 0. \tag{3.59}$$

Adding Equations (3.58) and (3.59) and taking the real part gives

$$\lim_{n\to\infty} (\operatorname{Re}\lambda_n + b[v_n]) = 0.$$

Thus

$$\lim_{n\to\infty} b[v_n] = -\lambda_0 \leq -\sigma < \frac{c_0}{2C_0^2 C_\ell}. \tag{3.60}$$

If we take the complex conjugate of Equation (3.58) we get

$$\lim_{n\to\infty} \left(\overline{\lambda_n} \|w_n\|_V^2 - \langle w_n, v_n\rangle_V\right) = 0. \tag{3.61}$$

Adding Equations (3.59) and (3.61) we have

$$\lim_{n\to\infty} \left(\lambda_n \|v_n\|_H^2 + b[v_n] + \overline{\lambda_n} \|w_n\|_V^2\right) = 0$$

and if we consider the imaginary part of the above limit we have

$$\lim_{n\to\infty}[(\text{Im }\lambda_n)(\|v_n\|_H^2 - \|w_n\|_V^2)] = 0.$$

Because $|\text{Im }\lambda_n| \geq 1$ for all $n \geq 1$ it follows that $\|v_n\|_H^2 - \|w_n\|_V^2 \to 0$ as $n \to \infty$ and combining this with the equation $\|v_n\|_H^2 + \|w_n\|_V^2 = 1$ we have $\|v_n\|_H^2 \to 1/2$ as $n \to \infty$. Recall that $\|v_n\|_V^2 \leq (C_0/c_0)b[v_n]$ and $\|v_n\|_H^2 \leq C_0 C_\ell \|v_n\|_V^2$. Thus

$$b[v_n] \geq \frac{c_0}{C_0^2 C_\ell}\|v_n\|_H^2.$$

Letting $n \to \infty$ we have

$$-\lambda_0 \geq \frac{c_0}{2C_0^2 C_\ell}$$

which contradicts (3.60). This contradiction completes the proof of (3.55). Therefore (3.51) follows immediately. □

The following result characterizes the spectral radius of $e^{\mathcal{A}t}$ for $t \geq 0$ according to the values of $\sigma_0(\mathcal{A})$.

Corollary 3.27. *Assume that* (H1) *and one of the three feedback schemes in Theorem 3.18 hold and let*

$$D = \frac{c_0^2}{2C_0^2(\ell^4 + \ell^2 + 1)}.$$

(1) *If $\sigma_0(\mathcal{A}) > -D$ then $r(e^{\mathcal{A}t}) > e^{-Dt}$ for all $t > 0$.*

(2) *If $\sigma_0(\mathcal{A}) = -D$ then $r(e^{\mathcal{A}t}) = e^{-Dt}$ for all $t \geq 0$.*

(3) *If $\sigma_0(\mathcal{A}) < -D$ then $r(e^{\mathcal{A}t}) \leq e^{-(\max\{\alpha,D\})t}$ for all $t \geq 0$, where α is the constant appearing in Theorem 3.18.*

Proof. (1) Assume that $\sigma_0(\mathcal{A}) > -D$. From Theorem 3.26 we have $\omega_0(\mathcal{A}) \leq \sigma_0(\mathcal{A})$ and so the spectral bound of $\mathcal{A}$ must be the same as its growth bound by Theorem 3.25, that is, $\omega_0(\mathcal{A}) = \sigma_0(\mathcal{A})$. Since the exponential function is strictly increasing we obtain

$$r(e^{\mathcal{A}t}) = e^{\omega_0(\mathcal{A})t} = e^{\sigma_0(\mathcal{A})t} > e^{-Dt}$$

for all $t > 0$. (2) Suppose that $\sigma_0(\mathcal{A}) = -D$. Again, from Theorems 3.25 and 3.26 we have $\omega_0(\mathcal{A}) = \sigma_0(\mathcal{A})$ and

$$r(e^{\mathcal{A}t}) = e^{-Dt}$$

for all $t \geq 0$. (3) If $\sigma_0(\mathcal{A}) < -D$ then $\omega_0(\mathcal{A}) \leq -D$. This implies that

$$r(e^{\mathcal{A}t}) \leq e^{-Dt}$$

for all $t \geq 0$. Recall that we also have $r(e^{\mathcal{A}t}) \leq e^{-\alpha t}$ for all $t \geq 0$. Thus, $r(e^{\mathcal{A}t}) \leq e^{-(\max\{\alpha,D\})t}$ for all $t \geq 0$. □

3.6 Exponential Decay of the Energy

In this section we show that the exponential decay of the energy of the system is equivalent to the exponential stabilizability of the system; and hence the energy of the system that we consider decay exponentially under the assumptions of Theorem 3.18. Before proving this result, we have the following weaker result. All throughout this section we assume that hypothesis (H1) is satisfied.

Theorem 3.28. *If $k_2, k_4 \geq 0$ then the energy $E(t)$ of the beam given by Equation* (3.2) *is not increasing.*

Proof. Differentiating the energy $E(t)$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = \int_0^\ell \left(EI\frac{\partial^2 y}{\partial x^2}\frac{\partial^3 y}{\partial x^2 \partial t} + \rho \frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial t^2}\right)\mathrm{d}x$$

Since

$$\rho\frac{\partial^2 y}{\partial t^2} = -\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2} + c_D I\frac{\partial^3 y}{\partial x^2\partial t}\right) - c_a\frac{\partial y}{\partial t}$$

we have

$$\begin{aligned}
\int_0^\ell \rho\frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial t^2}\mathrm{d}x \quad = \quad & -\int_0^\ell \frac{\partial y}{\partial t}\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2} + c_D I\frac{\partial^3 y}{\partial x^2\partial t}\right)\mathrm{d}x - \int_0^\ell c_a\left|\frac{\partial y}{\partial t}\right|^2\mathrm{d}x \\
= \quad & -\frac{\partial}{\partial x}\left(EI\frac{\partial^2 y}{\partial x^2} + c_D I\frac{\partial^3 y}{\partial x^2\partial t}\right)\bigg|_{x=\ell}\cdot\frac{\partial y(t,\ell)}{\partial t} \\
& + \int_0^\ell \frac{\partial^2 y}{\partial x\partial t}\frac{\partial}{\partial x}\left(EI\frac{\partial^2 y}{\partial x^2} + c_D I\frac{\partial^3 y}{\partial x^2\partial t}\right)\mathrm{d}x - \int_0^\ell c_a\left|\frac{\partial y}{\partial t}\right|^2\mathrm{d}x \\
= \quad & -k_4\left|\frac{\partial y(t,\ell)}{\partial t}\right|^2 + \left(EI\frac{\partial^2 y}{\partial x^2} + c_D I\frac{\partial^3 y}{\partial x^2\partial t}\right)\bigg|_{x=\ell}\cdot\frac{\partial^2 y(t,\ell)}{\partial t\partial x} \\
& - \int_0^\ell \frac{\partial^3 y}{\partial x^2\partial t}\left(EI\frac{\partial^2 y}{\partial x^2} + c_D I\frac{\partial^3 y}{\partial x^2\partial t}\right)\mathrm{d}x - \int_0^\ell c_a\left|\frac{\partial y}{\partial t}\right|^2\mathrm{d}x \\
= \quad & -k_4\left|\frac{\partial y(t,\ell)}{\partial t}\right|^2 - k_2\left|\frac{\partial^2 y(t,\ell)}{\partial t\partial x}\right|^2 - \int_0^\ell EI\frac{\partial^2 y}{\partial x^2}\frac{\partial^3 y}{\partial x^2\partial t}\mathrm{d}x \\
& - \int_0^\ell c_D I\left|\frac{\partial^3 y}{\partial x^2\partial t}\right|^2\mathrm{d}x - \int_0^\ell c_a\left|\frac{\partial y}{\partial t}\right|^2\mathrm{d}x.
\end{aligned}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -k_2\left|\frac{\partial^2 y(t,\ell)}{\partial t\partial x}\right|^2 - k_4\left|\frac{\partial y(t,\ell)}{\partial t}\right|^2 - \int_0^\ell\left(c_D I\left|\frac{\partial^3 y}{\partial x^2\partial t}\right|^2 + c_a\left|\frac{\partial y}{\partial t}\right|^2\right)\mathrm{d}x,$$

and since $c_a, k_2, k_4 \geq 0$, and $c_D I > 0$ the conclusion of the theorem follows immediately since $\mathrm{d}E(t)/\mathrm{d}t \leq 0$. □

For the next theorem, we need the following theorem in Ito and Kappel [20, Theorem 2.9].

Theorem 3.29. *Let $\{S(t)\}_{t\geq 0}$ be a C_0-semigroup on a Banach space X with infinitesimal generator A. Then, for any $x \in \mathcal{D}(A)$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} S(t)x = AS(t)x = S(t)Ax, \quad t \geq 0.$$

Theorem 3.30. *Assume that $k_2, k_4 \geq 0$. Let $[w_0\ v_0]^\mathsf{T} \in \mathcal{H}$ and*

$$\begin{bmatrix} y(t) \\ y_0(t) \end{bmatrix} = e^{t\mathcal{A}} \begin{bmatrix} w_0 \\ v_0 \end{bmatrix}.$$

Then $\mathrm{d}y(t)/\mathrm{d}t = y_0(t)$ and $y(\cdot) \in C([0,\infty);V) \cap C^1([0,\infty);H)$. Moreover, if $[w_0\ v_0]^\mathsf{T} \in \mathcal{D}(\mathcal{A})$ then $y(\cdot) \in C^1([0,\infty);V) \cap C^2([0,\infty);H)$.

Proof. First, let us assume that $[w_0\ v_0]^\mathsf{T} \in \mathcal{D}(\mathcal{A})$. Then, using Theorem 3.29 we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} y(t) \\ y_0(t) \end{bmatrix} &= \frac{\mathrm{d}}{\mathrm{d}t} \left(e^{t\mathcal{A}} \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} \right) \\ &= \mathcal{A} e^{t\mathcal{A}} \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} \\ &= \mathcal{A} \begin{bmatrix} y(t) \\ y_0(t) \end{bmatrix} \\ &= \begin{bmatrix} y_0(t) \\ -\frac{1}{\rho}(EIy''(t) + c_D I y_0''(t))'' - \frac{1}{\rho} c_a y_0(t) \end{bmatrix}. \end{aligned}$$

Thus $\mathrm{d}y(t)/\mathrm{d}t = y_0(t)$. By Theorem 3.8, if $S(t) = e^{t\mathcal{A}}$ then

$$S(\cdot) \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} \in C^1([0,\infty);\mathcal{H}) = C^1([0,\infty);V \times H)$$

so that

$$y(\cdot) = \pi_1 \left(S(\cdot) \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} \right) \in C^1([0,\infty);V)$$

and

$$y_0(\cdot) = \pi_2 \left(S(\cdot) \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} \right) \in C^1([0,\infty);H),$$

where π_1 and π_2 are the projection maps onto the first and second components, respectively. Using the fact that $\mathrm{d}y(t)/\mathrm{d}t = y_0(t)$ we have $y(\cdot) \in C^2([0,\infty);H)$. Therefore we have shown that $y(\cdot) \in C^1([0,\infty);V) \cap C^2([0,\infty);H)$ whenever $[w_0\ v_0]^\mathsf{T} \in \mathcal{D}(\mathcal{A})$.

Now let us assume that $[w_0\ v_0]^\mathsf{T} \in \mathcal{H}$. Since $\overline{\mathcal{D}(\mathcal{A})} = \mathcal{H}$, there exists a sequence $[w_{0n}\ v_{0n}]^\mathsf{T} \in \mathcal{D}(\mathcal{A})$ such that

$$\lim_{n\to\infty} \left\| \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} - \begin{bmatrix} w_{0n} \\ v_{0n} \end{bmatrix} \right\|_{\mathcal{H}} = 0.$$

For each positive integer n let

$$\begin{bmatrix} y_n(t) \\ y_{0n}(t) \end{bmatrix} = e^{t\mathcal{A}} \begin{bmatrix} w_{0n} \\ v_{0n} \end{bmatrix}.$$

Then $\mathrm{d}y_n(t)/\mathrm{d}t = y_{0n}(t)$ and $y_n(\cdot) \in C^1([0,\infty);V) \cap C^2([0,\infty);H)$. Since $\mathcal{A}$ generates a C_0-semigroup of contractions we have $\|e^{t\mathcal{A}}\|_{\mathcal{L}(\mathcal{H})} \le 1$ for all $t \ge 0$ and so

$$\begin{aligned} \left\| \begin{bmatrix} y(t) \\ y_0(t) \end{bmatrix} - \begin{bmatrix} y_n(t) \\ y_{0n}(t) \end{bmatrix} \right\|_{\mathcal{H}} &= \left\| e^{t\mathcal{A}} \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} - e^{t\mathcal{A}} \begin{bmatrix} w_{0n} \\ v_{0n} \end{bmatrix} \right\|_{\mathcal{H}} \\ &\le \left\| \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} - \begin{bmatrix} w_{0n} \\ v_{0n} \end{bmatrix} \right\|_{\mathcal{H}}. \end{aligned}$$

The above estimate shows that

$$\lim_{n\to\infty} \left\| \begin{bmatrix} y(t) \\ y_0(t) \end{bmatrix} - \begin{bmatrix} y_n(t) \\ y_{0n}(t) \end{bmatrix} \right\|_{\mathcal{H}} = 0$$

uniformly for $t \in [0,\infty)$. Hence

$$\lim_{n\to\infty} \|y_n(t) - y(t)\|_V = 0 \tag{3.62}$$

uniformly for $t \in [0,\infty)$ and

$$\lim_{n\to\infty} \|y_{0n}(t) - y_0(t)\|_H = 0 \tag{3.63}$$

uniformly for $t \in [0,\infty)$. We claim that $y(\cdot) \in C([0,\infty);V)$. Indeed, let $s, t \in [0,\infty)$ and $\epsilon > 0$. The continuity of $y_n(\cdot)$ implies that there exists a $\delta > 0$ such that

$$\|y_n(s) - y_n(t)\|_V < \frac{\epsilon}{3} \quad \text{whenever} \quad |s-t| < \delta. \tag{3.64}$$

Equation (3.62) implies the existence of a positive integer N such that

$$\|y_n(t) - y(t)\|_V < \frac{\epsilon}{3} \quad \text{whenever} \quad n \ge N, t \ge 0. \tag{3.65}$$

If $s, t \ge 0$, $|s-t| < \delta$ and $n \ge N$, then (3.64) and (3.65) give us

$$\|y(s) - y(t)\|_V \le \|y(s) - y_n(s)\|_V + \|y_n(s) - y_n(t)\|_V + \|y_n(t) - y(t)\|_V < \epsilon,$$

which proves that $y(\cdot) \in C([0,\infty);V)$. A similar argument, together with Equation (3.63), gives us $y_0(\cdot) \in C([0,\infty);H)$.

If we can show that $\mathrm{d}y(t)/\mathrm{d}t = y_0(t)$ then it also follows that $y(\cdot) \in C([0,\infty);V) \cap C^1([0,\infty);H)$. Note that, using the dot notation for the derivative with respect to t,

$$y_n(t) = y_n(0) + \int_0^t \dot{y}_n(s)\,\mathrm{d}s.$$

The above integral exists since $y_n \in C^2([0,\infty);V)$. Let $\epsilon > 0$ and $t \in [0,T]$ where $T > 0$ is fixed. Since $\|y_n(t) - y(t)\|_H \leq \sqrt{C_0 C_\ell}\|y_n(t) - y(t)\|_V$, Equation (3.62) implies that

$$\lim_{n\to\infty} \|y_n(t) - y(t)\|_H = 0$$

and so we can find a positive integer N_1 such that

$$\|y_n(t) - y(t)\|_H < \frac{\epsilon}{2} \quad \text{whenever} \quad n \geq N_1,$$

for all $t \in [0,T]$. Also, Equation (3.63) and $\dot{y}_n(t) = y_{0n}(t)$ implies that there exists a positive integer N_2 such that

$$\|\dot{y}_n(t) - y_0(t)\|_H < \frac{\epsilon}{2T} \quad \text{whenever} \quad n \geq N_2,$$

for all $t \in [0,T]$. Using these observations we get, for $n \geq \max\{N_1, N_2\}$,

$$\begin{aligned}
\left\| y_n(t) - \left(y(0) + \int_0^t y_0(s)\,\mathrm{d}s \right) \right\|_H &= \left\| (y_n(0) - y(0)) + \int_0^t (\dot{y}_n(s) - y_0(s))\,\mathrm{d}s \right\|_H \\
&\leq \|y_n(0) - y(0)\|_H + \int_0^t \|\dot{y}_n(s) - y_0(s)\|_H\,\mathrm{d}s \\
&< \frac{\epsilon}{2} + \frac{\epsilon t}{2T} \\
&= \frac{1}{2}\left(1 + \frac{t}{T}\right)\epsilon \\
&\leq \epsilon.
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} \left\| y_n(t) - \left(y(0) + \int_0^t y_0(s)\,\mathrm{d}s \right) \right\|_H = 0$$

uniformly for compact intervals $[0,T]$. The uniqueness of limits tells us that

$$y(t) = y(0) + \int_0^t y_0(s)\,\mathrm{d}s$$

for any $t \geq 0$. Consequently we have $\mathrm{d}y(t)/\mathrm{d}t = y_0(t)$. This establishes the theorem. $\square$

Theorem 3.31. *Assume that* $k_2, k_4 \geq 0$. *The exponential stability of the associated* C_0*-semigroup* $S(t) = e^{t\mathcal{A}}$ *generated by* $\mathcal{A}$ *is equivalent to the exponential decay of the energy* $E(t)$ *of the beam.*

Proof. Suppose that $[w_0\ v_0]^\mathsf{T} \in \mathcal{H}$ and that $y = y(t,x)$ is a solution of the system (3.9) with initial conditions (3.10) and boundary conditions (3.11). Then, from Theorem 3.30,

$$\begin{bmatrix} y \\ \partial y/\partial t \end{bmatrix} = e^{t\mathcal{A}} \begin{bmatrix} w_0 \\ v_0 \end{bmatrix}.$$

By definition

$$E(t) = \frac{1}{2} \left\| \begin{bmatrix} y \\ \partial y/\partial t \end{bmatrix} \right\|_{\mathcal{H}}^2,$$

for $t \geq 0$. If $S(t) = e^{t\mathcal{A}}$ is exponentially stable, then there exist constants $M \geq 1$ and $\alpha > 0$ such that $\|e^{t\mathcal{A}}\|_{\mathcal{L}(\mathcal{H})} \leq Me^{-\alpha t}$ for all $t \geq 0$. Then

$$\begin{aligned} E(t) &= \frac{1}{2} \left\| e^{t\mathcal{A}} \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} \right\|_{\mathcal{H}}^2 \\ &\leq \frac{1}{2} \|e^{t\mathcal{A}}\|_{\mathcal{L}(\mathcal{H})}^2 \left\| \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} \right\|_{\mathcal{H}}^2 \\ &\leq \frac{1}{2} M^2 e^{-2\alpha t} \left\| \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} \right\|_{\mathcal{H}}^2. \end{aligned}$$

Note that

$$\begin{aligned} \frac{1}{2} \left\| \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} \right\|_{\mathcal{H}}^2 &= \frac{1}{2} \int_0^\ell [EI(x)|w_0''(x)|^2 + \rho(x)|v_0(x)|^2]\, \mathrm{d}x \\ &= \frac{1}{2} \int_0^\ell \left[EI(x) \left| \frac{\partial^2 y(0,x)}{\partial x^2} \right|^2 + \rho(x) \left| \frac{\partial y(0,x)}{\partial t} \right|^2 \right] \mathrm{d}x \\ &= E(0). \end{aligned}$$

Hence

$$E(t) \leq M^2 e^{-2\alpha t} E(0)$$

for all $t \geq 0$, where $M^2 \geq 1$ and $2\alpha > 0$. Therefore the energy of the beam exponentially decays.

Conversely, assume that $E(t)$ exponentially decays and let $[w_0\ v_0]^\mathsf{T} \in \mathcal{H}$. Then $E(t) \leq Me^{-\alpha t} E(0)$ for some $M \geq 1$ and $\alpha > 0$. Since

$$\left\| e^{t\mathcal{A}} \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} \right\|_{\mathcal{H}}^2 = 2E(t) \leq 2Me^{-\alpha t} E(0)$$

we have

$$\left\| e^{t\mathcal{A}} \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} \right\|_{\mathcal{H}}^2 \leq Me^{-\alpha t} \left\| \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} \right\|_{\mathcal{H}}^2 .$$

Taking the square roots both sides give

$$\left\| e^{t\mathcal{A}} \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} \right\|_{\mathcal{H}} \leq \sqrt{M} e^{-(\alpha/2)t} \left\| \begin{bmatrix} w_0 \\ v_0 \end{bmatrix} \right\|_{\mathcal{H}},$$

and so, $\|e^{t\mathcal{A}}\|_{\mathcal{L}(\mathcal{H})} \leq \sqrt{M}e^{-(\alpha/2)t}$. Because $\sqrt{M} \geq 1$ and $\alpha/2 > 0$, the exponential stability of $e^{t\mathcal{A}}$ follows. □

From Theorem 3.31 it follows that under the assumptions of Theorem 3.18, the solution of the partial differential equation (3.9) with initial and boundary conditions (3.10) and (3.11), respectively, has derivatives decaying to zero in the sense that

$$\lim_{t\to\infty} \int_0^\ell (\rho(x)|\dot{y}(t,x)|^2 + EI(x)|y''(t,x)|^2)\, \mathrm{d}x = 0.$$

Moreover, the rate of decay is exponential.

3.7 Analyticity

We will prove that the infinitesimal generator $\mathcal{A}$ does not only generate an exponentially stable C_0-semigroup but also an analytic semigroup. The following theorem states that the semigroup generated by $\mathcal{A}$ can be extended analytically in a sector in the complex plane that contains the nonnegative real numbers.

Theorem 3.32. *Let $k_2, k_4 \geq 0$ and assume that* (H1) *holds. Then the semigroup $S(t) = e^{t\mathcal{A}}$ generated by $\mathcal{A}$ is a C_0-semigroup analytic for*

$$|\arg t| < \frac{\pi}{2} - \tan^{-1}\left(\frac{C_0}{c_0}\right).$$

Proof. First, we show that the operator $-\mathcal{A} + \frac{c_0}{C_0}I : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ belongs to the class $\mathscr{H}(\pi/2 - \tan^{-1}(C_0/c_0), 0)$. Since $\mathcal{A}$ is closed and densely defined, then $-\mathcal{A} + \frac{c_0}{C_0}I$ is also closed and densely defined. Let $[w\ v]^\mathsf{T}$ be a unit vector in $\mathcal{D}(\mathcal{A})$ and

$$\theta(w,v) = \left\langle \left(-\mathcal{A} + \frac{c_0}{C_0}I\right)[w\ v]^\mathsf{T}, [w\ v]^\mathsf{T} \right\rangle_{\mathcal{H}}.$$

From Theorem 3.5 we have

$$\begin{aligned} \theta(w,v) &= -\left\langle \mathcal{A}\begin{bmatrix} w \\ v \end{bmatrix}, \begin{bmatrix} w \\ v \end{bmatrix} \right\rangle_{\mathcal{H}} + \left\langle \frac{c_0}{C_0}\begin{bmatrix} w \\ v \end{bmatrix}, \begin{bmatrix} w \\ v \end{bmatrix} \right\rangle_{\mathcal{H}} \\ &= [|v'(\ell)|\ |v(\ell)|]\, F \begin{bmatrix} |v'(\ell)| \\ |v(\ell)| \end{bmatrix} + \int_0^\ell (c_D I|v''|^2 + c_a|v|^2)\, \mathrm{d}x + \frac{c_0}{C_0} - 2i\mathrm{Im}\, \langle v, w\rangle_V . \end{aligned}$$

Since $c_a, k_2, k_4 \geq 0$ it follows that

$$\begin{aligned}
\text{Re } \theta(w,v) &= [|v'(\ell)|\ |v(\ell)|]\, F \begin{bmatrix} |v'(\ell)| \\ |v(\ell)| \end{bmatrix} + \frac{c_0}{C_0} + \int_0^\ell (c_D I |v''|^2 + c_a |v|^2)\, \mathrm{d}x \\
&\geq \frac{c_0}{C_0} + \int_0^\ell c_D I |v''|^2 \, \mathrm{d}x \\
&\geq \frac{c_0}{C_0} + \frac{c_0}{C_0} \int_0^\ell EI |v''|^2 \, \mathrm{d}x \\
&= \frac{c_0}{C_0}(1 + \|v\|_V^2).
\end{aligned}$$

This shows that $-\mathcal{A} + \frac{c_0}{C_0} I$ is accretive. Let $\lambda \in \mathbb{C}$ be such that Re $\lambda < 0$. Because $-\mathcal{A}$ is m-accretive, then every $\zeta \in \mathbb{C}$ such that Re $\zeta < 0$ belongs to $\rho(-\mathcal{A})$ by Theorem 2.23. In particular, $\lambda - c_0/C_0 \in \rho(-\mathcal{A})$ since $\text{Re}(\lambda - c_0/C_0) < 0$. We know that $-\mathcal{A}$ is closed and so from Theorem 2.4 we have

$$\mathcal{R}\left(\left(-\mathcal{A} + \frac{c_0}{C_0} I\right) - \lambda I\right) = \mathcal{R}\left(-\mathcal{A} - \left(\lambda - \frac{c_0}{C_0}\right) I\right) = \mathcal{H}.$$

Therefore $-\mathcal{A} + \frac{c_0}{C_0} I$ is also m-accretive.

In order to use Theorem 2.26, it remains to show that $-\mathcal{A} + \frac{c_0}{C_0} I$ is sectorial with vertex 0, that is,

$$\Theta\left(-\mathcal{A} + \frac{c_0}{C_0} I\right) \subset \left\{\zeta \in \mathbb{C} \;\middle|\; |\text{arg } \zeta| \leq \frac{\pi}{2} - \omega\right\}$$

for some $\omega \in (0, \pi/2]$. Notice that we have

$$\|v\|_V^2 + 1 \leq \frac{C_0}{c_0} \text{Re } \theta(w,v). \tag{3.66}$$

Let us consider the imaginary part of $\theta(w,v)$. For simplicity, let $\alpha = \text{Re } w'', \beta = \text{Im } w'', \gamma = \text{Re } v''$ and $\delta = \text{Im } v''$. Then

$$\begin{aligned}
\text{Im } \theta(w,v) = 2\text{Im } \langle w, v \rangle_V &= 2\text{Im} \int_0^\ell EI w'' \overline{v''} \, \mathrm{d}x \\
&= 2\text{Im} \int_0^\ell EI(\alpha + i\beta)(\gamma - i\delta)\, \mathrm{d}x \\
&= \int_0^\ell EI(2\beta\gamma - 2\alpha\delta)\, \mathrm{d}x.
\end{aligned}$$

Expanding the inequality $-(|\beta| - |\gamma|)^2 \leq (|\alpha| - |\delta|)^2$ gives

$$-\beta^2 + 2|\beta\gamma| - \gamma^2 \leq \alpha^2 - 2|\alpha\delta| + \delta^2$$

which implies that
$$2|\beta\gamma| + 2|\alpha\delta| \leq \alpha^2 + \beta^2 + \gamma^2 + \delta^2.$$
Therefore
$$\begin{aligned}
|\mathrm{Im}\ \theta(w,v)| &\leq \int_0^\ell EI(2|\beta\gamma| + 2|\alpha\delta|)\,\mathrm{d}x \\
&\leq \int_0^\ell EI(\alpha^2+\beta^2)\,\mathrm{d}x + \int_0^\ell EI(\gamma^2+\delta^2)\,\mathrm{d}x \\
&= \int_0^\ell EI|w''|^2\,\mathrm{d}x + \int_0^\ell EI|v''|^2\,\mathrm{d}x \\
&= \|w\|_V^2 + \|v\|_V^2.
\end{aligned}$$
Using (3.66) and the inequality $\|w\|_V^2 \leq \|[w\ v]^{\mathsf{T}}\|_{\mathcal{H}}^2 = 1$ we have
$$|\mathrm{Im}\ \theta(w,v)| \leq 1 + \|v\|_V^2 \leq \frac{C_0}{c_0}\mathrm{Re}\ \theta(w,v).$$

If $\mathrm{Re}\ \theta(w,v) = 0$ then $\mathrm{Im}\ \theta(w,v) = 0$ by the above estimate, so that $\theta(w,v) = 0$. Thus $\mathrm{Re}\ \theta(w,v) = 0$ if and only if $\theta(w,v) = 0$. For $\theta(w,v) \neq 0$ we have $\mathrm{Re}\ \theta(w,v) > 0$ since $-\mathcal{A} + \frac{c_0}{C_0}I$ is accretive, and so
$$-\frac{C_0}{c_0} \leq \frac{\mathrm{Im}\ \theta(w,v)}{\mathrm{Re}\ \theta(w,v)} \leq \frac{C_0}{c_0}.$$
Hence it follows that $-\mathcal{A} + \frac{c_0}{C_0}I$ is sectorial with vertex 0 since
$$\Theta\left(-\mathcal{A} + \frac{c_0}{C_0}I\right) \subset \{\zeta \in \mathbb{C} \,|\, |\arg\zeta| \leq \tan^{-1}(C_0/c_0)\}.$$

From Theorem 2.26 it follows that $-\mathcal{A} + \frac{c_0}{C_0}I \in \mathscr{H}(\pi/2 - \tan^{-1}(C_0/c_0), 0)$. Since
$$-\mathcal{A} = \left(-\mathcal{A} + \frac{c_0}{C_0}I\right) - \frac{c_0}{C_0}I$$
we have $-\mathcal{A} \in \mathscr{H}(\pi/2 - \tan^{-1}(C_0/c_0), c_0/C_0)$ and by Theorem 2.25, $e^{t\mathcal{A}} = e^{-t(-\mathcal{A})}$ is analytic for
$$|\arg t| < \frac{\pi}{2} - \tan^{-1}\left(\frac{C_0}{c_0}\right).$$
This completes the proof of the theorem. $\square$

Chapter 4

Locally Damped Euler-Bernoulli Beam

In this chapter, we consider the case where the bending moment that is applied to the end $x = \ell$ of the beam is controlled by the linear feedback of the rotation angle $y'(t,\ell)$ and the angular velocity $\dot{y}'(t,\ell)$, and the shear force applied to the same end is controlled by the linear feedback of the displacement $y(t,\ell)$ and the velocity $\dot{y}(t,\ell)$. Thus, we consider the Euler-Bernoulli beam equation with Kelvin-Voigt and viscous damping

$$\rho(x)\frac{\partial^2 y(t,x)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y(t,x)}{\partial x^2}\right) + \frac{\partial^2}{\partial x^2}\left(c_D I(x)\frac{\partial^3 y(t,x)}{\partial x^2 \partial t}\right) + c_a(x)\frac{\partial y(t,x)}{\partial t} = 0,$$

where $(t,x) \in (0,\infty) \times (0,\ell)$, with initial conditions

$$y(0,x) \;=\; w_0(x), \qquad \left.\frac{\partial y(t,x)}{\partial t}\right|_{t=0} = v_0(x),$$

$x \in (0,\ell)$, under the boundary conditions

$$\begin{cases} \quad y(t,0) \;=\; \left.\dfrac{\partial y(t,x)}{\partial x}\right|_{x=0} \;=\; 0, \\ -\left.\left(EI(x)\dfrac{\partial^2 y(t,x)}{\partial x^2} + c_D I(x)\dfrac{\partial^3 y(t,x)}{\partial x^2 \partial t}\right)\right|_{x=\ell} \;=\; k_1\dfrac{\partial y(t,\ell)}{\partial x} + k_2\dfrac{\partial^2 y(t,\ell)}{\partial t \partial x}, \\ \dfrac{\partial}{\partial x}\left.\left(EI(x)\dfrac{\partial^2 y(t,x)}{\partial x^2} + c_D I(x)\dfrac{\partial^3 y(t,x)}{\partial x^2 \partial t}\right)\right|_{x=\ell} \;=\; k_3 y(t,\ell) + k_4\dfrac{\partial y(t,\ell)}{\partial t}, \end{cases}$$

for $t \in (0,\infty)$. The constants $k_1, k_2, k_3, k_4 \geq 0$ are called the **feedback coefficients**.

In this case, the energy at time t of the above elastic system is defined by

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^\ell \left(EI(x)\left|\frac{\partial^2 y(t,x)}{\partial x^2}\right|^2 + \rho(x)\left|\frac{\partial y(t,x)}{\partial t}\right|^2\right) \mathrm{d}x \\ &\quad + \frac{1}{2}\left[k_1\left|\frac{\partial y(t,\ell)}{\partial x}\right|^2 + k_3|y(t,\ell)|^2\right], \end{aligned} \tag{4.1}$$

where $k_1|\partial y(t,\ell)/\partial x|^2+k_3|y(t,\ell)|^2$ represents the energy of the rigid motion of the elastic system. All throughout this chapter, we will assume the following hypothesis.

(H2) Let $a \in (0,\ell)$, $c_0 > 0$ and the functions $\rho, EI, c_D I, c_a : [0,\ell] \to \mathbb{R}$ satisfies the following conditions: The functions ρ, EI and c_a are continuous except at the point $x = a$, with $\rho, EI \in C^2[0,a] \cap C_B^2(a,\ell)$, $c_a \in C^1[0,a] \cap C_B^1(a,\ell)$. The Kelvin-Voigt damping is locally distributed and $c_D I = D\chi_{(0,a)}$, $D \in C_B^1(0,a)$. Also, we assume that $\rho, EI \geq c_0 > 0$ on $[0,\ell]$, $D \geq c_0 > 0$ on $(0,a)$ and $c_a \geq 0$ on $[0,\ell]$.

Hypothesis (H2) implies the existence of a positive constant C_0 such that the supremum of ρ, EI, $c_D I$ and c_a on $[0,\ell]$ is less than or equal to C_0. In this assumption the boundary conditions at $x = \ell$ become

$$\begin{cases} -\left(EI(x)\dfrac{\partial^2 y(t,x)}{\partial x^2}\right)\Bigg|_{x=\ell} = k_1\dfrac{\partial y(t,\ell)}{\partial x} + k_2\dfrac{\partial^2 y(t,\ell)}{\partial t \partial x}, & t \in (0,\infty), \\ \dfrac{\partial}{\partial x}\left(EI(x)\dfrac{\partial^2 y(t,x)}{\partial x^2}\right)\Bigg|_{x=\ell} = k_3 y(t,\ell) + k_4\dfrac{\partial y(t,\ell)}{\partial t}, & t \in (0,\infty). \end{cases}$$

Also, hypothesis (H2) models the case where a pair of identical smart material patches of length a are bonded or embedded on opposite sides of the beam located at $0 \leq x \leq a$. These patches serve as passive or active dampers. This type of damping was also considered in Le Gall, Prieur and Rosier [23]. The presence of the patches affects the material properties of the beam such as the density and Young's modulus. As a result, jump discontinuities at the location of the edges of the patches are usually introduced into these properties. For the beam with clamped edge boundary conditions and patches bonded or embedded on the subinterval $(\alpha, \alpha+a) \subset (0,\ell)$, but with zero bending moment and shear force, the exponential stability has been shown by Liu and Liu [25]. We note that their method will not easily apply in our stabilization problem. They have assumed that outside the patches, the density and Young's modulus are constant, while in our case we assume that they depend on the spatial variable x.

The following theorem, which is analogous to Theorem 3.28, states that when the feedback coefficients k_2 and k_4 are nonnegative, then the energy of the locally damped beam will not increase.

Theorem 4.1. *If the feedback coefficients k_2 and k_4 are nonnegative then the energy $E(t)$ of the beam given by* (4.1) *is not increasing.*

Proof. Getting the derivative of the energy $E(t)$ with respect to time gives

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = \int_0^\ell \left(EI\frac{\partial^2 y}{\partial x^2}\frac{\partial^3 y}{\partial x^2 \partial t} + \rho\frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial t^2}\right)\mathrm{d}x + k_1\frac{\partial y(t,\ell)}{\partial x}\frac{\partial^2 y(t,\ell)}{\partial t\partial x} + k_3 y(t,\ell)\frac{\partial y(t,\ell)}{\partial t}.$$

Since

$$\rho\frac{\partial^2 y}{\partial t^2} = -\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2} + c_D I\frac{\partial^3 y}{\partial x^2\partial t}\right) - c_a\frac{\partial y}{\partial t}$$

we have

$$\begin{aligned}
\int_0^\ell \rho\frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial t^2}\mathrm{d}x &= -\int_0^\ell \frac{\partial y}{\partial t}\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2} + c_D I\frac{\partial^3 y}{\partial x^2\partial t}\right)\mathrm{d}x - \int_0^\ell c_a\left|\frac{\partial y}{\partial t}\right|^2\mathrm{d}x \\
&= -\left(k_3 y(t,\ell) + k_4\frac{\partial y(t,\ell)}{\partial t}\right)\frac{\partial y(t,\ell)}{\partial t} \\
&\quad + \int_0^\ell \frac{\partial^2 y}{\partial x\partial t}\frac{\partial}{\partial x}\left(EI\frac{\partial^2 y}{\partial x^2} + c_D I\frac{\partial^3 y}{\partial x^2\partial t}\right)\mathrm{d}x - \int_0^\ell c_a\left|\frac{\partial y}{\partial t}\right|^2\mathrm{d}x \\
&= -k_3 y(t,\ell)\frac{\partial y(t,\ell)}{\partial t} - k_4\left|\frac{\partial y(t,\ell)}{\partial t}\right|^2 - k_1\frac{\partial y(t,\ell)}{\partial x}\frac{\partial^2 y(t,\ell)}{\partial t\partial x} \\
&\quad - k_2\left|\frac{\partial^2 y(t,\ell)}{\partial t\partial x}\right|^2 - \int_0^\ell EI\frac{\partial^2 y}{\partial x^2}\frac{\partial^3 y}{\partial x^2\partial t}\mathrm{d}x - \int_0^a D\left|\frac{\partial^3 y}{\partial x^2\partial t}\right|^2\mathrm{d}x \\
&\quad - \int_0^\ell c_a\left|\frac{\partial y}{\partial t}\right|^2\mathrm{d}x.
\end{aligned}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -k_2\left|\frac{\partial^2 y(t,\ell)}{\partial t\partial x}\right|^2 - k_4\left|\frac{\partial y(t,\ell)}{\partial t}\right|^2 - \int_0^a D\left|\frac{\partial^3 y}{\partial x^2\partial t}\right|^2\mathrm{d}x - \int_0^\ell c_a\left|\frac{\partial y}{\partial t}\right|^2\mathrm{d}x,$$

and since $c_a, k_2, k_4 \geq 0$ and $D > 0$ on $(0,a)$ we have $\mathrm{d}E(t)/\mathrm{d}t \leq 0$. Therefore $E(t)$ is not increasing. □

4.1 Finite Energy State Space

As before we let $H = L^2_\rho(0,\ell)$ and now we consider the space $W = \{w \in H^2(0,\ell) \mid w(0) = w'(0) = 0\}$ equipped with the inner product

$$\langle w_1, w_2\rangle_W = k_1 w_1'(\ell)\overline{w_2'(\ell)} + k_3 w_1(\ell)\overline{w_2(\ell)} + \int_0^\ell EI w_1''\overline{w_2''}\,\mathrm{d}x.$$

The corresponding norm in W is

$$\begin{aligned}\|w\|_W &= (k_1|w'(\ell)|^2 + k_3|w(\ell)|^2 + \|w\|_V^2)^{\frac{1}{2}} \\ &= \left(k_1|w'(\ell)|^2 + k_3|w(\ell)|^2 + \int_0^\ell EI|w''|^2\,\mathrm{d}x\right)^{\frac{1}{2}}.\end{aligned}$$

It can be easily seen that $\|w\|_W \geq \|w\|_V \geq C_\ell^{-\frac{1}{2}}\|w\|_{H^2(0,\ell)}$. Now, from inequality (3.6) and (3.7) it follows that $|w(\ell)| \leq \sqrt{\ell}\|w'\|_{L^2(0,\ell)}$ and $|w'(\ell)| \leq \sqrt{\ell}\|w''\|_{L^2(0,\ell)}$. Thus

$$\begin{aligned}\|w\|_W^2 &\leq k_1\ell\|w''\|_{L^2(0,\ell)}^2 + k_3\ell\|w'\|_{L^2(0,\ell)}^2 + C_0\|w''\|_{L^2(0,\ell)}^2 \\ &\leq 3\max\{k_1\ell, k_3\ell, C_0\}\|w\|_{H^2(0,\ell)}^2.\end{aligned}$$

Therefore the Sobolev norm H^2 is also equivalent to the W-norm. Also, the embedding $W \hookrightarrow H$ is compact, continuous, and dense. The proof of this result is analogous to that of Theorem 3.3.

We define the finite energy state space to be the Hilbert space $\mathcal{H}_0 = W \times H$ with inner product

$$\left\langle \begin{bmatrix} w_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} w_2 \\ v_2 \end{bmatrix} \right\rangle_{\mathcal{H}_0} = \langle w_1, w_2\rangle_W + \langle v_1, v_2\rangle_H .$$

The corresponding norm is given by

$$\left\| \begin{bmatrix} w \\ v \end{bmatrix} \right\|_{\mathcal{H}_0} = \left(k_1|w'(\ell)|^2 + k_3|w(\ell)|^2 + \int_0^\ell (EI(x)|w''(x)|^2 + \rho(x)|v(x)|^2)\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

4.2 Semigroup Formulation

In this section, we reduce the partial differential equation

$$\begin{aligned}\rho(x)\frac{\partial^2 y(t,x)}{\partial t^2} &+ \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y(t,x)}{\partial x^2}\right) \\ &+ \frac{\partial^2}{\partial x^2}\left(c_D I(x)\frac{\partial^3 y(t,x)}{\partial x^2 \partial t}\right) + c_a(x)\frac{\partial y(t,x)}{\partial t} = 0 \end{aligned} \tag{4.2}$$

with initial conditions

$$\begin{aligned} y(0,x) &= w_0(x), \\ \dot{y}(0,x) &= v_0(x), \end{aligned} \tag{4.3}$$

and boundary conditions

$$\begin{aligned} y(t,0) &= y'(t,0) = 0 \\ -EI(\ell)y''(t,\ell) &= k_1 y'(t,\ell) + k_2\dot{y}'(t,\ell) \\ (EI(\ell)y''(t,\ell))' &= k_3 y(t,\ell) + k_4\dot{y}(t,\ell) \end{aligned} \tag{4.4}$$

as an abstract Cauchy problem on the Hilbert space $\mathcal{H}_0$. We consider the operator $\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subset \mathcal{H}_0 \to \mathcal{H}_0$, where

$$\mathcal{D}(\mathcal{A}_0) = \left\{ \begin{bmatrix} w \\ v \end{bmatrix} \in \mathcal{H}_0 \,\middle|\, \begin{array}{c} w, v \in V,\ EIw'' + c_D I v'' \in H^2(0,\ell), \\ \begin{bmatrix} -EI(\ell)w''(\ell) \\ ((EI(\ell)w''(\ell))' \end{bmatrix} = \begin{bmatrix} k_1 w'(\ell) + k_2 v'(\ell) \\ k_3 w(\ell) + k_4 v(\ell) \end{bmatrix} \end{array} \right\},$$

defined by

$$\mathcal{A}_0 z = \mathcal{A}_0 \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} v \\ -\frac{1}{\rho}(EIw'' + c_D I v'')'' - \frac{1}{\rho} c_a v \end{bmatrix}$$

for all $z = [w\ v]^\mathsf{T} \in \mathcal{D}(\mathcal{A}_0)$. Then the partial differential equation (4.2) with initial conditions (4.3) and boundary conditions (4.4) can be phrased as an abstract Cauchy problem on $\mathcal{H}_0$ as

$$\left\{ \begin{array}{rcl} \dfrac{\mathrm{d}z(t)}{\mathrm{d}t} & = & \mathcal{A}_0 z(t), \text{ for all } t > 0, \\ z(0) & = & z_0 \;=\; [w_0, v_0]^\mathsf{T}. \end{array} \right. \tag{4.5}$$

In the following lemma, we let

$$\alpha = 1 + k_1 \int_0^\ell \frac{\mathrm{d}s}{EI(s)}, \tag{4.6}$$

$$\beta = k_3 \int_0^\ell \frac{\ell - s}{EI(s)} \mathrm{d}s, \tag{4.7}$$

$$\gamma = k_1 \int_0^\ell \int_0^t \frac{\mathrm{d}s\, \mathrm{d}t}{EI(s)}, \tag{4.8}$$

$$\delta = 1 + k_3 \int_0^\ell \int_0^t \frac{\ell - s}{EI(s)} \mathrm{d}s\, \mathrm{d}t. \tag{4.9}$$

Lemma 4.2. *If the feedback coefficients k_1 and k_3 are nonnegative then the constant $\Delta_0 = \Delta_0(EI, k_1, k_3) = \alpha\delta - \beta\gamma$ is positive.*

Proof. Since $EI(s) > 0$ and $\ell - s \geq 0$ for all $s \in [0,\ell]$ it follows that $\alpha \geq 1$ and $\delta \geq 1$. Further, we have

$$\begin{aligned} \Delta_0 &= \alpha + (\delta - 1) + k_1 k_3 \left(\int_0^\ell \frac{\mathrm{d}s}{EI(s)} \right) \left(\int_0^\ell \int_0^t \frac{\ell - s}{EI(s)} \mathrm{d}s\, \mathrm{d}t \right) \\ &\quad - k_1 k_3 \left(\int_0^\ell \frac{\ell - s}{EI(s)} \mathrm{d}s \right) \left(\int_0^\ell \int_0^t \frac{\mathrm{d}s\, \mathrm{d}t}{EI(s)} \right) \\ &\geq 1 + k_1 k_3 \left(\int_0^\ell \int_0^\ell \int_0^t \frac{\ell - s}{EI(\tau)EI(s)} \mathrm{d}s\, \mathrm{d}t\, \mathrm{d}\tau - \int_0^\ell \int_0^\ell \int_0^t \frac{\ell - \tau}{EI(\tau)EI(s)} \mathrm{d}s\, \mathrm{d}t\, \mathrm{d}\tau \right) \\ &= 1 + k_1 k_3 \int_0^\ell \int_0^\ell \int_0^t \frac{\tau - s}{EI(\tau)EI(s)} \mathrm{d}s\, \mathrm{d}t\, \mathrm{d}\tau. \end{aligned}$$

Since
$$\int_0^\ell \int_0^\ell \int_0^t (\tau - s)\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\tau = \frac{\ell^4}{12}$$
we have
$$\Delta_0 \geq 1 + \frac{k_1 k_3 \ell^4}{12 C_0^2} > 0,$$
which completes the proof of the lemma. □

Lemma 4.3. *We have*
$$\begin{aligned} \langle \mathcal{A}_0 z, z\rangle_{\mathcal{H}_0} &= -\, k_2|v'(\ell)|^2 - k_4|v(\ell)|^2 - \int_0^\ell c_a|v|^2\,dx - \int_0^a D|v''|^2\,dx \\ &\quad + 2i\mathrm{Im}\,\langle v, w\rangle_W\,. \end{aligned}$$
for $z = [w\ v]^\mathsf{T} \in \mathcal{D}(\mathcal{A}_0)$. *In particular, if* $k_2, k_4 \geq 0$ *then* $\mathcal{A}_0$ *is dissipative.*

Proof. Let $[w\ v]^\mathsf{T} \in \mathcal{D}(\mathcal{A}_0)$. From the definition of the inner product in $\mathcal{H}_0$ we get
$$\begin{aligned} \left\langle \mathcal{A}_0 \begin{bmatrix} w \\ v \end{bmatrix}, \begin{bmatrix} w \\ v \end{bmatrix} \right\rangle_{\mathcal{H}_0} &= \left\langle \begin{bmatrix} v \\ -\frac{1}{\rho}(EIw'' + c_D I v'')'' - \frac{1}{\rho} c_a v \end{bmatrix}, \begin{bmatrix} w \\ v \end{bmatrix} \right\rangle_{\mathcal{H}_0} \\ &= \langle v, w\rangle_W - \int_0^\ell (EIw'' + c_D I v'')''\overline{v}\,\mathrm{d}x - \int_0^\ell c_a|v|^2\,\mathrm{d}x. \end{aligned}$$

Now, the inner product of v and w in W can be written, in terms of the inner product of v and w in V, as
$$\langle v, w\rangle_W = k_1 v'(\ell)\overline{w'(\ell)} + k_3 v(\ell)\overline{w(\ell)} + \langle v, w\rangle_V\,.$$

Integration by parts and the boundary conditions give us
$$\begin{aligned} \int_0^\ell (EIw'' + c_D I v'')''\overline{v}\,\mathrm{d}x &= (EIw'' + c_D I v'')'(\ell)\overline{v(\ell)} - \int_0^\ell (EIw'' + c_D I v'')'\overline{v'}\,\mathrm{d}x \\ &= k_3 w(\ell)\overline{v(\ell)} + k_4|v(\ell)|^2 - (EIw'' + c_D I v'')(\ell)\overline{v'(\ell)} \\ &\quad + \int_0^\ell (EIw'' + c_D I v'')\overline{v''}\,\mathrm{d}x \\ &= k_3 w(\ell)\overline{v(\ell)} + k_4|v(\ell)|^2 + k_1 w'(\ell)\overline{v'(\ell)} + k_2|v'(\ell)|^2 \\ &\quad + \langle w, v\rangle_V + \int_0^a D|v''|^2\,\mathrm{d}x \\ &= \overline{\langle v, w\rangle_W} + k_2|v'(\ell)|^2 + k_4|v(\ell)|^2 + \int_0^a D|v''|^2\,\mathrm{d}x. \end{aligned}$$

Hence we have the desired formula

$$\left\langle \mathcal{A}_0 \begin{bmatrix} w \\ v \end{bmatrix}, \begin{bmatrix} w \\ v \end{bmatrix} \right\rangle_{\mathcal{H}_0} = -k_2|v'(\ell)|^2 - k_4|v(\ell)|^2 - \int_0^\ell c_a|v|^2\,\mathrm{d}x - \int_0^a D|v''|^2\,\mathrm{d}x + 2i\mathrm{Im}\,\langle v, w\rangle_W .$$

Using the assumptions $c_a, k_2, k_4 \geq 0$ and $D \geq c_0 > 0$ on $(0,a)$ and by taking the real part of $\langle \mathcal{A}_0 z, z\rangle_{\mathcal{H}_0}$ we have

$$\mathrm{Re}\left\langle \mathcal{A}_0 \begin{bmatrix} w \\ v \end{bmatrix}, \begin{bmatrix} w \\ v \end{bmatrix} \right\rangle_{\mathcal{H}_0} = -k_2|v'(\ell)|^2 - k_4|v(\ell)|^2 - \int_0^\ell c_a|v|^2\,\mathrm{d}x - \int_0^a D|v''|^2\,\mathrm{d}x \leq 0.$$

Therefore $\mathcal{A}_0$ is dissipative. □

Lemma 4.4. *Assume that* $k_1, k_2, k_3, k_4 \geq 0$. *The linear operator* $\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subset \mathcal{H}_0 \to \mathcal{H}_0$ *is one-to-one and onto.*

Proof. For any $[f\ g]^\mathsf{T} \in \mathcal{H}_0$, we are going to find a unique $[w\ v]^\mathsf{T} \in \mathcal{D}(\mathcal{A}_0)$ such that $\mathcal{A}_0[u\ v]^\mathsf{T} = [f\ g]^\mathsf{T}$. This means that we are going to find a solution of the system

$$v = f \tag{4.10}$$

$$(EIw'' + c_D I v'')'' + c_a v = -\rho g. \tag{4.11}$$

Let us solve the differential equation (4.11) for w. Integrating the left hand side of Equation (4.11) twice gives

$$\int_x^\ell \int_\xi^\ell (EI(\tau)w''(\tau) + c_D I(\tau)v''(\tau))''\,\mathrm{d}\tau\,\mathrm{d}\xi$$
$$= k_1 w'(\ell) + k_3 w(\ell)(\ell - x) + [1\ (\ell - x)]F \begin{bmatrix} v'(\ell) \\ v(\ell) \end{bmatrix} + EI(x)w''(x) + c_D I(x) v''(x),$$

where $F = \mathrm{diag}(k_2, k_4)$. Solving for w'' in this equation gives us

$$w''(x) = -\frac{k_1 w'(\ell)}{EI(x)} - \frac{k_3(\ell - x)w(\ell)}{EI(x)} - \eta(x) \tag{4.12}$$

where $\eta(x) = \eta(f, g, x)$ is given by

$$\eta(x) = \frac{1}{EI(x)}\left([1\ (\ell - x)]F \begin{bmatrix} f'(\ell) \\ f(\ell) \end{bmatrix} + c_D I(x) f''(x) + \int_x^\ell \int_\xi^\ell (\rho(\tau)g(\tau) + c_a(\tau)f(\tau))\,\mathrm{d}\tau\,\mathrm{d}\xi\right). \tag{4.13}$$

Integrating (4.12) from 0 to x we get

$$w'(x) = -\left(k_1 \int_0^x \frac{\mathrm{d}s}{EI(s)}\right) w'(\ell) - \left(k_3 \int_0^x \frac{\ell - s}{EI(s)}\,\mathrm{d}s\right) w(\ell) - \int_0^x \eta(s)\,\mathrm{d}s. \tag{4.14}$$

Again, doing the same process in (4.14) yields

$$\begin{aligned} w(x) &= -\left(k_1 \int_0^x \int_0^t \frac{\mathrm{d}s\,\mathrm{d}t}{EI(s)}\right) w'(\ell) - \left(k_3 \int_0^x \int_0^t \frac{\ell - s}{EI(s)}\,\mathrm{d}s\,\mathrm{d}t\right) w(\ell) \\ &\quad - \int_0^x \int_0^t \eta(s)\,\mathrm{d}s\,\mathrm{d}t. \end{aligned} \tag{4.15}$$

Taking $x = \ell$ in (4.14) and (4.15) we obtain the following system

$$\begin{aligned} \left(1 + k_1 \int_0^\ell \frac{\mathrm{d}s}{EI(s)}\right) w'(\ell) + \left(k_3 \int_0^\ell \frac{\ell - s}{EI(s)}\,\mathrm{d}s\right) w(\ell) &= -\int_0^\ell \eta(s)\,\mathrm{d}s \\ \left(k_1 \int_0^\ell \int_0^t \frac{\mathrm{d}s\,\mathrm{d}t}{EI(s)}\right) w'(\ell) + \left(1 + k_3 \int_0^\ell \int_0^t \frac{\ell - s}{EI(s)}\,\mathrm{d}s\,\mathrm{d}t\right) w(\ell) &= -\int_0^\ell \int_0^t \eta(s)\,\mathrm{d}s\,\mathrm{d}t, \end{aligned}$$

and so we have

$$\begin{aligned} \alpha w'(\ell) + \beta w(\ell) &= -\int_0^\ell \eta(s)\,\mathrm{d}s \\ \gamma w'(\ell) + \delta w(\ell) &= -\int_0^\ell \int_0^t \eta(s)\,\mathrm{d}s\,\mathrm{d}t. \end{aligned}$$

From Lemma 4.2 it follows that the system has a unique solution $[w'(\ell)\ w(\ell)]^{\mathsf{T}} \in \mathbb{C}^2$ and it is given by

$$\begin{aligned} w'(\ell) &= -\frac{\delta}{\Delta_0} \int_0^\ell \eta(s)\,\mathrm{d}s + \frac{\beta}{\Delta_0} \int_0^\ell \int_0^t \eta(s)\,\mathrm{d}s\,\mathrm{d}t \\ w(\ell) &= -\frac{\alpha}{\Delta_0} \int_0^\ell \int_0^t \eta(s)\,\mathrm{d}s\,\mathrm{d}t + \frac{\gamma}{\Delta_0} \int_0^\ell \eta(s)\,\mathrm{d}s. \end{aligned}$$

If we let

$$\tau_1(x) = -k_1 \int_0^x \int_0^t \frac{\mathrm{d}s\,\mathrm{d}t}{EI(s)} \tag{4.16}$$

$$\tau_2(x) = -k_3 \int_0^x \int_0^t \frac{\ell - s}{EI(s)}\,\mathrm{d}s\,\mathrm{d}t, \tag{4.17}$$

then the function w is given by

$$\begin{aligned} w(x) &= \tau_1(x)\left(-\frac{\delta}{\Delta_0}\int_0^\ell \eta(s)\,\mathrm{d}s + \frac{\beta}{\Delta_0}\int_0^\ell\int_0^t \eta(s)\,\mathrm{d}s\,\mathrm{d}t\right) \\ &\quad + \tau_2(x)\left(-\frac{\alpha}{\Delta_0}\int_0^\ell\int_0^t \eta(s)\,\mathrm{d}s\,\mathrm{d}t + \frac{\gamma}{\Delta_0}\int_0^\ell \eta(s)\,\mathrm{d}s\right) - \int_0^x\int_0^t \eta(s)\,\mathrm{d}s\,\mathrm{d}t \\ &= \frac{\gamma\tau_2(x)-\delta\tau_1(x)}{\Delta_0}\int_0^\ell \eta(s)\,\mathrm{d}s + \frac{\beta\tau_1(x)-\alpha\tau_2(x)}{\Delta_0}\int_0^\ell\int_0^t \eta(s)\,\mathrm{d}s\,\mathrm{d}t \\ &\quad - \int_0^x\int_0^t \eta(s)\,\mathrm{d}s\,\mathrm{d}t. \end{aligned}$$

Hence, the required solution to the system (4.10) is

$$\begin{aligned} v(x) &= f(x) \\ w(x) &= \frac{\gamma\tau_2(x)-\delta\tau_1(x)}{\Delta_0}\int_0^\ell \eta(s)\,\mathrm{d}s + \frac{\beta\tau_1(x)-\alpha\tau_2(x)}{\Delta_0}\int_0^\ell\int_0^t \eta(s)\,\mathrm{d}s\,\mathrm{d}t \\ &\quad - \int_0^x\int_0^t \eta(s)\,\mathrm{d}s\,\mathrm{d}t, \end{aligned}$$

where the constants α, β, γ, and δ are given by Equations (4.6)−(4.9), $\Delta_0 = \alpha\delta-\beta\gamma$, and the functions τ_1, τ_2, and η are given by Equations (4.16), (4.17) and (4.13), respectively.

Since $f \in V$, then $v \in V$. The equalities $\tau_1(0) = \tau_2(0) = 0$ imply that $w(0) = w'(0) = 0$. Also, we have $(EIw'' + c_D I v'')'' = -\rho g - c_a v \in L^2(0,\ell)$. Moreover, we can see that

$$-EI(\ell)w''(\ell) = k_1 w'(\ell) + k_2 v'(\ell)$$

and

$$(EI(\ell)w''(\ell))' = k_3 w(\ell) + k_4 v(\ell).$$

Therefore we have found a vector $[w\ v]^\mathsf{T} \in \mathcal{D}(\mathcal{A}_0)$ such that $\mathcal{A}_0[w\ v]^\mathsf{T} = [f\ g]^\mathsf{T}$ and it can be easily seen that, using a similar argument as in Lemma 3.6, such a vector is unique. □

The previous lemma implies that $\mathcal{A}_0^{-1} : \mathcal{H}_0 \to \mathcal{H}_0$ exists. It is also continuous as shown in the next lemma.

Lemma 4.5. *Assume that $k_1, k_2, k_3, k_4 \geq 0$. The inverse $\mathcal{A}_0^{-1} : \mathcal{H}_0 \to \mathcal{H}_0$ of $\mathcal{A}_0$ is bounded.*

Proof. As before, we have the following estimates for the L_2-norm of η (see Equation

(4.13))

$$\begin{aligned}
\|\eta\|_{L^2(0,\ell)} &\leq \frac{1}{c_0}\left\|[1\ (\ell-x)]F\begin{bmatrix} f'(\ell) \\ f(\ell)\end{bmatrix}\right\|_{L^2(0,\ell)} + \frac{1}{c_0}\|c_D I f''\|_{L^2(0,\ell)} \\
&\quad + \frac{1}{c_0}\left\|\int_x^\ell\int_\xi^\ell \rho(\tau)g(\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi\right\|_{L^2(0,\ell)} + \frac{1}{c_0}\left\|\int_x^\ell\int_\xi^\ell c_a(\tau)f(\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi\right\|_{L^2(0,\ell)} \\
&\leq \frac{k_2\ell}{c_0}\|f''\|_{L^2(0,\ell)} + \frac{k_4\ell^2}{c_0\sqrt{3}}\|f'\|_{L^2(0,\ell)} + \frac{C_0}{c_0}\|f''\|_{L^2(0,\ell)} \\
&\quad + \frac{\ell^{\frac{3}{2}}}{c_0}\|\rho\|_{L^2(0,\ell)}\|g\|_{L^2(0,\ell)} + \frac{\ell^{\frac{3}{2}}}{c_0}\|c_a\|_{L^2(0,\ell)}\|f\|_{L^2(0,\ell)} \\
&\leq C_1(\|g\|_{L^2(0,\ell)} + \|f\|_{H^2(0,\ell)}),
\end{aligned}$$

where $C_1 = 3\max\{k_2\ell + C_0, k_4\ell^2/\sqrt{3}, \ell^{\frac{3}{2}}\|\rho\|_{L^2(0,\ell)}, \ell^{\frac{3}{2}}\|c_a\|_{L^2(0,\ell)}\}/c_0$. Now note that from (4.12) we obtain

$$\begin{aligned}
w''(x) &= -\frac{k_1}{EI(x)}\left(-\frac{\delta}{\Delta_0}\int_0^\ell \eta(s)\,\mathrm{d}s + \frac{\beta}{\Delta_0}\int_0^\ell\int_0^t \eta(s)\,\mathrm{d}s\,\mathrm{d}t\right) \\
&\quad - \frac{k_3(\ell-x)}{EI(x)}\left(-\frac{\alpha}{\Delta_0}\int_0^\ell\int_0^t \eta(s)\,\mathrm{d}s\,\mathrm{d}t + \frac{\gamma}{\Delta_0}\int_0^\ell \eta(s)\,\mathrm{d}s\right) - \eta(x) \\
&= -\eta(x) + \frac{k_1\delta - k_3\gamma(\ell-x)}{\Delta_0 EI(x)}\int_0^\ell \eta(s)\,\mathrm{d}s - \frac{k_1\beta - k_3\alpha(\ell-x)}{\Delta_0 EI(x)}\int_0^\ell\int_0^t \eta(s)\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

Observe that for all $x \in [0,\ell]$

$$\left|\frac{k_1\delta - k_3\gamma(\ell-x)}{\Delta_0 EI(x)}\right| \leq \frac{k_1\delta + k_3\gamma\ell}{\Delta_0 c_0} := \lambda_1. \tag{4.18}$$

Similarly,

$$\left|\frac{k_1\beta - k_3\alpha(\ell-x)}{\Delta_0 EI(x)}\right| \leq \frac{k_1\beta + k_3\alpha\ell}{\Delta_0 c_0} := \lambda_2, \tag{4.19}$$

for all $x \in [0,\ell]$. Using inequality (4.18) and with the aid of the Cauchy-Schwartz inequality we obtain

$$\begin{aligned}
\left\|\frac{k_1\delta - k_3\gamma(\ell-x)}{\Delta_0 EI(x)}\int_0^\ell \eta(s)\,\mathrm{d}s\right\|^2_{L^2(0,\ell)} &= \int_0^\ell \left|\frac{k_1\delta - k_3\gamma(\ell-x)}{\Delta_0 EI(x)}\int_0^\ell \eta(s)\,\mathrm{d}s\right|^2 \mathrm{d}x \\
&\leq \lambda_1^2\ell\left(\int_0^\ell |\eta(s)|\,\mathrm{d}s\right)^2 \\
&\leq \lambda_1^2\ell^2\|\eta\|^2_{L^2(0,\ell)}.
\end{aligned}$$

In a similar way, using (4.19) we obtain

$$\begin{aligned}
&\left\| \frac{k_1\beta - k_3\alpha(\ell - x)}{\Delta_0 EI(x)} \int_0^\ell \int_0^t \eta(s)\,\mathrm{d}s\,\mathrm{d}t \right\|^2_{L^2(0,\ell)} \\
&\qquad = \int_0^\ell \left| \frac{k_1\beta - k_3\alpha(\ell - x)}{\Delta_0 EI(x)} \int_0^\ell \int_0^t \eta(s)\,\mathrm{d}s\,\mathrm{d}t \right|^2 \mathrm{d}x \\
&\qquad \le \lambda_2^2 \int_0^\ell \left(\int_0^\ell \int_0^t |\eta(s)|\,\mathrm{d}s\,\mathrm{d}t \right)^2 \mathrm{d}x \\
&\qquad \le \lambda_2^2 \ell^3 \left(\int_0^\ell |\eta(s)|\,\mathrm{d}s \right)^2 \\
&\qquad \le \lambda_2^2 \ell^4 \|\eta\|^2_{L^2(0,\ell)}.
\end{aligned}$$

Therefore the L^2-norm of the second derivative of w can be estimated by

$$\begin{aligned}
\|w''\|_{L^2(0,\ell)} &\le \|\eta\|_{L^2(0,\ell)} + \lambda_1\ell\|\eta\|_{L^2(0,\ell)} + \lambda_2\ell^2\|\eta\|_{L^2(0,\ell)} \\
&= (1 + \lambda_1\ell + \lambda_2\ell^2)\|\eta\|_{L^2(0,\ell)} \\
&\le C_1(1 + \lambda_1\ell + \lambda_2\ell^2)(\|g\|_{L^2(0,\ell)} + \|f\|_{H^2(0,\ell)}) \\
&\le C_1(1 + \lambda_1\ell + \lambda_2\ell^2)(c_0^{-\frac{1}{2}}\|g\|_H + C_\ell^{\frac{1}{2}}\|f\|_V) \\
&= C_2(\|g\|_H + \|f\|_V) \\
&\le C_2(\|g\|_H + \|f\|_W),
\end{aligned}$$

where $C_2 = C_1(1 + \lambda_1\ell + \lambda_2\ell^2)\max\{c_0^{-\frac{1}{2}}, C_\ell^{\frac{1}{2}}\}$. Since $\|w\|_V \le \sqrt{C_0}\|w''\|_{L^2(0,\ell)}$ it follows that

$$\|w\|^2_W \le (k_1\ell + k_3\ell^3 + C_0)\|w''\|^2_{L^2(0,\ell)}$$

and this implies that

$$\|w\|^2_W \le C_3(\|g\|_H + \|f\|_W)^2$$

where $C_3 = C_2^2(k_1\ell + k_3\ell^3 + C_0)$. Hence, the square of the W-norm of w can be estimated by

$$\|w\|^2_W \le 2C_3 \left\| \begin{bmatrix} f \\ g \end{bmatrix} \right\|^2_{\mathcal{H}_0}.$$

Using this we can deduce that

$$\left\| \mathcal{A}_0^{-1} \begin{bmatrix} f \\ g \end{bmatrix} \right\|_{\mathcal{H}_0} = \left\| \begin{bmatrix} w \\ v \end{bmatrix} \right\|_{\mathcal{H}_0} \le \sqrt{2C_3 + C_\ell C_0} \left\| \begin{bmatrix} f \\ g \end{bmatrix} \right\|_{\mathcal{H}_0},$$

where the constants C_0, C_3 and C_ℓ are independent of f and g. This completes the proof that $\mathcal{A}_0^{-1} \in \mathcal{L}(\mathcal{H}_0)$. □

Theorem 4.6. *Assume that* $k_1, k_2, k_3, k_4 \geq 0$. *Then* $\mathcal{A}_0$ *generates a* C_0*-contraction semigroup* $S(t) = e^{t\mathcal{A}_0}$ *on* $\mathcal{H}_0$. *Moreover,* $\mathcal{A}_0^{-1} \in \mathcal{L}(\mathcal{H}_0)$ *is a closed operator.*

Proof. The fact that $\mathcal{A}_0$ is dissipative is shown in Lemma 4.3. From Lemma 4.5 we have shown that $\mathcal{A}_0^{-1} \in \mathcal{L}(\mathcal{H}_0)$ and so $0 \in \rho(\mathcal{A}_0)$. As a consequence, the operator $\mathcal{A}_0^{-1}$ is closed by Theorem 2.2. In a similar way, as in the proof of Theorem 3.4, we can show that $\mathcal{D}(\mathcal{A}_0)$ is dense in $\mathcal{H}_0$ and by Theorem 2.21 we can now conclude that $\mathcal{A}_0$ generates a C_0-contraction semigroup $S(t) = e^{t\mathcal{A}_0}$ on $\mathcal{H}_0$. □

The interpretation of the above theorem in terms of the system (4.2)−(4.4) is the following: For $z_0 = [w_0 \ v_0]^{\mathsf{T}} \in \mathcal{D}(\mathcal{A}_0)$ the partial differential equation (4.2) with initial conditions (4.3) and boundary conditions (4.4) admits a unique solution y such that

$$[y \ \partial y/\partial t]^{\mathsf{T}} \in C([0,\infty); \mathcal{D}(\mathcal{A}_0)) \cap C^1([0,\infty); \mathcal{H}_0).$$

4.3 Exponential Stability

4.3.1 Preliminary Results

Consider the form $b_0[\cdot,\cdot] : W \times W \to \mathbb{C}$ on W defined by

$$b_0[w,v] = k_2 w'(\ell)\overline{v(\ell)} + k_4 w(\ell)\overline{v(\ell)} + \int_0^\ell (c_D I w''\overline{v''} + c_a wv)\,\mathrm{d}x.$$

In a similar way, as in Theorem 3.11, we can show that b_0 is a nonnegative, closed, symmetric sesquilinear form on W and that $b_0[w] \leq K\|w\|_V^2$ for all $w \in W$ and for some positive constant K. Also, $\langle\cdot,\cdot\rangle_W$ is a positive definite, densely defined, closed, symmetric sesquilinear form in H. These and the Riesz representations theorems imply that there exist a self-adjoint, positive operator A_0 in H such that $\mathcal{D}(A_0^{\frac{1}{2}}) = W$ and $\langle w,v\rangle_W = \langle A_0^{\frac{1}{2}} w, A_0^{\frac{1}{2}} v\rangle_H$ for all $w, v \in W$ and a nonnegative, self-adjoint operator $S_0 \in \mathcal{L}(W)$ such that $b_0[w,v] = \langle S_0^{\frac{1}{2}} w, S_0^{\frac{1}{2}} v\rangle_W$ for all $w, v \in W$.

If y is a solution of the partial differential equation (4.2) with initial conditions (4.3) and boundary conditions (4.4) then

$$\begin{aligned}\langle y,\xi\rangle_W + b_0[\dot{y},\xi] &= \int_0^\ell (EIy'' + c_D I\dot{y}'')\overline{\xi''}\,\mathrm{d}x + k_1 y'(\ell)\overline{\xi'(\ell)} + k_3 y(\ell)\overline{\xi(\ell)} \\ &\quad + k_2\dot{y}'(\ell)\overline{\xi'(\ell)} + k_4\dot{y}(\ell)\overline{\xi(\ell)} + \int_0^\ell c_a\dot{y}\bar{\xi}\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= (EI(\ell)y''(\ell))\overline{\xi'(\ell)} - (EI(\ell)y''(\ell))'\overline{\xi(\ell)} + \int_0^\ell (EIy'' + c_D I\dot{y}'')''\overline{\xi}\,\mathrm{d}x \\
&\quad + k_1 y'(\ell)\overline{\xi'(\ell)} + k_2\dot{y}'(\ell)\overline{\xi'(\ell)} + k_3 y(\ell)\overline{\xi(\ell)} + k_4\dot{y}(\ell)\overline{\xi(\ell)} + \int_0^\ell c_a\dot{y}\overline{\xi}\,\mathrm{d}x \\
&= \left\langle \frac{1}{\rho}(EIy'' + c_D I\dot{y}'')'' + \frac{c_a\dot{y}}{\rho}, \xi \right\rangle_H .
\end{aligned}
$$

Therefore y satisfies the following variational evolution equation

$$\langle \ddot{y}(t), \xi\rangle_H + b_0[\dot{y}(t), \xi] + \langle y(t), \xi\rangle_W = 0$$

for all $\xi \in V$ and for all $t > 0$ with initial conditions

$$y(0) = w_0, \quad \dot{y}(0) = v_0.$$

Furthermore, the infinitesimal generator $\mathcal{A}_0$ can be represented by the operators A_0 and S_0 as

$$\mathcal{A}_0 \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} v \\ -A_0(w + S_0 v) \end{bmatrix}$$

for all $[w\ v]^\mathsf{T} \in \mathcal{D}(\mathcal{A}_0)$ and $\mathcal{D}(\mathcal{A}_0) = \{[w\ v]^\mathsf{T} \in \mathcal{H}_0 \,|\, v \in W, w + S_0 v \in \mathcal{D}(A_0)\}$.

As in the global damping case, the nonreal spectrum of $\mathcal{A}_0$ lies in the point spectrum of $\mathcal{A}_0$, in other words,

$$\{z \in \sigma(\mathcal{A}_0) \,|\, \mathrm{Im}\ z \neq 0\} \subset \sigma_p(\mathcal{A}_0). \tag{4.20}$$

Again, we consider the following unique continuation condition in the W setting.

(UCC2) If $w \in W$, $\langle w, \xi\rangle_W - \beta^2 \langle w, \xi\rangle_H = 0$ for all $\xi \in W$, $b_0[w] = 0$ and

$$w(\ell) = w'(\ell) = w''(\ell) = w'''(\ell) = 0$$

then $w = 0$ in W, for all $\beta \in \mathbb{R}$.

Lemma 4.7. *If $k_1, k_2, k_3, k_4 > 0$, (H2) and (UCC2) are satisfied then $i\mathbb{R} \subset \rho(\mathcal{A}_0)$.*

Proof. We will establish that $i\mathbb{R} \cap \sigma_p(\mathcal{A}_0) = \emptyset$. Because $0 \in \rho(\mathcal{A}_0)$ we can let $i\beta$ to be a nonzero complex number with $\beta \in \mathbb{R}$. Suppose that $[w\ v]^\mathsf{T} \in \mathcal{N}(\mathcal{A}_0 - i\beta I)$. From the definition of $\mathcal{A}_0$

$$\begin{bmatrix} v - i\beta w \\ -\frac{1}{\rho}(EIw'' + c_D I v'')'' - \frac{1}{\rho}c_a v - i\beta v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore

$$(EIw'' + c_D I v'')'' + c_a v - \beta^2\rho w = 0.$$

Getting the inner product of both sides of the above equation with $\xi \in W$ in $L^2(0,\ell)$ gives

$$\langle (EIw'' + c_D I v'')'', \xi \rangle_{L^2(0,\ell)} + \langle c_a v, \xi \rangle_{L^2(0,\ell)} - \beta^2 \langle w, \xi \rangle_H = 0. \tag{4.21}$$

The first inner product can be simplified into

$$\begin{aligned} \langle (EIw'' + c_D I v'')'', \xi \rangle_{L^2(0,\ell)} &= (k_1 w'(\ell) + k_2 v'(\ell))\overline{\xi'(\ell)} + (k_3 w(\ell) + k_4 v(\ell))\overline{\xi(\ell)} \\ &\quad + \langle EIw'' + c_D I v'', \xi'' \rangle_{L^2(0,\ell)} \\ &= \langle w, \xi \rangle_W + i\beta b_0[w,\xi] - \langle c_a v, \xi \rangle_{L^2(0,\ell)} . \end{aligned}$$

Using this in Equation (4.21) we obtain

$$\langle w, \xi \rangle_W + i\beta b_0[w,\xi] - \beta^2 \langle w, \xi \rangle_H = 0. \tag{4.22}$$

Letting $\xi = w$ in Equation (4.22) we get $\|w\|_W^2 - \beta^2 \|w\|_H^2 + i\beta b_0[w] = 0$ and thus $b_0[w] = 0$. Hence $S_0 w = 0$ and we have $b_0[w,\xi] = \langle S_0 w, \xi \rangle_W = 0$ for all $\xi \in W$. Therefore, Equation (4.22) is reduced to

$$\langle w, \xi \rangle_W - \beta^2 \langle w, \xi \rangle_H = 0. \tag{4.23}$$

Taking the inner product of $(\mathcal{A}_0 - i\beta I)[w\ v]^\mathsf{T}$ and $[w\ v]^\mathsf{T}$ in $\mathcal{H}_0$ gives

$$\begin{aligned} 0 &= \left\langle (\mathcal{A}_0 - i\beta I) \begin{bmatrix} w \\ v \end{bmatrix}, \begin{bmatrix} w \\ v \end{bmatrix} \right\rangle_{\mathcal{H}_0} \\ &= -k_2 |v'(\ell)|^2 - k_4 |v(\ell)|^2 - \int_0^a D|v''|^2 \,\mathrm{d}x - \int_0^\ell c_a |v|^2 \,dx \\ &\quad + i\left(2\mathrm{Im}\, \langle v, w \rangle_W - \beta \left\| \begin{bmatrix} w \\ v \end{bmatrix} \right\|_{\mathcal{H}_0}^2 \right). \end{aligned}$$

Since $k_2, k_4 > 0$ and $D > 0$ on $(0,a)$ it follows that $v'(\ell) = v(\ell) = 0$. Because $w = -iv/\beta$ we have $w(\ell) = w'(\ell) = 0$. Using the boundary conditions at $x = \ell$ we have

$$w''(\ell) = -\frac{k_1 w'(\ell) + k_2 v'(\ell)}{EI(\ell)} = 0.$$

Further, since $(EIw'')'(\ell) = EI'(\ell) w''(\ell) + EI(\ell) w'''(\ell) = EI(\ell) w'''(\ell)$,

$$w'''(\ell) = \frac{k_3 w(\ell) + k_4 v(\ell)}{EI(\ell)} = 0.$$

Thus, we have shown that $\langle w, \xi \rangle_W - \beta^2 \langle w, \xi \rangle_H = 0$ for all $\xi \in W$, $b_0[w] = 0$ and $w(\ell) = w'(\ell) = w''(\ell) = w'''(\ell) = 0$, and by (UCC2) we must have $w = 0$. It also follows that $v = 0$. Hence $\mathcal{N}(\mathcal{A}_0 - i\beta I) = \{0\}$, which implies that $i\beta \notin \sigma_p(\mathcal{A}_0)$ for all $\beta \in \mathbb{R}$ and from inclusion (4.20) we have $i\mathbb{R} \subset \rho(\mathcal{A}_0)$. $\square$

Lemma 4.8. *Assume that* $k_1, k_2, k_3, k_4 > 0$. *Then* (H2) *implies* (UCC2).

Proof. The fact that $w(\ell) = w'(\ell) = w''(\ell) = w'''(\ell) = \xi(0) = \xi'(0) = 0$ we have

$$\langle w, \xi \rangle_V = \langle (EIw'')'', \xi \rangle_{L^2(0,\ell)} .$$

Thus, the condition $\langle w, \xi \rangle_W - \beta^2 \langle w, \xi \rangle_H = 0$ for all $\xi \in W$ implies that

$$\left\langle (EIw'')'' - \beta^2 \rho w, \xi \right\rangle_{L^2(0,\ell)} = 0$$

for all $\xi \in W$. Since W is dense in $L^2(0, \ell)$, there exists a sequence $\{\xi_n\}_{n=1}^{\infty} \subset W$ such that

$$\lim_{n\to\infty} \|\xi_n - ((EIw'')'' - \beta^2 \rho w)\|_{L^2(0,\ell)} = 0.$$

Using the Cauchy-Schwartz inequality,

$$\lim_{n\to\infty} \left\langle (EIw'')'' - \beta^2 \rho w, \xi_n \right\rangle_{L^2(0,\ell)} = \|(EIw'')'' - \beta^2 \rho w\|^2_{L^2(0,\ell)}.$$

Since $\langle (EIw'')'' - \beta^2 \rho w, \xi_n \rangle_{L^2(0,\ell)} = 0$ for all $n \in \mathbb{N}$, the uniqueness of limits implies that $\|(EIw'')'' - \beta^2 \rho w\|^2_{L^2(0,\ell)} = 0$. Thus, w satisfies the ordinary differential equation

$$(EIw'')'' - \beta^2 \rho w = 0$$

in $(0, \ell)$, with initial conditions

$$w(\ell) = w'(\ell) = w''(\ell) = w'''(\ell) = 0.$$

The condition $w \in H^2(0, \ell)$ implies that $w \in C^1[0, \ell]$ by the Sobolev embedding theorem. Since $\rho, EI \in C^2_B(a, \ell)$, it follows that the above initial value problem has the zero solution $w = 0$ on the open interval (a, ℓ). Furthermore, $(EIw'')'' = \beta^2 \rho w \in L^2(0, \ell)$ implies $EIw'' \in H^2(0, \ell)$. Since $0 < EI \in C^2_B(a, \ell)$ then $0 < EI^{-1} \in C^2_B(a, \ell)$. Indeed, it is clear that $EI^{-1} \in C^2(a, \ell)$ is positive since $0 < c_0 \le EI \le C_0$. Hence

$$\sup_{x\in(a,\ell)} |EI^{-1}(x)| \le c_0^{-1}.$$

Because $(EI^{-1})' = -EI^{-2}EI'$ and

$$\begin{aligned}(EI^{-1})'' &= 2EI^{-3}(EI')^2 - EI^{-2}EI'' \\ &= EI^{-3}(2(EI')^2 - EI(EI''))\end{aligned}$$

we have

$$\sup_{x\in(a,\ell)} |(EI^{-1})'(x)| \le c_0^{-2}\|EI\|_{2,\infty}$$

and

$$\sup_{x\in(a,\ell)} |(EI^{-1})''(x)| \leq 3c_0^{-3}\|EI\|_{2,\infty}^2,$$

where

$$\|EI\|_{2,\infty} = \max_{0\leq i\leq 2}\sup_{a<x<\ell} |EI^{(i)}(x)|.$$

These estimates show that

$$\|EI^{-1}\|_{2,\infty} \leq c_0^{-1}\max\{1, c_0^{-1}\|EI\|_{2,\infty}, 3c_0^{-2}\|EI\|_{2,\infty}^2\} < \infty$$

and $EI^{-1} \in C_B^2(a,\ell)$. Thus $w'' \in H^2(a,\ell)$ by Theorem 2.8 and so $w \in H^4(a,\ell)$. Hence $w \in C^3[a,\ell]$ and

$$w(a) = w'(a) = w''(a) = w'''(a) = 0.$$

Since $0 < EI \in C^2[0,a]$, using a similar argument as above we also get $w \in H^4(0,a)$ and $w \in C^3[0,a]$.

Next, on the interval $(0,a)$ we consider the ordinary differential equation $(EIw'')'' - \beta^2\rho w = 0$ with initial conditions $w(a) = w'(a) = w''(a) = w'''(a) = 0$. Since $\rho, EI \in C^2[0,a]$, this differential equation has a unique solution and the initial conditions at the point a suggest that $w = 0$ on $(0,a)$. Combining this with the previous result and the fact that $w \in C^1[0,\ell]$ give $w = 0$ on $[0,\ell]$. Therefore the unique continuation condition holds. □

The previous two lemmas that we have just proved implies that if hypothesis (H2) and $k_1, k_2, k_3, k_4 > 0$ are satisfied then $i\mathbb{R} \subset \rho(\mathcal{A}_0)$. Now we are ready to state the main result of this chapter.

Theorem 4.9. *Assume that all the feedback coefficients k_1, k_2, k_3, k_4 are positive. If hypothesis* (H2) *holds, then the C_0-semigroup of contractions $S(t) = e^{t\mathcal{A}_0}$ generated by $\mathcal{A}_0$ is exponentially stable, that is, there exist constants $M \geq 1$ and $\alpha > 0$ such that*

$$\|e^{t\mathcal{A}_0}\|_{\mathcal{L}(\mathcal{H}_0)} \leq Me^{-\alpha t},$$

for all $t \geq 0$.

4.3.2 Proof of the Main Theorem

Now, we give the proof of Theorem 4.9. We use a contradiction argument to prove this theorem. Assume that the C_0-semigroup of contractions $S(t) = e^{t\mathcal{A}_0}$ generated by $\mathcal{A}_0$ is not exponentially stable. Then by the Resonance Theorem and Theorem 2.22, we can find a sequence $\{\beta_n\}_{n=1}^\infty$ of real numbers with the property that $|\beta_n| \to \infty$ as $n \to \infty$ and a sequence of unit vectors $\{z_n\}_{n=1}^\infty \subset \mathcal{D}(\mathcal{A}_0)$ such that

$$\lim_{n\to\infty} \|(i\beta_n I - \mathcal{A}_0)z_n\|_{\mathcal{H}_0} = 0.$$

For each $n \in \mathbb{N}$, let $z_n = [w_n \ v_n]^\mathsf{T}$ and $(i\beta_n I - \mathcal{A}_0)z_n = h_n = [f_n \ g_n]^\mathsf{T}$. Then we have the system

$$\begin{aligned} f_n &= i\beta_n w_n - v_n && (4.24)\\ g_n &= i\beta_n v_n + \frac{1}{\rho}(EIw_n'' + c_D I v_n'')'' + \frac{c_a v_n}{\rho}, && (4.25)\end{aligned}$$

and $f_n \to 0$ in W and $g_n \to 0$ in H as $n \to \infty$. Since

$$\|f_n\|_W^2 = k_1|f_n'(\ell)|^2 + k_3|f_n(\ell)|^2 + \int_0^\ell EI|f_n''|^2 \,\mathrm{d}x,$$

$k_1, k_3 > 0$, $EI \geq c_0 > 0$ and $\|f_n\|_W \to 0$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} f_n'(\ell) = \lim_{n\to\infty} f_n(\ell) = 0. \tag{4.26}$$

Because

$$\langle f_n, w_n\rangle_W = i\beta_n\|w_n\|_W^2 - \langle v_n, w_n\rangle_W \to 0$$

as $n \to \infty$ by Cauchy-Schwartz inequality, by taking the real part we obtain

$$\lim_{n\to\infty} \mathrm{Re}\,\langle v_n, w_n\rangle_W = \lim_{n\to\infty} \mathrm{Re}\,\langle w_n, v_n\rangle_W = 0$$

and taking the imaginary part gives us

$$\lim_{n\to\infty} (\beta_n\|w_n\|_W^2 + \mathrm{Im}\,\langle w_n, v_n\rangle_W) = 0.$$

Since $|\beta_n| \to \infty$ as $n \to \infty$, then for some $N \in \mathbb{N}$ we have $|\beta_n| > 0$ for all $n \geq N$.

Dividing by β_n for $n \geq N$ and letting n tend to infinity give us

$$\lim_{n\to\infty}\left(\|w_n\|_W^2 + \frac{1}{\beta_n}\mathrm{Im}\,\langle w_n, v_n\rangle_W\right) = \lim_{n\to\infty}\left(\frac{\beta_n\|w_n\|_W^2 + \mathrm{Im}\,\langle w_n, v_n\rangle_W}{\beta_n}\right) = 0. \tag{4.27}$$

The inner product of g_n and v_n in H is given by

$$\begin{aligned} \langle g_n, v_n\rangle_H &= \left\langle i\beta_n v_n + \frac{1}{\rho}(EIw_n'' + c_D I v_n'')'' + \frac{c_a v_n}{\rho}, v_n\right\rangle_H\\ &= i\beta_n\|v_n\|_H^2 + \langle (EIw_n'' + c_D I v_n'')'' + c_a v_n, v_n\rangle_{L^2(0,\ell)}\\ &= i\beta_n\|v_n\|_H^2 + \overline{\langle v_n, w_n\rangle_W} + k_2|v_n'(\ell)|^2 + k_4|v_n(\ell)|^2 + \int_0^\ell c_a|v_n|^2\,\mathrm{d}x\\ &\quad + \int_0^a D|v_n''|^2\,\mathrm{d}x. \end{aligned} \tag{4.28}$$

From the equality $(i\beta_n I - \mathcal{A}_0)z_n = h_n$ we get $\langle \mathcal{A}_0 z_n, z_n\rangle_{\mathcal{H}_0} = i\beta_n - \langle h_n, z_n\rangle_{\mathcal{H}_0}$. Hence $\mathrm{Re}\,\langle \mathcal{A}_0 z_n, z_n\rangle_{\mathcal{H}_0} = -\mathrm{Re}\,\langle h_n, z_n\rangle_{\mathcal{H}_0}$. By the Cauchy-Schwartz inequality $\langle h_n, z_n\rangle_{\mathcal{H}_0} \to 0$ as $n \to \infty$ and hence $\mathrm{Re}\,\langle \mathcal{A}_0 z_n, z_n\rangle_{\mathcal{H}_0} \to 0$ as $n \to \infty$. But

$$\mathrm{Re}\,\langle \mathcal{A}_0 z_n, z_n\rangle_{\mathcal{H}_0} = -k_2|v_n'(\ell)|^2 - k_4|v_n(\ell)|^2 - \int_0^\ell c_a|v|^2\,\mathrm{d}x - \int_0^a D|v_n''|^2\,\mathrm{d}x.$$

Because $k_2, k_4 > 0$, $c_a(x) \geq 0$ for all $x \in [0,\ell]$ and $D(x) > 0$ for all $x \in (0,a)$ we have

$$\lim_{n\to\infty} v_n'(\ell) = \lim_{n\to\infty} v_n(\ell) = 0 \tag{4.29}$$

and

$$\lim_{n\to\infty} \int_0^a D|v_n''|^2\,\mathrm{d}x = 0. \tag{4.30}$$

From (4.26) and (4.29) we have the limits

$$\begin{aligned} \lim_{n\to\infty} \beta_n w_n(\ell) &= -i\lim_{n\to\infty}(f_n(\ell) + v_n(\ell)) = 0, \\ \lim_{n\to\infty} \beta_n w_n'(\ell) &= -i\lim_{n\to\infty}(f_n'(\ell) + v_n'(\ell)) = 0, \end{aligned}$$

and dividing by β_n for $n \geq N$ we get

$$\lim_{n\to\infty} w_n(\ell) = \lim_{n\to\infty} w_n'(\ell) = 0. \tag{4.31}$$

The boundary conditions at the free end together with (4.29) and (4.31) imply that

$$\lim_{n\to\infty} w_n''(\ell) = \frac{1}{EI(\ell)} \lim_{n\to\infty}(k_1 w_n'(\ell) + k_2 v_n'(\ell)) = 0 \tag{4.32}$$

and

$$\lim_{n\to\infty}(EI(\ell)w_n''(\ell))' = \lim_{n\to\infty}(k_3 w_n(\ell) + k_4 v_n(\ell)) = 0. \tag{4.33}$$

On the other hand, from (4.28),

$$\lim_{n\to\infty}(\beta_n\|v_n\|_H^2 + \mathrm{Im}\,\langle w_n, v_n\rangle_W) = \lim_{n\to\infty} \mathrm{Im}\,\langle g_n, v_n\rangle_H = 0$$

and dividing by β_n for $n \geq N$ and letting $n \to \infty$ we have

$$\lim_{n\to\infty}\left(\|v_n\|_H^2 + \frac{1}{\beta_n}\mathrm{Im}\,\langle w_n, v_n\rangle_W\right) = 0. \tag{4.34}$$

Subtracting Equation (4.34) from Equation (4.27) gives us the limit

$$\lim_{n\to\infty}(\|w_n\|_W^2 - \|v_n\|_H^2) = 0.$$

Let $\lim_{n\to\infty}\|v_n\|_H^2 = \gamma_0$. Since $\|w_n\|_W^2 + \|v_n\|_H^2 = \|z_n\|_{\mathcal{H}_0}^2 = 1$ it follows that $\|w_n\|_W^2 \to 1-\gamma_0$ as $n\to\infty$. Therefore we have $\gamma_0 = 1/2$ and

$$\lim_{n\to\infty}\|w_n\|_W^2 = \lim_{n\to\infty}\|v_n\|_H^2 = \frac{1}{2}. \tag{4.35}$$

Now, let us consider the sequence of functions $\{\beta_n w_n\}_{n=1}^{\infty}$. Although $\{\beta_n\}_{n=1}^{\infty}$ is unbounded we show that the limit of the H-norm of $\beta_n w_n$ exists as $n\to\infty$. First, let us observe that

$$\|\beta_n w_n\|_H^2 = \|f_n + v_n\|_H^2 = \|f_n\|_H^2 + 2\mathrm{Re}\,\langle f_n, v_n\rangle_H + \|v_n\|_H^2.$$

Since $\|v_n\|_H \le 1$ we have $|\langle f_n, v_n\rangle_H| \le \|f_n\|_H \to 0$ as $n\to\infty$. Thus

$$\lim_{n\to\infty}\|\beta_n w_n\|_H^2 = \lim_{n\to\infty}\|v_n\|_H^2 = \frac{1}{2}. \tag{4.36}$$

Because $D \ge c_0 > 0$ on the interval $(0,a)$ we have

$$\|v_n''\|_{L^2(0,a)} \le \frac{1}{c_0}\int_0^a D|v_n''|^2\,\mathrm{d}x \tag{4.37}$$

and this implies that $\|v_n''\|_{L^2(0,a)} \to 0$ as $n\to\infty$. Using the fact that $v_n(0) = v_n'(0) = 0$ we get

$$\|v_n\|_{H^2(0,a)} \le (a^4 + a^2 + 1)^{\frac{1}{2}}\|v_n''\|_{L^2(0,a)}.$$

Hence $\|v_n\|_{H^2(0,a)} \to 0$ as $n\to\infty$.

Integrating twice both sides of $\rho g_n = i\beta_n\rho v_n + (EIw_n'' + c_D I v_n'')'' + c_a v_n$ over the triangular region

$$\{(\xi,\tau)\,|\,x\le\xi\le\ell, \xi\le\tau\le\ell\},$$

where $x\in[0,\ell]$, we have

$$\begin{aligned}
\int_x^\ell\int_\xi^\ell \rho(\tau)g_n(\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi \;=\;& i\beta_n\int_x^\ell\int_\xi^\ell \rho(\tau)v_n(\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi + EI(x)w_n''(x) + c_D I(x)v_n''(x) \\
&+ [1\ (\ell-x)]\begin{bmatrix} k_1 w_n'(\ell) + k_2 v_n'(\ell) \\ k_3 w_n(\ell) + k_4 v_n(\ell)\end{bmatrix} \\
&+ \int_x^\ell\int_\xi^\ell c_a(\tau)v_n(\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi.
\end{aligned}$$

We introduce a new sequence $\{u_n\}_{n=1}^{\infty}$ of functions defined by

$$\begin{aligned}
u_n(x) \;=\;& \frac{1}{i\beta_n}\Bigg(-EI(x)w_n''(x) - c_D I(x)v_n''(x) + \int_x^\ell\int_\xi^\ell \rho(\tau)g_n(\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi \\
&- [1\ (\ell-x)]\begin{bmatrix} k_1 w_n'(\ell) + k_2 v_n'(\ell) \\ k_3 w_n(\ell) + k_4 v_n(\ell)\end{bmatrix} - \int_x^\ell\int_\xi^\ell c_a(\tau)v_n(\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi\Bigg).
\end{aligned}$$

Then the second derivative of u_n is given by

$$u_n'' = \frac{-(EIw_n'' + c_D I v_n'')'' - c_a v_n + \rho g_n}{i\beta_n} = \rho v_n.$$

We will characterize the sequence $\{u_n\}_{n=1}^{\infty}$ in various function spaces in the following claims.

Claim 1. As $n \to \infty$, $\beta_n u_n \to 0$ in $L^2(0,a)$.

Proof of Claim 1. Since $\|g_n\|_H \to 0$ and $\|v_n\|_{L^2(0,\ell)} \to 0$ as $n \to \infty$ we have

$$\left\| \int_x^\ell \int_\xi^\ell \rho(\tau) g_n(\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi \right\|_{L^2(0,a)} \leq \ell^{\frac{3}{2}} \|\rho\|_{L^2(0,\ell)} \|g_n\|_{L^2(0,\ell)} \to 0$$

and

$$\left\| \int_x^\ell \int_\xi^\ell c_a(\tau) v_n(\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi \right\|_{L^2(0,a)} \leq \ell^{\frac{3}{2}} \|c_a\|_{L^2(0,\ell)} \|v_n\|_{L^2(0,\ell)} \to 0$$

as $n \to \infty$. Also, Equations (4.29) and (4.31) give

$$\begin{aligned} \left\| [1 \ (\ell - x)] \begin{bmatrix} k_1 w_n'(\ell) + k_2 v_n'(\ell) \\ k_3 w_n(\ell) + k_4 v_n(\ell) \end{bmatrix} \right\|_{L^2(0,a)} &\leq (k_3 |w_n(\ell)| + k_4 |v_n(\ell)|) \|\ell - x\|_{L^2(0,a)} \\ &\quad + \sqrt{\ell}(k_1 |w_n'(\ell)| + k_2 |v_n'(\ell)|) \to 0, \end{aligned}$$

as $n \to \infty$. Therefore it remains to show that $EIw_n'' + c_D I v_n'' \to 0$ in $L^2(0,a)$ as $n \to \infty$. Since $w_n'' = (v_n'' + f_n'')/(i\beta_n)$ and $v_n'', f_n'' \to 0$ in $L^2(0,a)$ we have $w_n'' \to 0$ in $L^2(0,a)$. Using the inequality $\|EIw_n'' + c_D I v_n''\|_{L^2(0,a)} \leq C_0(\|w_n''\|_{L^2(0,a)} + \|v_n''\|_{L^2(0,a)})$ we get the desired limit $\|EIw_n'' + c_D I v_n''\|_{L^2(0,a)} \to 0$ as $n \to \infty$. This completes the proof of Claim 1.

Claim 2. As $n \to \infty$, $u_n \to 0$ in $H^4(0,a)$.

Proof of Claim 2. From Claim 1, since $|\beta_n| \to \infty$, we also have $u_n = (\beta_n u_n)/\beta_n \to 0$ in $L^2(0,a)$. Because $v_n \in H^2(0,a)$ and $\rho \in C^2[0,a]$, from Theorem 2.8 we have $\rho v_n \in H^2(0,a)$ and $\|\rho v_n\|_{H^2(0,a)} \leq 4\sqrt{3}\|v_n\|_{H^2(0,a)}\|\rho\|_{2,\infty}$. Since $\|v_n\|_{H^2(0,a)} \to 0$ we have $\|\rho v_n\|_{H^2(0,a)} \to 0$. In particular, $\|u_n^{(4)}\|_{L^2(0,a)} = \|(\rho v_n)''\|_{L^2(0,a)} \to 0$. Combining the above remarks we have

$$\|u_n\|_{L^2(0,a)}^2 + \|u_n^{(4)}\|_{L^2(0,a)}^2 \to 0$$

as $n \to \infty$ and Theorem 2.12 implies that $\|u_n\|_{H^4(0,a)} \to 0$.

Claim 3. As $n \to \infty$, $\sqrt{|\beta_n|}u_n \to 0$ in $H^2(0,a)$.

Proof of Claim 3. Using the interpolation inequality, specifically Theorem 2.13, we get

$$\begin{aligned} \|\sqrt{|\beta_n|}u_n\|_{H^2(0,a)} &\leq K\|\sqrt{|\beta_n|}u_n\|_{H^4(0,a)}^{1/2}\|\sqrt{|\beta_n|}u_n\|_{L^2(0,a)}^{1/2} \\ &\leq K\|u_n\|_{H^4(0,a)}^{1/2}\|\beta_n u_n\|_{L^2(0,a)}^{1/2}. \end{aligned}$$

This inequality together with Claim 1 and Claim 2 prove Claim 3.

Claim 4. For each $n \in \mathbb{N}$, $w_n \in H^4(a,\ell)$. Further we have

$$\lim_{n\to\infty} \beta_n w_n'(a) = \lim_{n\to\infty} \frac{(EIw_n'')(a^+)}{\sqrt{|\beta_n|}} = 0. \tag{4.38}$$

Proof of Claim 4. Since $w_n, v_n \in H^2(0,\ell)$, Sobolev's embedding theorem implies that $w_n, v_n \in C^1[0,\ell]$. By the triangle inequality we have

$$\begin{aligned} \|\beta_n w_n\|_{H^2(0,a)} &\leq \|f_n\|_{H^2(0,a)} + \|v_n\|_{H^2(0,a)} \\ &\leq \|f_n\|_{H^2(0,\ell)} + \|v_n\|_{H^2(0,a)} \\ &\leq \sqrt{C_\ell}\|f_n\|_W + \|v_n\|_{H^2(0,a)}. \end{aligned}$$

Thus

$$\lim_{n\to\infty} \|\beta_n w_n\|_{H^2(0,a)} = 0. \tag{4.39}$$

Hence $\beta_n w_n$ converges to 0 in $C^1[0,a]$ which implies that $\beta_n w_n(a) \to 0$ and $\beta_n w_n'(a) \to 0$ as $n \to \infty$. On the interval (a,ℓ) we have $EIw_n'' + c_D I v_n'' = EIw_n''$ and from the definition of the domain of $\mathcal{A}_0$ we have $EIw_n'' \in H^2(a,\ell)$. Because $EI \in C_B^2(a,\ell)$ and $EI \geq c_0 > 0$ it follows that $EI^{-1} \in C_B^2(a,\ell)$ and hence $w_n'' \in H^2(a,\ell)$ by Theorem 2.8. Therefore $w_n \in H^4(a,\ell)$.

Now note that $c_D I(a) = 0$ and so

$$\begin{aligned} (EIw_n'')(a^+) &= \int_a^\ell \int_\xi^\ell (\rho(\tau)g_n(\tau) - c_a(\tau)v_n(\tau))\,\mathrm{d}\tau\,\mathrm{d}\xi \\ &\quad - [1\ (\ell - a)]\begin{bmatrix} k_1 w_n'(\ell) + k_2 v_n'(\ell) \\ k_3 w_n(\ell) + k_4 v_n(\ell) \end{bmatrix} - i\beta_n u_n(a). \end{aligned}$$

An upper bound for the integral in the above equality is given by

$$\begin{aligned}\left|\int_a^\ell \int_\xi^\ell (\rho(\tau)g_n(\tau) - c_a(\tau)v_n(\tau))\,\mathrm{d}\tau\,\mathrm{d}\xi\right| &\leq \int_a^\ell \int_\xi^\ell |\rho(\tau)g_n(\tau) - c_a(\tau)v_n(\tau)|\,\mathrm{d}\tau\,\mathrm{d}\xi \\ &\leq \int_0^\ell \int_0^\ell |\rho(\tau)g_n(\tau) - c_a(\tau)v_n(\tau)|\,\mathrm{d}\tau\,\mathrm{d}\xi \\ &\leq \ell \int_0^\ell (\rho(\tau)|g_n(\tau)| + c_a(\tau)|v_n(\tau)|)\,\mathrm{d}\tau \\ &= \ell(\|\rho g_n\|_{L^1(0,\ell)} + \|c_a v_n\|_{L^1(0,\ell)}).\end{aligned}$$

Hölder's inequality gives us

$$\begin{aligned}\|\rho g_n\|_{L^1(0,\ell)} &\leq \|\rho\|_{L^2(0,\ell)}\|g_n\|_{L^2(0,\ell)} \\ &\leq c_0^{-\frac{1}{2}}\|\rho\|_{L^2(0,\ell)}\|g_n\|_H\end{aligned}$$

and

$$\|c_a v_n\|_{L^1(0,\ell)} \leq c_0^{-\frac{1}{2}}\|c_a\|_{L^2(0,\ell)}\|v_n\|_H.$$

Using these we have

$$\begin{aligned}\left|\frac{(EIw_n'')(a^+)}{\sqrt{|\beta_n|}}\right| &\leq \frac{\ell(\|\rho\|_{L^2(0,\ell)}\|g_n\|_H + \|c_a\|_{L^2(0,\ell)}\|v_n\|_H)}{\sqrt{c_0|\beta_n|}} \\ &+ \frac{k_1|w_n'(\ell)| + k_2|v_n'(\ell)| + (\ell-a)(k_3|w_n(\ell)| + k_4|v_n(\ell)|)}{\sqrt{|\beta_n|}} \\ &+ \sqrt{|\beta_n|}u_n(a).\end{aligned}$$

Since $\sqrt{|\beta_n|}u_n \to 0$ as $n \to \infty$ in $H^2(0,a)$ it follows that $\sqrt{|\beta_n|}u_n \to 0$ as $n \to \infty$ in $C^1[0,a]$ by the Sobolev embedding theorem. In particular, $\sqrt{|\beta_n|}u_n(a) \to 0$ as $n \to \infty$. This together with Equations (4.29), (4.31), (4.35) we obtain the second part of (4.38).

Note that from (4.24) we have $v_n = i\beta_n w_n - f_n$ and substituting this in (4.25) yields

$$\begin{aligned}g_n &= i\beta_n(i\beta_n w_n - f_n) + \rho^{-1}(EIw_n'' + c_D I v_n'')'' + \rho^{-1}c_a(i\beta_n w_n - f_n) \\ &= (i\rho^{-1}c_a\beta_n - \beta_n^2)w_n - (\rho^{-1}c_a + i\beta_n)f_n + \rho^{-1}(EIw_n'' + c_D I v_n'')''.\end{aligned}$$

Let us introduce the multiplier $q \in C^2[0,\ell]$ such that $q(a) = 0$. Such function will be constructed later. Therefore, if we take the real part of the inner product of the above equation with $\rho q w_n'$ in $L^2(a,\ell)$ and since $EIw_n'' + c_D I v_n'' = EIw_n''$ on (a,ℓ) we have

$$\begin{aligned}\operatorname{Re}\langle g_n, \rho q w_n'\rangle_{L^2(a,\ell)} &= \operatorname{Re}\left\langle (i\rho^{-1}c_a\beta_n - \beta_n^2)w_n, \rho q w_n'\right\rangle_{L^2(a,\ell)} \\ &+ \operatorname{Re}\left\langle -(\rho^{-1}c_a + i\beta_n)f_n, \rho q w_n'\right\rangle_{L^2(a,\ell)} \\ &+ \operatorname{Re}\langle (EIw_n'')'', q w_n'\rangle_{L^2(a,\ell)}.\end{aligned} \tag{4.40}$$

In what follows, we will study the convergence of each term of both sides of Equation (4.40) as $n \to \infty$.

Left Hand Side of Equation (4.40). Cauchy-Schwartz inequality gives us the following upper bound

$$\begin{aligned}\mathrm{Re}\,\langle g_n, \rho q w_n'\rangle_{L^2(a,\ell)} &\leq |\,\langle g_n, \rho q w_n'\rangle_{L^2(a,\ell)}\,| \\ &\leq \|g_n\|_{L^2(a,\ell)}\|\rho q w_n'\|_{L^2(a,\ell)} \\ &\leq C_0 Q_0 \|g_n\|_{L^2(0,\ell)}\|w_n'\|_{L^2(0,\ell)}\end{aligned}$$

where $Q_0 = \max_{x\in[0,\ell]}|q(x)|$. Using the limit $\|\beta_n w_n\|_H^2 \to 1/2$ as $n \to \infty$ we get $\|w_n\|_H \to 0$ and the equivalence of the H-norm and the L^2-norm implies $\|w_n\|_{L^2(0,\ell)} \to 0$ as $n \to \infty$. By the Gagliardo-Nirenberg inequality, there exist positive constants K_1 and K_2 such that

$$\|w_n'\|_{L^2(0,\ell)} \leq K_1 \|w_n''\|_{L^2(0,\ell)}^{1/2}\|w_n\|_{L^2(0,\ell)}^{1/2} + K_2\|w_n\|_{L^2(0,\ell)}.$$

Thus

$$\|w_n'\|_{L^2(0,\ell)} \leq K_1 c_0^{-\frac{1}{4}}\|w_n\|_W^{1/2}\|w_n\|_{L^2(0,\ell)}^{1/2} + K_2\|w_n\|_{L^2(0,\ell)}$$

and this implies that $\|w_n'\|_{L^2(0,\ell)} \to 0$ because $\|w_n\|_W^2 \to 1/2$ and $\|w_n\|_{L^2(0,\ell)} \to 0$. Since $\|g_n\|_H \to 0$ we also have $\|g_n\|_{L^2(0,\ell)} \to 0$. Therefore the left hand side of Equation (4.40) vanishes as n tends to infinity, that is,

$$\lim_{n\to\infty} \mathrm{Re}\,\langle g_n, \rho q w_n'\rangle_{L^2(a,\ell)} = 0. \tag{4.41}$$

First Term of Right Hand Side of Equation (4.40). For simplicity, we let $\alpha_n = \mathrm{Re}\; w_n$ and $\beta_n = \mathrm{Im}\; w_n$. A simple calculation gives us $w_n\overline{w_n'} = (\alpha_n\alpha_n' + \beta_n\beta_n') + i(\beta_n\alpha_n' - \alpha_n\beta_n')$. Thus

$$\begin{aligned}\mathrm{Re}\left\langle -\beta_n^2 w_n, \rho q w_n'\right\rangle_{L^2(a,\ell)} &= -\int_a^\ell \beta_n^2\rho q \mathrm{Re}(w_n\overline{w_n'})\,\mathrm{d}x \\ &= -\int_a^\ell \beta_n^2 \rho q(\alpha_n\alpha_n' + \beta_n\beta_n')\,\mathrm{d}x \\ &= -\beta_n^2\rho q\left(\frac{\alpha_n^2}{2} + \frac{\beta_n^2}{2}\right)\Big|_a^\ell + \int_a^\ell \beta_n^2(\rho q)'\left(\frac{\alpha_n^2}{2} + \frac{\beta_n^2}{2}\right)\mathrm{d}x \\ &= \frac{1}{2}(\rho(a)q(a)|\beta_n w_n(a)|^2 - \rho(\ell)q(\ell)|\beta_n w_n(\ell)|^2) \\ &\quad + \frac{1}{2}\int_a^\ell (\rho q)'|\beta_n w_n|^2\,\mathrm{d}x.\end{aligned}$$

Since the value of the multiplier q at a is 0 we have

$$\mathrm{Re}\left\langle -\beta_n^2 w_n, \rho q w_n' \right\rangle_{L^2(a,\ell)} = -\frac{1}{2}\rho(\ell)q(\ell)|\beta_n w_n(\ell)|^2 + \frac{1}{2}\int_a^\ell (\rho q)'|\beta_n w_n|^2 \,\mathrm{d}x.$$

Because $|\beta_n w_n(\ell)|^2 \to 0$ as $n \to \infty$, letting $n \to \infty$ in the above equality yields

$$\lim_{n\to\infty} \mathrm{Re}\left\langle -\beta_n^2 w_n, \rho q w_n' \right\rangle_{L^2(a,\ell)} = \lim_{n\to\infty} \frac{1}{2}\int_a^\ell (\rho q)'|\beta_n w_n|^2 \,\mathrm{d}x. \tag{4.42}$$

Now, let us observe that

$$\begin{array}{rcl} |\mathrm{Re}\left\langle i\rho^{-1}c_a\beta_n w_n, \rho q w_n' \right\rangle_{L^2(a,\ell)}| & \leq & |\left\langle ic_a\beta_n w_n, q w_n' \right\rangle_{L^2(a,\ell)}| \\ & = & |\left\langle c_a\beta_n w_n, q w_n' \right\rangle_{L^2(a,\ell)}| \\ & = & |\left\langle c_a q w_n', \beta_n w_n \right\rangle_{L^2(a,\ell)}| \\ & \leq & C_0 Q_0 \|w_n'\|_{L^2(a,\ell)} \|\beta_n w_n\|_{L^2(a,\ell)} \\ & \leq & \dfrac{C_0 Q_0}{\sqrt{c_0}} \|w_n'\|_{L^2(0,\ell)} \|\beta_n w_n\|_H \end{array}$$

Since $\|w'\|_{L^2(0,\ell)} \to 0$ and $\|\beta_n w_n\|_H \to 1/\sqrt{2}$, this estimate gives us

$$\lim_{n\to\infty} \mathrm{Re}\left\langle i\rho^{-1}c_a\beta_n w_n, \rho q w_n' \right\rangle_{L^2(a,\ell)} = 0. \tag{4.43}$$

For the first term of the right hand side of Equation (4.40) we have, by Equations (4.42) and (4.43),

$$\lim_{n\to\infty} \mathrm{Re}\left\langle (i\rho^{-1}c_a\beta_n - \beta_n^2) w_n, \rho q w_n' \right\rangle_{L^2(a,\ell)} = \lim_{n\to\infty} \frac{1}{2}\int_a^\ell (\rho q)'|\beta_n w_n|^2 \,\mathrm{d}x. \tag{4.44}$$

Second Term of Right Hand Side of Equation (4.40). Integration by parts and $q(a) = 0$ implies that

$$\begin{array}{rcl} \langle -i\beta_n f_n, \rho q w_n' \rangle_{L^2(a,\ell)} & = & -i\displaystyle\int_a^\ell \beta_n \rho q f_n \overline{w_n'} \,\mathrm{d}x \\ & = & -i\rho(\ell)q(\ell)f_n(\ell)\overline{\beta_n w_n(\ell)} + i\displaystyle\int_a^\ell (\rho q f_n)' \,\overline{\beta_n w_n} \,\mathrm{d}x. \end{array}$$

If we take the real part we get

$$\mathrm{Re}\,\langle -i\beta_n f_n, \rho q w_n' \rangle_{L^2(a,\ell)} = \rho(\ell)q(\ell)f_n(\ell)\mathrm{Im}(\beta_n w_n(\ell)) - \mathrm{Im}\int_a^\ell (\rho q f_n)' \,\overline{\beta_n w_n} \,\mathrm{d}x. \tag{4.45}$$

Since $\beta_n w_n(\ell) \to 0$, and $f_n(\ell) \to 0$ as $n \to \infty$ we have

$$\lim_{n\to\infty} \rho(\ell)q(\ell)f_n(\ell)\mathrm{Im}(\beta_n w_n(\ell)) = 0. \tag{4.46}$$

Let $M = \sup_{x\in(a,\ell)} |(\rho q)'(x)|$. We are sure that $M < \infty$ since $\rho \in C_B^2(a,\ell)$ and $q \in C^2[0,\ell]$. Moreover, the integral in Equation (4.45) satisfies the following inequalities

$$\begin{aligned}
\left|\mathrm{Im}\int_a^\ell (\rho q f_n)'\,\overline{\beta_n w_n}\,\mathrm{d}x\right| &\leq \left|\int_a^\ell (\rho q f_n)'\,\overline{\beta_n w_n}\,\mathrm{d}x\right| \\
&\leq |\left\langle (\rho q)' f_n, \beta_n w_n\right\rangle_{L^2(a,\ell)}| + |\left\langle (\rho q) f_n', \beta_n w_n\right\rangle_{L^2(a,\ell)}| \\
&\leq \|(\rho q)' f_n\|_{L^2(a,\ell)}\|\beta_n w_n\|_{L^2(a,\ell)} + \|(\rho q) f_n'\|_{L^2(a,\ell)}\|\beta_n w_n\|_{L^2(a,\ell)} \\
&\leq (M + C_0 Q_0)\|f\|_{H^2(a,\ell)}\|\beta_n w_n\|_{L^2(a,\ell)} \\
&\leq \frac{M + C_0 Q_0}{\sqrt{c_0}}\|f\|_{H^2(0,\ell)}\|\beta_n w_n\|_H.
\end{aligned}$$

Utilizing these, we obtain the limit

$$\lim_{n\to\infty} \mathrm{Im}\int_a^\ell (\rho q f_n)'\,\overline{\beta_n w_n}\,\mathrm{d}x = 0. \tag{4.47}$$

From Equations (4.45)−(4.47) we therefore have

$$\lim_{n\to\infty} \mathrm{Re}\left\langle -i\beta_n f_n, \rho q w_n'\right\rangle_{L^2(a,\ell)} = 0. \tag{4.48}$$

Similarly the estimate

$$\begin{aligned}
|\left\langle -\rho^{-1} c_a f_n, \rho q w_n'\right\rangle_{L^2(a,\ell)}| &\leq Q_0 C_0 \|f_n\|_{L^2(a,\ell)}\|w_n'\|_{L^2(a,\ell)} \\
&\leq Q_0 C_0 \|f_n\|_{H^2(0,\ell)}\|w_n'\|_{L^2(0,\ell)}
\end{aligned}$$

implies that

$$\lim_{n\to\infty} \left\langle -\rho^{-1} c_a f_n, \rho q w_n'\right\rangle_{L^2(a,\ell)} = 0. \tag{4.49}$$

It follows from Equations (4.48) and (4.49) that

$$\lim_{n\to\infty} \mathrm{Re}\left\langle -(\rho^{-1} c_a + i\beta_n) f_n, \rho q w_n'\right\rangle_{L^2(a,\ell)} = 0, \tag{4.50}$$

that is, the second term of the right hand side of (4.40) vanishes as n increases.

Third Term of Right Hand Side of Equation (4.40). Integration by parts and expanding $(q\overline{w_n'})' = q'\overline{w_n'} + q\overline{w_n''}$ give us

$$\begin{aligned}
\langle (EIw_n'')'', qw_n' \rangle_{L^2(a,\ell)} &= \int_a^\ell (EIw_n'')'' q\overline{w_n'}\,\mathrm{d}x \\
&= (EIw_n'')'(\ell)q(\ell)\overline{w_n'(\ell)} - \int_a^\ell (EIw_n'')'(q\overline{w_n'})'\,\mathrm{d}x \\
&= (EIw_n'')'(\ell)q(\ell)\overline{w_n'(\ell)} - \int_a^\ell (EIw_n'')'q'\overline{w_n'}\,\mathrm{d}x \\
&\quad - \int_a^\ell (EIw_n'')'q\overline{w_n''}\,\mathrm{d}x.
\end{aligned}$$

We will rewrite the above two integrals of the last expression. For the first integral, integration by parts yields

$$\begin{aligned}
\int_a^\ell (EIw_n'')'q'\overline{w_n'}\,\mathrm{d}x &= (EIw_n'')(\ell)q'(\ell)\overline{w_n'(\ell)} - (EIw_n'')(a^+)q'(a)\overline{w_n'(a)} \\
&\quad - \int_a^\ell EIw_n''(q'\overline{w_n'})'\,\mathrm{d}x \\
&= (EIw_n'')(\ell)q'(\ell)\overline{w_n'(\ell)} - (EIw_n'')(a^+)q'(a)\overline{w_n'(a)} \\
&\quad - \int_a^\ell EIq''w_n''\overline{w_n'}\,\mathrm{d}x - \int_a^\ell EIq'|w_n''|^2\,\mathrm{d}x.
\end{aligned}$$

For the second integral, we expand $(EIw_n'')' = EI'w_n'' + EIw_n'''$ to get

$$\begin{aligned}
\int_a^\ell (EIw_n'')'q\overline{w_n''}\,\mathrm{d}x &= \int_a^\ell (EI'w_n'' + EIw_n''')q\overline{w_n''}\,\mathrm{d}x \\
&= \int_a^\ell EI'q|w_n''|^2\,\mathrm{d}x + \int_a^\ell EIqw_n'''\,\overline{w_n''}\,\mathrm{d}x.
\end{aligned}$$

If we take the real part we obtain

$$\begin{aligned}
\mathrm{Re}\int_a^\ell (EIw_n'')'q\overline{w_n''}\,\mathrm{d}x &= \int_a^\ell EI'q|w_n''|^2\,\mathrm{d}x + \int_a^\ell EIq\mathrm{Re}(w_n'''\,\overline{w_n''})\,\mathrm{d}x \\
&= \int_a^\ell EI'q|w_n''|^2\,\mathrm{d}x + \frac{EI(\ell)q(\ell)|w_n''(\ell)|^2}{2} - \frac{1}{2}\int_a^\ell (EIq)'|w_n''|^2\,\mathrm{d}x \\
&= \frac{EI(\ell)q(\ell)|w_n''(\ell)|^2}{2} + \frac{1}{2}\int_a^\ell EI'q|w_n''|^2\,\mathrm{d}x - \frac{1}{2}\int_a^\ell EIq'|w_n''|^2\,\mathrm{d}x
\end{aligned}$$

Combining these expressions for the two integrals that we have obtained it follows that

$$\begin{aligned}\operatorname{Re}\langle (EIw_n'')'', qw_n'\rangle_{L^2(a,\ell)} &= \operatorname{Re}\left[((EIw_n'')'(\ell)q(\ell) - (EIw_n'')(\ell)q'(\ell))\,\overline{w_n'(\ell)}\right] \\ &+ \operatorname{Re}\left[(EIw_n'')(a^+)q'(a)\overline{w_n'(a)}\right] - \frac{EI(\ell)q(\ell)|w_n''(\ell)|^2}{2} \\ &+ \frac{1}{2}\int_a^\ell (3EIq' - EI'q)|w_n''|^2\,\mathrm{d}x + \operatorname{Re}\int_a^\ell EIq''w_n''\overline{w_n'}\,\mathrm{d}x.\end{aligned}$$

Note that we can rewrite the second term of the right hand side of the above equality as

$$\operatorname{Re}\left[(EIw_n'')(a^+)q'(a)\overline{w_n'(a)}\right] = \operatorname{Re}\left(\frac{q'(a)}{\sqrt{|\beta_n|}}\frac{(EIw_n'')(a^+)}{\sqrt{|\beta_n|}}\overline{|\beta_n|w_n'(a)}\right). \tag{4.51}$$

From Equations (4.31), (4.32) and (4.33) we have

$$\lim_{n\to\infty}\operatorname{Re}\left[((EIw_n'')'(\ell)q(\ell) - (EIw_n'')(\ell)q'(\ell))\,\overline{w_n'(\ell)}\right] = 0$$

and

$$\lim_{n\to\infty}\frac{EI(\ell)q(\ell)|w_n''(\ell)|^2}{2} = 0.$$

Now, (4.38), (4.51), and $|\beta_n| \to \infty$ implies that

$$\lim_{n\to\infty}\operatorname{Re}\left[(EIw_n'')(a^+)q'(a)\overline{w_n'(a)}\right] = 0. \tag{4.52}$$

Next, let us consider the last term of $\operatorname{Re}\langle (EIw_n'')'', qw_n'\rangle_{L^2(a,\ell)}$. If $Q_2 = \max_{x\in[0,\ell]}|q''(x)|$ then we have the following estimate

$$\begin{aligned}\left|\operatorname{Re}\int_a^\ell EIq''w_n''\overline{w_n'}\,\mathrm{d}x\right| &\leq |\langle EIq''w_n'', w_n'\rangle_{L^2(a,\ell)}| \\ &\leq \|EIq''w_n''\|_{L^2(a,\ell)}\|w_n'\|_{L^2(a,\ell)} \\ &\leq C_0Q_2\|w_n''\|_{L^2(a,\ell)}\|w_n'\|_{L^2(a,\ell)} \\ &\leq C_0Q_2\|w_n\|_{H^2(0,\ell)}\|w_n'\|_{L^2(0,\ell)}.\end{aligned}$$

Hence, this estimate, together with $\|w_n\|_{H^2(0,\ell)} \to 0$ and $\|w_n'\|_{L^2(0,\ell)} \to 0$, gives us

$$\lim_{n\to\infty}\operatorname{Re}\int_a^\ell EIq''w_n''\overline{w_n'}\,\mathrm{d}x = 0. \tag{4.53}$$

Thus, the third term of the right hand side of Equation (4.40) satisfies

$$\lim_{n\to\infty}\operatorname{Re}\langle (EIw_n'')'', qw_n'\rangle_{L^2(a,\ell)} = \lim_{n\to\infty}\frac{1}{2}\int_a^\ell (3EIq' - EI'q)|w_n''|^2\,\mathrm{d}x.$$

Consequently, from (4.41), (4.44), (4.50), (4.52) and (4.53) we have

$$\lim_{n\to\infty}\left[\int_a^\ell (\rho q)'|\beta_n w_n|^2\,\mathrm{d}x + \int_a^\ell (3EIq' - EI'q)|w_n''|^2\,\mathrm{d}x\right] = 0. \tag{4.54}$$

As promised, we will now construct a function q having the stated properties. This construction is provided on the next claim.

Claim 5. There exists a function $q \in C^2[0,\ell]$ such that $q(a) = 0$, $(\rho q)' > 1$ and $3EIq' - EI'q > 0$ in (a,ℓ).

Proof of Claim 5. Consider the function $q : [0,\ell] \to \mathbb{R}$ defined by $q(x) = e^{\nu(x-a)} - 1$, where

$$0 < \max\left\{\frac{2\|\rho\|_{2,\infty}+1}{c_0}, \frac{2\|EI\|_{2,\infty}}{3c_0}\right\} < \nu < \infty,$$

with

$$\|\rho\|_{2,\infty} = \max_{0\le i\le 2}\sup_{x\in(a,\ell)} |\rho^{(i)}(x)|$$

and

$$\|EI\|_{2,\infty} = \max_{0\le i\le 2}\sup_{x\in(a,\ell)} |EI^{(i)}(x)|.$$

It is clear that $q \in C^2[0,\ell]$ and $q(a) = 0$. From the definition of ν it follows that

$$\nu c_0 - \|\rho\|_{2,\infty} > 1 + \|\rho\|_{2,\infty} > 0.$$

Also, if $x > a$ then $e^{\nu(x-a)} > 1$ and so

$$(\nu c_0 - \|\rho\|_{2,\infty})e^{\nu(x-a)} > \nu c_0 - \|\rho\|_{2,\infty}. \tag{4.55}$$

If $x \in (a,\ell)$ then from (4.55) we have

$$\begin{aligned}
(\rho q)'(x) &= \nu\rho(x)e^{\nu(x-a)} + \rho'(x)(e^{\nu(x-a)} - 1)\\
&= (\nu\rho(x) + \rho'(x))e^{\nu(x-a)} - \rho'(x)\\
&\ge (\nu c_0 - \|\rho\|_{2,\infty})e^{\nu(x-a)} - \|\rho\|_{2,\infty}\\
&> \nu c_0 - 2\|\rho\|_{2,\infty}\\
&> 1.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
3EI(x)q'(x) - EI'(x)q(x) &= 3\nu EI(x)e^{\nu(x-a)} - EI'(x)(e^{\nu(x-a)} - 1)\\
&= (3\nu EI(x) - EI'(x))e^{\nu(x-a)} + EI'(x)\\
&\ge (3\nu c_0 - \|EI\|_{2,\infty})e^{\nu(x-a)} - \|EI\|_{2,\infty}\\
&> 3\nu c_0 - 2\|EI\|_{2,\infty}\\
&> 0,
\end{aligned}$$

which are the desired inequalities.

Using the properties of the multiplier q stated in the above claim, we obtain the inequality

$$\int_a^\ell |\beta_n w_n|^2 \,\mathrm{d}x \quad \leq \quad \int_a^\ell (\rho q)'|\beta_n w_n|^2 \,\mathrm{d}x + \int_a^\ell (3EIq' - EI'q)|w_n''|^2 \,\mathrm{d}x$$

and from Equation (4.54) we have

$$\lim_{n\to\infty} \|\beta_n w_n\|_{L^2(a,\ell)} = 0. \tag{4.56}$$

From Equation (4.39) we have $\|\beta_n w_n\|_{L^2(0,a)} \to 0$ and combining this with Equation (4.56) we get $\|\beta_n w_n\|_{L^2(0,\ell)} \to 0$. Moreover, the equivalence of the H-norm and the L^2-norm implies that $\|\beta_n w_n\|_H \to 0$ as $n \to \infty$, which contradicts Equation (4.36). Therefore, the C_0-semigroup of contractions $S(t) = e^{t\mathcal{A}_0}$ generated by $\mathcal{A}_0$ must be exponentially stable. This completes the proof of Theorem 4.9.

Bibliography

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] H. T. Banks and R. C. H. del Rosario, *Convergence of approximations in feedback control of structures*, Mathematical and Computer Modeling, 33 (2001), pp. 65–78.

[3] H. T. Banks, R. C. Smith, and Y. Wang, *Smart Material Structures: Modeling, Estimation and Control*, John Wiley & Sons, 1996.

[4] L. Berrahmoune, *Stabilization of beams with nonlinear feedback*, SIAM J. Control Optim., 41 (2002), pp. 1166–1193.

[5] J. Cagnol and J. P. Zolésio, *Control and Boundary Analysis*, Marcel Dekker/Taylor & Francis Ltd., 2004.

[6] G. Chen, S. A. Fulling, F. J. Narcowich, and S. Sun, *Exponential decay of energy of evolution equations with locally distributed damping*, SIAM J. Appl. Math., 51 (1991), pp. 266–301.

[7] S. Chen, K. Liu, and Z. Liu, *Spectrum and stability for elastic systems with global or local Kelvin–Voigt damping*, SIAM J. Appl. Math., 59 (1998), pp. 651–668.

[8] R. C. H. del Rosario, *Convergence of approximations in feedback control of structures*, NSRI Technical Report, University of the Philippines, 2001.

[9] ——, *On the stability of the semigroup associated with a non-homogeneous flexible beam with boundary damping*, Matimyas Matematika, 30 (2007), pp. 1–12.

[10] K. J. Engel and R. Nagel, *One–Parameter Semigroups for Linear Evolution Equations*, Springer–Verlag, New York, 2000.

[11] L. M. Gearhart, *Spectral theory for contraction semigroups on Hilbert space*, Trans. Amer. Math. Soc., 236 (1978), pp. 385–394.

[12] G. M. L. Gladwell, *Inverse Problems in Vibration*, Springer, 2nd ed., 2004.

[13] B. Guo, *Riesz basis approach to the stabilization of a flexible beam with a tip mass*, SIAM J. Control Optim., 39 (2001), pp. 1736–1747.

[14] ——, *Riesz basis property and exponential stability of controlled Euler-Bernoulli beam equations with variable coefficients*, SIAM J. Control Optim., 40 (2002), pp. 1905–1923.

[15] B. Guo and Q. Song, *Tracking control of a flexible beam by nonlinear boundary feedback*, Journal of Applied Mathematics and Stochastic Analysis, 8 (1995), pp. 47–58.

[16] B. Guo, J. Wang, and S. Yung, *On the C_0-semigroup generation and exponential stability resulting from a shear force feedback on a rotating beam*, Systems & Control Letters, Elsevier, 54 (2005), pp. 557–574.

[17] F. Guo and F. Huang, *Boundary feedback stabilization of the undamped Euler–Bernoulli beam with both ends free*, SIAM J. Control Optim., 43 (2004), pp. 341–356.

[18] E. Hille, *Functional Analysis and Semi-Groups*, Colloq. Publ. Amer. Math. Soc., 1948.

[19] F. L. Huang, *Characteristic condition for exponential stability of linear dynamical systems in Hilbert spaces*, Ann. Differential Equations, 1 (1985), pp. 43–56.

[20] K. Ito and F. Kappel, *Evolution Equations and Approximations*, World Scientific, River Edge, New Jersey, 2002.

[21] J. Jurgen, *Postmodern Analysis*, Springer-Verlag, 2005.

[22] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 2nd ed., 1980.

[23] P. Le Gall, C. Prieur, and L. Rosier, *Output feedback stabilization of a clamped-free beam*, International Journal of Control, 80 (2007), pp. 1201–1216.

[24] K. Liu, *Locally distributed control and damping for the conservative system*, SIAM J. Control Optim., 35 (1997), pp. 1574–1590.

[25] K. Liu and Z. Liu, *Exponential decay of energy of the Euler-Bernoulli beam with locally distributed Kelvin-Voigt damping*, SIAM J. Control Optim., 36 (1998), pp. 1086–1098.

[26] K. Liu, Z. Liu, and B. Rao, *Exponential stability of an abstract nondissipative linear system*, SIAM J. Control Optim., 40 (2001), pp. 149–165.

[27] Z. Liu and S. Zheng, *Uniform exponential stability of approximations in linear viscoelasticity*, Journal of Mathematical Systems, Estimation, and Control, 8 (1998), pp. 1–17.

[28] ——, *Semigroups Associated with Dissipative Systems*, Chapman and Hall/CRC, Washington D.C., 1999.

[29] M. Miklavčič, *Applied Functional Analysis and Partial Differential Equations*, World Scientific, Singapore, 1998.

[30] J. C. Oostveen, *Strongly Stabilizable Infinite-Dimensional Systems*, PhD thesis, University of Groningen, 1999. Published by SIAM, ISBN 90-367-1018-9.

[31] J. C. Oostveen, R. F. Curtain, and K. Ito, *An approximation theory for strongly stabilizing solutions to the operator LQ Riccati equations*, SIAM J. Control Optim., 38 (2000), pp. 1909–1937.

[32] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[33] J. Prüss, *On the spectrum of C_0-semigroups*, Trans. Amer. Math. Soc., 284 (1984), pp. 847–857.

[34] M. Renardy and R. C. Roberts, *An Introduction to Partial Differential Equations*, Springer-Verlag, New York, 2nd ed., 2004.

[35] F. G. Tricomi, *Integral Equations*, Interscience, New York, 1957.

[36] M. Tucsnak and G. Weiss, *Observation and Control for Operator Semigroups*, Birkhäuser Verlag, Germany, 2009.

[37] K. Yosida, *On the differentiability and the representation of one-parameter semigroups of linear operators*, J. Math. Soc. Japan, 1 (1948), pp. 15–21.

[38] ——, *Functional Analysis*, Springer-Verlag, New York, 6th ed., 1980.

[39] H. L. Zhao, K. Liu, and C. G. Zhang, *Stability for the Timoshenko beam system with local Kelvin-Voigt damping*, Acta Mathematica Sinica, English Series, 21 (2005), pp. 655–666.