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A fluid–structure interaction model with interior damping and delay in the structure

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Abstract. A coupled system of partial differential equations modeling the interaction of a fluid and a structure with delay in the feedback is studied. The model describes the dynamics of an elastic body immersed in a fluid that is contained in a vessel, whose boundary is made of a solid wall. The fluid component is modeled by the linearized Navier-Stokes equation, while the solid component is given by the wave equation neglecting transverse elastic force. Spectral properties and exponential or strong stability of the interaction model under appropriate conditions on the damping factor, delay factor and the delay parameter are established using a generalized Lax-Milgram method.

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Keywords. Fluid–structure model · Feedback delay · Stability · Generalized Lax-Milgram method.

1. Introduction

Consider an elastic body occupying a bounded domain $\Omega_s \subset \mathbb{R}^d$, where $d = 2$ or $d = 3$, and it is immersed in a fluid that is contained in a vessel. Suppose that the boundary Γ_f of the vessel is made of a solid wall. We denote by $\Omega_f \subset \mathbb{R}^d$ the region where the fluid is occupied and Γ_s the interface between the solid and the fluid. All throughout this paper, we assume that Γ_s and Γ_f are sufficiently smooth and that $\overline{\Gamma}_s \cap \overline{\Gamma}_f = \emptyset$. Let $u : (0,\infty) \times \Omega_f \to \mathbb{R}^d$, $p : (0,\infty) \times \Omega_f \to \mathbb{R}$ and $w : (0,\infty) \times \Omega_s \to \mathbb{R}^d$ represent the velocity field of the fluid, the pressure in the fluid and the displacement of the structure, respectively. A linear model describing the interaction of the fluid and the structure is given by the coupled linearized incompressible Navier-Stokes wave system

$$\begin{cases} u_t(t,x) - \Delta u(t,x) + \nabla p(t,x) = 0, & \text{in } (0,\infty) \times \Omega_f, \\ \operatorname{div} u(t,x) = 0, & \text{in } (0,\infty) \times \Omega_f, \\ u(t,x) = 0, & \text{on } (0,\infty) \times \Gamma_f, \\ u(t,x) = w_t(t,x), & \text{on } (0,\infty) \times \Gamma_s, \\ w_{tt}(t,x) - \Delta w(t,x) = F(t,x), & \text{in } (0,\infty) \times \Omega_s, \\ \dfrac{\partial w}{\partial \nu}(t,x) = \dfrac{\partial u}{\partial \nu}(t,x) - p(t,x)\nu(x), & \text{on } (0,\infty) \times \Gamma_s, \\ u(0,x) = u_0(x), & \text{in } \Omega_f, \\ w(0,x) = w_0(x),\ w_t(0,x) = w_1(x), & \text{in } \Omega_s. \end{cases} \tag{1.1}$$

Here, F can be viewed as a source or control on the structure. The unit vector ν is outward normal to the fluid domain Ω_f, and hence, it will be inward to the structure domain Ω_s. In this model, the boundary of the solid is stationary and as mentioned in [8], this assumption is suitable under small and rapid oscillations, that is, when the displacement of the solid is small compared to its velocity. The boundary conditions on the interface Γ_s represent the continuity of the velocities and stresses for the fluid and solid components. On the other hand, on Γ_f we have the no-slip boundary condition.

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In this paper, we study the system (1.1) using the velocity of the structure as the feedback law

$$F(t,x) = -k_0 w_t(t-\tau,x) - k_1 w_t(t,x), \quad \text{in } (0,\infty)\times\Omega_s, \tag{1.2}$$

where $k_1 > 0$ is the damping factor, $k_0 > 0$ is the delay factor, and $\tau > 0$ is a constant delay. Physically, this means that a fraction of the feedback will be felt by the system after some time and the constant τ signifies the extent of the delay. The constants k_1 and k_0 quantify the strengths of damping and delay in the feedback, respectively. The initial history for the velocity of the structure is denoted by

$$w_t(\theta,x) = g(\theta,x), \quad \text{in } (-\tau,0)\times\Omega_s. \tag{1.3}$$

Recent interests in fluid–structure models include numerical and experimental studies, and lately, there are works that lean toward rigorous mathematical analysis. The model (1.1) is based on the works of Avalos and Trigianni [3,6]. Their system is similar to the one considered earlier by Du et al. [14]. Nonlinear versions have been also considered by Barbu et al. [8,9], Lasiecka and Lu [17,18] and Lu [22,23]. Without any external force F and with transversal elastic force in the wave component, i.e., with the wave equation $w_{tt} - \Delta w + w = 0$, it was shown in [3] using semigroup methods that the solutions of (1.1) are strongly asymptotically stable. The result holds for every initial data in the state space excluding those that lie in the kernel of the associated generator and also under additional conditions, which is related to the geometry of the structure. It relies on whether a certain over-determined boundary value problem has a solution. This hypothesis, named condition (H) below, will be also utilized in this study. Later, the authors studied the same model in [6] but with internal damping in the structure. This additional dissipative mechanism allows the energy of the solution to decay to zero exponentially.

Systems that are stable may turn into an unstable one if there is delay, see for example the classical works of Datko et al. [11] and Datko [10]. This is because delay induces a transport phenomena in the system that generate oscillations which may lead into instability. Since then, several authors studied the effect of delay in various multidimensional wave equations and as well in heat and Schrödinger equations. In the absence of the fluid and with homogeneous Dirichlet condition on a part of the boundary, the stability and instability properties of the wave equation with the feedback law (1.2) was considered by Nicaise and Pignotti [24]. It is shown in their work that if the damping factor is larger than the delay factor, then the energy of the system decays to zero exponentially. On the other hand, if these coefficients are equal, it was established that there is a sequence of delays that yield solutions with constant energies. Even when the damping and delay factors are equal, the presence of other dissipative mechanisms such as viscoelasticity can provide asymptotic stability for the wave equation, see for example the work of Kirane and Said-Houari [16]. We would like to extend the study to the fluid–structure model (1.1)–(1.3) and analyze for the influence of the fluid on the wave equation using the framework and methods in [3].

Due to the absence of the displacement term, the wave equation will be formulated as a first-order system in terms of the velocity w_t and stress ∇w, in contrast to the formulation in terms of the displacement and velocity in [3]. The same first-order formulation has been used for wave equations with viscoelasticity by Desch et al. [12] and for fluid–structure models in [17,18,22,23], where the works [22,23] were based on the earlier paper by Lasiecka and Seidman [19] that deals with the stabilization of a structural acoustics model. Moreover, this is also a familiar way of writing the multidimensional wave equation in the entire space as a first-order symmetric hyperbolic system. The basis for this particular setup stems from the fact that the energy contains only the L^2-norm of the gradient of the structure's displacement. Nevertheless, the displacement can be recovered by integrating the velocity with respect to time.

The said formulation requires a different state space representation of the interaction model and leads to a different structure on the kernel of the corresponding generator, the space of steady states, and different analysis and tools will come in place. The construction of the semigroup and the well-posedness for (1.1)–(1.3) will be discussed in Sect. 2. It will be shown in Sect. 3 that under the condition $k_1 > k_0$, the energy of the solutions decays to zero exponentially (Theorem 3.5) using the frequency domain method. Under the case $k_1 = k_0$, together with an additional *geometric condition* or except possibly for a countably infinite number of delays which is related to the spectrum of the Dirichlet Laplacian on Ω_s, the energy decays asymptotically to zero (Theorem 3.4). This will be done using a generalized Lax-Milgram method as in [12] and applying the classical Tauberian-type theorems for the stability of semigroups [1,21]. Thus, under certain circumstances, the dissipative effect of the fluid due to diffusion is strong enough to stabilize the coupled system even when the damping and delay factors are the same.

Our asymptotic stability result Theorem 3.4, under the condition (H) stated below, has been already shown for both linear [3,5] and nonlinear [17,23] problems in the case where there is no damping ($k_1 = 0$) and no delay ($k_0 = 0$). In fact, rational or polynomial decay rates have been provided for a heat–structure model by Avalos and Trigianni [5] and for a fluid–structure model by Avalos and Bucci [2]. On the other hand, the exponential stability Theorem 3.5 has been established for system (1.1)–(1.2) with linear damping in [6] and with nonlinear damping in [23], however, without the delay term. In these references, the treatment for linear problems relies on spectral analysis, while for nonlinear problems, they are obtained through the multiplier method. In the current work, we will also use spectral analysis to prove our results.

2. Semigroup construction and well-posedness

The first step in writing the system (1.1)–(1.3) into an abstract evolution equation is to eliminate the pressure term p. In accordance to the non-homogeneous Neumann boundary condition on the interface Γ_s, the typical Leray projection method used in eliminating the pressure in the Navier-Stokes equation with no-slip boundary condition cannot be applied. A novel approach, introduced and successfully applied in [3], of eliminating the pressure is to write it in terms of the fluid velocity and normal stress of the structure. This is done thanks to the realization that p satisfies an elliptic problem with Neumann condition on Γ_f and Dirichlet condition on Γ_s. The same idea has been used, at least at the *formal* level, in the numerical approximations of the solutions for the linear Stokes problem through pressure matrix methods, see for instance [26, Section 9.6.1].

The above strategy leads to a non-standard formulation of the definition for the semigroup generator including its domain, which implicitly incorporates the pressure term. In the present paper, we shall also use this strategy for the coupled system (1.1)–(1.3) with the first-order formulation of the wave component. Broadly speaking, we will follow the theoretical framework and methods presented in [3]. Accordingly, the first step is to write p in terms of u and ∇w. To do this, we first recall the notations in [3]. Define the Dirichlet map $D_s : H^{\frac{1}{2}}(\Gamma_s) \to H^1(\Omega_f)$ and the Neumann map $N_f : H^{\frac{3}{2}}(\Gamma_f) \to H^1(\Omega_f)$ as follows. Given $g \in H^{\frac{1}{2}}(\Gamma_s)$, let $h = D_s g$ be the weak solution of the elliptic problem

$$\begin{cases} \Delta h = 0, & \text{in } \Omega_f, \\ \dfrac{\partial h}{\partial \nu} = 0, & \text{on } \Gamma_f, \\ h = g, & \text{on } \Gamma_s. \end{cases}$$

Given $h \in H^{\frac{3}{2}}(\Gamma_f)$, let $g = N_f h$ be the weak solution of

$$\begin{cases} \Delta g = 0, & \text{in } \Omega_f, \\ \dfrac{\partial g}{\partial \nu} = h, & \text{on } \Gamma_f, \\ g = 0, & \text{on } \Gamma_s. \end{cases}$$

From the classical elliptic regularity in [20], we can see that $D_s \in \mathcal{L}(H^r(\Gamma_s), H^{r+\frac{1}{2}}(\Omega_f))$ and $N_f \in \mathcal{L}(H^r(\Gamma_f), H^{r+\frac{3}{2}}(\Omega_f))$ for every real number r. If the pressure term p, along with u and w satisfies (1.1), then taking the divergence of the first equation in (1.1) and using the boundary conditions yield

$$\begin{cases} \Delta p = 0, & \text{in } (0,\infty)\times\Omega_f, \\ \dfrac{\partial p}{\partial \nu} = \Delta u \cdot \nu, & \text{on } (0,\infty)\times\Gamma_f, \\ p = \dfrac{\partial u}{\partial \nu}\cdot \nu - (\nabla w \cdot \nu)\cdot \nu, & \text{on } (0,\infty)\times\Gamma_s. \end{cases}$$

In terms of the Dirichlet and Neumann maps defined above, the pressure can be written in terms of ∇w and u as

$$p = -D_s((\nabla w \cdot \nu)\cdot \nu) + D_s\left(\frac{\partial u}{\partial \nu}\cdot \nu\right) + N_f(\Delta u \cdot \nu).$$

Let $v(t,x) = w_t(t,x)$, $\sigma(t,x) = \nabla w(t,x)$ for $(t,x) \in (0,T)\times\Omega_s$ and $z(\theta,t,x) = w_t(t+\theta,x)$ for $(\theta,t,x)\in(-\tau,0)\times(0,T)\times\Omega_s$. The fluid–structure system will be posed in the state space

$$H := L^2(\Omega_s)^d \times L^2(\Omega_s)^{d\times d} \times L^2(-\tau,0;L^2(\Omega_s)^d)\times H_f$$

where $H_f := \{u \in L^2(\Omega_f)^d : \text{div } u = 0 \text{ in } \Omega_f,\ u\cdot\nu = 0 \text{ on } \Gamma_f\}$. The space H is equipped with the inner product

$$\begin{aligned}&((v_1,\sigma_1,z_1,u_1),(v_2,\sigma_2,z_2,u_2))_H \\ &:= \int\limits_{\Omega_s}(v_1\cdot v_2 + \sigma_1\cdot\sigma_2)\,\mathrm{d}x + k_0\int\limits_{-\tau}^{0}\int\limits_{\Omega_s} z_1\cdot z_2\,\mathrm{d}x\,\mathrm{d}\theta + \int\limits_{\Omega_f} u_1\cdot u_2\,\mathrm{d}x\end{aligned}$$

with the dot representing the inner product in $\mathbb{C}^d$ or $\mathbb{C}^{d\times d}$ where it is applicable.

Let $L^2_{\text{div}}(\Omega_s)^{d\times d} = \{\sigma\in L^2(\Omega_s)^{d\times d} : \text{div }\sigma \in L^2(\Omega_s)^d\}$, where div denotes the distributional divergence and is endowed with the graph norm. There is a generalized normal trace operator $\sigma\mapsto\sigma\cdot\nu$ which is continuous from $L^2_{\text{div}}(\Omega_s)^{d\times d}$ into $H^{-\frac12}(\Gamma_s)^d$. Moreover, the following generalized Green's identity

$$\int\limits_{\Omega_s}\text{div}\,\sigma\cdot u\,\mathrm{d}x = -\langle\sigma\cdot\nu,u\rangle_{H^{-\frac12}(\Gamma_s)^d\times H^{\frac12}(\Gamma_s)^d} - \int\limits_{\Omega_s}\sigma\cdot\nabla u\,\mathrm{d}x$$

holds for all $\sigma\in L^2_{\text{div}}(\Omega_s)^{d\times d}$ and $u\in H^1(\Omega_s)^d$. Recall that ν is inward to Γ_s. The space

$$Y(\Omega_s) := \{\sigma\in L^2(\Omega_s)^{d\times d} : \text{div}\,\sigma = 0 \text{ in } \Omega_s,\ \sigma\cdot\nu = 0 \text{ on } \Gamma_s\}$$

is a closed subspace of $L^2(\Omega_s)^{d\times d}$ and there holds the Helmholtz orthogonal decomposition

$$L^2(\Omega_s)^{d\times d} = Y(\Omega_s)\oplus G(\Omega_s)$$

where

$$G(\Omega_s) = \{\sigma\in L^2(\Omega_s)^{d\times d} : \sigma = \nabla\varrho \text{ for some } \varrho\in H^1(\Omega_s)^d\},$$

see for example [27].

Consider the operators $L_1 : L^2_{\text{div}}(\Omega_s)^{d\times d}\to L^2(\Omega_f)^d$ and $L_2 : H^1(\Omega_f)^d\cap\{u\in H_f : \frac{\partial u}{\partial\nu}\in H^{-\frac12}(\Gamma_s)^d,$ $\Delta u\cdot\nu\in H^{-\frac32}(\Gamma_f)\}\to L^2(\Omega_f)^d$ defined as follows

$$\begin{aligned} L_1\sigma &= -D_s((\sigma\cdot\nu)\cdot\nu), \\ L_2 u &= D_s\left(\frac{\partial u}{\partial\nu}\cdot\nu\right) + N_f(\Delta u\cdot\nu).\end{aligned}$$

These operators are well defined from the elliptic regularity stated above. Define the linear operator $A : D(A) \subset H \to H$ by

$$A = \begin{pmatrix} -k_1 I & \operatorname{div} & -k_0\gamma|_{\theta=-\tau} & 0 \\ \nabla & 0 & 0 & 0 \\ 0 & 0 & \partial_\theta & 0 \\ 0 & -\nabla L_1 & 0 & \Delta - \nabla L_2 \end{pmatrix}$$

with domain $D(A)$ comprising of all elements $(v, \sigma, z, u) \in H$ such that $v \in H^1(\Omega_s)^d$, $\sigma \in L^2_{\text{div}}(\Omega_s)^{d\times d}$, $z \in H^1(-\tau, 0; L^2(\Omega_s)^d)$, $u \in H^1(\Omega_f)^d \cap H_f$, $u = 0$ on Γ_f, $u = v$ on Γ_s, $z(0) = v$ in Ω_s, $\frac{\partial u}{\partial \nu} - \sigma \cdot \nu = \pi\nu$ in $H^{-\frac{1}{2}}(\Gamma_s)^d$, $\Delta u \cdot \nu \in H^{-\frac{3}{2}}(\Gamma_f)$, and $\Delta u - \nabla\pi \in H_f$ where $\pi = L_1\sigma + L_2 u$. Here, $\gamma|_{\theta=-\tau}$ is the trace operator. The system (1.1)–(1.3) can now be phrased as a first-order evolution equation in H

$$\begin{cases} \dot{X}(t) = AX(t) & \text{for } t > 0, \\ X(0) = X_0, \end{cases} \tag{2.1}$$

where $X_0 = (w_1, \nabla w_0, g, u_0)$.

In characterizing the kernel $N(A)$ of A, we need the following result.

Proposition 2.1. *For every $f = (f_1, \dots, f_d) \in L^2(\Omega_s)^d$ and $\phi \in H^{-\frac{1}{2}}(\Gamma_s)^d$ satisfying the compatibility condition*

$$\int_{\Omega_s} f_j \, \mathrm{d}x + \langle \phi, e_j \rangle_{H^{-\frac{1}{2}}(\Gamma_s)^d \times H^{\frac{1}{2}}(\Gamma_s)^d} = 0, \quad \text{for } j = 1, \dots, d,$$

where e_j is the canonical unit vector in $\mathbb{R}^d$, the boundary value problem

$$\begin{cases} \operatorname{div}\sigma = f, & \text{in } \Omega_s, \\ \sigma \cdot \nu = \phi, & \text{on } \Gamma_s, \end{cases} \tag{2.2}$$

admits a unique solution $\sigma \in L^2_{\text{div}}(\Omega_s)^{d\times d} \cap G(\Omega_s)$. This solution is given by $\sigma = \nabla\psi$ where $\psi \in H^1(\Omega_s)^d$ is a solution of the Neumann problem

$$\begin{cases} \Delta\psi = f, & \text{in } \Omega_s, \\ \dfrac{\partial\psi}{\partial\nu} = \phi, & \text{on } \Gamma_s. \end{cases} \tag{2.3}$$

Moreover, σ satisfies the estimate

$$\|\sigma\|_{L^2_{\text{div}}(\Omega_s)^{d\times d}} \le C(\|f\|_{L^2(\Omega_s)^d} + \|\phi\|_{H^{-\frac{1}{2}}(\Gamma_s)^d}). \tag{2.4}$$

In particular, all solutions of (2.2) take the form $\sigma = \nabla\psi + \rho$ for some $\rho \in Y(\Omega_s)$.

Proof. With the above compatibility condition, problem (2.3) admits a solution $\psi \in H^1(\Omega_s)^d$ unique up to an additive constant vector and it satisfies the stability estimate

$$\|\psi\|_{H^1(\Omega_s)^d/\mathbb{R}^d} \le C(\|f\|_{L^2(\Omega_s)^d} + \|\phi\|_{H^{-\frac{1}{2}}(\Gamma_s)^d}). \tag{2.5}$$

Clearly, $\sigma = \nabla\psi$ lies in $L^2_{\text{div}}(\Omega_s)^{d\times d} \cap G(\Omega_s)$ and it satisfies (2.2). The estimate (2.4) follows from (2.5) and the fact that $\operatorname{div}\sigma = f$. If $\tilde{\sigma} \in L^2_{\text{div}}(\Omega_s)^{d\times d} \cap G(\Omega_s)$ is also a solution of (2.2), then $\sigma - \tilde{\sigma} \in G(\Omega_s) \cap Y(\Omega_s) = \{0\}$, and hence, the solution is unique in $L^2_{\text{div}}(\Omega_s)^{d\times d} \cap G(\Omega_s)$. □

Theorem 2.2. *Assume that $k_1 \ge 0$ and $k_0 > 0$. Let I_d be the $d\times d$ identity matrix and $\langle I_d \rangle = \{cI_d : c \in \mathbb{C}\}$. Then*

$$N(A) = \{0\} \times (\langle I_d \rangle \oplus Y(\Omega_s)) \times \{0\} \times \{0\} \tag{2.6}$$

and in particular

$$N(A)^\perp = L^2(\Omega_s)^d \times (G(\Omega_s)/\langle I_d \rangle) \times L^2(-\tau, 0; L^2(\Omega_s)^d) \times H_f \tag{2.7}$$

where $G(\Omega_s)/\langle I_d\rangle$ denotes the orthogonal complement of $\langle I_d\rangle$ in $G(\Omega_s)$ and it is given by

$$G(\Omega_s)/\langle I_d\rangle = \left\{ \sigma \in G(\Omega_s) : \int_{\Omega_s} \mathrm{Tr}(\sigma)\,\mathrm{d}x = 0 \right\}.$$

Here, Tr *denotes the trace of a matrix.*

Proof. Denote by N_0 the set on the right-hand side of (2.6). Assume that $\sigma \in \langle I_d\rangle \oplus Y(\Omega_s)$. To prove that $(0,\sigma,0,0) \in D(A)$, we only need to show that $\sigma\cdot\nu = -\pi\nu$ on Γ_s and $\nabla\pi \in H_f$ where $\pi = L_1\sigma$. By assumption, $\sigma = cI_d + \rho$ for some constant c and $\rho \in Y(\Omega_s)$. Thus, on Γ_s

$$\sigma\cdot\nu = (cI_d + \rho)\cdot\nu = c\nu$$

since $\rho\cdot\nu = 0$ on Γ_s. However, we have $\pi = -(\sigma\cdot\nu)\cdot\nu = c$ on Γ_s and thus $\pi\nu = -\sigma\cdot\nu$. The equation $\pi = c$ on Γ_s and the fact that $\pi = L_1\sigma$ imply that π is constant and hence $\nabla\pi = 0 \in H_f$. It is obvious that $A(0,\sigma,0,0) = 0$, and therefore, $N_0 \subset N(A)$.

Conversely, suppose that $(v,\sigma,z,u) \in N(A)$. From the definition of A, we immediately see that $z(\theta) = v$ for every $\theta \in (-\tau,0)$, v is constant, σ satisfies the boundary value problem

$$\begin{cases} \operatorname{div}\sigma = (k_0+k_1)v, & \text{in } \Omega_s,\\ \sigma\cdot\nu = \dfrac{\partial u}{\partial\nu} - \pi\nu, & \text{on } \Gamma_s, \end{cases} \tag{2.8}$$

and u satisfies the Stokes equation

$$\begin{cases} \Delta u - \nabla\pi = 0, & \text{in } \Omega_f,\\ \operatorname{div} u = 0, & \text{in } \Omega_f,\\ u = 0, & \text{on } \Gamma_f,\\ u = v, & \text{on } \Gamma_s. \end{cases} \tag{2.9}$$

Taking the inner product of the differential equation in (2.8) with v, applying the divergence theorem and using the boundary condition $u = v$ on Γ_s yield

$$(k_0+k_1)\int_{\Omega_s} |v|^2\,\mathrm{d}x = -\left\langle \frac{\partial u}{\partial\nu} - \pi\nu, u \right\rangle_{H^{-\frac12}(\Gamma_s)^d \times H^{\frac12}(\Gamma_s)^d}. \tag{2.10}$$

Recall that ν is inward to Ω_s. Multiplying the Stokes equation by u, integrating over Ω_f and then using Green's identity, one can see that (2.10) becomes

$$(k_0+k_1)\int_{\Omega_s} |v|^2\,\mathrm{d}x + \int_{\Omega_f} |\nabla u|^2\,\mathrm{d}x = 0.$$

Since k_0+k_1 is nonnegative, it follows that u is constant, and according to the boundary condition on Γ_f in (2.9), this constant must be zero. As a consequence, the boundary condition on Γ_s of the same system implies that v must be also zero, and so is z.

We can see that σ satisfies the problem (2.2) with $f = 0$ and $\phi = -\pi\nu$. From the divergence theorem, $\int_{\Gamma_s} \nu_j\,\mathrm{d}s = 0$ for $j = 1,\ldots,d$, and so the pair $(0,-\pi\nu)$ is compatible. According to Proposition 2.1, all solutions to this problem are of the form $\sigma = -\pi I_d + \rho$ where $\rho \in Y(\Omega_s)$, which is an element of $\langle I_d\rangle \oplus Y(\Omega_s)$. This proves the other inclusion $N(A) \subset N_0$. Therefore, $N(A) = N_0$, and since $I_d \in G(\Omega_s)$, we have indeed a direct sum in the second component of $N(A)$.

If $G(\Omega_s)/\langle I_d\rangle$ is the orthogonal complement of $\langle I_d\rangle$ in $G(\Omega_s)$, then one can easily see that

$$(\langle I_d\rangle \oplus Y(\Omega_s))^\perp = G(\Omega_s)/\langle I_d\rangle \tag{2.11}$$

where the left-hand side is taken with respect to $L^2(\Omega_s)^{d\times d}$. Indeed, given $\sigma \in G(\Omega_s)/\langle I_d\rangle$ and $\kappa \in \langle I_d\rangle \oplus Y(\Omega_s)$, so that $\kappa = cI_d + \rho$ for some constant c and $\rho \in Y(\Omega_s)$, we have

$$\int_{\Omega_s} \kappa \cdot \sigma \,\mathrm{d}x = c \int_{\Omega_s} I_d \cdot \sigma \,\mathrm{d}x + \int_{\Omega_s} \rho \cdot \sigma \,\mathrm{d}x = 0$$

since I_d is orthogonal to σ and $G(\Omega_s)$ is orthogonal to $Y(\Omega_s)$. Thus, $\sigma \in (\langle I_d\rangle \oplus Y(\Omega_s))^\perp$ and we have one inclusion. For the reverse inclusion, note that if $\kappa \in (\langle I_d\rangle \oplus Y(\Omega_s))^\perp$ and $\rho \in \langle I_d\rangle \subset \langle I_d\rangle \oplus Y(\Omega_s)$ then κ lies in $G(\Omega_s)$ and it is orthogonal to ρ, which means that $\kappa \in G(\Omega_s)/\langle I_d\rangle$. This completes the proof of (2.11), and hence (2.7). The characterization of $G(\Omega_s)/\langle I_d\rangle$ is a direct consequence of the fact that $\sigma \cdot I_d$ is the trace of σ. □

The notation $G(\Omega_s)/\langle I_d\rangle$ for the orthogonal complement of $\langle I_d\rangle$ in $G(\Omega_s)$ is motivated from the fact that the latter space is isomorphic to the former when viewed as a factor space. Now we prove the invariance of $N(A)^\perp$ under A. This space will be the state space for our stability problem.

Theorem 2.3. *The space $N(A)^\perp$ is invariant under A, i.e., $A(D(A) \cap N(A)^\perp) \subset N(A)^\perp$.*

Proof. Let $(v, \sigma, z, u) \in D(A) \cap N(A)^\perp$. In order to $A(v, \sigma, z, u) \in N(A)^\perp$, the component v must satisfy

$$\int_{\Omega_s} \operatorname{div} v \,\mathrm{d}x = \int_{\Omega_s} \operatorname{Tr}(\nabla v) \,\mathrm{d}x = 0,$$

or equivalently, by the divergence theorem

$$\int_{\Gamma_s} v \cdot \nu \,\mathrm{d}s = 0. \tag{2.12}$$

Since u is divergence free in Ω_f and it vanishes on Γ_f, we have

$$\int_{\Gamma_s} u \cdot \nu \,\mathrm{d}s = \int_{\Omega_f} \operatorname{div} u \,\mathrm{d}x = 0$$

and hence, (2.12) holds because $u = v$ on Γ_s. □

Define $\tilde{A}$ to be the part of A in $N(A)^\perp$, i.e., the operator $\tilde{A} : D(A) \cap N(A)^\perp \to N(A)^\perp$ given by $\tilde{A}X = AX$ for $X \in N(A)^\perp$. This operator is well defined according to Theorem 2.3.

Theorem 2.4. *Suppose that $k_1 \geq k_0 > 0$. The linear operator $\tilde{A}$ is dissipative and generates a strongly continuous semigroup of contractions on $N(A)^\perp$.*

The corresponding result in the case where $k_1 = k_0 = 0$ and $k_0 = 0$ has been established in [3] and [6], respectively. In order to prove the theorem, we need to solve certain Stokes equations. For this, we recall the following classical result whose proof can be found in [28].

Proposition 2.5. *Let $m \geq -1$ be an integer and $\Omega \subset \mathbb{R}^d$ be a bounded C^r-domain, where $d = 2, 3$ and $r = \max(2, m+2)$. For every $f \in H^m(\Omega)^d$ and $\phi \in H^{m+\frac{3}{2}}(\partial\Omega)^d$ such that $\int_{\partial\Omega} \phi \cdot \nu \,\mathrm{d}s = 0$, where ν is the unit normal outward to Ω, the system*

$$\begin{cases} \Delta u - \nabla p = f, & \text{in } \Omega, \\ \operatorname{div} u = 0, & \text{in } \Omega, \\ u = \phi, & \text{on } \partial\Omega, \end{cases} \tag{2.13}$$

has a unique solution $(u, p) \in H^{m+2}(\Omega)^d \times (H^{m+1}(\Omega)/\mathbb{R})$ satisfying the estimate

$$\|u\|_{H^{m+2}(\Omega)^d} + \|p\|_{H^{m+1}(\Omega)/\mathbb{R}} \leq C(\|f\|_{H^m(\Omega)^d} + \|\phi\|_{H^{m+\frac{3}{2}}(\partial\Omega)^d})$$

for some $C > 0$ independent of u, p, f and ϕ.

Proof of Theorem 2.4. The first step is to show that $\tilde{A}$ is dissipative. Let $X = (v, \sigma, z, u)$ be an arbitrary element of $D(A)$ and $\pi = L_1\sigma + L_2 u$ be the associated pressure. Using Green's identity and the divergence theorem, we have

$$\mathrm{Re} \int_{\Omega_f} (\Delta u - \nabla \pi) \cdot u \, \mathrm{d}x = - \int_{\Omega_f} |\nabla u|^2 \, \mathrm{d}x + \mathrm{Re} \left\langle \frac{\partial u}{\partial \nu} - \pi\nu, u \right\rangle_{H^{-\frac{1}{2}}(\Gamma_s)^d \times H^{\frac{1}{2}}(\Gamma_s)^d}. \tag{2.14}$$

Here, we used the fact that u is divergence free and $u = 0$ on Γ_f. On the other hand, applying the divergence theorem with respect to the domain Ω_s, we obtain

$$\begin{aligned} &\mathrm{Re} \int_{\Omega_s} (\operatorname{div} \sigma - k_1 v - k_0 z(-\tau)) \cdot v + \nabla v \cdot \sigma \, \mathrm{d}x \\ &= -\mathrm{Re}\langle \sigma \cdot \nu, v\rangle_{H^{-\frac{1}{2}}(\Gamma_s)^d \times H^{\frac{1}{2}}(\Gamma_s)^d} - k_1 \int_{\Omega_s} |v|^2 \, \mathrm{d}x - \mathrm{Re}\, k_0 \int_{\Omega_s} z(-\tau) \cdot v \, \mathrm{d}x. \end{aligned} \tag{2.15}$$

For the delay variable, we integrate by parts, use the condition $z_{|\theta=0} = v$ and take the real part to get

$$\mathrm{Re}\, k_0 \int_{-\tau}^{0} \int_{\Omega_s} \partial_\theta z \cdot z \, \mathrm{d}x \, \mathrm{d}\theta = \frac{k_0}{2} \int_{\Omega_s} |v|^2 \, \mathrm{d}x - \frac{k_0}{2} \int_{\Omega_s} |z(-\tau)|^2 \, \mathrm{d}x. \tag{2.16}$$

Taking the sum of (2.14)–(2.16), using the boundary conditions $\sigma \cdot \nu = \frac{\partial u}{\partial \nu} - \pi\nu$ and $u = v$ on Γ_s so that the boundary terms will be canceled, and then applying the Cauchy-Schwarz inequality and then the elementary inequality $|ab| \le \frac{1}{2}(a^2 + b^2)$ to the last term on the right-hand side of (2.15), we obtain

$$\mathrm{Re}(AX, X)_H \le - \int_{\Omega_f} |\nabla u|^2 \, \mathrm{d}x - (k_1 - k_0) \int_{\Omega_s} |v|^2 \, \mathrm{d}x. \tag{2.17}$$

This means that A and $\tilde{A}$ are dissipative whenever $k_1 \ge k_0$.

The next step is to prove the invertibility of $\tilde{A}$. It is clear that $\tilde{A}$ is injective. Let us show that $\tilde{A}$ is surjective, first for *sufficiently large* k_1. Given $(\eta, \kappa, \zeta, \varphi) \in H$, the equation $\tilde{A}(v, \sigma, z, u) = (\eta, \kappa, \zeta, \varphi)$ with unknown $(v, \sigma, z, u) \in D(\tilde{A})$ is equivalent to the system where v satisfies

$$\nabla v = \kappa, \quad \text{in } \Omega_s, \tag{2.18}$$

u is the solution of the Stokes equation

$$\begin{cases} \Delta u - \nabla \pi = \varphi, & \text{in } \Omega_f, \\ \operatorname{div} u = 0, & \text{in } \Omega_f, \\ u = 0, & \text{on } \Gamma_f, \\ u = v, & \text{on } \Gamma_s, \end{cases} \tag{2.19}$$

and $\sigma \in G(\Omega_s)/\langle I_d \rangle$ satisfies the boundary value problem

$$\begin{cases} \operatorname{div} \sigma = k_0 z(-\tau) + k_1 v + \eta, & \text{in } \Omega_s, \\ \sigma \cdot \nu = \dfrac{\partial u}{\partial \nu} - \pi\nu, & \text{in } \Gamma_s, \end{cases} \tag{2.20}$$

where the delay variable z is given by

$$z(\theta) = v - \int_{\theta}^{0} \zeta(\vartheta) \, \mathrm{d}\vartheta, \quad \text{in } L^2(\Omega_s)^d. \tag{2.21}$$

Recall that π is the solution of the elliptic problem

$$\begin{cases} \Delta\pi = 0, & \text{in } \Omega_f, \\ \pi = \dfrac{\partial u}{\partial \nu}\cdot\nu - (\sigma\cdot\nu)\cdot\nu, & \text{on } \Gamma_s, \\ \dfrac{\partial \pi}{\partial \nu} = \Delta u\cdot\nu, & \text{on } \Gamma_f. \end{cases} \tag{2.22}$$

From (2.21), it is clear that $z\in H^1(-\tau,0;L^2(\Omega_s)^d)$. On the other hand, since $\kappa\in G(\Omega_s)/\langle I_d\rangle\subset G(\Omega_s)$, it follows that $(\kappa,\rho)_{L^2(\Omega_s)^d}=0$ for every divergence-free vector field $\rho\in C_0^\infty(\Omega_s)^d$. By a classical result, there exists $\tilde v\in H^1(\Omega_s)^d$, which is unique up to an additive constant vector, that satisfies (2.18), see [27, Lemma 2.2.2] for example. Applying the divergence theorem, we obtain

$$\int\limits_{\Gamma_s}\tilde v\cdot\nu\,\mathrm{d}s = -\int\limits_{\Omega_s}\operatorname{div}\tilde v\,\mathrm{d}x = -\int\limits_{\Omega_s}\operatorname{Tr}(\kappa)\,\mathrm{d}x = 0. \tag{2.23}$$

As been said, $v=\tilde v+v^*$, where v^* is a constant vector, also satisfies (2.18) and hence (2.23) where $\tilde v$ is replaced by v. The vector v^* will be chosen so that the data in (2.20) are compatible.

Taking $m=-1$ in Proposition 2.5, the Stokes equation (2.19) admits a solution pair $(u,\tilde\pi)\in(H^1(\Omega_f)^d\cap H_f)\times L^2(\Omega_f)$. The function $\tilde\pi$ is harmonic since

$$\Delta\tilde\pi = \operatorname{div}(\varphi-\Delta u) = \Delta(\operatorname{div} u) = 0.$$

Therefore, $\tilde\pi$ has the following traces $\tilde\pi|_{\Gamma_s}\in H^{-\frac12}(\Gamma_s)$ and $\frac{\partial\tilde\pi}{\partial\nu}|_{\Gamma_f}\in H^{-\frac32}(\Gamma_f)$ while u satisfies $\frac{\partial u}{\partial \nu}|_{\Gamma_s}\in H^{-\frac12}(\Gamma_s)^d$ and $\Delta u\cdot\nu$ in $H^{-\frac32}(\Gamma_f)$, refer to [4, Lemma 3.1]. For every constant π^*, (u,π) with $\pi=\tilde\pi+\pi^*$ is also a solution pair for (2.19). The constant π^* will be determined below by imposing the condition $\sigma\in G(\Omega_s)/\langle I_d\rangle$ where σ solves (2.20).

Consider the decomposition $u=\tilde u+\sum_{j=1}^d v_j^* w_j$ and $\pi=\tilde\pi_0+\sum_{j=1}^d v_j^*\varrho_j$, where $v^*=(v_1^*,\dots,v_d^*)\in\mathbb{C}^d$ and the pairs $(\tilde u,\tilde\pi_0),(w_j,\varrho_j)\in(H^1(\Omega_f)^d\cap H_f)\times L^2(\Omega_f)$ satisfy the following Stokes equations

$$\begin{cases} \Delta\tilde u-\nabla\tilde\pi_0=\varphi, & \text{in } \Omega_f, \\ \operatorname{div}\tilde u = 0, & \text{in } \Omega_f, \\ \tilde u = 0, & \text{on } \Gamma_f, \\ \tilde u = \tilde v, & \text{on } \Gamma_s, \end{cases} \tag{2.24}$$

and

$$\begin{cases} \Delta w_j-\nabla\varrho_j=0, & \text{in } \Omega_f, \\ \operatorname{div} w_j = 0, & \text{in } \Omega_f, \\ w_j = 0, & \text{on } \Gamma_f, \\ w_j = e_j, & \text{on } \Gamma_s, \end{cases} \tag{2.25}$$

respectively. The boundary data in (2.24) and (2.25) are admissible according to (2.23) and $\int\limits_{\Gamma_s}\nu\cdot e_j\,\mathrm{d}s = \int\limits_{\Gamma_s}\nu_j\,\mathrm{d}s=0$, respectively. The compatibility condition for (2.20) is given by, for $l=1,\dots,d$

$$\begin{aligned} 0 = (k_0+k_1)\int\limits_{\Omega_s}(\tilde v_l+v_l^*)\,\mathrm{d}x - k_0\int\limits_{-\tau}^{0}\int\limits_{\Omega_s}\zeta_l(\vartheta)\,\mathrm{d}\vartheta + \int\limits_{\Omega_s}\eta_l\,\mathrm{d}x \\ + \left\langle\frac{\partial u}{\partial\nu}-\tilde\pi\nu, e_l\right\rangle_{H^{-\frac12}(\Gamma_s)^d\times H^{\frac12}(\Gamma_s)^d}, \end{aligned} \tag{2.26}$$

where we used $\pi^* \int_{\Gamma_s} \nu \cdot e_l \, \mathrm{d}s = 0$. Using the above decomposition and Green's identity, the last term in the above equation can be written as

$$\int_{\Omega_f} \left(\sum_{j=1}^{d} v_j^* \nabla w_j \cdot \nabla w_l + \nabla \tilde{u} \cdot \nabla w_l + \varphi \cdot w_l \right) \mathrm{d}x = \left\langle \frac{\partial u}{\partial \nu} - \tilde{\pi}\nu, e_l \right\rangle_{H^{-\frac{1}{2}}(\Gamma_s)^d \times H^{\frac{1}{2}}(\Gamma_s)^d}. \tag{2.27}$$

Equations (2.26) and (2.27) provide us a $d \times d$ system of equations

$$Mv^* = F \tag{2.28}$$

for some vector $F = F(\eta, \kappa, \zeta, \varphi)$ independent of v^*, and the matrix M has the entries

$$M_{jl} = \begin{cases} (k_0 + k_1)|\Omega_s| + \|\nabla w_j\|^2_{L^2(\Omega_f)^{d\times d}}, & \text{if } l = j, \\ (\nabla w_l, \nabla w_j)_{L^2(\Omega_f)^{d\times d}}, & \text{if } l \neq j, \end{cases}$$

for $j = 1, \ldots, d$. Here, $|\Omega_s|$ denotes the Lebesgue measure of Ω_s. For sufficiently large k_1, the matrix M is strictly diagonally dominant, that is, $M_{jj} > \sum_{l\neq j} M_{jl}$ for every $j = 1, \ldots, d$. To see this, we first apply Proposition 2.5 with $m = -1$ to (2.25) in order to obtain the estimate

$$\|w_j\|_{H^1(\Omega_f)^d} \leq C\|e_j\|_{H^{-1}(\Omega_f)^d} \leq C\|e_j\|_{L^2(\Omega_f)^d} \leq C|\Omega_f|^{\frac{1}{2}} \tag{2.29}$$

for every j. Then, one may take, for example, $k_1 > (d-1)C^2|\Omega_f||\Omega_s|^{-1}$, where C is the positive constant in (2.29). Indeed, by applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \sum_{l\neq j} M_{jl} &\leq \sum_{l\neq j} \|\nabla w_j\|_{L^2(\Omega_f)^d} \|\nabla w_l\|_{L^2(\Omega_f)^d} \\ &= (d-1)C^2|\Omega_f| < k_1|\Omega_s| \leq M_{jj} \end{aligned}$$

for every $j = 1, \ldots, d$. Therefore, for sufficiently large k_1 the matrix M is invertible according to the well-known Levy-Desplanques Theorem, see [15] for instance. Thus, we can solve for v^* in the linear system (2.28).

Let $f^* \in L^2(\Omega_s)^d$ denote the right-hand side of (2.20), i.e.,

$$f^* = (k_0 + k_1)v - k_0 \int_{\theta}^{0} \zeta(\vartheta) \, \mathrm{d}\vartheta + \eta.$$

From Proposition 2.1, the function $\sigma = \nabla\psi - \pi^* I_d \in L^2_{\mathrm{div}}(\Omega_s)^{d\times d} \cap G(\Omega_s)$, where ψ satisfies the Neumann problem (2.3) with $f = f^*$ and $\phi = \frac{\partial u}{\partial \nu} - \tilde{\pi}\nu$, is a solution of (2.20). In order for σ to be an element of $G(\Omega_s)/\langle I_d\rangle$, we must have

$$\int_{\Omega_s} \mathrm{Tr}(\nabla\psi) \, \mathrm{d}x - d\pi^*|\Omega_s| = \int_{\Omega_s} \mathrm{Tr}(\nabla\psi - \pi^* I_d) \, \mathrm{d}x = 0.$$

Choosing $\pi^* = -(d|\Omega_s|)^{-1} \int_{\Omega_s} \psi \cdot \nu \, \mathrm{d}s$ yields $\sigma \in G(\Omega_s)/\langle I_d\rangle$.

It remains to show that π satisfies (2.22), i.e., $\pi = L_1\sigma + L_2 u$. We already know that π is harmonic. The second line in (2.22) holds in $H^{-\frac{1}{2}}(\Omega_s)$ by taking the inner product, in the sense of traces, of the second line in (2.20) with ν. Also, $\varphi \in H_f$ and the first equation of (2.22) imply that $\frac{\partial \pi}{\partial \nu} = \nabla\pi \cdot \nu = \Delta u \cdot \nu$ in $H^{-\frac{3}{2}}(\Gamma_f)$. Hence, $\pi = L_1 v + L_2\sigma$, and therefore, $(v, \sigma, z, u) \in D(\tilde{A})$ satisfies $\tilde{A}(v, \sigma, z, u) = (\eta, \kappa, \zeta, \varphi)$.

The operator $\tilde{A}$ is therefore bijective, and by the closed graph theorem, 0 lies in the resolvent set of $\tilde{A}$. By the Lumer-Phillips Theorem, $\tilde{A}$ generates a strongly continuous semigroup of contractions on $N(A)^\perp$. This completes the proof of the theorem in the case where k_1 is sufficiently large. However, by the bounded perturbation theorem for semigroups, this implies that the conclusion of the theorem also holds for every $k_1 \geq k_0$. □

Corollary 2.6. *Suppose that $k_1 \geq k_0 > 0$. The operators A and A^* generate strongly continuous semigroups of contractions on H. In particular, the Cauchy problem (2.1) admits a unique weak solution $X \in C([0,\infty); H)$ for every initial data $X_0 \in H$.*

Proof. It is enough to prove the range conditions $R(I - A) = H = R(I - A^*)$. Given $Y \in H$, write $Y = Y_1 + Y_2$ where $Y_1 \in N(A)^\perp$ and $Y_2 \in N(A)$. From Theorem 2.4, it follows that there exists $X_1 \in D(\tilde{A})$ such that $(I - \tilde{A})X_1 = Y_1$. If $X = X_1 + Y_2$ then $X \in D(A)$ and

$$(I - A)X = (I - \tilde{A})X_1 + Y_2 = Y.$$

Therefore, $I - A$ is surjective. The case of A^* is analogous. □

As in [3], it can be shown that $p \in C([0,\infty); L^2(\Omega_f))$ where $p = L_1\sigma + L_2 u$ and $(v, \sigma, z, u) = e^{tA}X_0$ for a given data $X_0 \in D(A)$. To close this section, we determine the adjoint of the closed operator A.

Theorem 2.7. *The adjoint $A^* : D(A^*) \to H$ of A is given by*

$$A^* = \begin{pmatrix} -k_1 I & -\mathrm{div} & k_0\gamma|_{\theta=0} & 0 \\ -\nabla & 0 & 0 & 0 \\ 0 & 0 & -\partial_\theta & 0 \\ 0 & \nabla L_1 & 0 & \Delta - \nabla L_2 \end{pmatrix}. \tag{2.30}$$

The domain $D(A^*)$ of A^* is the set of all elements in H such that

$$(\eta, \kappa, \zeta, \varphi) \in H^1(\Omega_s)^d \times L^2_{\mathrm{div}}(\Omega_s)^{d\times d} \times H^1(-\tau, 0; L^2(\Omega_s)^d) \times (H^1(\Omega_f)^d \cap H_f)$$

with the properties $\varphi = 0$ on Γ_f, $\varphi = \eta$ on Γ_s, $\zeta(-\tau) = -\eta$ in Ω_s, $\frac{\partial\varphi}{\partial\nu} + \kappa\cdot\nu = p\nu$ in $H^{-\frac{1}{2}}(\Gamma_s)^d$, $\Delta\varphi\cdot\nu \in H^{-\frac{3}{2}}(\Gamma_f)$ and $\Delta\varphi - \nabla p \in H_f$ where $p = -L_1\kappa + L_2\varphi$. Moreover, the kernels of A and A^* coincide.

Proof. Define the operator $B : D(B) \to H$ by the right-hand side of (2.30) where the domain $D(B)$ is the set in the description of $D(A^*)$. With the isometric isomorphism $J : H \to H$ defined by

$$J(v, \sigma, z(\theta), u) = (-v, \sigma, z(-\theta - \tau), -u),$$

which satisfies $J^{-1} = J$, the operators A and B are similar, that is, $JAJ = B$ and $D(JAJ) = D(B)$. This implies that B is m-dissipative and $N(A) = N(B)$. We show that A^* is an extension of B and since A^* is the adjoint of a generator of a strongly continuous semigroup of contractions, A^* does not contain a strict m-dissipative operator and so we must have $A^* = B$.

We show that

$$(AX, Y)_H = (X, BY)_H \tag{2.31}$$

holds whenever $X = (v, \sigma, z, u) \in D(A)$ and $Y = (\eta, \kappa, \zeta, \varphi) \in D(B)$, so that $Y \in D(A^*)$ and consequently A^* is an extension of B. By definition, we have

$$(AX, Y)_H = -\int_{\Omega_s} (k_1 v - \mathrm{div}\,\sigma + k_0 z(-\tau))\cdot\eta\,\mathrm{d}x + \int_{\Omega_s} \nabla v\cdot\kappa\,\mathrm{d}x \tag{2.32}$$
$$+ k_0 \int_{-\tau}^{0}\int_{\Omega_s} \partial_\theta z(\theta)\cdot\zeta(\theta)\,\mathrm{d}x\,\mathrm{d}\theta + \int_{\Omega_f} (\Delta u - \nabla\pi)\cdot\varphi\,\mathrm{d}x.$$

Integrating by parts, using Green's identities, the divergence theorem and $\zeta(-\tau) = -\eta$, we obtain

$$\int_{\Omega_s} \operatorname{div}\sigma\cdot\eta\,\mathrm{d}x = -\langle\sigma\cdot\nu,\eta\rangle_{H^{-\frac{1}{2}}(\Gamma_s)^d\times H^{\frac{1}{2}}(\Gamma_s)^d} - \int_{\Omega_s}\sigma\cdot\nabla\eta\,\mathrm{d}x$$

$$\int_{\Omega_s}\nabla v\cdot\kappa\,\mathrm{d}x = -\overline{\langle\kappa\cdot\nu,v\rangle}_{H^{-\frac{1}{2}}(\Gamma_s)^d\times H^{\frac{1}{2}}(\Gamma_s)^d} - \int_{\Omega_s} v\cdot\operatorname{div}\kappa\,\mathrm{d}x$$

$$\int_{-\tau}^{0}\int_{\Omega_s}\partial_\theta z(\theta)\cdot\zeta(\theta)\,\mathrm{d}x\,\mathrm{d}\theta = \int_{\Omega_s}(v\cdot\zeta(0)+z(-\tau)\cdot\eta)\,\mathrm{d}x - \int_{-\tau}^{0}\int_{\Omega_s} z(\theta)\cdot\partial_\theta\zeta(\theta)\,\mathrm{d}x\,\mathrm{d}\theta$$

$$\int_{\Omega_f}(\Delta u-\nabla\pi)\cdot\varphi\,\mathrm{d}x = \left\langle\frac{\partial u}{\partial\nu}-\pi\nu,\varphi\right\rangle_{H^{-\frac{1}{2}}(\Gamma_s)^d\times H^{\frac{1}{2}}(\Gamma_s)^d} - \overline{\left\langle\frac{\partial\varphi}{\partial\nu}-p\nu,u\right\rangle}_{H^{-\frac{1}{2}}(\Gamma_s)^d\times H^{\frac{1}{2}}(\Gamma_s)^d} + \int_{\Omega_f} u\cdot(\Delta\varphi-\nabla p)\,\mathrm{d}x.$$

Using these equations in (2.32) together with the boundary conditions $u = v$ on Γ_s, $\varphi = \eta$ on Γ_s, $\frac{\partial\varphi}{\partial\nu}+\kappa\cdot\nu = p\nu$ in $H^{-\frac{1}{2}}(\Gamma_s)^d$ and $\frac{\partial u}{\partial\nu}-\sigma\cdot\nu = \pi\nu$ in $H^{-\frac{1}{2}}(\Gamma_s)^d$, it can be seen that (2.31) is satisfied. □

3. Spectral properties and stability

In the absence of delay, it was shown in [3] the partial compactness of the resolvents of the operator A. More precisely, the projection of a resolvent onto the state space corresponding to the velocity fields for the fluid and structure components is compact. Here, we will show that even though the operator A does not have compact resolvents, the spectrum comprises of only eigenvalues except possibly on the negative real axis. This will be established in a more straightforward manner through a variational method, deviating from the methods provided in [3]. To this end, we introduce the following Hilbert spaces

$$H_0 := L^2(\Omega_s)^d \times H_f,$$
$$H_1 := \{(v,u)\in H^1(\Omega_s)^d\times(H^1(\Omega_f)^d\cap H_f) : u = 0 \text{ on } \Gamma_f \text{ and } v = u \text{ on } \Gamma_s\},$$

equipped with the inner products

$$((v,u),(w,\psi))_{H_0} := \int_{\Omega_s} v\cdot w\,\mathrm{d}x + \int_{\Omega_f} u\cdot\psi\,\mathrm{d}x$$
$$((v,u),(w,\psi))_{H_1} := \int_{\Omega_s}(v\cdot w+\nabla v\cdot\nabla w)\,\mathrm{d}x + \int_{\Omega_f}\nabla u\cdot\nabla\psi\,\mathrm{d}x,$$

respectively. The embedding $H_1 \subset H_0$ is continuous, dense and compact.

For each nonzero complex number λ, define the sesquilinear form $a_\lambda : H_1\times H_1 \to \mathbb{C}$ by

$$a_\lambda((v,u),(w,\psi)) := q(\lambda)\int_{\Omega_s} v\cdot w\,\mathrm{d}x + \frac{1}{\lambda}\int_{\Omega_s}\nabla v\cdot\nabla w\,\mathrm{d}x + \lambda\int_{\Omega_f} u\cdot\psi\,\mathrm{d}x + \int_{\Omega_f}\nabla u\cdot\nabla\psi\,\mathrm{d}x$$

where $q(\lambda) = \lambda + k_1 + k_0 e^{-\lambda\tau}$. For a given $Y = (\eta, \kappa, \zeta, \varphi) \in H$ and $\lambda \in \mathbb{C} \setminus \{0\}$, define the anti-linear form $F_{Y,\lambda} : H_1 \times H_1 \to \mathbb{C}$ by

$$F_{Y,\lambda}(w, \psi) := \int_{\Omega_s} \left(\eta \cdot w - \frac{1}{\lambda}\kappa \cdot \nabla w\right) \mathrm{d}x - k_0 \int_{-\tau}^{0} \int_{\Omega_s} e^{-\lambda(\tau+\theta)} \zeta(\theta) \cdot w \, \mathrm{d}x \, \mathrm{d}\theta + \int_{\Omega_f} \varphi \cdot \psi \, \mathrm{d}x.$$

In the sequel, $\rho(A)$, $\sigma(A)$ and $\sigma_p(A)$ denote the resolvent set, spectrum and point spectrum of a closed operator A, respectively.

Theorem 3.1. *The spectrum of A in $\mathbb{C}\setminus(-\infty, 0]$ consists of only eigenvalues, that is, $\sigma(A) \cap (\mathbb{C}\setminus(-\infty, 0]) = \sigma_p(A)$. The same property holds for A^*.*

The proof of this theorem is based on the following result whose proof can be found in [12, Theorem 3] or [25, Lemma 2.1].

Lemma 3.2. [Lax-Milgram-Fredholm] *Let H_1 and H_0 be Hilbert spaces such that the embedding $H_1 \subset H_0$ is compact and dense. Suppose that $a_1 : H_1 \times H_1 \to \mathbb{C}$ and $a_2 : H_0 \times H_0 \to \mathbb{C}$ are two bounded sesquilinear forms such that a_1 is H_1-coercive and $F : H_1 \to \mathbb{C}$ is a continuous conjugate linear form. The variational equation*

$$a_1(u, v) + a_2(u, v) = F(v), \quad \forall v \in H_1,$$

has either a unique solution $u \in H_1$ for all $F \in H_1'$ or has a non-trivial solution for $F = 0$.

Proof of Theorem 3.1. The fact that A and A^* are generators of strongly continuous semigroups of contractions implies that $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$ lies in their respective resolvent sets. Let $\lambda \neq 0$ with $\operatorname{Re}\lambda \leq 0$. The equation

$$(\lambda I - A)(v, \sigma, z, u) = (\eta, \kappa, \zeta, \varphi) \tag{3.1}$$

for $(v, \sigma, z, u) \in D(A)$ and $Y := (\eta, \kappa, \zeta, \varphi) \in H$ is equivalent to the system of differential equations

$$(\lambda + k_1)v - \operatorname{div}\sigma + k_0 z(-\tau) = \eta, \tag{3.2}$$

$$\lambda\sigma - \nabla v = \kappa, \tag{3.3}$$

$$\lambda z(\theta) - \partial_\theta z(\theta) = \zeta(\theta), \tag{3.4}$$

$$\lambda u - \Delta u + \nabla\pi = \varphi, \tag{3.5}$$

and supplied with the boundary conditions listed in the definition of $D(A)$. Applying the variation of parameter formula to (3.4) yields the following equation in $L^2(\Omega_s)^d$

$$z(\theta) = e^{\lambda\theta} v + \int_{\theta}^{0} e^{\lambda(\theta-\vartheta)} \zeta(\vartheta) \, \mathrm{d}\vartheta, \quad \theta \in (-\tau, 0). \tag{3.6}$$

Let $w \in H^1(\Omega_s)^d$. Multiplying (3.2) by w, integrating over Ω_s, applying the divergence theorem and then rearranging the terms give us

$$q(\lambda) \int_{\Omega_s} v \cdot w \, \mathrm{d}x + \int_{\Omega_s} \sigma \cdot \nabla w \, \mathrm{d}x + \langle \sigma \cdot \nu, w \rangle_{H^{-\frac{1}{2}}(\Gamma_s)^d \times H^{\frac{1}{2}}(\Gamma_s)^d}$$

$$= \int_{\Omega_s} \eta \cdot w \, \mathrm{d}x - k_0 \int_{-\tau}^{0} \int_{\Omega_s} e^{-\lambda(\tau+\theta)} \zeta(\theta) \cdot w \, \mathrm{d}x \, \mathrm{d}\theta. \tag{3.7}$$

Taking the inner product of (3.3) with ∇w yields

$$\lambda \int_{\Omega_s} \sigma \cdot \nabla w \,\mathrm{d}x - \int_{\Omega_s} \nabla v \cdot \nabla w \,\mathrm{d}x = \int_{\Omega_s} \kappa \cdot \nabla w \,\mathrm{d}x. \tag{3.8}$$

Suppose that $\psi \in H^1(\Omega_f)^d \cap H_f$ and $\psi = 0$ on Γ_f. Taking the inner product of (3.5) with ψ and using the divergence theorem, we have

$$\lambda \int_{\Omega_f} u \cdot \psi \,\mathrm{d}x + \int_{\Omega_f} \nabla u \cdot \nabla \psi \,\mathrm{d}x - \left\langle \frac{\partial u}{\partial \nu} - \pi\nu, \psi \right\rangle_{H^{-\frac{1}{2}}(\Gamma_s)^d \times H^{\frac{1}{2}}(\Gamma_s)^d} = \int_{\Omega_f} \varphi \cdot \psi \,\mathrm{d}x. \tag{3.9}$$

If $\psi = w$ on Γ_s, then dividing (3.8) by $-\lambda$ and then adding the result to (3.7) and (3.9), it can be seen that the boundary terms cancel, which leads to the variational equation

$$a_\lambda((v, u), (w, \psi)) = F_{Y,\lambda}(w, \psi), \tag{3.10}$$

where a_λ and $F_{Y,\lambda}$ are the forms stated preceding the theorem. We have shown that if (3.1) holds, then (3.10) is satisfied for every $(w, \psi) \in H_1$.

Let us verify the other direction. Assume that there exists $(u, v) \in H_1$ such that (3.10) is true for all $(w, \psi) \in H_1$. Taking $w = 0$ and $\psi \in H_0^1(\Omega_f)^d \cap H_f$ leads to the equation (3.9) without the duality pairing. This implies that $u \in H_f$ satisfies (3.5) for some $\tilde{\pi} \in L^2(\Omega_f)^d$. For every constant π^*, the pair (u, π) where $\pi = \tilde{\pi} + \pi^*$ also satisfies (3.5). As in the proof of Theorem 2.4, $\frac{\partial u}{\partial \nu} - \pi\nu \in H^{-\frac{1}{2}}(\Gamma_s)^d$.

Define $z \in H^1(-\tau, 0; L^2(\Omega_s)^d)$ by (3.6) and $\sigma \in L^2(\Omega_s)^{d\times d}$ by

$$\sigma = \frac{1}{\lambda}(\kappa + \nabla v).$$

By construction, σ and z satisfy (3.3) and (3.4), respectively. Setting $\psi = 0$ and $w \in H_0^1(\Omega)$ in (3.10) and rearranging the terms

$$\int_{\Omega_s} \sigma \cdot \nabla w \,\mathrm{d}x = \int_{\Omega_s} (\eta - (\lambda + k_1)v - k_0 z(-\tau)) \cdot w \,\mathrm{d}x.$$

This implies that (3.2) is satisfied in $H^{-1}(\Omega_s)^d$, and *a posteriori* in $L^2(\Omega_s)^d$ since the right-hand side lies in $L^2(\Omega_s)^d$. As a result, $\sigma \in L^2_{\mathrm{div}}(\Omega_s)^{d\times d}$. Now, we choose the constant π^* according to

$$\pi^* = \frac{1}{|\Gamma_s|} \left\langle \frac{\partial u}{\partial \nu} - \tilde{\pi}\nu - \sigma \cdot \nu, \nu \right\rangle_{H^{-\frac{1}{2}}(\Gamma_s)^d \times H^{\frac{1}{2}}(\Gamma_s)^d}$$

and from this choice, we have

$$\left\langle \frac{\partial u}{\partial \nu} - \pi\nu - \sigma \cdot \nu, \nu \right\rangle_{H^{-\frac{1}{2}}(\Gamma_s)^d \times H^{\frac{1}{2}}(\Gamma_s)^d} = 0. \tag{3.11}$$

Given $\phi \in H^{\frac{1}{2}}(\Gamma_s)^d$, let $\varphi = \phi - \overline{\phi}\nu \in H^{\frac{1}{2}}(\Gamma_s)^d$ where $\overline{\phi}$ is the average of $\phi \cdot \nu$ on Γ_s, i.e.,

$$\overline{\phi} = \frac{1}{|\Gamma_s|} \int_{\Gamma_s} \phi \cdot \nu \,\mathrm{d}s.$$

By construction, it holds that $\int_{\Gamma_s} \varphi \cdot \nu \,\mathrm{d}s = 0$. We know from trace theory that there exists $w \in H^1(\Omega_s)^d$ such that $w = \varphi$ on Γ_s. On the other hand, from Proposition 2.5, the Stokes equation

$$\begin{cases} -\Delta\psi + \nabla\varrho = 0, & \text{in } \Omega_f, \\ \operatorname{div}\psi = 0, & \text{in } \Omega_f, \\ \psi = 0, & \text{on } \Gamma_f, \\ \psi = \varphi, & \text{on } \Gamma_s. \end{cases}$$

admits a solution $(\psi,\varrho)\in(H^1(\Omega_f)^d\cap H_f)\times L^2(\Omega_f)$. Choosing the pair $(w,\psi)\in H_1$ in (3.10) and then using Green's identity and the divergence theorem, we have

$$\left\langle \frac{\partial u}{\partial\nu}-\pi\nu-\sigma\cdot\nu,\varphi\right\rangle_{H^{-\frac{1}{2}}(\Gamma_s)^d\times H^{\frac{1}{2}}(\Gamma_s)^d}=0.$$

From (3.11) and the equation $\phi=\varphi+\overline{\phi}\nu$, we can see that this equality is also true if we replace the function φ by ϕ. Since $\phi\in H^{\frac{1}{2}}(\Gamma_s)^d$ is arbitrary, we obtain $\frac{\partial u}{\partial\nu}-\pi\nu-\sigma\cdot\nu=0$ in $H^{-\frac{1}{2}}(\Gamma_s)^d$. Using the same argument as in the proof of Theorem 2.4, it can be shown that $\pi=L_1\sigma+L_2u$. Combining the above observations shows that $(v,\sigma,z,u)\in D(A)$ and (3.1) holds.

Decompose a_λ into $a_\lambda=a_\lambda^1+a_\lambda^2$ where the sesquilinear forms $a_\lambda^1:H_1\times H_1\to\mathbb{C}$ and $a_\lambda^2:H_0\times H_0\to\mathbb{C}$ are defined by

$$a_\lambda^1((v,u),(w,\psi)):=\int_{\Omega_s}v\cdot w\,\mathrm{d}x+\frac{1}{\lambda}\int_{\Omega_s}\nabla v\cdot\nabla w\,\mathrm{d}x+\int_{\Omega_f}\nabla u\cdot\nabla\psi\,\mathrm{d}x,$$

$$a_\lambda^2((v,u),(w,\psi)):=(q(\lambda)-1)\int_{\Omega_s}v\cdot w\,\mathrm{d}x+\lambda\int_{\Omega_f}u\cdot\psi\,\mathrm{d}x.$$

Notice that the form a_λ^2 is bounded. On the other hand, for every nonzero element of (v,u) in H_1 there holds

$$\frac{|a_\lambda^1((v,u),(v,u))|}{\|(v,u)\|_{H_1}^2}=\left|1+\left(\frac{1}{\lambda}-1\right)\int_{\Omega_s}\frac{|\nabla v|^2}{\|(v,u)\|_{H_1}^2}\,\mathrm{d}x\right|.$$

Thus, a_λ^1 is H_1-coercive if $\inf_{\varepsilon\geq0}|1+(\frac{1}{\lambda}-1)\varepsilon|>0$ holds. This inequality is satisfied provided that $\operatorname{Im}\lambda\neq0$. From the compactness of the embedding $H_1\subset H_0$, it follows from Lemma 3.2 that $\lambda\neq0$ with $\operatorname{Re}\lambda\leq0$ is either in the resolvent set or an eigenvalue of A. Combined with the earlier remark that the right-half part of the complex plane lies in $\rho(A)$, this is equivalent to what the theorem stated.

For the operator A^*, notice that it is almost the same with A except for a change of signs on its definition as well as on its domain. These differences in signs will not affect the applicability of the analysis presented above. □

We would like to note that the method and results presented in the previous theorem can be adapted to the original fluid–structure system presented in [3,6], with or without delay.

The spectrum of the generator A on the imaginary axis and the stability of the corresponding semigroup are connected to the solvability of the over-determined boundary value problem on the structure domain

$$\begin{cases}-\Delta\varphi=\mu\varphi, & \text{in }\Omega_s,\\ \varphi=0, & \text{on }\Gamma_s,\\ \dfrac{\partial\varphi}{\partial\nu}=k\nu, & \text{on }\Gamma_s,\end{cases}\tag{3.12}$$

where $\mu\in\sigma(-\Delta_D)$, $k\in\mathbb{R}$ and $-\Delta_D:H^2(\Omega_s)^d\cap H_0^1(\Omega_s)^d\to L^2(\Omega_s)^d$ is the Dirichlet Laplacian. The spectrum of $-\Delta_D$ consists of only a countable number of positive eigenvalues, and we let $\sigma(-\Delta_D)=\{\mu_n\}_{n=1}^\infty$ arranged in increasing order so that $\mu_n\to\infty$. If $k=0$, then the unique continuation condition for elliptic operators in [29, Corollary 15.2.2] implies that $\varphi=0$. We consider the following hypothesis.

(H) The over-determined problem (3.12) has the trivial solution $\varphi=0$ and hence $k=0$.

Condition (H) imposed on the over-determined problem (3.12) is not new, and it was first introduced in [3], and later in [6,18,22], in the context of the stabilization of certain fluid–structure interaction models without delay. This condition depends on the geometry of the structure domain, and it has been studied also in [6] under certain domains. In fact they considered the over-determined problem where the

Neumann boundary condition appears only on a subset of the boundary. Condition (H) is satisfied for certain partially flat domains; however, this is not the case for spherical domains.

Theorem 3.3. *Let $\tau > 0$ be fixed.*

1. *If $k_1 > k_0$, then A and A^* have no purely imaginary eigenvalues, that is,*
$$\sigma(A) \cap i\mathbb{R} = \sigma(A^*) \cap i\mathbb{R} = \{0\}. \tag{3.13}$$
2. *Suppose that $k_1 = k_0$. If condition* (H) *holds, then* (3.13) *is satisfied.*
3. *Assume that $k_1 = k_0$ and* (3.12) *has non-trivial solutions φ_{n_j}, $j = 1, \ldots, J$ where possibly $J = \infty$. Let M be the set of all $m \in \mathbb{N}$ such that $\mu_m = \frac{\pi^2}{\tau^2}(2n+1)^2$ for some nonnegative integer n. Then,*
$$\sigma(A) \cap i\mathbb{R} = \sigma(A^*) \cap i\mathbb{R} = \{\pm i\sqrt{\mu_m}\}_{m\in M}. \tag{3.14}$$
Eigenfunctions of A corresponding to $\pm i\sqrt{\mu_m}$ for $m \in M$ are
$$X_{m,j} = \begin{pmatrix} \varphi_{n_j} \\ (\pm i\sqrt{\mu_m})^{-1}\nabla\varphi_{n_j} \\ e^{\pm i\theta\sqrt{\mu_m}}\varphi_{n_j} \\ 0 \end{pmatrix}, \quad j = 1, \ldots, J. \tag{3.15}$$
Similarly, eigenfunctions of A^* associated with $\pm i\sqrt{\mu_m}$ for $m \in M$ are
$$X^*_{m,j} = \begin{pmatrix} -\varphi_{n_j} \\ (\pm i\sqrt{\mu_m})^{-1}\nabla\varphi_{n_j} \\ e^{\mp i(\theta+\tau)\sqrt{\mu_m}}\varphi_{n_j} \\ 0 \end{pmatrix}, \quad j = 1, \ldots, J. \tag{3.16}$$

Proof. Let us determine the nonzero purely imaginary eigenvalues, if there are any. Take $X = (v, \sigma, z, u) \in D(A)$ with $AX = irX$ where $r \neq 0$ is a real number. Then, $(AX, X)_H = ir\|X\|_H^2$, and from (2.13), we have
$$\int_{\Omega_f} |\nabla u|^2 \,\mathrm{d}x + (k_1 - k_0)\int_{\Omega_s} |v|^2 \,\mathrm{d}x \leq -\mathrm{Re}(AX, X)_H = 0.$$
It follows that u is constant, and from the boundary condition on Γ_f, this constant must be zero. If $k_1 > k_0$, then the latter inequality implies that v is zero. Consequently, $\sigma = (ir)^{-1}\nabla v = 0$ and $z(\theta) = 0$ for every $\theta \in (-\tau, 0)$. This proves the first part.

The equation $AX = irX$ is equivalent to the systems (3.2)–(3.5) with $\lambda = ir$ together with the boundary conditions stated in the domain of A, which is in turn equivalent to the variational equality (3.10), where the right-hand side is equal to zero. Using these, it is not hard to see that $\varphi = -\frac{v}{ir}$ satisfies the over-determined problem
$$\begin{cases} -\Delta\varphi = -ir(ir + k_1 + k_0e^{-ir\tau})\varphi, & \text{in } \Omega_s, \\ \varphi = 0, & \text{in } \Gamma_s, \\ \dfrac{\partial\varphi}{\partial\nu} = \pi\nu, & \text{in } \Gamma_s. \end{cases} \tag{3.17}$$

Suppose that $k_1 = k_0$. Let $\lambda = -ir(ir + k_1 + k_0e^{-ir\tau})$. If $\lambda \notin \sigma(-\Delta_D)$ then the first two equations in (3.17) can be written as $(\lambda I - \Delta_D)\varphi = 0$ and hence $\varphi = 0$. Therfore $v = 0$, $\sigma = 0$, and $z = 0$, and we established the second part.

Finally, suppose that $k_1 = k_0$ and $\lambda = \mu_m$ for some integer m. For this to hold then, necessarily we must have $\cos r\tau = -1$ and $r^2 = \mu_m$. These imply that $r\tau = (2n+1)\pi$, and hence, $\frac{\pi^2}{\tau^2}(2n+1)^2 \in \sigma(-\Delta_D)$. This proves (3.14) in the case of A. The representation of the eigenfunctions in (3.15) can be obtained from (3.2)–(3.5). According to the isomorphism J given in the proof of Theorem 2.7, the eigenvectors for A^* are given by (3.16). Indeed, we have

$$A^* X^*_{m,j} = A^* J X_{m,j} = J A X_{m,j} = J(\pm i\sqrt{\mu_m} X_{m,j}) = \pm i\sqrt{\mu_m} X^*_{m,j}.$$

This proves the last part of the theorem. □

We can see from the above theorem that in the case where the damping and delay factors coincide and (3.12) has non-trivial solutions, there are sequences of delays converging to zero or at infinity for which the corresponding energy is constant. For example, if $\tau_{m,n} = \frac{\pi}{\sqrt{\mu_m}}(2n+1)$, then $\tau_{m,n} \to \infty$ as $n \to \infty$ and m fixed, while $\tau_{m,n} \to 0$ as $m \to \infty$ and n fixed. From the previous theorem and the classical result of Arendt-Batty [1] and Lyubich-Phong [21], we have the following strong stability result.

Theorem 3.4. [Asymptotic Stability] *Suppose that $k_1 = k_0 > 0$ and $\tau > 0$. The semigroup generated by $\tilde{A}$ is strongly stable, that is, $e^{t\tilde{A}} X_0 \to 0$ in H as $t \to \infty$ for every $X_0 \in N(A)^\perp$, if one of the following properties is satisfied.*

1. *The condition* (H) *holds.*
2. *It holds that $\tau \neq \frac{\pi}{\sqrt{\mu}}(2n+1)$ for every integer $n \geq 0$ and $\mu \in \sigma(-\Delta_D)$.*

Moreover, if Π is the orthogonal projection of H onto $N(A)$, then $e^{tA} X_0 \to \Pi X_0$ in H as $t \to \infty$ for every $X_0 \in H$.

This means that even though (H) is not satisfied, the system is still stable except for a countable number of delays. We would like to point out that in the case where $k_1 = k_0 = 0$, this stability property has been already proved in [22] for the nonlinear case and in [3,6] for the linear case with the displacement term in the wave equation. For decay rates in the non-delayed, case we refer to [2,5].

If $k_1 > k_0$, then we expect to have exponential stability. This is the content of the following theorem whose proof is based on the frequency domain method. Again, we mention that this has been already established in the non-delayed case, see [6,22]. The method in [6] is to show the uniform boundedness of the resolvents on the imaginary axis. The proof we provide below uses the Gearhart-Prüss Theorem.

Theorem 3.5. [Exponential Stability] *If $k_1 > k_0$, then the semigroup generated by $\tilde{A}$ is uniformly exponentially stable, that is, there are constants $M \geq 1$ and $\alpha > 0$ such that $\|e^{t\tilde{A}} X_0\|_H \leq M e^{-\alpha t} \|X_0\|_H$ for every $X_0 \in N(A)^\perp$ and $t \geq 0$. In particular, for each $t \geq 0$ and $X_0 \in H$, we have $\|e^{tA} X_0 - \Pi X_0\|_H \leq M e^{-\alpha t} \|X_0\|_H$.*

Proof. Assume on the contrary that the semigroup generated by $\tilde{A}$ is not exponentially stable. According to the Gearhart-Prüss Theorem, see [13, Theorem V.1.11], we have $\sup\{\|(\lambda I - \tilde{A})^{-1}\|_{\mathcal{L}(H)} : \operatorname{Re}\lambda > 0\} = \infty$. By the Banach-Steinhaus Theorem and the uniform boundedness of the resolvents on compact sets, there exists a sequence of complex numbers $(\lambda_n)_n$ with $\operatorname{Re}\lambda_n > 0$ such that $|\lambda_n| \to \infty$ and a sequence of unit vectors $X_n := (v_n, \sigma_n, z_n, u_n) \in D(\tilde{A})$ such that $\|(\lambda_n I - \tilde{A}) X_n\|_H \to 0$. Let $Y_n := (\eta_n, \kappa_n, \zeta_n, \varphi_n) = (\lambda_n I - \tilde{A}) X_n$. The latter equation is equivalent to the system (3.2)–(3.5) with λ, (v, σ, z, u) and $(\eta, \kappa, \zeta, \varphi)$ replaced by λ_n, $(v_n, \sigma_n, z_n, u_n)$ and $(\eta_n, \kappa_n, \zeta_n, \varphi_n)$, respectively.

From the dissipativity of the operator $\tilde{A}$, we have

$$\begin{aligned} \operatorname{Re}(Y_n, X_n) &= \operatorname{Re}(\lambda_n - (\tilde{A} X_n, X_n)_H) \\ &\geq \operatorname{Re}\lambda_n + \int_{\Omega_f} |\nabla u_n|^2 \, \mathrm{d}x + (k_1 - k_0) \int_{\Omega_s} |v_n|^2 \, \mathrm{d}x. \end{aligned}$$

Since $\operatorname{Re}\lambda_n > 0$ and $k_1 > k_0$ we have $\operatorname{Re}\lambda_n \to 0$,

$$v_n \to 0 \quad \text{strongly in } L^2(\Omega_s)^d, \tag{3.18}$$

$$u_n \to 0 \quad \text{strongly in } H^1(\Omega_f)^d, \tag{3.19}$$

where the second limit is due to the Poincaré inequality. Consequently, $|\mathrm{Im}\,\lambda_n| \to \infty$. The delay variable z_n satisfies the estimate

$$\int_{-\tau}^{0}\int_{\Omega_s} |z_n(\theta)|^2\,\mathrm{d}x\,\mathrm{d}\theta \le C_\tau \left(\int_{-\tau}^{0}\int_{\Omega_s} |\zeta_n(\theta)|^2\,\mathrm{d}x\,\mathrm{d}\theta + \int_{\Omega_s} |v_n|^2\,\mathrm{d}x \right) \tag{3.20}$$

for some constant $C_\tau > 0$. Using (3.18) and the fact that $\zeta_n \to 0$ in $L^2(-\tau, 0; L^2(\Omega_s)^d)$, we obtain

$$z_n \to 0 \quad \text{strongly in } L^2(-\tau, 0; L^2(\Omega_s)^d). \tag{3.21}$$

Taking the inner product in H, both sides of $Y_n = (\lambda_n I - \tilde{A})X_n$ with X_n yield the following set of equations

$$\int_{\Omega_s} \eta_n \cdot v_n\,\mathrm{d}x = (\lambda_n + k_1)\int_{\Omega_s} |v_n|^2\,\mathrm{d}x + \int_{\Omega_s} \sigma_n \cdot \nabla v_n\,\mathrm{d}x + \langle \sigma_n \cdot \nu, v_n\rangle_{H^{-\frac{1}{2}}(\Gamma_s)^d \times H^{\frac{1}{2}}(\Gamma_s)^d} + k_0 \int_{\Omega_s} z_n(-\tau)\cdot v_n\,\mathrm{d}x \tag{3.22}$$

$$\int_{\Omega_s} \kappa_n \cdot \sigma_n\,\mathrm{d}x = \lambda_n \int_{\Omega_s} |\sigma_n|^2\,\mathrm{d}x - \int_{\Omega_s} \nabla v_n \cdot \sigma_n\,\mathrm{d}x \tag{3.23}$$

$$\int_{-\tau}^{0}\int_{\Omega_s} \zeta_n(\theta)\cdot z_n(\theta)\,\mathrm{d}x\,\mathrm{d}\theta = \lambda_n \int_{-\tau}^{0}\int_{\Omega_s} |z_n(\theta)|^2\,\mathrm{d}x\,\mathrm{d}\theta - \int_{-\tau}^{0}\int_{\Omega_s} z_{n\theta}(\theta)\cdot z_n(\theta)\,\mathrm{d}x\,\mathrm{d}\theta \tag{3.24}$$

$$\int_{\Omega_f} \varphi_n \cdot u_n\,\mathrm{d}x = \lambda_n \int_{\Omega_f} |u_n|^2\,\mathrm{d}x + \int_{\Omega_f} |\nabla u_n|^2\,\mathrm{d}x - \left\langle \frac{\partial u_n}{\partial \nu} - \pi_n \nu, u_n \right\rangle_{H^{-\frac{1}{2}}(\Gamma_s)^d \times H^{\frac{1}{2}}(\Gamma_s)^d}. \tag{3.25}$$

Since X_n is bounded and $Y_n \to 0$ in H, each of these terms tends to 0 as $n \to \infty$.

Dividing (3.25) by $\mathrm{Im}\,\lambda_n$, taking the imaginary part and applying (3.19) yield

$$\frac{1}{\mathrm{Im}\,\lambda_n}\mathrm{Im}\left\langle \frac{\partial u_n}{\partial \nu} - \pi_n \nu, u_n \right\rangle_{H^{-\frac{1}{2}}(\Gamma_s)^d \times H^{\frac{1}{2}}(\Gamma_s)^d} \to 0. \tag{3.26}$$

Similarly, if we divide (3.24) by $\mathrm{Im}\,\lambda_n$, take the imaginary part and use (3.21), then we obtain

$$\frac{1}{\mathrm{Im}\,\lambda_n}\int_{-\tau}^{0}\int_{\Omega_s} \mathrm{Im}(z_{n\theta}(\theta)\cdot z_n(\theta))\,\mathrm{d}x\,\mathrm{d}\theta \to 0. \tag{3.27}$$

On the other hand, if we take the real part of (3.24) and pass to the limit, then we get

$$\mathrm{Re}\,\lambda_n \int_{-\tau}^{0}\int_{\Omega_s} |z_n(\theta)|^2\,\mathrm{d}x\,\mathrm{d}\theta - \frac{1}{2}\int_{\Omega_s} (|v_n|^2 - |z_n(-\tau)|^2)\,\mathrm{d}x \to 0$$

and by applying (3.18) and (3.21), we have

$$z_n(-\tau) \to 0 \quad \text{strongly in } L^2(\Omega_s)^d. \tag{3.28}$$

Now, if we take the sum of (3.23)–(3.25), subtract the result from (3.22) and use the equations $v_n = u_n$ on Γ_s and $\sigma_n \cdot \nu = \frac{\partial u_n}{\partial \nu} - \pi_n \nu$ in $H^{-\frac{1}{2}}(\Gamma_s)^d$, then we have

$$\lambda_n \left(1 - 2\int_{\Omega_s} |v_n|^2 \, \mathrm{d}x\right) - k_1 \int_{\Omega_s} |v_n|^2 \, \mathrm{d}x - 2\int_{\Omega_s} \mathrm{Re}(\nabla v_n \cdot \sigma_n) \, \mathrm{d}x$$
$$- k_0 \int_{\Omega_s} z_n(-\tau) \cdot v_n \, \mathrm{d}x - \int_{-\tau}^{0} \int_{\Omega_s} z_{n\theta}(\theta) \cdot z_n(\theta) \, \mathrm{d}x \, \mathrm{d}\theta + \int_{\Omega_f} |\nabla u_n|^2 \, \mathrm{d}x$$
$$- 2 \left\langle \frac{\partial u_n}{\partial \nu} - \pi_n \nu, u_n \right\rangle_{H^{-\frac{1}{2}}(\Gamma_s)^d \times H^{\frac{1}{2}}(\Gamma_s)^d} \to 0.$$

Dividing by $\mathrm{Im}\,\lambda_n$, taking the imaginary part and using (3.18), (3.26)–(3.28) give us $\|v_n\|^2_{L^2(\Omega_s)^d} \to \frac{1}{2}$, which is a contradiction to (3.18). Therefore, the semigroup generated by $\tilde{A}$ must be exponentially stable. This completes the proof of the theorem. □

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