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RESEARCH ARTICLE

Stabilization of the wave equation with acoustic and delay boundary conditions

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Abstract In this paper, we consider the wave equation on a bounded domain with mixed Dirichlet-impedance type boundary conditions coupled with oscillators on the Neumann boundary. The system has either a delay in the pressure term of the wave component or the velocity of the oscillator component. Using the velocity as a boundary feedback it is shown that if the delay factor is less than that of the damping factor then the energy of the solutions decays to zero exponentially. The results are based on the energy method, a compactness-uniqueness argument and an appropriate weighted trace estimate. In the critical case where the damping and delay factors are equal, it is shown using variational methods that the energy decays to zero asymptotically.

Keywords Wave equations · Acoustic boundary conditions · Feedback delays · Stabilization · Energy method

1 Introduction

Consider an open and bounded domain $\Omega \subset \mathbb{R}^n$ with C^2-boundary. This regularity condition on the boundary of Ω is assumed so that classical theory for elliptic boundary value problems is applicable. Suppose that $\partial\Omega$ is the disjoint union of Γ_D and Γ_N, that is, $\Gamma_D \cup \Gamma_N = \partial\Omega$ and $\overline{\Gamma}_N \cap \overline{\Gamma}_D = \emptyset$, and Γ_D, Γ_N are nonempty. Also, assume that there exists a strictly convex function $m \in C^2(\overline{\Omega})$ such that $\nabla m \cdot \nu \leq 0$ on Γ_N. We

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analyze the well-posedness and stability property of the wave equation with acoustic boundary conditions and boundary delay

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) = 0, & \text{in } (0,\infty)\times\Omega, \\ u(t,x) = 0, & \text{on } (0,\infty)\times\Gamma_D, \\ \dfrac{\partial u}{\partial \nu}(t,x) - \delta_t(t,x) = -au_t(t-\tau,x) - ku_t(t,x), & \text{on } (0,\infty)\times\Gamma_N, \\ M\delta_{tt}(t,x) + D\delta_t(t,x) + K\delta(t,x) + u_t(t,x) = 0, & \text{on } (0,\infty)\times\Gamma_N, \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & \text{in } \Omega, \\ u_t(t,x) = f(t,x), & \text{in } (-\tau,0)\times\Gamma_N, \\ \delta(0,x) = \delta_0(x), \qquad \delta_t(0,x) = \delta_1(x), & \text{on } \Gamma_N. \end{cases} \tag{1.1}$$

Here, $\tau > 0$ is a constant delay parameter and $a, k \geq 0$. All throughout this paper, we assume that $M, D, K > 0$. For simplicity of exposition we assume that M, D and K are constants. The case where they depend on x and uniformly bounded away from zero can be done in a similar manner as in the case when they are constants. The system (1.1) models the evolution of the velocity potential u of a fluid contained in the domain Ω where the speed of sound and fluid density are normalized to one. The boundary of the domain is not rigid, however, each point reacts like a harmonic oscillator. Assuming that the surface is locally reacting, the normal displacement δ of the boundary into the domain satisfies the above second order differential equation. For the wave motion on the boundary Γ_N, we assume an impedance-type boundary condition with a delay in the pressure term u_t. For more details in the absence of delay we refer to [2].

We can think of the term $-ku_t - au_t(\cdot - \tau)$ as a boundary feedback law where the second term represents delay. Our goal is to prove the exponential stability of the system when $a < k$. For the critical case $k = a$, it is shown that the energy of the solutions decay asymptotically to zero. This means that the mechanical dissipation in the oscillator component is strong enough to stabilize the system (1.1). In the absence of oscillators, stability and instability properties of this model was analyzed by Nicaise and Pignotti [11]. If there is no delay, the well-posedness of (1.1) using semigroup theory and the spectral properties of the generator has been studied in [2]. Works related to the stability or instability properties of wave equations with interior or boundary delay can be found in [1,3–5,8,12,13,16].

We will also consider the case where the delay occurs at the oscillator and the feedback law is given by $-D\delta_t$ in the oscillator equation

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) = 0, & \text{in } (0,\infty)\times\Omega, \\ u(t,x) = 0, & \text{on } (0,\infty)\times\Gamma_D, \\ \dfrac{\partial u}{\partial \nu}(t,x) + u_t(t,x) - \delta_t(t,x) = 0, & \text{on } (0,\infty)\times\Gamma_N, \\ M\delta_{tt}(t,x) + D_0\delta_t(t-\tau,x) + K\delta(t,x) & \\ \qquad\qquad + u_t(t,x) = -D\delta_t(t,x), & \text{on } (0,\infty)\times\Gamma_N, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & \text{in } \Omega, \\ \delta(0,x) = \delta_0(x), \quad \delta_t(0,x) = \delta_1(x), & \text{on } \Gamma_N, \\ \delta_t(t,x) = g(t,x), & \text{in } (-\tau,0)\times\Gamma_N. \end{cases} \tag{1.2}$$

If $D_0 > D$ then we show that the energy of the solution decays to zero exponentially. In the case $D_0 = D$, the solutions have an asymptotically decaying energy. This is due to the mechanical dissipation that is present on the Neumann boundary of the wave motion.

The classical energy of the solutions of (1.1) and (1.2) is defined by

$$E_0(t) = E_w(t) + E_b(t) \tag{1.3}$$

where

$$E_w(t) = \frac{1}{2}\int_\Omega (|u_t(t,x)|^2 + |\nabla u(t,x)|^2)\,\mathrm{d}x$$

represents the kinetic and potential energies of the wave motion while

$$E_b(t) = \frac{1}{2}\int_{\Gamma_N} (K|\delta(t,x)|^2 + M|\delta_t(t,x)|^2)\,\mathrm{d}x$$

are the potential and kinetic energies of the boundary motion. Under the assumption $k > a$ for (1.1) or $D > D_0$ in the case of (1.2), it is shown that the the energy of the solutions decay exponentially. The main difficulty in our problem is to have an observability estimate regarding the initial data for the oscillators, that is,

$$\int_{\Gamma_N} (|\delta_0(x)|^2 + |\delta_1(x)|^2)\,\mathrm{d}x \le CE(0).$$

for some $C > 0$, where E represents the total energy of the system (1.1) or (1.2). Refer to Sects. 4 and 5 for the exact formulation of the energy E in each of these cases. This observability estimate is proved thanks to an appropriate weighted trace estimate, see Lemma 3.3 below.

2 Well-posedness through semigroup theory

In this section, we prove the existence and uniqueness of solutions of (1.1) and (1.2) using semigroup theory for bounded linear operators. First let us consider the case of (1.1). Let $H^1_{\Gamma_D}(\Omega) = \{u \in H^1(\Omega) : u_{|\Gamma_D} = 0\}$ be equipped with the inner product

$$(u_1, u_2)_{H^1_{\Gamma_D}(\Omega)} = \int_\Omega \nabla u_1 \cdot \nabla u_2 \, \mathrm{d}x.$$

With the Poincaré inequality, the norm corresponding to this inner product is equivalent to the full Sobolev H^1-norm.

Define the graph space $E(\Delta) = \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}$ where Δ represents the distributional Laplacian. Equipped with the inner product

$$(v_1, v_2)_{E(\Delta)} = \int_\Omega (v_1 v_2 + \nabla v_1 \cdot \nabla v_2 + \Delta v_1 \, \Delta v_2) \, \mathrm{d}x,$$

$E(\Delta)$ becomes a Hilbert space. From [9, Section 1.5] there is a generalized trace operator $u \mapsto \frac{\partial u}{\partial \nu} \in \mathcal{L}(E(\Delta); H^{-\frac{1}{2}}(\Gamma_N))$ and we have the following Green's identity

$$\left\langle \frac{\partial u}{\partial \nu}, w \right\rangle_{H^{-\frac{1}{2}}(\Gamma_N) \times H^{\frac{1}{2}}(\Gamma_N)} = \int_\Omega (\Delta u) w \, \mathrm{d}x + \int_\Omega \nabla u \cdot \nabla w \, \mathrm{d}x, \quad \forall w \in H^1_{\Gamma_D}(\Omega),$$

where $\langle \cdot, \cdot \rangle$ represents the duality pairing between $H^{-\frac{1}{2}}(\Gamma_N)$ and $H^{\frac{1}{2}}(\Gamma_N)$.

Define $v(t, x) = u_t(t, x)$ for $(t, x) \in (0, \infty) \times \Omega$, $z(t, \theta, x) = u_t(t + \theta, x)$ for $(t, \theta, x) \in (0, \infty) \times (-\tau, 0) \times \Gamma_N$ and $\sigma(t, x) = \delta_t(t, x)$ for $(t, x) \in (0, \infty) \times \Gamma_N$. Then (1.1) is equivalent to the system

$$\begin{cases} u_t(t, x) - v(t, x) = 0, & \text{in } (0, \infty) \times \Omega, \\ v_t(t, x) - \Delta u(t, x) = 0, & \text{in } (0, \infty) \times \Omega, \\ z_t(t, \theta, x) - z_\theta(t, \theta, x) = 0, & \text{in } (0, \infty) \times (-\tau, 0) \times \Gamma_N, \\ u(t, x) = 0, & \text{on } (0, \infty) \times \Gamma_D, \\ \dfrac{\partial u}{\partial \nu}(t, x) - \sigma(t, x) = -az(t, -\tau, x) - kv(t, x), & \text{on } (0, \infty) \times \Gamma_N, \\ \delta_t(t, x) - \sigma(t, x) = 0, & \text{on } (0, \infty) \times \Gamma_N, \\ M\sigma_t(t, x) + D\sigma(t, x) + K\delta(t, x) + v(t, x) = 0, & \text{on } (0, \infty) \times \Gamma_N, \\ u(0, x) = u_0(x), \quad v(0, x) = u_1(x), & \text{in } \Omega, \\ z(0, \theta, x) = f(\theta, x), & \text{in } (-\tau, 0) \times \Gamma_N, \\ \delta(0, x) = \delta_0(x), \quad \sigma(0, x) = \delta_1(x), & \text{on } \Gamma_N. \end{cases} \tag{2.1}$$

The system (2.1) is posed in the space of data with finite energies including the delay term. In this respect, we consider the Hilbert space

$$X = H^1_{\Gamma_D}(\Omega) \times L^2(\Omega) \times L^2((-\tau, 0) \times \Gamma_N) \times L^2(\Gamma_N) \times L^2(\Gamma_N)$$

with the inner product

$$\begin{aligned} ((u_1, v_1, z_1, \delta_1, \sigma_1), (u_2, v_2, z_2, \delta_2, \sigma_2))_X &= \int_\Omega (\nabla u_1(x) \cdot \nabla u_2(x) + v_1(x) v_2(x)) \, \mathrm{d}x \\ &+ \int_{-\tau}^0 \int_{\Gamma_N} z_1(\theta, x) z_2(\theta, x) \, \mathrm{d}x \, \mathrm{d}\theta + \int_{\Gamma_N} (K\delta_1(x)\delta_2(x) + M\sigma_1(x)\sigma_2(x)) \, \mathrm{d}x. \end{aligned}$$

as the state space. Define the linear operator $A : D(A) \subset X \to X$ by

$$A \begin{pmatrix} u \\ v \\ z \\ \delta \\ \sigma \end{pmatrix} = \begin{pmatrix} v \\ \Delta u \\ z_\theta \\ \sigma \\ -M^{-1}(D\sigma + K\delta + v) \end{pmatrix}$$

where

$$D(A) = \{(u, v, z, \delta, \sigma) \in X : u \in E(\Delta),\ v \in H^1_{\Gamma_D}(\Omega),\ z \in H^1((-\tau, 0); L^2(\Gamma_N)) \\ \frac{\partial u}{\partial \nu} = -kv - az(-\tau) + \sigma \text{ on } \Gamma_N,\ z(0) = v\}.$$

Then (1.1) can now be phrased as a first order evolution equation in X

$$\begin{cases} U'(t) = AU(t), \quad t > 0, \\ U(0) = (u_0, u_1, f, \delta_0, \delta_1). \end{cases} \tag{2.2}$$

Theorem 2.1 *The operator A generates a strongly continuous semigroup on X. For every $U_0 \in D(A)$ the Cauchy problem (2.2) admits a unique solution*

$$U \in C([0, \infty); D(A)) \cap C^1([0, \infty); X)$$

where $D(A)$ is equipped with the graph norm.

Proof To prove the theorem, we use the Lumer–Phillips Theorem in reflexive Banach spaces, see [7] for example. Let $U = (u, v, z, \delta, \sigma) \in D(A)$. Using Green's identity and the boundary conditions

$$\begin{aligned} (AU, U)_X = & \int_\Omega (\nabla v \cdot \nabla u + (\Delta u)v)\, \mathrm{d}x + \int_{-\tau}^0 \int_{\Gamma_N} z_\theta z\, \mathrm{d}x\, \mathrm{d}\theta \\ & + \int_{\Gamma_N} K\sigma\delta\, \mathrm{d}x - \int_{\Gamma_N} (D\sigma + K\delta + v)\sigma\, \mathrm{d}x \\ = & - k \int_{\Gamma_N} |v|^2\, \mathrm{d}x - a \int_{\Gamma_N} z(-\tau)v + \frac{1}{2} \int_{\Gamma_N} |v|^2\, \mathrm{d}x \\ & - \frac{1}{2} \int_{\Gamma_N} |z(-\tau)|^2\, \mathrm{d}x - D \int_{\Gamma_N} |\sigma|^2\, \mathrm{d}x. \end{aligned}$$

Applying Cauchy–Schwarz inequality we obtain

$$(AU, U)_X \le \kappa \int_{\Gamma_N} |v|^2\, \mathrm{d}x - D \int_{\Gamma_N} |\sigma|^2\, \mathrm{d}x$$

where $\kappa = -k + \frac{a^2}{2} + \frac{1}{2}$. Thus $A - \kappa I$ is dissipative.

The next step is to show the range condition $R(\lambda I - A) = X$ for $\lambda > 0$. Fix $(f, g, h, \varphi, \phi) \in X$ and $\lambda > 0$. The equation $(\lambda I - A)(u, v, z, \delta, \sigma) = (f, g, h, \varphi, \phi)$ for $(u, v, z, \delta, \sigma) \in D(A)$ is equivalent to the system

$$\lambda u - v = f \tag{2.3}$$

$$\lambda v - \Delta u = g \tag{2.4}$$

$$\lambda z(\theta) - z_\theta(\theta) = h(\theta) \tag{2.5}$$

$$\lambda\delta - \sigma = \varphi \tag{2.6}$$

$$\lambda\sigma + M^{-1}(D\sigma + K\delta + v) = \phi \tag{2.7}$$

together with the conditions $z(0) = v$ and $\frac{\partial u}{\partial \nu} = -kv - az(-\tau) + \sigma$. Using (2.5) we obtain immediately from the variation of parameters formula that

$$z(\theta) = e^{\lambda\theta}v + \int_\theta^0 e^{\lambda(\theta-\vartheta)}h(\vartheta)\,\mathrm{d}\vartheta. \tag{2.8}$$

From (2.3) we also have

$$u = \frac{1}{\lambda}(v + f). \tag{2.9}$$

Multiplying (2.7) by M and using $\sigma = \lambda\delta + \varphi$, which follows from (2.6), and then solving for δ we arrive at

$$\delta = \frac{M}{p(\lambda)}\phi - \frac{M\lambda + D}{p(\lambda)}\varphi - \frac{1}{p(\lambda)}v \tag{2.10}$$

where $p(\lambda) = M\lambda^2 + D\lambda + K > 0$ for $\lambda > 0$. Thus

$$\sigma = \frac{M\lambda}{p(\lambda)}\phi - \left(\frac{(M\lambda + D)\lambda}{p(\lambda)} - 1\right)\varphi - \frac{\lambda}{p(\lambda)}v. \tag{2.11}$$

Taking the inner product of (2.4) with λw in $L^2(\Omega)$, where $w \in H^1_{\Gamma_D}(\Omega)$, yields

$$\lambda^2 \int_\Omega vw\,\mathrm{d}x - \lambda \int_\Omega (\Delta u)w\,\mathrm{d}x = \lambda \int_\Omega gw\,\mathrm{d}x. \tag{2.12}$$

Using Green's identity, the boundary condition $\frac{\partial u}{\partial \nu} = -kv - az(-\tau) + \sigma$, the Eqs. (2.8) and (2.11), and then rearranging the terms we obtain the variational equation

$$\begin{aligned}\lambda^2 \int_\Omega vw\,\mathrm{d}x &+ \int_\Omega \nabla v \cdot \nabla w\,\mathrm{d}x + q(\lambda)\int_{\Gamma_N} vw\,\mathrm{d}x \\ &= \int_\Omega (\lambda g w - \nabla f \cdot \nabla w)\,\mathrm{d}x + \int_{\Gamma_N} Fw\,\mathrm{d}x, \quad \forall w \in H^1_{\Gamma_D}(\Omega),\end{aligned} \tag{2.13}$$

where

$$q(\lambda) = \lambda\left(k + ae^{-\lambda\tau} + \frac{\lambda}{p(\lambda)}\right)$$
$$F = -\, a\lambda \int_{-\tau}^{0} e^{-\lambda(\tau+\vartheta)} h(\vartheta)\, \mathrm{d}\vartheta - \lambda\left(\frac{(M\lambda + D)\lambda}{p(\lambda)} - 1\right)\varphi + \frac{M\lambda^2}{p(\lambda)}\phi.$$

Since $q(\lambda) > 0$ and $F \in L^2(\Gamma_N)$, Lax–Milgram Lemma implies that there exists a unique $v \in H^1_{\Gamma_D}(\Omega)$ satisfying (2.13). With this v in hand, we define z, u, δ and σ by (2.8), (2.9), (2.10) and (2.11), respectively. It is clear that $z \in H^1((-\tau, 0); L^2(\Gamma_N))$ and $z(0) = v$. Choosing $w \in \mathcal{D}(\Omega)$ in (2.13), it follows that

$$\Delta u = -\lambda^2 u + \lambda f + g$$

in the sense of distributions and therefore $u \in E(\Delta) \cap H^1_{\Gamma_D}(\Omega)$. Applying Green's identity and by reversing the passage from (2.12) to (2.13) we have

$$\left\langle \frac{\partial u}{\partial \nu}, w \right\rangle_{H^{-\frac{1}{2}}(\Gamma_N)\times H^{\frac{1}{2}}(\Gamma_N)} = \int_{\Gamma_N} (-kv - az(-\tau) + \sigma) w \,\mathrm{d}x$$

for every $w \in H^1_{\Gamma_D}(\Omega)$. Because the trace operator maps $H^1_{\Gamma_D}(\Omega)$ onto $H^{\frac{1}{2}}(\Gamma_N)$, it follows that $\frac{\partial u}{\partial \nu} = -kv - az(-\tau) + \sigma$ in $H^{-\frac{1}{2}}(\Gamma_N)$. Hence $(u, v, z, \delta, \sigma) \in D(A)$.

Therefore $R(\lambda I - A) = X$ for all $\lambda > 0$ so that $R(\lambda I - (A - \kappa I)) = R((\lambda + \kappa)I - A) = X$ for all $\lambda > \max(0, \kappa)$. Thus, by the Lumer–Philipps Theorem $A - \kappa I$ generates a strongly semigroup of contractions in X, and consequently A generates a strongly continuous semigroup by the perturbation theorem. □

We turn to the well-posedness of (1.2). Let $\zeta(t, \theta, x) = \delta_t(t + \theta, x)$ for $(t, \theta, x) \in (0, \infty) \times (-\tau, 0) \times \Gamma_N$ so that $\zeta_t(t, \theta, x) = \zeta_\theta(t, \theta, x)$. Define the operator $A_0 : D(A_0) \subset X \to X$ by

$$A_0 \begin{pmatrix} u \\ v \\ \zeta \\ \delta \\ \sigma \end{pmatrix} = \begin{pmatrix} v \\ \Delta u \\ \zeta_\theta \\ \sigma \\ -M^{-1}(D_0\zeta(-\tau) + K\delta + v + D\sigma) \end{pmatrix}$$

where

$$D(A_0) = \{(u, v, z, \delta, \sigma) \in X : u \in E(\Delta),\ v \in H^1(\Omega),\ \zeta \in H^1((-\tau, 0); L^2(\Gamma_N))$$
$$\frac{\partial u}{\partial \nu} = -v + \sigma \text{ on } \Gamma_N,\ \zeta(0) = \sigma\}.$$

Then (1.2) can be written as a first order Cauchy problem in X

$$\begin{cases} U'(t) = A_0 U(t), & t > 0, \\ U(0) = U_0 := (u_0, u_1, f, \delta_0, \delta_1). \end{cases} \tag{2.14}$$

Furthermore, we have the following theorem whose proof follows the same steps as in the proof of the previous theorem, and so the details are omitted.

Theorem 2.2 *The operator A_0 generates a strongly continuous semigroup on X. In particular, for every $U_0 \in D(A_0)$ the Cauchy problem (2.14) admits a unique solution $U \in C([0,\infty); D(A_0)) \cap C^1([0,\infty); X)$.*

3 Stability for the system with delay in the wave component

We will use the observability result in Lasiecka et al. [10] for wave equations with mixed Dirichlet–Neumann boundary conditions and a uniqueness–compactness argument to prove the exponential decay of the energy of the solutions of (1.1) and (1.2). Again, let us first consider (1.1) and introduce the total energy

$$E(t) = E_0(t) + \frac{a}{2}\int_{-\tau}^{0}\int_{\Gamma_N} |u_t(t+\theta, x)|^2 \,\mathrm{d}x\,\mathrm{d}\theta =: E_0(t) + E_d(t).$$

For convenience, we introduce the shorthand $f(t) = f(t, \cdot)$.

Theorem 3.1 *Suppose that $k > a$. Then there exists a constant $C > 0$ independent of t such that for every data in $D(A)$ it holds that*

$$E'(t) \leq -CD(t), \quad t > 0, \tag{3.1}$$

where

$$D(t) = \int_{\Gamma_N} (|u_t(t,x)|^2 + |u_t(t-\tau, x)|^2 + |\delta_t(t,x)|^2)\,\mathrm{d}x.$$

The map $U_0 = (u_0, u_1, \delta_0, \delta_1, f_0) \mapsto (u_t, u_t(\cdot - \tau))$ from $D(A)$ to $L^2(0,T; L^2(\Gamma_N)^2)$ admits a unique extension to X.

Proof Taking the derivative of the energy, applying Green's identity, the boundary conditions and the differential equation for δ we have

$$\begin{aligned} E'(t) &= \int_\Omega (u_t(t)u_{tt}(t) + \nabla u(t)\cdot\nabla u_t(t))\,\mathrm{d}x + \int_{\Gamma_N} K\delta(t)\delta_t(t)\,\mathrm{d}x \\ &\quad + \int_{\Gamma_N} M\delta_t(t)\delta_{tt}(t)\,\mathrm{d}x + a\int_{-\tau}^{0}\int_{\Gamma_N} u_t(t+\theta)u_{tt}(t+\theta)\,\mathrm{d}x\,\mathrm{d}\theta \\ &= \int_{\Gamma_N} u_t(t)(\delta_t(t) - ku_t(t) - au_t(t-\tau))\,\mathrm{d}x + \int_{\Gamma_N} K\delta(t)\delta_t(t)\,\mathrm{d}x \end{aligned}$$

$$+\int_{\Gamma_N} \delta_t(t)(-D\delta_t(t) - K\delta(t) - u_t(t))\,\mathrm{d}x$$
$$+\,a\int_{-\tau}^{0}\int_{\Gamma_N} u_\theta(t+\theta)u_{\theta\theta}(t+\theta)\,\mathrm{d}x\,\mathrm{d}\theta.$$

The last integral can be simplified, using Fubini's Theorem, to

$$\int_{-\tau}^{0}\int_{\Gamma_N} u_\theta(t+\theta)u_{\theta\theta}(t+\theta)\,\mathrm{d}x\,\mathrm{d}\theta = \frac{1}{2}\int_{\Gamma_N}\int_{-\tau}^{0}\frac{\mathrm{d}}{\mathrm{d}\theta}|u_\theta(t+\theta)|^2\,\mathrm{d}\theta\,\mathrm{d}x$$
$$= \frac{1}{2}\int_{\Gamma_N}(|u_\theta(t)|^2 - |u_\theta(t-\tau)|^2)\,\mathrm{d}x \;=\; \frac{1}{2}\int_{\Gamma_N}(|u_t(t)|^2 - |u_t(t-\tau)|^2)\,\mathrm{d}x.$$

Therefore from Cauchy–Schwarz inequality we obtain

$$E'(t) \le -\frac{1}{2}(k-a)\int_{\Gamma_N}|u_t(t)|^2\,\mathrm{d}x - \frac{a}{2k}(k-a)\int_{\Gamma_N}|u_t(t-\tau)|^2\,\mathrm{d}x$$
$$-\,D\int_{\Gamma_N}|\delta_t(t)|^2\,\mathrm{d}x$$

and the result follows since $k > a$.

The rest of the theorem is a direct consequence of the estimate

$$\int_0^T\int_{\Gamma_N}(|u_t(t,x)|^2 + |u_t(t-\tau,x)|^2)\,\mathrm{d}x\,\mathrm{d}t \le -C^{-1}(E(T)-E(0)) \le C^{-1}E(0),$$

obtained by integrating (3.1), and the fact that $E(0)$ is equivalent to $\|U_0\|_H^2$. □

Corollary 3.2 *The map* $U_0 \mapsto \frac{\partial u}{\partial \nu} : D(A) \to L^2(0,T;L^2(\Gamma_N))$ *admits a unique extension to* X.

Proof If $U_0 \in D(A)$ then $U(t) := e^{tA}U_0 \in C([0,T];D(A)) \cap C^1([0,T];X)$. In particular, for each $t \in [0,T]$ we have $\frac{\partial}{\partial \nu}u(t) = -u_t(t) - ku_t(t-\tau) - \delta_t(t)$ in $L^2(\Gamma_N)$. The corollary follows immediately from the previous theorem. □

The following lemma plays a crucial role in the proof of the observability estimate for the oscillator component. The proof is based on the multiplier method.

Lemma 3.3 (Weighted-Trace Estimate) *For every* $T > 0$, $\vartheta > 0$ *and* $w \in H^1((0,T);L^2(\Gamma_N))$ *we have*

$$\int_{\Gamma_N}|w(0,x)|^2\,\mathrm{d}x + \int_{\Gamma_N}|w(T,x)|^2\,\mathrm{d}x$$
$$\le \vartheta\int_0^T\int_{\Gamma_N}|w_t(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t + \left(\frac{2}{T}+\frac{1}{\vartheta}\right)\int_0^T\int_{\Gamma_N}|w(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t. \quad (3.2)$$

Proof Note that the left hand side of (3.2) makes sense due to the embedding $H^1((0,T); L^2(\Gamma_N)) \hookrightarrow C([0,T]; L^2(\Gamma_N))$. By a standard density argument, we may take without loss of generality that $w \in C^1([0,T] \times \Gamma_N)$. Define the multiplier $\chi : [0,T] \to [-1,1]$ by

$$\chi(t) = \frac{2t}{T} - 1.$$

By Young's inequality $ab \leq \frac{\vartheta}{2}a^2 + \frac{1}{2\vartheta}b^2$ for $a, b \geq 0$ and $\vartheta > 0$, and the fact that $\|\chi\|_{L^\infty[0,T]} = 1$ we have

$$\begin{aligned}
|w(T,x)|^2 + |w(0,x)|^2 &= \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}(\chi(t)|w(t,x)|^2)\,\mathrm{d}t \\
&= \int_0^T (\frac{2}{T}|w(t,x)|^2 + 2\chi(t)w_t(t,x)w(t,x))\,\mathrm{d}t \\
&\leq \left(\frac{2}{T} + \frac{1}{\vartheta}\right)\int_0^T |w(t,x)|^2\,\mathrm{d}t + \vartheta\int_0^T |w_t(t,x)|^2\,\mathrm{d}t.
\end{aligned}$$

Integrating over Γ_N proves the desired estimate. □

Theorem 3.4 *There exists $T^* > 0$ depending only on M, D and K such that for all $T > T^*$ there is a constant $C > 0$ independent of T such that every regular solution of*

$$M\delta_{tt} + D\delta_t + K\delta = f \qquad in\ (0,\infty) \times \Gamma_N \tag{3.3}$$

with $f \in L^2((0,T) \times \Gamma_N)$ satisfies the estimate

$$\begin{aligned}
&\int_{\Gamma_N} (|\delta(0)|^2 + |\delta_t(0)|^2)\,\mathrm{d}x + \int_0^T\int_{\Gamma_N} |\delta(t)|^2\,\mathrm{d}t \\
&\quad \leq C\int_0^T\int_{\Gamma_N} (|\delta_t(t)|^2 + |f(t)|^2)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.4}$$

Proof In this proof, C will denote a generic positive constant depending only on M, D and K. Multiplying the Eq. (3.3) by δ and then integrating over $(0,T) \times \Gamma_N$ we have

$$\begin{aligned}
&M\int_{\Gamma_N} (\delta_t(T)\delta(T) - \delta_t(0)\delta(0))\,\mathrm{d}x - M\int_0^T\int_{\Gamma_N} |\delta_t(t)|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{D}{2}\int_{\Gamma_N} (|\delta(T)|^2 - |\delta(0)|^2)\,\mathrm{d}x + K\int_0^T\int_{\Gamma_N} |\delta(t)|^2\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\Gamma_N} f(t)\delta(t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Using Young's inequality we have

$$\int_0^T\int_{\Gamma_N} |\delta(t)|^2 \,\mathrm{d}x\,\mathrm{d}t \le C\int_{\Gamma_N} (|\delta_t(T)|^2 + |\delta_t(0)|^2 + |\delta(T)|^2 + |\delta(0)|^2)\,\mathrm{d}x + C\int_0^T\int_{\Gamma_N} (|\delta_t(t)|^2 + |f(t)|^2)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.5}$$

According to Lemma 3.3, for all $\vartheta_0 > 0$ and $\vartheta_1 > 0$ it holds that

$$\int_{\Gamma_N} |\delta(0)|^2\,\mathrm{d}x + \int_{\Gamma_N} |\delta(T)|^2\,\mathrm{d}x \le \vartheta_0 \int_0^T\int_{\Gamma_N} |\delta_t(t)|^2\,\mathrm{d}x\,\mathrm{d}t + \left(\frac{2}{T} + \frac{1}{\vartheta_0}\right)\int_0^T\int_{\Gamma_N} |\delta(t)|^2\,\mathrm{d}x\,\mathrm{d}t \tag{3.6}$$

and

$$\int_{\Gamma_N} |\delta_t(0)|^2\,\mathrm{d}x + \int_{\Gamma_N} |\delta_t(T)|^2\,\mathrm{d}x \le \vartheta_1 \int_0^T\int_{\Gamma_N} |\delta_{tt}(t)|^2\,\mathrm{d}x\,\mathrm{d}t + \left(\frac{2}{T} + \frac{1}{\vartheta_1}\right)\int_0^T\int_{\Gamma_N} |\delta_t(t)|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{3.7}$$

Using (3.6) and (3.7) in (3.5) yields

$$\begin{aligned}\int_0^T\int_{\Gamma_N} |\delta(t)|^2\,\mathrm{d}x\,\mathrm{d}t \le{}& \left(\frac{2C}{T} + \frac{C}{\vartheta_0}\right)\int_0^T\int_{\Gamma_N} |\delta(t)|^2\,\mathrm{d}x\,\mathrm{d}t \\ &+ C\vartheta_1\int_0^T\int_{\Gamma_N} |\delta_{tt}(t)|^2\,\mathrm{d}x\,\mathrm{d}t \\ &+ C\left(1 + \vartheta_0 + \frac{2}{T} + \frac{1}{\vartheta_1}\right)\int_0^T\int_{\Gamma_N} (|\delta_t(t)|^2 + |f(t)|^2)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.8}$$

The second term on the right hand side of (3.8) can be absorbed by the other two terms. Indeed, from the Eq. (3.3) once more we have

$$\int_0^T\int_{\Gamma_N} |\delta_{tt}(t)|^2\,\mathrm{d}x\,\mathrm{d}t \le C\int_0^T\int_{\Gamma_N} |\delta(t)|^2\,\mathrm{d}x\,\mathrm{d}t + C\int_0^T\int_{\Gamma_N} (|\delta_t(t)|^2 + |f(t)|^2)\,\mathrm{d}x\,\mathrm{d}t \tag{3.9}$$

and therefore

$$\int_0^T\int_{\Gamma_N}|\delta(t)|^2\,\mathrm{d}x\,\mathrm{d}t \le \left(\frac{2C}{T}+\frac{C}{\vartheta_0}+C\vartheta_1\right)\int_0^T\int_{\Gamma_N}|\delta(t)|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+C\left(1+\vartheta_0+\vartheta_1+\frac{2}{T}+\frac{1}{\vartheta_1}\right)\int_0^T\int_{\Gamma_N}(|\delta_t(t)|^2+|f(t)|^2)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.10}$$

Choosing $T^*=8C$, $\vartheta_1=\frac{1}{4C}$ and $\vartheta_0=4C$ we obtain from (3.10) that

$$\int_0^T\int_{\Gamma_N}|\delta(t)|^2\,\mathrm{d}t \le C\int_0^T\int_{\Gamma_N}(|\delta_t(t)|^2+|f(t)|^2)\,\mathrm{d}x\,\mathrm{d}t$$

for all $T>T^*$. Consequently, (3.4) follows from this estimate together with (3.6), (3.7) and (3.9). □

Before we proceed, we recall the following compactness result in [14].

Theorem 3.5 (Aubin–Lions–Simon) *Let X, B and Y be Banach spaces such that the embeddings $X\subset B\subset Y$ are continuous and the embedding $X\subset B$ is compact. If $(f_n)_n$ is bounded in $L^\infty(0,T;X)$ and $(f_n')_n$ is bounded in $L^r(0,T;Y)$ for some $r>1$ then $(f_n)_n$ is relatively compact in $C(0,T;B)$.*

Theorem 3.6 *There exists $T^*>0$ such that for every $T>T^*$, there is a constant $C_T>0$ so that every solution of* (1.1) *with initial data in $D(A)$ satisfies*

$$E(0)\le C_T\int_0^T D(t)\,\mathrm{d}t.$$

Proof From the observability estimate in [10], there exists $\tilde{T}>0$ such that for every $T>\tilde{T}$ there is a constant $c_T>0$ with

$$E_w(0)\le c_T\int_0^T\int_{\Gamma_N}\left(\left|\frac{\partial u}{\partial\nu}\right|^2+|u_t|^2\right)\mathrm{d}x\,\mathrm{d}t+c_T\|u\|_{H^{\frac{1}{2}+\varepsilon}((0,T)\times\Omega)}$$

whenever $\varepsilon>0$. Using the boundary condition on Γ_N we have

$$E_w(0)\le c_T\int_0^T D(t)\,\mathrm{d}t+c_T\|u\|_{H^{\frac{1}{2}+\varepsilon}((0,T)\times\Omega)},\qquad\forall\,T>\tilde{T}.$$

From Theorem 3.4 with $f=-u_t$, we can see that there exists a constant $T_0^*>0$ such that

$$E_b(0)\le C\int_0^T\int_{\Gamma_N}(|u_t(t)|^2+|\delta_t(t)|^2)\,\mathrm{d}x\,\mathrm{d}t.$$

for all $T > T_0^*$. The change of variable $t = \theta + \tau$ implies that

$$E_d(0) = \frac{a}{2}\int_{-\tau}^{0}\int_{\Gamma_N} |u_t(\theta)|^2 \,\mathrm{d}x\,\mathrm{d}\theta = \frac{a}{2}\int_{0}^{\tau}\int_{\Gamma_N} |u_t(t-\tau)|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Therefore for all $T > T^* := \max(\tilde{T}, T_0^*, \tau)$ we have

$$\begin{aligned} E(0) &= E_w(0) + E_b(0) + E_d(0) \\ &\le C_T \int_0^T D(t)\,\mathrm{d}t + C_T \|u\|_{H^{\frac{1}{2}+\varepsilon}((0,T)\times\Omega)}. \end{aligned} \tag{3.11}$$

The second term can be absorbed by the first term using a compactness-uniqueness argument, for example see [11]. Indeed, by contradiction, suppose that there exists a sequence $(U_{0n})_n \subset D(A)$ such that

$$E^{(n)}(0) > n\int_0^T D^{(n)}(t)\,\mathrm{d}t, \tag{3.12}$$

where $E^{(n)}$ and $D^{(n)}$ represents the total energy and dissipation term of the system corresponding to the solution $(u_n(t), u_{nt}(t), u_{nt}(t-\tau), \delta_n(t), \delta_{nt}(t)) = U_n(t) = e^{tA}U_{0n}$. By normalizing u_n, we can assume without loss of generality that

$$\|u_n\|_{H^{\frac{1}{2}+\varepsilon}((0,T)\times\Omega)} = 1. \tag{3.13}$$

Using (3.11) we have

$$E^{(n)}(0) \le C_T \int_0^T D^{(n)}(t)\,\mathrm{d}t + C_T. \tag{3.14}$$

From (3.12) and (3.14) we can see that

$$\int_0^T D^{(n)}(t)\,\mathrm{d}t < \frac{C_T}{n - C_T} \tag{3.15}$$

for all $n > C_T$.

Since the energy is decreasing one can see from (3.14) that

$$\begin{aligned} \|u_n\|^2_{H^1((0,T)\times\Omega)} &= \int_0^T (\|u_{nt}(t)\|^2_{L^2(\Omega)} + \|\nabla u_n(t)\|^2_{[L^2(\Omega)]^n})\,\mathrm{d}t \\ &\le \int_0^T E^{(n)}(t)\,\mathrm{d}t \;\le\; \int_0^T E^{(n)}(0)\,\mathrm{d}t \\ &\le \frac{TC_T^2}{n - C_T} + TC_T. \end{aligned}$$

Thus, $(u_n)_n$ is bounded in $H^1((0,T)\times\Omega)$. By the compactness of the embedding $H^{\frac{1}{2}+\varepsilon}((0,T)\times\Omega)\subset H^1((0,T)\times\Omega)$, with $0<\varepsilon<\frac{1}{2}$, we have

$$u_n \to u \quad \text{strongly in } H^{\frac{1}{2}+\varepsilon}((0,T)\times\Omega) \tag{3.16}$$

after extracting an appropriate subsequence. Thus, $u_n \to u$ in $L^2(0,T;L^2(\Omega))$.

The sequence $(U_{0n})_n$ is bounded in X according to (3.14), and hence up to a subsequence, U_{0n} converges weakly to some element U_0 in X. Let $\tilde{U}(t)=e^{tA}U_0$. Then

$$U_n \to \tilde{U} \quad \text{weakly-star in } L^\infty(0,T;X). \tag{3.17}$$

Indeed, for $v\in L^1(0,T;X)$ we have

$$\left|\int_0^T (U_n(t)-\tilde{U}(t),v(t))_X\,\mathrm{d}t\right| \le \int_0^T |(U_{0n}-U_0,(e^{tA})^*v(t))_X|\,\mathrm{d}t \to 0$$

by the dominated convergence theorem and the uniform boundedness of $t\mapsto e^{tA}$ on compact intervals. The limit (3.17) implies that

$$\begin{aligned} &u_n \to u \quad \text{weakly-star in } L^\infty(0,T;H^1(\Omega)),\\ &u_{nt} \to u_t \quad \text{weakly-star in } L^\infty(0,T;L^2(\Omega)). \end{aligned}$$

Consequently,

$$\|u_n\|_{L^\infty(0,T;H^1(\Omega))}+\|u_{nt}\|_{L^\infty(0,T;L^2(\Omega))}\le C_T,$$

for some constant $C_T>0$ independent of n.

Since the embedding $H^1(\Omega)\subset H^{1-\varepsilon}(\Omega)$ is compact and $H^1(\Omega)\subset H^{1-\varepsilon}(\Omega)\subset L^2(\Omega)$ are continuous, we can apply the Theorem 3.5 to get that, after taking a subsequence,

$$u_n \to u \quad \text{strongly in } L^\infty(0,T;H^{1-\varepsilon}(\Omega)).$$

By trace theory, $0=u_{n|\Gamma_D}\to u_{|\Gamma_D}$ in $L^\infty(0,T;L^2(\Gamma_D))$ so that $u=0$ on Γ_D. From (3.15) one can see that $(u_{nt},\delta_{nt},u_{nt}(\cdot-\tau))\to 0$ in $L^2(0,T;L^2(\Gamma_N)^3)$ and thus $\frac{\partial u_n}{\partial \nu}\to 0$ in $L^2(0,T;L^2(\Gamma_N))$.

According to Corollary 3.2, $\frac{\partial u_n}{\partial \nu}\to\frac{\partial u}{\partial \nu}$ weakly in $L^2(0,T;L^2(\Gamma_N))$, which implies that $\frac{\partial u}{\partial \nu}=0$ on $(0,T)\times\Gamma_N$ by uniqueness of weak limits.

Let $v=u_t$. In the following, we show that v is the very weak solution of the wave equation

$$\begin{cases} v_{tt}-\Delta v=0, & \text{in } (0,T)\times\Omega,\\ v=0, & \text{on } (0,T)\times\partial\Omega,\\ \dfrac{\partial v}{\partial \nu}=0, & \text{on } (0,T)\times\Gamma_N,\\ v(0)=u_1\in L^2(\Omega), & v_t(0)=\Delta u_0\in H^1_{\Gamma_D}(\Omega)' \end{cases} \tag{3.18}$$

where

$$\langle \Delta u_0, w \rangle_{H^1_{\Gamma_D}(\Omega)' \times H^1_{\Gamma_D}(\Omega)} = -\int_\Omega \nabla u_0 \cdot \nabla w \,\mathrm{d}x, \qquad \forall w \in H^1_{\Gamma_D}(\Omega).$$

Given $f \in L^2(0,T; L^2(\Omega))$, let $\varphi \in C^1([0,T]; L^2(\Omega)) \cap C([0,T]; H^1(\Omega_s))$ be the weak solution of the wave equation

$$\begin{cases} \varphi_{tt} - \Delta\varphi = f, & \text{in } (0,T) \times \Omega, \\ \varphi = 0, & \text{on } (0,T) \times \Gamma_D, \\ \dfrac{\partial \varphi}{\partial \nu} = 0, & \text{on } (0,T) \times \Gamma_N, \\ \varphi(T) = \varphi(T) = 0, & \text{in } \Omega. \end{cases}$$

Integrating by parts in time and using Green's identity in space we have

$$\begin{aligned} 0 &= \int_0^T \int_\Omega (u_{ntt} - \Delta u_n)\varphi_t \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_\Omega u_{n1}\varphi_t(0)\,\mathrm{d}x - \int_0^T \int_{\Gamma_N} \frac{\partial u_n}{\partial \nu}\varphi_t \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega (u_{nt}\varphi_{tt} - \nabla u_n \cdot \nabla \varphi_t)\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_\Omega u_{n1}\varphi_t(0)\,\mathrm{d}x - \int_\Omega \nabla u_{n0} \cdot \nabla\varphi(0)\,\mathrm{d}x - \int_0^T \int_{\Gamma_N} \frac{\partial u_n}{\partial \nu}\varphi_t \,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_0^T \int_\Omega u_{nt} f \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Taking the limit as $n \to \infty$ and invoking the facts that $u_{nt} \to u_t$ weakly in $L^2(0,T; L^2(\Omega))$, $u_{n0} \to u_0$ weakly in $H^1_{\Gamma_D}(\Omega)$, $u_{n1} \to u_1$ weakly in $L^2(\Omega)$ and $\frac{\partial u_n}{\partial \nu} \to 0$ strongly in $L^2(0,T; L^2(\Gamma_N))$ we obtain

$$\int_0^T \int_\Omega v f \,\mathrm{d}x\,\mathrm{d}t = -\int_\Omega u_1\varphi_t(0)\,\mathrm{d}x - \int_\Omega \nabla u_0 \cdot \nabla\varphi(0)\,\mathrm{d}x. \tag{3.19}$$

Using an analogous argument, it can be shown that

$$\int_0^T \int_\Omega v f \,\mathrm{d}x\,\mathrm{d}t = -\int_\Omega u_1\phi_t(0)\,\mathrm{d}x - \int_\Omega \nabla u_0 \cdot \nabla\phi(0)\,\mathrm{d}x, \tag{3.20}$$

where $\phi \in C^1([0,T]; L^2(\Omega)) \cap C([0,T]; H^1(\Omega_s))$ is the weak solution of the wave equation

$$\begin{cases} \phi_{tt} - \Delta\phi = f, & \text{in } (0,T) \times \Omega, \\ \phi = 0, & \text{on } (0,T) \times \partial\Omega, \\ \phi(T) = \phi_t(T) = 0, & \text{in } \Omega. \end{cases}$$

According to (3.19) and (3.20), we conclude that v is indeed the very weak solution to (3.18).

By Holmgren's uniqueness principle we must have $v \equiv 0$ in $(0, T) \times \Omega$ and therefore u must be independent of t. Thus, u satisfies the elliptic boundary value problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_N, \end{cases}$$

in the distributional sense, and thus $u \equiv 0$ in Ω. This is a contradiction to (3.13) and (3.16). This completes the proof of the theorem. □

Now we are ready to prove our stabilization result for (1.1).

Theorem 3.7 *Suppose that $k > a$. Then there exist constants $M \geq 1$ and $\alpha > 0$ such that the energy of the solutions of* (1.1) *satisfies*

$$E(t) \leq Me^{-\alpha t} E(0), \qquad \forall t \geq 0.$$

Proof Combining the previous results, it can be seen that there is a constant $T^* > 0$ such that

$$E(T) \leq E(0) \leq C_T \int_0^T D(t)\, \mathrm{d}t \leq C_T (E(0) - E(T))$$

for every $T > T^*$, and therefore

$$E(T) \leq \frac{C_T}{C_T + 1} E(0) =: \tilde{C} E(0), \qquad \forall T > T^*.$$

Since $\tilde{C} < 1$, the result follows from standard semigroup theory. □

4 Stability for the system with delay in the oscillator component

In the case of (1.2), the total energy is defined by

$$E_1(t) = E_0(t) + \frac{D_0}{2} \int_{-\tau}^0 \int_{\Gamma_N} |\delta_t(t + \theta, x)|^2 \, \mathrm{d}x \, \mathrm{d}\theta.$$

The results of the previous section can be adapted to the present case. Indeed, we have the following theorem.

Theorem 4.1 *Suppose that $D > D_0$. There is a constant $C > 0$ independent of t such that for every initial data in $D(A_0)$ we have*

$$E_1'(t) \leq -CD_1(t), \qquad t > 0.$$

Furthermore, there exists $T^* > 0$ *such that for every* $T > 0$

$$E_1(0) \leq C_T \int_0^T D_1(t)\,\mathrm{d}t.$$

for some $C_T > 0$ *where*

$$D_1(t) = \int_{\Gamma_N} (|u_t(t,x)|^2 + |\delta_t(t-\tau,x)|^2 + |\delta_t(t,x)|^2)\,\mathrm{d}x.$$

Therefore, for some constants $M_1 \geq 1$ *and* $\alpha_1 > 0$ *we have*

$$E_1(t) \leq M_1 e^{-\alpha_1 t} E(0), \qquad t \geq 0.$$

Proof The proof is similar as in the previous section, now using Theorem 3.4 with $f(t,x) = -u_t(t,x) - D_0\delta_t(t-\tau,x)$. The details are left to the reader. □

5 The cases $k = a$ and $D = D_0$

In this section, we will study the stability properties of systems (1.1) and (1.2) in the case where $k = a$ and $D = D_0$, respectively. We start with the wave Eq. (1.1).

Lemma 5.1 *Suppose that* $a = k \geq 0$ *and let*

$$q(\lambda) := a\lambda(1 + e^{-\lambda\tau}) + \frac{\lambda^2}{M\lambda^2 + D\lambda + K}, \qquad \lambda \in \mathbb{C} \setminus \{\lambda_\pm\},$$

where $\lambda_\pm$ *are the complex roots of the quadratic equation* $M\lambda^2 + D\lambda + K = 0$*. Then*

$$\{\lambda \in \mathbb{C} \setminus \{\lambda_\pm\} : \Im q(\lambda) \neq 0\} \cap \sigma(A) = \sigma_p(A). \tag{5.1}$$

In particular, $\sigma_p(A) \cap i\mathbb{R} = \emptyset$*.*

Proof With the same reasoning as in the proof of Theorem 2.1, it can be shown that the equation $(\lambda I - A)(u, v, z, \delta, \sigma) = (f, g, h, \varphi, \phi)$, where $\lambda \neq \lambda_\pm$, $(u, v, z, \delta, \sigma) \in D(A)$ and $(f, g, h, \varphi, \phi) \in X$, is equivalent to the variational Eq. (2.13). We can write (2.13) as

$$a_1(v,w) + a_2(v,w) = f_0(w), \qquad \forall\, w \in H^1_{\Gamma_D}(\Omega), \tag{5.2}$$

where $a_1 : H^1_{\Gamma_D}(\Omega) \times H^1_{\Gamma_D}(\Omega) \to \mathbb{C}$, $a_2 : L^2(\Omega) \times L^2(\Omega) \to \mathbb{C}$ and $f_0 : H^1_{\Gamma_D}(\Omega) \to \mathbb{C}$ are defined by

$$\begin{aligned}
a_1(v,w) &= \int_\Omega (vw + \nabla v \cdot \nabla w)\,\mathrm{d}x + q(\lambda)\int_{\Gamma_N} uw\,\mathrm{d}x\\
a_2(v,w) &= (\lambda^2 - 1)\int_\Omega vw\,\mathrm{d}x\\
f_0(w) &= \int_\Omega (\lambda g w - \nabla f \cdot \nabla w)\,\mathrm{d}x + \int_{\Gamma_N} Fw\,\mathrm{d}x.
\end{aligned}$$

If the inequality

$$\inf_{\varepsilon\geq 0} |1+\varepsilon q(\lambda)| > 0 \tag{5.3}$$

holds then the generalized Lax–Milgram method in [6] applied to the variational Eq. (5.2) yields either $\lambda \in \rho(A)$ or $\lambda \in \sigma_p(A)$, and this implies (5.1). It is not hard to see that $\Im q(\lambda) \neq 0$ implies (5.3).

Now let us show that A does not have purely imaginary eigenvalues. A direct calculation shows that

$$q(ib) = b\left(a\sin b\tau - \frac{b(K-Mb^2)}{(K-Mb^2)^2+D^2b^2}\right) + ib\left(a + a\cos b\tau + \frac{Db^2}{(K-Mb^2)^2+D^2b^2}\right)$$

and thus $\Im q(ib) \neq 0$ for $b \in \mathbb{R}\setminus\{0\}$. This means that $ib \in \rho(A)$ or $ib \in \sigma_p(A)$. We show that the second case does not hold. The eigenvalue problem $A(u, v, z, \delta, \sigma) = ib(u, v, z, \delta, \sigma)$ is equivalent to the following elliptic problem with mixed Dirichlet-Neumann boundary conditions

$$\begin{cases} \Delta u = b^2 u, & \text{in } \Omega,\\ u = 0, & \text{on } \Gamma_D,\\ \dfrac{\partial u}{\partial \nu} = -q(ib)u, & \text{on } \Gamma_N, \end{cases} \tag{5.4}$$

Multiplying the first equation of (5.4) by u and using Green's identity we have

$$-b^2\int_\Omega |u|^2\,\mathrm{d}x = \int_\Omega |\nabla u|^2\,\mathrm{d}x + q(ib)\int_{\Gamma_N} |u|^2\,\mathrm{d}x.$$

Taking the imaginary part and invoking the fact that $\Im q(ib) \neq 0$, one can see that $u = 0$ on Γ_N and hence $u = 0$ on $\partial\Omega$ and $\frac{\partial u}{\partial \nu} = 0$ on Γ_N. Using this in (2.8)–(2.11) yields $v = 0$ on Γ_N, $\delta = \sigma = 0$ on Γ_N and $z = 0$ on $(-\tau, 0)\times\Gamma_N$. By standard elliptic regularity theory, the function u lies in $H^2(\Omega)\cap H_0^1(\Omega)$ and satisfies $-\Delta u + b^2 u = 0$ in Ω and $\frac{\partial u}{\partial \nu} = 0$ on Γ_N. Applying a unique continuation theorem for elliptic operators, see [15, Corollary 15.2.2] for instance, we have $u = 0$ in Ω. Therefore ib is not an eigenvalue of A for every nonzero real number b. It can be checked directly that $0 \notin \sigma_p(A)$. Therefore $\sigma_p(A)\cap i\mathbb{R} = \emptyset$. □

Theorem 5.2 *If $k = a$ then the energy $E(t)$ associated with (1.1) decays to zero asymptotically as $t \to \infty$.*

Proof The result is a direct consequence of Lemma 5.1 and the Arendt–Batty–Lyubic–Vu Theorem [7, Corollary V.2.22]. □

Now let us turn our attention to the system (1.2).

Theorem 5.3 *If $D = D_0$ then the solution of (1.2) is asymptotically stable.*

Proof The existence of a nontrivial solution $(u, v, \zeta, \delta, \sigma) \in D(A_0)$ of the equation

$$A_0(u, v, \zeta, \delta, \sigma) = \lambda(u, v, \zeta, \delta, \sigma)$$

is equivalent to the existence of a nontrivial solution $u \in H^1_{\Gamma_D}(\Omega)$ of the elliptic problem with mixed Dirichlet-Neumann boundary condition

$$\begin{cases} \Delta u = \lambda^2 u, & \text{in } \Omega \\ u = 0, & \text{on } \Gamma_D \\ \dfrac{\partial u}{\partial \nu} = -q_0(\lambda)u, & \text{on } \Gamma_N. \end{cases} \tag{5.5}$$

where

$$q_0(\lambda) = \lambda + \frac{\lambda^2}{M\lambda^2 + \lambda D e^{-\lambda\tau} + K + \lambda D},$$

as long as the denominator does not vanish.

The same argument as in Lemma 5.1 shows that

$$\{\lambda \in \mathbb{C} \setminus \{\lambda_\pm\} : \Im q_0(\lambda) \neq 0\} \cap \sigma(A_0) = \sigma_p(A_0). \tag{5.6}$$

If $\lambda = ib$ for some nonzero real number b then

$$q_0(ib) = -\frac{b^2(K - Mb^2 + Db\sin b\tau)}{(K - Mb^2 + Db\sin b\tau)^2 + b^2D^2(1 + \cos b\tau)^2} + ib\Big(1 + \frac{b^2D(1 + \cos b\tau)}{(K - Mb^2 + Db\sin b\tau)^2 + b^2D^2(1 + \cos b\tau)^2}\Big).$$

Because $D > 0$, it follows that $\Im q(ib) \neq 0$ for every real number $b \neq 0$. With this information, we may now proceed as in the proofs of Lemma 5.1 and Theorem 5.2 to establish the theorem. □

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