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Convergence, Isometry and Quasiconvexity in the Modular Sequence Space $w_0^0(\Phi)$

Gilbert R. Peralta
Department of Mathematics and Computer Science, University of the Philippines Baguio, Governor Pack Road, Baguio City 2600, Philippines
Current Address: Institut für Mathematik und Wissenschaftliches Rechnen, Karl-Franzens-Universität Graz, Heinrichstrasse 36, 8010 Graz, Austria
Email: grperalta@upb.edu.ph; gilbert.peralta@edu.uni-graz.at

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Abstract. We consider a modular sequence space generated by a sequence of Orlicz functions. Sufficient conditions were imposed in this sequence to guarantee that the modular of the space generates a Luxemburg norm. The (Δ_2, δ_2) condition was used to prove the equivalence of modular convergence and Luxemburg norm convergence. Using a linear operator with certain properties, we will generate another modular sequence space and establish an isometry between this space and the previous one. Also, the quasiconvexity of the modular will be studied.

Keywords: Isometry; Luxemburg norms; Modular sequence spaces; Orlicz functions; Quasiconvexity; (Δ_2, δ_2) condition.

1. Introduction

In the early 1930's, Birnbaum and Orlicz [1] tried to generalize the classical function spaces of the Lebesgue type L^p in connection with orthogonal expansions. They have considered functions which behave similarly to the power function $f(t) = t^p$. Another generalization was obtained by Luxemburg [4] in 1955. These generalizations found their applications in the theory of integral equations having kernels of nonpower types. In this paper, we consider modular sequence spaces generated by a sequence of extended real valued functions, and we note that we will follow the formulation provided by Kozlowski [3]. Other papers studying sequence spaces can be found in [2], [6] and [7].

This paper is organized as follows. In Section 2, we will define the modular sequence space $w_0^0(\Phi)$ generated by a sequence of Orlicz functions Φ. Sufficient conditions are given in Section 3 so that the modular of $w_0^0(\Phi)$ generates a norm, which is usually called the Luxemburg norm. We prove that modular convergence and Luxemburg norm convergence is equivalent provided that Φ satisfies certain conditions. In Section 4, we will use a linear operator to generate another modular sequence space and discuss some properties of this new modular sequence space in relation to $w_0^0(\Phi)$. Finally, we will obtain a necessary and sufficient condition for the quasiconvexity of the modular in $w_0^0(\Phi)$ in Section 5.

2. Definitions and Preliminaries

Let X be a real or complex vector space. A function $\rho : X \to [0, \infty]$ is called a *modular* if it satisfies the following properties:

(1) $\rho(x) = 0$ if and only if $x = 0$,
(2) $\rho(\alpha x) = \rho(x)$ for all $x \in X$ and for all scalars α such that $|\alpha| = 1$,
(3) $\rho(\alpha x + (1 - \alpha)y) \leq \rho(x) + \rho(y)$ for all $x, y \in X$ and scalar $\alpha \in [0, 1]$.

If X is a complex vector space, then condition (2) is equivalent to $\rho(e^{it}x) = \rho(x)$ for all $x \in X$ and for all $t \in [0, 2\pi]$. On the other hand, if X is a real vector space, then condition (2) is equivalent to $\rho(-x) = \rho(x)$ for all $x \in X$.

Let ρ be a modular on X. Then

$$\rho(\alpha x) \leq \rho(\beta x), \quad 0 \leq \alpha \leq \beta. \tag{1}$$

If $\alpha = \beta$ then equality holds. If $\alpha < \beta$ then using the third property of the modular we have

$$\rho(\alpha x) = \rho((\alpha/\beta)(\beta x) + (1 - \alpha/\beta)0) \leq \rho(\beta x).$$

A function $\varphi : \mathbb{R} \to [0, \infty]$ is called an *Orlicz function* if it is a continuous even function which is increasing on $[0, \infty)$ with $\varphi(x) = 0$ if and only if $x = 0$ and $\varphi(x) \to \infty$ as $x \to \infty$. Let $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ be a family of Orlicz function and let S be the space of real sequences $x = \{x_k\}_{k=1}^{\infty}$ considered as a vector space over $\mathbb{R}$. For each $r \in \mathbb{N}$, let $\rho_r : S \to [0, \infty]$ be defined by

$$\rho_r(x) = \frac{1}{2^{r-1}} \sum_{k=2^{r-1}}^{2^r - 1} \varphi_k(|x_k|).$$

Define $\rho : S \to [0, \infty]$ by $\rho(x) = \sup_{r \geq 1} \rho_r(x)$. It follows from the properties of the Orlicz function that ρ satisfies the first two properties of a modular. For the third property, let $x, y \in S$ and $\alpha \in [0, 1]$. If $x_k \leq y_k$ then that the set of all linear combinations $\alpha x_k + (1-\alpha)y_k$ is just the interval $[x_k, y_k]$. Thus we have $|\alpha x_k + (1-\alpha)y_k| \le \max\{|x_k|, |y_k|\}$ and so

$$\varphi_k(|\alpha x_k + (1-\alpha)y_k|) \le \varphi_k(\max\{|x_k|, |y_k|\}) \le \varphi_k(|x_k|) + \varphi_k(|y_k|).$$

Therefore, for each $r \in \mathbb{N}$ it follows that

$$\frac{1}{2^{r-1}} \sum_{k=2^{r-1}}^{2^r-1} \varphi_k(|\alpha x_k + (1-\alpha)y_k|) \le \frac{1}{2^{r-1}} \sum_{k=2^{r-1}}^{2^r-1} \varphi_k(|x_k|) + \frac{1}{2^{r-1}} \sum_{k=2^{r-1}}^{2^r-1} \varphi_k(|y_k|),$$

and so $\rho_r(\alpha x + (1-\alpha)y) \le \rho_r(x) + \rho_r(y)$, taking the supremum shows that $\rho(\alpha x + (1-\alpha)y) \le \rho(x) + \rho(y)$. Consider the sectionally modular sequence space $w_0^0(\Phi)$ defined as

$$w_0^0(\Phi) = \{x = \{x\}_{k=1}^{\infty} \mid \rho(\alpha_n x) \to 0 \text{ whenever } \alpha_n \to 0\}.$$

Representation theorems for modularly continuous orthogonally additive functionals on $w_0^0(\Phi)$ were given by Paredes [5].

3. Convergence in $w_0^0(\Phi)$

In this section, we will impose sufficient conditions on the sequence Φ of Orlicz functions so that the modulars ρ and ρ_r generate a Luxembourg norm.

Theorem 3.1. *The modular sequence space $w_0^0(\Phi)$ is a linear subspace of S.*

Proof. It is clear that $0 \in w_0^0(\Phi)$. Let $x \in w_0^0(\Phi)$ and α be a scalar. Assume that $\alpha_n \to 0$ as $n \to \infty$. Then $\alpha\alpha_n \to 0$ and so $\rho(\alpha_n(\alpha x)) = \rho((\alpha\alpha_n)x) \to 0$. This shows that $\alpha x \in w_0^0(\Phi)$. Now let $x, y \in w_0^0(\Phi)$. Since $2\alpha_n \to 0$ we have

$$\rho(\alpha_n(x+y)) = \rho\left(\frac{1}{2}(2\alpha_n x) + \frac{1}{2}(2\alpha_n y)\right) \le \rho(2\alpha_n x) + \rho(2\alpha_n y) \to 0.$$

Hence $x + y \in w_0^0(\Phi)$. Therefore $w_0^0(\Phi)$ is a subspace of S. ∎

Theorem 3.2. *Let $\Phi = \{\varphi_k\}_{k=1}^{\infty}$ be a family of Orlicz functions such that for each $k \in \mathbb{N}$,*

$$\varphi_k\left(\frac{a+b}{2}\right) \le \frac{1}{2}\varphi_k(a) + \frac{1}{2}\varphi_k(b), \quad \text{for all } a, b \ge 0. \tag{2}$$

Define $\|\cdot\|_\rho : w_0^0(\Phi) \to [0, \infty]$ by

$$\|x\|_\rho = \inf\{u > 0 \mid \rho(x/u) \le 1\}.$$

Then $(w_0^0(\Phi), \|\cdot\|_\rho)$ is a normed space.

Proof. We claim that $\varphi_k(\lambda a+(1-\lambda)b)\leq \lambda\varphi_k(a)+(1-\lambda)\varphi_k(b)$ for all $a,b\geq 0$, and $\lambda\in[0,1]$ of the form $\lambda=m/2^n$ with $m\in\mathbb{N}\cup\{0\}$ and $n\in\mathbb{N}$. We proceed by induction on n. If $n=1$, then the possible values for m are $0,1,2$ and so λ takes the values $0,1/2,1$. We can directly check that the said property holds for these values of λ. For the induction step, assume that the said property holds for $n-1$. Then, for $m\in\{0,1,\ldots,2^{n-1}\}$ we have

$$\begin{aligned}\varphi_k\left(\frac{m}{2^n}a+\left(1-\frac{m}{2^n}\right)b\right) &= \varphi_k\left(\frac{1}{2}\left(\frac{m}{2^{n-1}}a\right)+\frac{1}{2}\left(1-\frac{m}{2^{n-1}}\right)b+\frac{1}{2}b\right)\\ &\leq \frac{1}{2}\varphi_k\left(\frac{m}{2^{n-1}}a+\left(1-\frac{m}{2^{n-1}}\right)b\right)+\frac{1}{2}\varphi_k(b)\\ &\leq \frac{1}{2}\left[\frac{m}{2^{n-1}}\varphi_k(a)+\left(1-\frac{m}{2^{n-1}}\right)\varphi_k(b)\right]+\frac{1}{2}\varphi_k(b)\\ &= \frac{m}{2^n}\varphi_k(a)+\left(1-\frac{m}{2^n}\right)\varphi_k(b).\end{aligned}$$

If $m\in\{2^{n-1}+1,\ldots,2^n\}$ then we have

$$\begin{aligned}\varphi_k\left(\frac{m}{2^n}a+\left(1-\frac{m}{2^n}\right)b\right) &= \varphi_k\left(\frac{1}{2}\left(1+\frac{m-2^{n-1}}{2^{n-1}}\right)a+\frac{1}{2}\left(2-\frac{m}{2^{n-1}}\right)b\right)\\ &\leq \frac{1}{2}\varphi_k(a)+\frac{1}{2}\varphi_k\left(\frac{m-2^{n-1}}{2^{n-1}}a+\frac{2^n-m}{2^{n-1}}b\right).\end{aligned}$$

Since $m-2^{n-1}\in\{1,2,\ldots,2^{n-1}\}$ and

$$\frac{2^n-m}{2^{n-1}}=1-\frac{m-2^{n-1}}{2^{n-1}},$$

we have by the previous case

$$\varphi_k\left(\frac{m-2^{n-1}}{2^{n-1}}a+\frac{2^n-m}{2^{n-1}}b\right)\leq\frac{m-2^{n-1}}{2^{n-1}}\varphi_k(a)+\frac{2^n-m}{2^{n-1}}\varphi_k(b).$$

Using this we obtain

$$\begin{aligned}\varphi_k\left(\frac{m}{2^n}a+\left(1-\frac{m}{2^n}\right)b\right) &\leq \frac{1}{2}\varphi_k(a)+\frac{m-2^{n-1}}{2^n}\varphi_k(a)+\frac{2^n-m}{2^n}\varphi_k(b)\\ &= \frac{m}{2^n}\varphi_k(a)+\left(1-\frac{m}{2^n}\right)\varphi_k(b).\end{aligned}$$

for $m-2^{n-1}\in\{1,2,\ldots,2^{n-1}\}$. This establishes our claim.

Next we will show that the set $Q=\{m/2^n\in[0,1]\,|\,m\in\mathbb{N}\cup\{0\},n\in\mathbb{N}\}$ is dense in $[0,1]$. Indeed, let $x,y\in(0,1)$ be such that $x<y$ and p be a positive integer. Then there exists a positive integer n such that $p<2^n(y-x)$ and $\lfloor 2^n x\rfloor\geq 1$. Thus $2^n x<p+2^n x<2^n y$. Note that $\lfloor 2^n x\rfloor\leq 2^n x$ and $2^n x<p+\lfloor 2^n x\rfloor$. Hence

$$2^n x<p+\lfloor 2^n x\rfloor<2^n y.$$

It follows that $x < m/2^n < y$ where $m = p + \lfloor 2^n x \rfloor$ and so $\overline{Q} = [0,1]$.

Let $\lambda \in [0,1]$. Then we can find a sequence $\{\lambda_n\}_{n=1}^{\infty} \subset Q$ such that $\lambda_n \to \lambda$ and using the continuity of φ_k we obtain

$$\begin{aligned}\varphi_k(\lambda a + (1-\lambda)b) &= \lim_{n\to\infty} \varphi_k(\lambda_n a + (1-\lambda_n)b) \\ &\le \lim_{n\to\infty} [\lambda_n \varphi_k(a) + (1-\lambda_n)\varphi_k(b)] \\ &= \lambda\varphi_k(a) + (1-\lambda)\varphi_k(b),\end{aligned}$$

which shows that φ_k is convex.

Now let us show that $\|\cdot\|_\rho$ is a norm on $w_0^0(\Phi)$. For each $x \in w_0^0(\Phi)$ we let $S_x = \{u > 0 \,|\, \rho(x/u) \le 1\}$. Since $S_0 = (0,\infty)$ it follows that $\|0\|_\rho = 0$. Conversely, assume that $\|x\|_\rho = 0$. Then there exists a sequence of positive integers $\{u_n\}_{n=1}^{\infty}$ such that $\rho(x/u_n) \le 1$ and $u_n \to 0$. This implies that there exists a positive integer N such that $0 < u_n < 1$ whenever $n \ge N$. Thus, for $n \ge N$ we have, by the convexity of φ_k,

$$\begin{aligned}\rho(x) &= \sup_{r\in\mathbb{N}} \frac{1}{2^{r-1}} \sum_{k=2^{r-1}}^{2^r-1} \varphi_k(|x_k|) \\ &= \sup_{r\in\mathbb{N}} \frac{1}{2^{r-1}} \sum_{k=2^{r-1}}^{2^r-1} \varphi_k\left(u_n(|x_k|/u_n) + (1-u_n)0\right) \\ &\le \sup_{r\in\mathbb{N}} \frac{1}{2^{r-1}} \sum_{k=2^{r-1}}^{2^r-1} u_n\varphi_k(|x_k|/u_n) \\ &= u_n\rho(x/u_n) \\ &\le u_n \;\to\; 0.\end{aligned}$$

Consequently $\rho(x) = 0$, so that $x = 0$. Therefore $\|x\|_\rho = 0$ if and only if $x = 0$.

For homogeneity, let $x \in w_0^0(\Phi)$ and α be a nonzero scalar. Let $u \in S_{\alpha x}$ and so $u \in S_{|\alpha| x}$. Hence $\rho(x/(u|\alpha|^{-1})) \le 1$, that is, $u|\alpha|^{-1} \in S_x$. Thus $\|x\|_\rho \le u|\alpha|^{-1}$. Because u is an arbitrary element of $S_{\alpha x}$ we get $|\alpha|\|x\|_\rho \le \|\alpha x\|_\rho$. For the other inequality, let $u \in S_x$. Then $\rho(\alpha x/(u|\alpha|)) \le 1$, showing that $u|\alpha| \in S_{\alpha x}$. Therefore $\|\alpha x\|_\rho \le |\alpha| u$ whenever $u \in S_x$. Hence $\|\alpha x\|_\rho \le |\alpha|\|x\|_\rho$.

Finally, for the triangle inequality, let $\epsilon > 0$. Note that we can find a sequence $\{u_n\}_{n=1}^{\infty} \subset S_x$ such that $u_n \to \|x\|_\rho$ and a sequence $\{v_n\}_{n=1}^{\infty} \subset S_y$ such that $v_n \to \|y\|_\rho$. Therefore, for sufficiently large N, we have $u_n < \|x\|_\rho + \epsilon/2$ and $v_n < \|y\|_\rho + \epsilon/2$ for $n \ge N$. From the fact that each of the φ_k is increasing and convex, we have

$$\begin{aligned}\rho\left(\frac{x+y}{u_n+v_n}\right) &= \sup_{r\in\mathbb{N}} \frac{1}{2^{r-1}} \sum_{k=2^{r-1}}^{2^r-1} \varphi_k\left(\frac{|x_k+y_k|}{u_n+v_n}\right) \\ &\le \sup_{r\in\mathbb{N}} \frac{1}{2^{r-1}} \sum_{k=2^{r-1}}^{2^r-1} \varphi_k\left(\frac{u_n}{u_n+v_n}\frac{|x_k|}{u_n} + \frac{v_n}{u_n+v_n}\frac{|y_k|}{v_n}\right)\end{aligned}$$

$$\leq \frac{u_n}{u_n+v_n}\rho\left(\frac{x_k}{u_n}\right)+\frac{v_n}{u_n+v_n}\rho\left(\frac{y_k}{v_n}\right)$$
$$\leq 1.$$

Hence $u_n + v_n \in S_{x+y}$ and so

$$\|x+y\|_\rho \leq u_n + v_n < \|x\|_\rho + \|y\|_\rho + \epsilon,$$

whenever $n \geq N$. Letting $\epsilon \to 0^+$ gives us the desired inequality $\|x+y\|_\rho \leq \|x\|_\rho + \|y\|_\rho$. Therefore $(w_0^0(\Phi), \|\cdot\|_\rho)$ is a normed space. ■

We note that the functional $\|\cdot\|_\rho$ defined in the previous theorem is called the *Luxemburg norm* generated by the modular ρ. Let

$$v_0^0(\Phi) = \{x = \{x_k\}_{k=1}^\infty \mid \rho(\lambda x) < \infty \text{ for some } \lambda \neq 0\}.$$

If the family Φ satisfies the property (2), then we have $w_0^0(\Phi) = v_0^0(\Phi)$. Indeed, if $x \in w_0^0(\Phi)$ then $\rho(\alpha_n x) \to 0$ whenever $\alpha_n \to 0$. In particular, $\rho(x/n) \to 0$ as $n \to \infty$. For some $N \in \mathbb{N}$ we have $\rho(x/n) < 1$ for all $n \geq N$. Taking $\lambda = 1/N$ shows that $x \in v_0^0(\Phi)$ and so $w_0^0(\Phi) \subset v_0^0(\Phi)$. On the other hand, let $x \in v_0^0(\Phi)$ so that $\rho(\lambda x) < \infty$ for some nonzero real number λ. If $\alpha_n \to 0$ then $|\alpha_n| < |\lambda|$ for sufficiently large values of n. From the proof of the previous theorem we have

$$\begin{aligned}\rho_r(\alpha_n x) &= \frac{1}{2^{r-1}}\sum_{k=2^{r-1}}^{2^r-1}\varphi_k\left(\frac{|\alpha_n|}{|\lambda|}(|\lambda x_k|)+\left(1-\frac{|\alpha_n|}{|\lambda|}\right)0\right)\\ &\leq \frac{|\alpha_n|}{|\lambda|}\rho_r(\lambda x)\end{aligned}$$

for sufficiently large values of n. The above estimate shows that $\rho(\alpha_n x) \leq |\alpha_n|\rho(\lambda x)/|\lambda|$ and so $\rho(\alpha_n x) \to 0$. This limit proves that $v_0^0(\Phi) \subset w_0^0(\Phi)$. Therefore $v_0^0(\Phi) = w_0^0(\Phi)$ provided that the family Φ satisfies inequality (2). From this equality, assuming that Φ satisfies (2), we may characterize the elements of $w_0^0(\Phi)$ as those vectors x in S such that the function value of some nonzero scalar multiple of x under ρ is finite.

We can see that ρ_r is also a modular on $w_0^0(\Phi)$ and using the same argument as above, $\|\cdot\|_{\rho_r}$ is a norm on the modular space $w_0^0(\Phi)$ provided that Φ satisfies (2). For each $x, y \in w_0^0(\Phi)$, let $\|x\|_\infty = \sup_{r\in\mathbb{N}} \|x\|_{\rho_r}$. The family Φ is said to satisfy the (Δ_2, δ_2) *condition* if there exists a constant M (independent of k) such that for each k, $\varphi_k(2u) \leq \varphi_k(u)$ for all $u \geq 0$. The (Δ_2, δ_2) condition implies that $\rho(2x) \to 0$ whenever $\rho(x) \to 0$.

Lemma 3.3. *If Φ satisfies (2), $x \in w_0^0(\Phi)$ and $\|x\|_\rho < 1$ then $\rho(x) \leq \|x\|_\rho$. Moreover, if $\|x\|_{\rho_r} < 1$ then $\rho_r(x) \leq \|x\|_{\rho_r}$.*

Proof. If $\|x\|_\rho = 0$ then we have equality. Let $\epsilon > 0$. If $\|x\|_\rho \in (0,1)$ then we can find a sequence $\{u_n\}_{n=1}^\infty \subset S_x$ and a positive integer N such that

$$0 < u_n < \|x\|_\rho + \epsilon, \quad n \geq N.$$

If $\epsilon \in (0, 1 - \|x\|_\rho)$ then

$$\rho(x) = \rho(u_n x/u_n) \leq (\|x\|_\rho + \epsilon)\rho(x/u_n) \leq \|x\|_\rho + \epsilon.$$

Letting $\epsilon \to 0^+$ we have $\rho(x) \leq \|x\|_\rho$. The second statement of the lemma can be shown using a similar argument. ∎

We are now ready to prove the main result of this section. The following theorem states that modular convergence, Luxemburg norm convergence and $\|\cdot\|_\infty$-convergence are equivalent provided that the sequence Φ satisfies (2) and the (Δ_2, δ_2) condition.

Theorem 3.4. *Let Φ be a family of Orlicz function satisfying (2) and the (Δ_2, δ_2) condition. Then the following are equivalent*

(i) $\rho(x^{(n)} - x) \to 0$ *as* $n \to \infty$,
(ii) $\|x^{(n)} - x\|_\rho \to 0$ *as* $n \to \infty$,
(iii) $\|x^{(n)} - x\|_\infty \to 0$ *as* $n \to \infty$.

Proof. Assume that (i) holds, that is, $\rho(x^{(n)} - x) \to 0$. Using an induction argument and the (Δ_2, δ_2) condition we have $\rho(2^m(x^{(n)} - x)) \to 0$ for all $m \in \mathbb{N}$. Thus, for some $N \in \mathbb{N}$ we have $\rho(2^m(x^{(n)} - x)) < \epsilon$ for all $n \geq N$, where $\epsilon \in (0, 1)$. Note that $2^M > (\epsilon/2)^{-1}$ for some positive integer $M \geq N$ and so $\rho((x^{(n)} - x)/(\epsilon/2)) \leq \rho(2^M(x^{(n)} - x)) < 1$. This shows that $\|x^{(n)} - x\|_\rho \leq \epsilon/2 < \epsilon$ for all $n \geq N$. Therefore $\|x^{(n)} - x\|_\rho \to 0$.

Next, let us assume that (ii) holds. If

$$A = \{u > 0 \,|\, \rho((x^{(n)} - x)/u) \leq 1\}$$

and

$$B_r = \{u > 0 \,|\, \rho_r((x^{(n)} - x)/u) \leq 1\}$$

then $A \subset B_r$ for all $r \in \mathbb{N}$. Thus $\inf A \geq \inf B_r$ for all $r \in \mathbb{N}$ and so $\|x^{(n)} - x\|_{\rho_r} \leq \|x^{(n)} - x\|_\rho$ for all $r \in \mathbb{N}$. Hence $\|x^{(n)} - x\|_\infty \leq \|x^{(n)} - x\|_\rho$ and this inequality proves (iii).

If (iii) is satisfied, then there exists an N such that $\|x^{(n)} - x\|_{\rho_r} \leq \|x^{(n)} - x\|_\infty < \epsilon/2$ for all $r \in \mathbb{N}$ and for all $n \geq N$. If $\epsilon/2 < 1$ then using the previous lemma we get $\rho_r(x^{(n)} - x) \leq \|x^{(n)} - x\|_{\rho_r}$ for all $r \in \mathbb{N}$. Taking the supremum we obtain $\rho(x^{(n)} - x) \leq \epsilon/2 < \epsilon$ for all $n \geq N$. Thus (i) holds. ∎

4. Isometric Modular Spaces

Using an injective linear operator from S into itself, we will generate another modular sequence space. We prove that this new modular sequence space and $w_0^0(\Phi)$ are isometric provided that the linear operator in S is bijective.

Theorem 4.1. *Let S_0 be a subspace of S and $T : S_0 \to S$ be an injective linear operator. Then the composition ρT is a modular on S_0.*

Proof. Since T is linear it follows that $\rho T(0) = 0$. On the other hand, if $\rho T(x) = 0$ then $Tx = 0$ and because T is injective we have $x = 0$. Since each φ_k is even we have $\rho T(-x) = \rho(-Tx) = \rho T(x)$ for all $x \in S_0$. Finally, if $\alpha \in [0, 1]$ then using the linearity of T we obtain

$$\begin{aligned} \rho T(\alpha x + (1-\alpha)y) &= \rho(\alpha Tx + (1-\alpha)Ty) \\ &\leq \rho(Tx) + \rho(Ty), \end{aligned}$$

for all $x, y \in S_0$. Hence, ρT is a modular on S_0. ∎

Taking $S_0 = S$ it follows that ρT is also a modular on S provided that T is an injective linear operator from S into itself. In this case, we let

$$w_0^0(\Phi)_T = \left\{ x \in S \,\middle|\, \sup_{r \in \mathbb{N}} \frac{1}{2^{r-1}} \sum_{k=2^{r-1}}^{2^r - 1} \varphi_k(\alpha_n |(Tx)_k|) \to 0 \quad \text{whenever} \quad \alpha_n \to 0 \right\}.$$

Similarly, it can be shown that $w_0^0(\Phi)_T$ is a subspace of S and if the sequence Φ of Orlicz functions satisfies (2) then the functional

$$\|x\|_{\rho T} = \inf\{u > 0 \,|\, \rho T(x/u) \leq 1\}$$

is a norm on $w_0^0(\Phi)_T$. Note that if I is the identity operator on S then $w_0^0(\Phi)_I = w_0^0(\Phi)$. Suppose that $T : S \to S$ is linear and injective. If $w_0^0(\Phi)$ is T-invariant, that is, $T[w_0^0(\Phi)] \subset w_0^0(\Phi)$ then it follows that

$$w_0^0(\Phi) \subset w_0^0(\Phi)_T. \tag{3}$$

Indeed, if $x \in w_0^0(\Phi)$ then $Tx \in w_0^0(\Phi)$ so that we have

$$\rho T(\alpha_n x) = \rho(\alpha_n Tx) \to 0$$

as $\alpha_n \to 0$, thus $x \in w_0^0(\Phi)_T$.

Theorem 4.2. *Let S_0 be a subspace of the space S of all real sequences and let $T : S_0 \to S$ be an injective linear operator. Then*

$$T[w_0^0(\Phi)_T \cap S_0] = w_0^0(\Phi) \cap T[S_0]. \tag{4}$$

If in addition, the sequence Φ of Orlicz function satisfies inequality (2) then the normed spaces $(w_0^0(\Phi)_T \cap S_0, \|\cdot\|_{\rho T})$ and $(w_0^0(\Phi) \cap T[S_0], \|\cdot\|_\rho)$ are isometric.

Proof. Let $x = \{x_k\}_{k=1}^\infty \in T[w_0^0(\Phi)_T \cap S_0]$ so that $x = Ty$ for some $y \in w_0^0(\Phi)_T \cap S_0$. In particular, we have $y \in w_0^0(\Phi)_T$ so that

$$\lim_{n \to \infty} \rho T(\alpha_n y) = 0$$

as $\alpha_n \to 0$. Hence

$$\lim_{n\to\infty} \rho(\alpha_n x) = \lim_{n\to\infty} \rho(\alpha_n Ty) = \lim_{n\to\infty} \rho T(\alpha_n y) = 0,$$

whenever $\alpha_n \to 0$. It follows that $x \in w_0^0(\Phi)$. Notice that we also have $x = Ty \in T[S_0]$ and so $T[w_0^0(\Phi)_T \cap S_0] \subset w_0^0(\Phi) \cap T[S_0]$. Conversely, let $x \in w_0^0(\Phi) \cap T[S_0]$. Then $x \in w_0^0(\Phi)$ and $x = Ty$ for some $y \in S_0$. Hence

$$\lim_{n\to\infty} \rho T(\alpha_n y) = \lim_{n\to\infty} \rho(\alpha_n x) = 0,$$

as $\alpha_n \to 0$. Thus $y \in w_0^0(\Phi)_T$. Consequently, we have $T[w_0^0(\Phi)_T \cap S_0] \supset w_0^0(\Phi) \cap T[S_0]$ and this completes the proof of (4).

Let $T_1 : w_0^0(\Phi)_T \cap S_0 \to w_0^0(\Phi) \cap T[S_0]$ be defined by $T_1 x = Tx$, that is, T_1 is the restriction of T on $w_0^0(\Phi)_T \cap S_0$. Because T is linear and injective, then so is T_1. Equation (4) shows that $T_1[w_0^0(\Phi)_T \cap S_0] = w_0^0(\Phi) \cap T[S_0]$, that is, T_1 is surjective. Therefore in order to show that $w_0^0(\Phi)_T \cap S_0$ and $w_0^0(\Phi) \cap T[S_0]$ are isometric, it remains to show that T_1 preserves norm. If $x \in w_0^0(\Phi)_T \cap S_0$ then then

$$\|T_1 x\|_\rho = \inf\{u > 0 \,|\, \rho(T_1 x/u) \le 1\} = \inf\{u > 0 \,|\, \rho T_1(x/u) \le 1\} = \|x\|_{\rho T}.$$

This completes the proof of the theorem. ■

An immediate consequence of this theorem is the following corollary.

Corollary 4.3. *If $T : S \to S$ is a bijective linear operator and $\Phi = \{\varphi_k\}_{k=1}^\infty$ is a family of Orlicz functions satisfying (2) then $(w_0^0(\Phi)_T, \|\cdot\|_{\rho T})$ is isometric to $(w_0^0(\Phi), \|\cdot\|_\rho)$.*

Note that if $T : S \to S$ is an injective linear operator then T^n is also injective and linear for all positive integer n. Hence it follows that ρT^n is also a modular on S for all positive integer n. If $w_0^0(\Phi)_{T^m}$ is T-invariant for all $m = 0, 1, \ldots, n-1$ then we have the following increasing chain of subspaces of S

$$\{0\} \subset w_0^0(\Phi) \subset w_0^0(\Phi)_T \subset w_0^0(\Phi)_{T^2} \subset \cdots \subset w_0^0(\Phi)_{T^n} \subset S$$

for all positive integer n. This can be easily seen using (3) together with an induction argument.

5. Quasiconvexity

Let X be a real vector space and $f : X \to [0, \infty]$. The function f is said to be *quasiconvex* with constant $M \ge 1$, if for any positive integer n, for any elements $x_1, x_2, \ldots, x_n \in X$ and nonnegative numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ satisfying

$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, we have

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \le M \sum_{i=1}^{n} \alpha_i f(M x_i).$$

We note that a quasiconvex function with constant $M = 1$ is convex. The following theorem states that the quasiconvexity of each of the elements of Φ is a necessary and sufficient condition for the quasiconvexity of the modular ρ.

Theorem 5.1. *The modular ρ on S is quasiconvex with constant $M \ge 1$ if and only if each element of Φ is quasiconvex with the same constant M.*

Proof. Suppose that the constants $\alpha_1, \alpha_2, \cdots, \alpha_n \ge 0$ satisfy $\alpha_1+\alpha_2+\cdots+\alpha_n = 1$. Assume that φ_k is quasiconvex with constant $M \ge 1$ for each $k \in \mathbb{N}$. If $x^{(1)}, x^{(2)}, \ldots, x^{(n)} \in S$ then, by the triangle inequality and the fact that each φ_k is increasing, we have

$$\begin{aligned}
\rho_r\left(\sum_{i=1}^{n} \alpha_i x^{(i)}\right) &= \frac{1}{2^{r-1}} \sum_{k=2^{r-1}}^{2^r-1} \varphi_k\left(\left|\sum_{i=1}^{n} \alpha_i (x^{(i)})_k\right|\right) \\
&\le \frac{1}{2^{r-1}} \sum_{k=2^{r-1}}^{2^r-1} \varphi_k\left(\sum_{i=1}^{n} \alpha_i |(x^{(i)})_k|\right) \\
&\le \frac{M}{2^{r-1}} \sum_{k=2^{r-1}}^{2^r-1} \sum_{i=1}^{n} \alpha_i \varphi_k(|(M x^{(i)})_k|) \\
&= M \sum_{i=1}^{n} \alpha_i \rho_r(M x^{(i)})
\end{aligned}$$

for each $r \in \mathbb{N}$. Taking the supremum shows that ρ is quasiconvex with constant M.

For the converse, let $u_1, u_2, \ldots, u_n \in \mathbb{R}$. Let $k \in \mathbb{N}$ be fixed. Denote e_k to be the sequence in S with 1 in the kth position and 0 otherwise. We let $x^{(i)} = u_i e_k$. Note that there exists a unique positive integer r_k such that $2^{r_k-1} \le k \le 2^{r_k} - 1$. By definition we have $\rho_r(x^{(i)}) = 2^{-r_k+1}\varphi_k(|(x^{(i)})_k|) = 2^{-r_k+1}\varphi_k(|u_i|)$ if $r = r_k$ and $\rho_r(x^{(i)}) = 0$ if $r \ne r_k$. Hence $\rho(x^{(i)}) = 2^{-r_k+1}\varphi_k(|u_i|)$. Similarly we can show that $\rho(Mx^{(i)}) = 2^{-r_k+1}\varphi_k(M|u_i|)$ and

$$\rho\left(\sum_{i=1}^{n} \alpha_i x^{(i)}\right) = 2^{-r_k+1}\varphi_k\left(\left|\sum_{i=1}^{n} \alpha_i u_i\right|\right).$$

The quasiconvexity of ρ and the fact that φ_k is even imply

$$\varphi_k\left(\sum_{i=1}^{n} \alpha_i u_i\right) = 2^{r_k-1}\rho\left(\sum_{i=1}^{n} \alpha_i x^{(i)}\right)$$

$$\leq M \sum_{i=1}^{n} \alpha_i 2^{r_k - 1} \rho(Mx^{(i)})$$
$$= M \sum_{i=1}^{n} \alpha_i \varphi_k(Mu_i).$$

This completes the proof of the theorem. ∎

Corollary 5.2. *If there exist constants $C_2, C_1 > 0$ and $m \geq 1$ such that $C_2 m \geq C_1$ and*

$$C_1 m^{-1} |u|^m \leq \varphi_k(u) \leq C_2 |u|^m \tag{5}$$

for all $u \in \mathbb{R}$ and for all $k \in \mathbb{N}$ then ρ is a quasiconvex modular on S with constant $(C_2 m / C_1)^{1/(m+1)} \geq 1$.

Proof. From the previous theorem it suffices to show that φ_k is quasiconvex with constant $(C_2 m/C_1)^{1/(m+1)}$ for all $k \in \mathbb{N}$. Using (5) and the convexity of the function $g(t) = t^m$ we have

$$\begin{aligned}
\varphi_k\left(\sum_{i=1}^{n} \alpha_i u_i\right) &\leq C_2 \left|\sum_{i=1}^{n} \alpha_i u_i\right|^m \leq C_2 \sum_{i=1}^{n} \alpha_i |u_i|^m \\
&= \left(\frac{C_2 m}{C_1}\right)^{1/(m+1)} \sum_{i=1}^{n} \alpha_i C_1 m^{-1} \left| u_i \left(\frac{C_2 m}{C_1}\right)^{1/(m+1)} \right|^m \\
&\leq \left(\frac{C_2 m}{C_1}\right)^{1/(m+1)} \sum_{i=1}^{n} \alpha_i \varphi_k \left(u_i \left(\frac{C_2 m}{C_1}\right)^{1/(m+1)} \right)
\end{aligned}$$

for all $u_1, u_2, \ldots, u_n \in \mathbb{R}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$. ∎

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