Feedback Stabilization of a Linear Fluid–Membrane System with Time Delay

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Abstract A coupled parabolic–hyperbolic system of partial differential equations modelling the interaction of a fluid and a membrane is considered. The model is reformulated as an abstract Cauchy problem and thereby constructing a semigroup for the evolution. This is done by eliminating the pressure. The system is stabilized through a feedback force applied to the membrane incorporating a time delay. The spectral properties and stability are considered under suitable conditions on the fluid viscosity, damping coefficient and delay coefficient.

Keywords Fluid-Membrane system · Stability · Feedback law · Delay

1 Introduction

Let us consider a sufficiently smooth bounded domain Ω in two- or three-dimensional space. Denote by Γ the boundary of the fluid domain Ω and $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$ where Γ_0 and Γ_1 are nonempty open subsets of Γ, both with positive surface measure. On the boundary Γ_1 we have a solid wall while on Γ_0 we have a membrane. Let Σ_0 be the boundary of Γ_0. A linear model describing the above situation is given by the following coupled Navier–Stokes-wave system

$$\begin{cases} u_t - \mu \Delta u + \nabla p = 0, & \text{in } (0,\infty) \times \Omega, \\ \operatorname{div} u = 0, & \text{in } (0,\infty) \times \Omega, \\ u = 0, & \text{on } (0,\infty) \times \Gamma_1, \\ u = w_t \nu, & \text{on } (0,\infty) \times \Gamma_0, \\ w_{tt} - \Delta w = p - \mu \nu \cdot \partial_\nu u + F, & \text{in } (0,\infty) \times \Gamma_0, \\ w = 0, & \text{in } (0,\infty) \times \Sigma_0, \\ \int_{\Gamma_s} w \, \mathrm{d}x = 0, & \text{in } (0,\infty), \end{cases} \tag{1}$$

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supplied with the initial conditions $u(0, x) = u_0(x)$ in Ω and $w(0, x) = w_0(x)$, $w_t(0, x) = w_1(x)$ in Γ_0. In (1), u and p are the velocity field and pressure for the fluid, w is the transversal displacement of the membrane, and F is the feedback force. Moreover, μ is the fluid viscosity and ν is the unit normal outward to Ω. In this paper, we consider small but rapid oscillations, under which we can assume that the domain occupied by the membrane is fixed. Unlike in the Navier–Stokes equation with no-slip boundary condition where the pressure is determined up to a constant, the pressure in (1) is unique due to the Neumann-type boundary condition on Γ_0. In fact, for sufficiently smooth solutions, p satisfies an elliptic problem with mixed Neumann–Robin boundary conditions.

Without the feedback force F, the above system is stable due to the diffusion of the fluid component. This dissipative mechanism, through the interface boundary condition, produces a dissipation for the membrane component. On the other hand, to stabilize the system faster one could add a dissipative mechanism for the membrane by introducing a feedback force. One of the common interior feedback force for the wave equation is using the velocity. This feedback may not be felt instantaneously by the evolution; that is, delay may take place. This consideration gives us the following form of the linear feedback law

$$F(t, x) = -\alpha w_t(t, x) - \beta w_t(t - \tau, x),$$

for $t > 0$ and $x \in \Gamma_0$, where $\alpha, \beta \geq 0$. The constant $\tau > 0$ represents the extent of delay, while the constants α and β represent the strengths of damping and delay, respectively. To have a well-posed system, one must incorporate an initial history

$$w(\theta, x) = z_0(x) \text{ in } (-\tau, 0) \times \Gamma_0.$$

It is well known that delay induces a transport phenomenon in the system creating oscillations that may lead into instability; see [10] and the references therein. In the absence of delay, models similar to (1) where the membrane is replaced by a plate has been studied in [4, 5, 9]. Typically, if the damping factor dominates the delay factor, then the system will be stable, i.e. as if delay is not present, however, with a possible slower decay rate. If such terms are equal then the system may not be stable; see [10]. If there are other dissipative mechanisms in the system then we may obtain stability under appropriate conditions, for instance, viscoelasticity in wave equations in [7] and fluid viscosity in a fluid–structure system in [11]. In this paper, we shall also see that viscosity plays a role in deriving sufficient conditions for exponential stability.

The plan of this paper is as follows. In Sect. 2, we introduce generalized trace results that are needed in the elimination of the pressure in the semigroup formulation. In Sect. 3, we write (1) as an abstract Cauchy problem in a suitable state space and prove that it generates a contraction semigroup under a suitable assumption on α, β and μ. The spectral properties and uniform exponential stability of the semigroup will be discussed in Sects. 4 and 5, respectively.

2 Generalized Traces for Some Graph Spaces

Let $\Sigma \subset \Gamma$ be sufficiently smooth. For $s = m + \sigma$ where m is a nonnegative integer and $\sigma \in (0, 1)$, let $H^s_{00}(\Sigma) = \{w \in H^s(\Sigma) : \|w\|_{s,\Sigma} < \infty\}$ where

$$\|w\|^2_{s,\Sigma} = \|u\|^2_{H^s(\Sigma)} + \sum_{|\alpha|=m} \int_\Sigma \frac{|D^\alpha u(x)|^2}{d(x, \partial\Sigma)^{2\sigma}} \, \mathrm{d}x$$

and $d(x, \partial\Sigma)$ denotes the distance of x from $\partial\Sigma$. For each nonnegative integer m, $C_0^\infty(\Sigma)$ is dense in $H_{00}^{m+1/2}(\Sigma)$ and we have $(H_{00}^{m+1/2}(\Sigma))' = H^{-m-1/2}(\Sigma)$, see [8] for more details.

Consider the Hilbert space $L^2_{\mathrm{div}}(\Omega) = \{u \in L^2(\Omega) : \operatorname{div} u \in L^2(\Omega)\}$ with the graph norm. Recall that elements of $L^2_{\mathrm{div}}(\Omega)$ admit generalized normal traces $u \cdot \nu|_\Sigma$ and the corresponding mapping is a continuous linear operator from $L^2_{\mathrm{div}}(\Omega)$ into $H^{-1/2}(\Sigma)$. Moreover, for every $\varphi \in H^1(\Omega)$ with trace in $H^{1/2}_{00}(\Sigma)$ we have

$$\langle u \cdot \nu|_\Sigma, \varphi\rangle = \int_\Omega (\operatorname{div} u)\varphi \, \mathrm{d}x + \int_\Omega u \cdot \nabla\varphi \, \mathrm{d}x.$$

For the fluid component we shall use the function spaces

$$\begin{aligned} H &= \{u \in L^2(\Omega) : \operatorname{div} u = 0 \text{ in } \Omega,\ u \cdot \nu|_{\Gamma_1} = 0\}, \\ V &= \{u \in H^1(\Omega) : \operatorname{div} u = 0 \text{ in } \Omega,\ u|_{\Gamma_1} = 0\}. \end{aligned}$$

Recall that the trace maps $\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma)$ and $\gamma_1 : H^2(\Omega) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ defined by $\gamma_0 u = u|_\Gamma$ and $\gamma_1 u = (u|_\Gamma, \partial_\nu u|_\Gamma)$ are surjective bounded linear operators. It follows that the operators $\gamma_0\gamma_0^*$ and $\gamma_1\gamma_1^*$ are strictly positive definite and thus invertible. Given $\varphi \in H^{1/2}_{00}(\Sigma)$, we extend it by zero to the whole boundary Σ to obtain an element in $H^{1/2}(\Gamma)$, which we shall still denote by φ. We do a similar construction for $\psi \in H^{3/2}_{00}(\Sigma)$. Consider the lifting operators $\ell : H^{3/2}_{00}(\Sigma) \times H^{1/2}_{00}(\Sigma) \to H^2(\Omega)$ and $\kappa : H^{1/2}_{00}(\Sigma) \to H^1(\Omega)$ given by

$$\ell(\psi, \varphi) = \gamma_1^*(\gamma_1\gamma_1^*)^{-1}(\psi, \varphi), \qquad \kappa\varphi = \gamma_0^*(\gamma_0\gamma_0^*)^{-1}\varphi.$$

Let ℓ_1 and ℓ_2 be the coordinate functions of ℓ, that is, $\ell_1\psi = \ell(\psi, 0)$ and $\ell_2\varphi = \ell(0, \varphi)$. It follows that ℓ_1, ℓ_2 and κ are bounded linear operators.

Let $\mathscr{D} = \{p \in H^1(\Omega) : \Delta p \in L^2(\Omega)\}$ and $\mathscr{W} = \{\pi \in L^2(\Omega) : \Delta\pi \in H^{-1}(\Omega)\}$ be equipped with the corresponding graph norms. Given $\pi \in \mathscr{W}$ and $p \in \mathscr{D}$, we define $\pi|_\Sigma$ and $\partial_\nu p|_\Sigma$ by

$$\langle \pi|_\Sigma, \varphi\rangle = \int_\Omega \pi\,\Delta(\ell_2\varphi)\,\mathrm{d}x - \langle \Delta\pi, \ell_2\varphi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}$$
$$\langle \partial_\nu p|_\Sigma, \psi\rangle = \int_\Omega (\Delta p)\ell_1\psi\,\mathrm{d}x + \int_\Omega (\nabla p)\cdot\nabla(\ell_1\psi)\,\mathrm{d}x \tag{2}$$

for every $\varphi \in H_{00}^{1/2}(\Sigma)$ and $\psi \in H_{00}^{3/2}(\Sigma)$. From the definition and the properties of the above extension operators, one can immediately see that $\pi \mapsto \pi|_\Sigma \in \mathscr{L}(\mathscr{W}, H^{-1/2}(\Sigma))$ and $p \mapsto \partial_\nu p|_\Sigma \in \mathscr{L}(\mathscr{D}, H^{-3/2}(\Sigma))$. If $\pi \in H^1(\Omega)$ and $p \in H^2(\Omega)$ then these traces coincide with the usual traces. This remark follows immediately from the above definitions and Green's identities.

Now consider the subspace $\mathscr{Y} = \{\pi \in L^2(\Omega) : \Delta\pi \in L^2(\Omega)\}$ of $\mathscr{W}$ with the associated graph norm. Given $\pi \in \mathscr{Y}$, define $\pi|_\Sigma$ and $\partial_\nu\pi|_\Sigma$ as follows

$$\langle \pi|_\Sigma, \varphi\rangle = \int_\Omega \pi\,\Delta(\ell_2\varphi)\,\mathrm{d}x - \int_\Omega (\Delta\pi)\ell_2\varphi\,\mathrm{d}x$$
$$\langle \partial_\nu\pi|_\Sigma, \psi\rangle = \int_\Omega (\Delta\pi)\ell_1\psi\,\mathrm{d}x - \int_\Omega \pi\,\Delta(\ell_1\psi)\,\mathrm{d}x$$

for every $\varphi \in H_{00}^{1/2}(\Sigma)$ and $\psi \in H_{00}^{3/2}(\Sigma)$. Again these traces are bounded, more precisely, $\pi \mapsto \pi|_\Sigma \in \mathscr{L}(\mathscr{Y}, H^{-1/2}(\Sigma))$ and $\pi \mapsto \partial_\nu\pi|_\Sigma \in \mathscr{L}(\mathscr{Y}, H^{-3/2}(\Sigma))$. Notice that the definition of $\pi|_\Sigma$ is the same whether it is viewed as an element of $\mathscr{W}$ or $\mathscr{Y}$. Likewise, if $p \in H^1(\Omega) \cap \mathscr{Y} \subset \mathscr{D}$ then the definition of $\partial_\nu p|_\Sigma$ coincides with the earlier formulation (2).

Let us consider the graph space $G = \{(u, p) \in V \times L^2(\Omega) : -\mu\Delta u + \nabla p \in L^2_{\mathrm{div}}(\Omega)\}$ endowed with the graph norm. Notice that $\Delta p = \mathrm{div}(\nabla p - \mu\Delta u) \in L^2(\Omega)$ so that $p \in \mathscr{Y}$ and hence it admits traces such that

$$\|p\|_{H^{-1/2}(\Sigma)} + \|\partial_\nu p\|_{H^{-3/2}(\Sigma)} \le C(\|p\|_{L^2(\Omega)} + \|\,\mathrm{div}(\nabla p - \mu\Delta u)\|_{L^2(\Omega)}).$$

For $(u, p) \in G$, we define the following

$$\langle \mu\partial_\nu u|_\Sigma, \varphi\rangle = \langle p\nu|_\Sigma, \varphi\rangle - \mu\int_\Omega \nabla u\cdot\nabla(\kappa\varphi)\,\mathrm{d}x + \int_\Omega p\,\mathrm{div}(\kappa\varphi)\,\mathrm{d}x$$
$$+ \int_\Omega (-\mu\Delta u + \nabla p)\cdot\kappa\varphi\,\mathrm{d}x$$
$$\langle \mu\Delta u\cdot\nu|_\Sigma, \psi\rangle = \langle \partial_\nu p|_\Sigma, \psi\rangle + \int_\Omega (-\mu\Delta u + \nabla p)\cdot\nabla(\ell_1\psi)\,\mathrm{d}x$$

for every $\varphi \in H_{00}^{1/2}(\Sigma)$ and $\psi \in H_{00}^{3/2}(\Sigma)$. Again, one can see immediately that these generalized traces are bounded; that is, $(u, p) \mapsto \partial_\nu u|_\Sigma \in \mathscr{L}(G, H^{-1/2}(\Sigma))$ and $(u, p) \mapsto \Delta u\cdot\nu \in \mathscr{L}(G, H^{-3/2}(\Sigma))$. In fact, we have

$$\|\partial_\nu u\|_{H^{-1/2}(\Sigma)} \le C(\|p\|_{H^{-1/2}(\Sigma)} + \|u\|_V + \|p\|_{L^2(\Omega)} + \|\mu\Delta u - \nabla p\|_{L^2(\Omega)})$$
$$\|\Delta u \cdot \nu\|_{H^{-3/2}(\Sigma)} \le C(\|\mu\Delta u - \nabla p\|_H + \|\partial_\nu p\|_{H^{-3/2}(\Sigma)}).$$

From the above discussion note that $-\mu\Delta u + \nabla p$ admits a generalized normal trace on Σ. In the case $\operatorname{div}(-\mu\Delta u + \nabla p) = 0$, it follows from the divergence theorem that

$$(-\mu\Delta u + \nabla p)\cdot \nu|_\Sigma = \mu\Delta u \cdot \nu|_\Sigma - \partial_\nu p|_\Sigma.$$

In particular, we have the following generalized integration by parts formula

$$\int_\Omega (\mu\Delta u - \nabla p) f \,\mathrm{d}x = \langle \mu\partial_\nu u - p\nu|_\Sigma, f\rangle - \mu\int_\Omega \nabla u \cdot \nabla f \,\mathrm{d}x$$

for every $(u, p) \in G$ and $f \in V$. We refer to [2, 13] for similar discussions.

3 Abstract Formulation and Well-Posedness of the System

The coupled system (1) will be expressed as an evolution equation in a suitable state space. Using the divergence theorem, it can be seen that we need to factor the constants in the space for the states associated with the membrane. Let

$$X = \{(u, w, v, z) \in H \times \widehat{H}^1_0(\Gamma_0) \times \widehat{L}^2(\Gamma_0) \times L^2(-\tau, 0; \widehat{L}^2(\Gamma_0)) : u\cdot\nu = v \text{ in } \Gamma_0\},$$

where $\widehat{L}^2(\Gamma_0) = \{w \in L^2(\Gamma_0) : \int_{\Gamma_0} w \,\mathrm{d}x = 0\}$ and $\widehat{H}^1_0(\Gamma_0) = H^1(\Gamma_0) \cap \widehat{L}^2(\Gamma_0)$, be equipped with the norm

$$\|(u, w, v, z)\|_X^2 = \int_\Omega |u|^2 \,\mathrm{d}x + \int_{\Gamma_0} |\nabla w|^2 + |v|^2 \,\mathrm{d}x + \beta\int_{-\tau}^0\int_{\Gamma_0} |z|^2 \,\mathrm{d}x\,\mathrm{d}\theta.$$

Following [1], we eliminate p in the system by rewriting it as an elliptic problem with boundary data involving the fluid velocity and the displacement of the membrane. Define the mixed Neumann–Robin map $M : H^{-3/2}(\Gamma_1) \times H^{-3/2}(\Gamma_0) \to L^2(\Omega)$ according to

$$\pi = M(\varphi, \psi) \Longleftrightarrow \begin{cases} \Delta\pi = 0 & \text{in } \Omega, \\ \partial_\nu\pi = \varphi & \text{on } \Gamma_1, \\ \partial_\nu\pi + \pi = \psi & \text{on } \Gamma_0. \end{cases}$$

For smooth solutions we can see that p satisfies the boundary value problem

$$\begin{cases} \Delta p = 0 & \text{in } \Omega, \\ \partial_\nu p = \mu \Delta u \cdot \nu & \text{on } \Gamma_1, \\ \partial_\nu p + p = -\Delta w + \alpha v + \beta z(-\tau) + \mu \nu \cdot \partial_\nu u + \mu \Delta u \cdot \nu & \text{on } \Gamma_0. \end{cases}$$

Hence, we can represent p in terms of the map M as follows

$$p = L(u, w, v, z) := M(\mu \Delta u \cdot \nu, -\Delta w + \alpha v + \beta z(-\tau) + \mu \nu \cdot \partial_\nu u + \mu \Delta u \cdot \nu).$$

To keep track of the retarded term in (1), let us introduce the delay variable $z(t, \theta, x) = w_t(t + \theta, x)$, which satisfies the following transport equation in $(-\tau, 0)$ with parameter $x \in \Gamma_0$

$$\begin{cases} z_t(t, \theta, x) - z_\theta(t, \theta, x) = 0, & \text{in } (0, \infty) \times (-\tau, 0) \times \Gamma_0, \\ z(t, 0, x) = w_t(t, x), & \text{in } (0, \infty) \times \Gamma_0, \\ z(0, \theta, x) = z_0(\theta, x), & \text{in } (-\tau, 0) \times \Gamma_0. \end{cases} \tag{3}$$

The fluid–membrane system (1) can now be rewritten as an evolution equation in X

$$\frac{d}{dt}(u, w, v, z) = A(u, w, v, z)$$

where $A : D(A) \to X$ is the linear operator defined by

$$A(u, w, v, z) := (\mu \Delta u - \nabla p, v, \Delta w - \alpha v - \beta z(-\tau) + p - \mu \nu \cdot \partial_\nu u, \partial_\theta z)$$

with domain $D(A)$ consisting of all elements $(u, w, v, z) \in X$ such that $u \in V$, $v \in \widehat{H}_0^1(\Gamma_0)$, $z \in H^1(-\tau, 0; \widehat{L}^2(\Gamma_0))$, $u = v\nu$ on Γ_0, $z_{|\theta=0} = v$ on Γ_0, $\mu \Delta u - \nabla p \in H$ and $\Delta w - \alpha v - \beta z(-\tau) + p - \mu \nu \cdot \partial_\nu u \in \widehat{L}^2(\Gamma_0)$, where $p = L(u, w, v, z) \in L^2(\Omega)$.

Let C_P be the constant in the following inequality obtained from trace theory and the Poincaré inequality

$$\int_{\Gamma_0} |u \cdot \nu|^2 \, \mathrm{d}x \le C_P \int_\Omega |\nabla u|^2 \, \mathrm{d}x, \qquad \forall u \in V. \tag{4}$$

Theorem 1. *If $\alpha + \frac{\mu}{C_P} \ge \beta$, where C_P is the constant in (4), then A is the generator of a strongly continuous semigroup of contractions on X.*

Proof. We apply the Lumer–Phillips Theorem and hence we must show that A is m-dissipative. Given $X_0 = (u, w, v, z) \in D(A)$, by applying generalized Green's identity for the membrane component, divergence theorem to the fluid component and Cauchy–Schwarz inequality we have

$$\begin{aligned}\mathrm{Re}(AX_0, X_0)_X &= -\mu\int_\Omega |\nabla u|^2\,\mathrm{d}x - \left(\alpha - \frac{|\beta|}{2}\right)\int_{\Gamma_0} |v|^2\,\mathrm{d}x - |\beta|\int_{\Gamma_0} z(-\tau)v\,\mathrm{d}x\\ &\quad -\frac{|\beta|}{2}\int_{\Gamma_0} |z(-\tau)|^2\,\mathrm{d}x \;\le\; -\left(\alpha - |\beta| + \frac{\mu}{C_P}\right)\int_{\Gamma_0} |v|^2\,\mathrm{d}x, \qquad (5)\end{aligned}$$

establishing the dissipativity of A.

To prove maximality, it is enough to prove that $0 \in \rho(A)$, where $\rho(A)$ denotes the resolvent set of A. In order to show this, we need to find $(u, w, v, z) \in D(A)$ such that $A(u, w, v, z) = (f, g, h, \zeta)$ for a given $(f, g, h, \zeta) \in X$ and $\|(u, w, v, z)\|_X \le C\|(f, g, h, \zeta)\|_X$ for some constant $C > 0$ independent of (u, w, v, z) and (f, g, h, ζ). The equation to solve is equivalent to $v = g$, $z_\theta = \zeta$, $z_{|\theta=0} = v$, the Stokes problem

$$\begin{cases} -\mu\Delta u + \nabla p = -f, & \text{in } \Omega,\\ \operatorname{div} u = 0, & \text{in } \Omega,\\ u = 0, & \text{on } \Gamma_1,\\ u = g\nu, & \text{on } \Gamma_0, \end{cases} \qquad (6)$$

and the elliptic equation with homogeneous Dirichlet condition

$$\begin{cases} -\Delta w = -\alpha g - \beta z(-\tau) + p - \mu\nu\cdot\partial_\nu u - h, & \text{in } \Gamma_0,\\ w = 0, & \text{on } \Sigma_0. \end{cases} \qquad (7)$$

We can see immediately that the delay variable is given by $z(\theta) = v - \int_\theta^0 \zeta(\vartheta)\,\mathrm{d}\vartheta$ from which we have $z \in H^1(-\tau, 0; \widehat{L}^2(\Gamma_0))$ and

$$\|z\|_{L^2(-\tau,0;L^2(\Gamma_0))} + \|z(-\tau)\|_{L^2(\Gamma_0)} \le C_\tau(\|g\|_{L^2(\Gamma_0)} + \|\zeta\|_{L^2(-\tau,0;L^2(\Gamma_0))}). \qquad (8)$$

The Stokes equation (6) admits a solution pair $(u, \tilde{p}) \in V \times L^2(\Omega)$; see, for instance [12]. Given a constant p^*, the pair (u, p) where $p = \tilde{p} + p^*$ is also a solution pair and we have

$$\|u\|_V + \|p\|_{L^2(\Omega)} \le C(\|f\|_H + \|g\|_{H_0^1(\Gamma_0)}),$$

and consequently we have the following trace estimate

$$\|p\|_{H^{-1/2}(\Gamma_0)} + \|\partial_\nu u\|_{H^{-1/2}(\Gamma_0)} \le C(\|f\|_H + \|g\|_{H_0^1(\Gamma_0)}). \qquad (9)$$

Since the right-hand side for the elliptic equation (7) lies in $H^{-1}(\Gamma_0)$, by standard elliptic theory we have a solution $w \in H^1(\Gamma_0)$ and from (8) and (9) it is not hard to see that $\|w\|_{H_0^1(\Gamma_0)} \le C\|(f, g, h, \zeta)\|_X$ for some constant $C > 0$.

The final step is to choose the constant p^* in such a way that w has average zero. Let $\psi \in H_0^1(\Gamma_0)$ be the solution of the Poisson equation $-\Delta\psi = 1$ on Γ_0 with boundary condition $\psi = 0$ on Σ_0. A straightforward calculation yields that $w \in \widehat{L}^2(\Gamma_0)$ if and only if

$$p^* = \|\nabla\psi\|^{-2}_{L^2(\Gamma_0)}\left((\alpha+\beta)\int_{\Gamma_0} v\psi\,\mathrm{d}x - \beta\int_{-\tau}^{0}\int_{\Gamma_0}\zeta(\vartheta)\psi\,\mathrm{d}x\,\mathrm{d}\vartheta - \langle\tilde{p} - \mu\nu\cdot\partial_\nu u, \psi\rangle\right).$$

Finally one can check that $p = L(u, w, v, z)$ and $\|(u, w, v, z)\|_X \le C\|(f, g, h, \zeta)\|_X$.

4 Spectral Properties

First, let us present the adjoint of the generator A. To describe the said operator, we consider the isomorphism $J : X \to X$

$$J(f, g, h, \zeta(\theta)) = (-f, g, -h, z(-\theta-\tau)).$$

Theorem 2. *The X-adjoint $A^* : D(A^*) \to X$ of the closed operator A is given by*

$$A^*(f, g, h, \zeta) = (\mu\Delta f - \nabla\pi, -h, -\Delta g - \alpha h + \beta\zeta(0) + \pi - \mu\nu\cdot\partial_\nu f, -\partial_\theta\zeta)$$

with domain $D(A^*)$ comprising of all elements $(f, g, h, \zeta) \in X$ such that $f \in V$, $h \in \widehat{H}^1_0(\Gamma_0)$, $\zeta \in H^1(-\tau, 0; \widehat{L}^2(\Gamma_0))$, $f = h\nu$ on Γ_0, $\zeta(-\tau) = -h$ on Γ_0, $\mu\Delta f - \nabla\pi \in H$ and $-\Delta g - \alpha h + \beta\zeta(0) + \pi - \mu\nu\cdot\partial_\nu f \in \widehat{L}^2(\Gamma_0)$ where $\pi = -LJ(f, g, h, \zeta)$.

Proof. The proof is similar to [11, Theorem 2.7] and therefore we omit it here.

In the following, we shall show that the spectrum of A consists of eigenvalues only, except possibly on the negative real axis. This will be done by rewriting the resolvent equation in variational form on a suitable space and then applying the Fredholm alternative and Lax–Milgram Lemma. For this direction, we introduce the following function spaces

$$W_0 = H \times \widehat{L}^2(\Gamma_0), \qquad W_1 = \{(u, v) \in V \times \widehat{H}^1_0(\Gamma_0) : u = v\nu \text{ on } \Gamma_0\}.$$

The embedding $W_1 \subset W_0$ is compact, dense and continuous.

Given a nonzero complex number λ and $Y = (f, g, h, \varphi) \in X$ we define the sesquilinear form $a_\lambda : W_1 \times W_1 \to \mathbb{C}$

$$\begin{aligned} a_\lambda((u, v), (\phi, \psi)) = &\lambda\int_\Omega u\cdot\phi\,\mathrm{d}x + \mu\int_\Omega \nabla u\cdot\nabla\phi\,\mathrm{d}x + q(\lambda)\int_{\Gamma_0} v\psi\,\mathrm{d}x \\ &+ \frac{1}{\lambda}\int_{\Gamma_0}\nabla v\cdot\nabla\psi\,\mathrm{d}x \end{aligned}$$

where $q(\lambda) = \lambda + \alpha + \beta e^{-\lambda\tau}$ and the antilinear form $F_{Y,\lambda} : W_1 \to \mathbb{C}$ by

$$F_{Y,\lambda}(\phi,\psi)=\int_{\Omega} f\cdot\phi\,\mathrm{d}x-\frac{1}{\lambda}\int_{\Gamma_0}\nabla g\cdot\nabla\psi\,\mathrm{d}x+\int_{\Gamma_0} h\psi\,\mathrm{d}x$$
$$-\beta\int_{-\tau}^{0}\int_{\Gamma_0} e^{-\lambda(\theta+\tau)}\varphi(\theta)\psi\,\mathrm{d}x\,\mathrm{d}\theta.$$

Theorem 3. *Let $\sigma(A)$ and $\sigma_p(A)$ be the spectrum and point spectrum of A, respectively. If $\alpha+\frac{\mu}{C_P}\geq|\beta|$ then $\sigma(A)\cap(\mathbb{C}\setminus(-\infty,0])=\sigma_p(A)$ and $\sigma(A^*)\cap(\mathbb{C}\setminus(-\infty,0])=\sigma_p(A^*)$.*

Proof. Let $\lambda\in\mathbb{C}\setminus(-\infty,0]$. For a given $Y=(f,g,h,\varphi)\in X$, suppose that there exists $(u,w,v,z)\in D(A)$ such that

$$(\lambda I-A)(u,w,v,z)=(f,g,h,\zeta). \tag{10}$$

This equation is equivalent to the condition that $\lambda w-v=g$, $\lambda z-\partial_\theta z=h$, $z(0)=v$, u satisfies the Stokes equation

$$\begin{cases}\lambda u-\mu\Delta u+\nabla p=f, & \text{in } \Omega,\\ \operatorname{div} u=0, & \text{in } \Omega,\\ u=0, & \text{on } \Gamma_1,\\ u=v\nu, & \text{on } \Gamma_0,\end{cases} \tag{11}$$

and (v,w) satisfies the boundary value problem

$$\begin{cases}(\lambda+\alpha)v-\Delta w=-\beta z(-\tau)-p+\mu\nu\cdot\partial_\nu u+h, & \text{in } \Gamma_0,\\ w=0, & \text{on } \Sigma_0.\end{cases} \tag{12}$$

Applying the variation of parameters formula to the equation for z, we obtain

$$z(\theta)=e^{\lambda\theta}v+\int_{\theta}^{0} e^{\lambda(\theta-\vartheta)}\varphi(\vartheta)\,\mathrm{d}\vartheta. \tag{13}$$

Using this and the fact that $w=\frac{1}{\lambda}(v+g)$ we can see that the variational form of the elliptic equation (12) is given by

$$q(\lambda)\int_{\Gamma_0} v\psi\,\mathrm{d}x+\frac{1}{\lambda}\int_{\Gamma_0}\nabla v\cdot\nabla\psi\,\mathrm{d}x=-\frac{1}{\lambda}\int_{\Gamma_0}\nabla g\cdot\nabla\psi\,\mathrm{d}x+\int_{\Gamma_0} h\psi\,\mathrm{d}x$$
$$-\beta\int_{-\tau}^{0}\int_{\Gamma_0} e^{-\lambda(\theta+\tau)}\varphi(\theta)\psi\,\mathrm{d}x\,\mathrm{d}\theta-\langle\mu\partial_\nu u-p\nu,\psi\nu\rangle \tag{14}$$

for every $\psi\in H_0^1(\Gamma_0)$. Also, the weak form of the Stokes equation (11) is given by

$$\lambda \int_{\Omega} u \cdot \phi \, \mathrm{d}x + \mu \int_{\Omega} \nabla u \cdot \nabla \phi \, \mathrm{d}x = \langle \mu \partial_\nu u - p\nu, \phi \rangle + \int_{\Omega} f \cdot \phi \, \mathrm{d}x \tag{15}$$

for every $\phi \in V$. Therefore if $(\phi, \psi) \in W_1$, taking the sum of (14) and (15) so that the duality pairing vanishes, we obtain the variational equation

$$a_\lambda((u, v), (\phi, \psi)) = F_{Y,\lambda}(\phi, \psi), \qquad \forall\, (\phi, \psi) \in W_1. \tag{16}$$

Conversely suppose that the variational Eq. (16) is satisfied. Define z and w as above. Choosing $\psi = 0$ we can see that u satisfies the Stokes equation (11) in the sense of distributions. Using Green's identity the elliptic equation (12) holds in the distributional sense as well. We choose $p = \tilde{p} + p^*$ where

$$p^* = \langle \mu\nu \partial_\nu u - \tilde{p} - \beta z(-\tau) - (\lambda + \alpha) v, \psi_0 \rangle$$

and $\{\psi_0\}$ is a basis of $\{\psi \in H_0^1(\Gamma_0) : \Delta u \text{ is constant}\}$, which has dimension 1, and is the orthogonal complement of $\widehat{H}_0^1(\Gamma_0)$ in $H_0^1(\Gamma_0)$. Split the sesquilinear form a_λ as $a_\lambda = a_{0,\lambda} + a_{1,\lambda}$ where the sesquilinear forms $a_{i,\lambda} : W_i \times W_i \to \mathbb{C}$ for $i = 0, 1$ are given by

$$\begin{aligned}
a_{1,\lambda}((u, v), (\phi, \psi)) &= \mu \int_{\Omega} \nabla u \cdot \nabla \phi \, \mathrm{d}x + \frac{1}{\lambda} \int_{\Gamma_0} \nabla v \cdot \nabla \psi \, \mathrm{d}x \\
a_{0,\lambda}((u, v), (\phi, \psi)) &= \lambda \int_{\Omega} u \cdot \phi \, \mathrm{d}x + q(\lambda) \int_{\Gamma_0} v\psi \, \mathrm{d}x.
\end{aligned}$$

The form $a_{1,\lambda}$ is W_1-coercive provided that $\operatorname{Im} \lambda \neq 0$ and $a_{0,\lambda}$ is bounded. From the Lax–Milgram–Fredholm Lemma (see [6]) we obtain the desired result. The corresponding result for the adjoint can be done in a similar way.

5 Uniform Exponential Stability

In this section we prove that the energy of the solutions for the fluid–membrane interaction model decays to zero exponentially under the condition $\alpha + \frac{\mu}{C_P} > \beta$. The result will be shown using the Lyapunov method. The success of this method to the system (1) relies on the following theorem in [5].

Theorem 4. *Let S be the Stokes map defined in the following way*

$$u = Sv \Longleftrightarrow \begin{cases} -\mu\Delta u + \nabla p = 0, & \textit{in } \Omega, \\ \operatorname{div} u = 0, & \textit{in } \Omega, \\ u = 0, & \textit{on } \Gamma_1, \\ u = v\nu, & \textit{on } \Gamma_0. \end{cases}$$

Then it holds that $S \in \mathscr{L}(\widehat{L}^2(\Gamma_0), H^{1/2}(\Omega) \cap H) \cap \mathscr{L}(\widehat{H}_0^1(\Gamma_0), H^{3/2}(\Omega) \cap H)$.

Theorem 5. *Suppose that* $\alpha + \frac{\mu}{C_P} > \beta$. *The semigroup generated by* A *is uniformly exponentially stable; that is, there exist* $\sigma > 0$ *and* $M \geq 1$ *such that* $\|e^{tA} X_0\|_X \leq Me^{-\sigma t}\|X_0\|_X$ *for every* $X_0 \in X$ *and* $t \geq 0$.

Proof. By a standard density argument it is enough to consider initial data in the domain of A. For this purpose, let $Y(t) = (u(t), v(t), w(t), z(t)) = e^{tA}(u_0, w_0, v_0, z_0)$ where $(u_0, w_0, v_0, z_0) \in D(A)$. We define the Lyapunov functional L as follows

$$L(t) = \frac{1}{2}\|(u(t), v(t), w(t), z(t))\|_X^2 + \varepsilon_1 \int_{-\tau}^{0} \int_{\Gamma_0} e^{a\theta}|z(t,\theta)|^2 \,\mathrm{d}x \,\mathrm{d}\theta$$
$$+ \varepsilon_2 \int_{\Omega} u(t) \cdot Sw(t) \,\mathrm{d}x + \varepsilon_2 \int_{\Gamma_0} w(t)v(t) \,\mathrm{d}x.$$

The positive constants a, ε_1 and ε_2 will be chosen below. Note that for sufficiently small ε_1 and ε_2, the functional $L(t)$ and the energy $E(t) := \frac{1}{2}\|(u(t), v(t), w(t), z(t))\|_X^2$ are equivalent, that is, there exist constants $c_1, c_2 > 0$ independent of t such that $c_1 E(t) \leq L(t) \leq c_2 E(t)$ for every $t \geq 0$.

Revising the dissipativity estimate (5) we have

$$\frac{d}{dt}E(t) \leq -\varepsilon \int_{\Omega} |\nabla u(t)|^2 \,\mathrm{d}x - \left(\alpha - \beta + \frac{\mu - \varepsilon}{C_P}\right) \int_{\Gamma_0} |v(t)|^2 \,\mathrm{d}x \tag{17}$$

where $\varepsilon > 0$ is small enough so that $k := \alpha - \beta + \frac{\mu - \varepsilon}{C_P} > 0$. On the other hand, taking the derivative of the second term of L and then using the transport equation for z we have

$$\frac{d}{dt} \int_{-\tau}^{0} \int_{\Gamma_0} e^{a\theta}|z(t,\theta)|^2 \,\mathrm{d}x \,\mathrm{d}\theta = \int_{-\tau}^{0} \int_{\Gamma_0} e^{a\theta} \partial_\theta(|z(t,\theta)|^2) \,\mathrm{d}x \,\mathrm{d}\theta$$
$$= \int_{\Gamma_0} (|v(t)|^2 - e^{-a\tau}|z(t,-\tau)|^2) \,\mathrm{d}x - a \int_{-\tau}^{0} \int_{\Gamma_0} e^{a\theta}|z(t,\theta)|^2 \,\mathrm{d}x \,\mathrm{d}\theta. \tag{18}$$

Taking the derivative of the third term of L and using the fact that $\operatorname{div} Sw = 0$, $Sw = 0$ on Γ_1 and $Sw = w\nu$ on Γ_0 we have

$$\frac{d}{dt} \int_{\Omega} u(t) \cdot Sw(t) \,\mathrm{d}x \tag{19}$$
$$= \int_{\Omega} (\mu \Delta u(t) - \nabla p(t)) \cdot Sw(t) \,\mathrm{d}x + \int_{\Omega} u(t) \cdot Sv(t) \,\mathrm{d}x$$
$$= -\mu \int_{\Omega} \nabla u(t) \cdot \nabla Sw(t) \,\mathrm{d}x + \langle \mu\nu \cdot \partial_\nu u(t) - p(t), w(t) \rangle + \int_{\Omega} u(t) \cdot Sv(t) \,\mathrm{d}x.$$

From Theorem 4 and the Poincaré inequality, we have the estimates

$$\left|\mu \int_{\Omega} \nabla u(t) \cdot \nabla S w(t)\,\mathrm{d}x\right| \le C_{\mu,\varepsilon_3} \int_{\Omega} |\nabla u(t)|^2\,\mathrm{d}x + \varepsilon_3 \int_{\Gamma_0} |\nabla w(t)|^2\,\mathrm{d}x \tag{20}$$

$$\left|\int_{\Omega} u(t) \cdot S v(t)\,\mathrm{d}x\right| \le C_{\mu} \int_{\Omega} |\nabla u(t)|^2\,\mathrm{d}x + C \int_{\Gamma_0} |v(t)|^2\,\mathrm{d}x. \tag{21}$$

Let C_{Γ_0} be the Poincaré constant corresponding to the domain Γ_0. Then we have

$$\begin{aligned}
&\langle \mu\nu \cdot \partial_\nu u(t) - p(t), w(t)\rangle + \frac{d}{dt}\int_{\Gamma_0} v(t)w(t)\,\mathrm{d}x - \int_{\Gamma_0} |v(t)|^2\,\mathrm{d}x \\
&= -\int_{\Gamma_0} |\nabla w(t)|^2\,\mathrm{d}x - \int_{\Gamma_0} (\alpha v(t) - \beta z(t,-\tau))w(t)\,\mathrm{d}x \\
&\le -(1-\varepsilon_3 C_{\Gamma_0})\int_{\Gamma_0} |\nabla w(t)|^2\,\mathrm{d}x - C_{\alpha,\beta,\varepsilon_3}\int_{\Gamma_0} (|v(t)|^2 + |z(t,-\tau)|^2)\,\mathrm{d}x. \qquad (22)
\end{aligned}$$

Therefore, if we choose the positive constants ε_i for $i = 1, 2, 3$ in such a way that $\varepsilon - \varepsilon_2(C_\mu + C_{\mu,\varepsilon_3}) > 0, k - \varepsilon_1 - (1 + C + C_{\alpha,\beta,\varepsilon_3})\varepsilon_2 > 0, \varepsilon_1 e^{-a\tau} - C_{\alpha,\beta,\varepsilon_3}\varepsilon_2 > 0$ and $1 - \varepsilon_3(1 + C_{\Gamma_0}) > 0$, then from (17) to (22) we can see that there exists a positive constant $C > 0$ such that $L'(t) \le -CL(t)$. Using the equivalence of L and E, we obtain the desired result.

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