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INTERFACE STABILIZATION OF A PARABOLIC-HYPERBOLIC PDE SYSTEM WITH DELAY IN THE INTERACTION

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Abstract. A coupled parabolic-hyperbolic system of partial differential equations modeling the interaction of a structure submerged in a fluid is studied. The system being considered incorporates delays in the interaction on the interface between the fluid and the solid. We study the stability properties of the interaction model under suitable assumptions between the competing strengths of the delays and the feedback controls.

1. **Introduction.** This paper is devoted to the study of stability properties of a coupled linear parabolic-hyperbolic PDE system with delay in the interaction. It addresses asymptotic, rational and exponential stability. The system being considered is a simplified model describing the interaction of an elastic body that is completely submerged in a fluid. Delay between the interaction on the interface is being considered.

First, let us set-up the notation and the geometrical configuration of the problem. Let Ω_s be a bounded smooth domain in $\mathbb{R}^d$ occupied by the structure. The relevant physical scenarios are $d = 2$ or $d = 3$, however, we shall consider the general case where $d \geq 2$ in the analysis. Denote the boundary of Ω_s by Γ_s. Let $\Omega_f \subset \mathbb{R}^d$ be the region occupied by the heat component. We also assume that that Ω_f is sufficiently smooth, and that its boundary consists of two parts $\partial\Omega_f = \Gamma_s \cup \Gamma_f$ where $\overline{\Gamma}_f = \Gamma_f$ and $\overline{\Gamma}_s = \Gamma_s$ have no common points. This means that the interface Γ_s between the solid and the fluid does not touch the part Γ_f of the boundary of Ω_f.

Let $u(t,x) = (u_1(t,x),\dots,u_d(t,x))$ and $w(t,x) = (w_1(t,x),\dots,w_d(t,x))$ be the velocity and displacement of the heat and structure at time t and position x, respectively. Then a linear model describing the dynamics of the above system is given

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by the coupled heat-wave system

$$\begin{cases} u_t(t,x) - \Delta u(t,x) = 0, & \text{in } (0,\infty)\times\Omega_f, \\ u(t,x) = 0, & \text{on } (0,\infty)\times\Gamma_f, \\ u(t,x) = \alpha w_t(t-\tau_s,x) + F(t,x), & \text{on } (0,\infty)\times\Gamma_s, \\ w_{tt}(t,x) - \Delta w(t,x) + w(t,x) = 0, & \text{in } (0,\infty)\times\Omega_s, \\ \dfrac{\partial w}{\partial \nu}(t,x) = \beta \dfrac{\partial u}{\partial \nu}(t-\tau_f,x) + G(t,x), & \text{in } (0,\infty)\times\Gamma_s, \\ u(0,x) = u_0(x), & \text{in } \Omega_f, \\ w(0,x) = w_0(x),\ w_t(0,x) = w_1(x), & \text{in } \Omega_s. \end{cases} \tag{1}$$

We shall make the convention that the unit normal vector ν is outward to Ω_s on Γ_s and is outward to Ω_f on Γ_f. In particular, ν on Γ_s will be inward with respect to fluid domain Ω_f. The terms F and G represent feedback controls that will be specified below.

In the case of small but rapid oscillations, the assumption that the interface Γ_s is stationary is reasonable. The system (1) is based on the one studied in [5], in the cases where there is no delay ($\tau_f = \tau_s = 0$) and no feedbacks ($F = G = 0$). The boundary conditions in (1) on Γ_s are obtained by matching the velocities and stresses of the fluid and solid components. However, in the current problem, these are not equal. The velocity of the structure serves as a source term at the boundary for the heat equation, and its effect is not instantaneous but a delay takes place. In this case, the constant $\tau_s > 0$ represents the extent of the delay while the constant $\alpha > 0$ signifies the strength of the delayed-velocity term. On the other hand, the normal stress of the fluid on the interface enters as a source term for the wave equation and delay is also being considered, with $\tau_f > 0$ measuring the extent of delay while $\beta > 0$ denotes the strength of the delayed-stress term. Such delay may occur if for instance there is a small boundary layer on the interface that impedes the instantaneous interaction between the fluid and the solid.

The fluid-structure model (1) without delay and several variants both in the linear and nonlinear settings have been studied extensively for the past years. This includes the case where instead of the heat equation for the fluid component, the Navier-Stokes equation takes place, and the Lamé equation is used instead of the wave equation for the solid component. We refer the reader to [2, 3, 4, 5, 6, 7, 9, 10, 15, 19, 20, 29, 30] and the references therein for the analysis of such variations. These investigations include the well-posedness, regularity and stability of the interaction models. With regards to a fluid-structure model with delay only on the interior feedback for the structure, see [34].

Without the feedback controls and in the case where $\alpha = \beta$ and $F = G = 0$ one can easily check that the system is dissipative, that is, the energy is decreasing. The source of dissipation is from the diffusion of the fluid. In this work, we consider the following feedback controls

$$\begin{cases} F(t,x) = \gamma \dfrac{\partial u}{\partial \nu}(t,x), & \text{on } (0,\infty)\times\Gamma_s, \\ G(t,x) = -\delta w_t(t,x), & \text{on } (0,\infty)\times\Gamma_s, \end{cases} \tag{2}$$

where γ and δ are positive constants. To achieve a well-posed system, the initial histories for the stress and velocity of the fluid and the solid on the interface, respectively, shall be specified

$$\begin{cases} \dfrac{\partial u}{\partial \nu}(\theta, x) = y_0(\theta, x), & \text{in } (-\tau_f, 0) \times \Gamma_s, \\ w_t(\theta, x) = z_0(\theta, x), & \text{in } (-\tau_s, 0) \times \Gamma_s. \end{cases} \tag{3}$$

In Section 2, the well-posedness of (1)–(3) based on semigroup methods will be discussed.

We would like to point out that Neumann-type boundary feedbacks in terms of the solid component have been considered in [20], whereas in the present paper we consider a Neumann-type boundary feedback in terms of the fluid component. Technically this will serve as a regularization in terms of the Neumann trace for the fluid component. In effect, this has an advantage when one needs to define the appropriate function spaces for the delay variables. These particular feedbacks turns the system (1) into a coupled parabolic-hyperbolic system with Robin-Neumann boundary conditions on the interface.

It is well-known that delays have a destabilizing effect [12, 13]. This is due to the fact that delay induces a transport phenomenon in the system that creates oscillations which can lead to instability. The goal is then to determine the relationship among the parameters α, β, γ and δ for which the energy associated with the above model decays to zero asymptotically, or better, to have uniform exponential decay rates. Under the assumption $\alpha\beta \leq \gamma\delta$, we show in Section 3 that the system is asymptotically stable using spectral methods and a generalized Lax-Milgram Lemma. If $\alpha\beta < \gamma\delta$ then we prove in Section 4 that the energy decays exponentially to zero through the energy method, following the approach given in [6, 21, 22, 23, 24].

Our result will be valid for positive delay parameters. A direct application of the Cauchy-Schwarz inequality will give us a sufficient condition for stability, however this is *not optimal*. The results will be based on the positivity of a quadratic form induced by the boundary terms and the corresponding result obtained by merely applying the Cauchy-Schwarz inequality is just a specific case. The said quadratic form is a special case of the one considered in [37] for coupled wave equations in the entire space with delay. It is worth noting that the condition $\alpha\beta \leq \gamma\delta$ for the stability of our problem coincides with the one given in [28] in the case of a coupled ordinary differential equations with delay.

Under the critical case, the dissipation terms induced by the feedbacks are cancelled, however the system still posseses dissipation due to the diffusion of the fluid. Without delays and feedbacks and with $\alpha = \beta = 1$, the rational stability of the system (1) has been proved in [5] using a resolvent-based approach. This particular method relies on establishing a polynomial or algebraic growth of the resolvents on the imaginary axis. The corresponding decay estimate follows from a theorem in [11]. The same strategy has been also applied to a fluid-structure interaction model in [2]. Other relevant references for the rational decay rates of coupled heat-wave systems without delay are provided in [5]. We would like to point out that the decay rate $O(t^{-\frac{1}{3}})$ obtained in the present case when delay is incorporated is weaker than the one obtained based on the original model without delay. This is due to the additional terms in the interface, which are only square integrable. In the absence of delay, a decay rate $O(t^{-\frac{1}{2}})$ was established in [5] and recently improved to $O(t^{-1})$ using a microlocal analysis argument in [8]. Nevertheless, we will show that the system is asymptotically stable under the critical case and with an additional geometric condition, the decay rate $O(t^{-\frac{1}{3}})$ will be established in Section 5. Further related problems will be mentioned in Section 6.

2. **The evolution system.** In this section, we recast (1)–(3) as a first order evolution system on a certain Hilbert space and prove its well-posedness through semigroup theory. The regularity of solutions for smooth and compatible data will be provided using elliptic regularity. Before we proceed with the formulation, we first recall in the following subsection the traces for a graph space. We follow the usual notations $H^k(\Omega)$ and $L^p(\Omega)$ for the Sobolev and Lebesgue spaces, respectively. For simplicity, the product of m copies of a Banach space X will be denoted by the same notation X instead of X^m.

2.1. **Traces for a graph space.** It is known that if a certain function satisfies an elliptic problem, then a generalize boundary trace for that function can be defined, see [26, Chapter 2]. In the following discussion, we take the formulation in [38, pp. 432-433]. Let $\mathcal{W}(\Omega_f) = \{u \in H^1(\Omega_f) : \Delta u \in L^2(\Omega_f)\}$ be equipped with the graph norm

$$\|u\|_{\mathcal{W}(\Omega_f)} = (\|u\|^2_{H^1(\Omega_f)} + \|\Delta u\|^2_{L^2(\Omega_f)})^{\frac{1}{2}}.$$

Endowed with the inner product associated with this norm, $\mathcal{W}(\Omega_f)$ is a Hilbert space. For a given $\phi \in H^{\frac{1}{2}}(\Gamma_s)$, we extend it by zero to $\partial\Omega_f$ and obtain an element in $H^{\frac{1}{2}}(\partial\Omega_f)$ and denote this extension by the same notation.

Recall that the trace map $\gamma_0 : H^1(\Omega_f) \to H^{\frac{1}{2}}(\partial\Omega_f)$ is onto, and thus $\gamma_0\gamma_0^*$ is invertible, where $\gamma_0^* : H^{\frac{1}{2}}(\partial\Omega_f) \to H^1(\Omega_f)$ is the adjoint of γ_0. Consider the bounded linear operator $\ell : H^{\frac{1}{2}}(\partial\Omega_f) \to H^1(\Omega_f)$ given by

$$\ell = \gamma_0^*(\gamma_0\gamma_0^*)^{-1}$$

so that $\gamma_0\ell\phi = \phi$ for every $\phi \in H^{\frac{1}{2}}(\partial\Omega_f)$. Define $\frac{\partial u}{\partial \nu}|_{\Gamma_s}$ by

$$\left\langle \frac{\partial u}{\partial \nu}\bigg|_{\Gamma_s}, \phi \right\rangle = \int_{\Omega_f} (\Delta u)\ell\phi \,\mathrm{d}x + \int_{\Omega_f} \nabla u \cdot \nabla(\ell\phi)\,\mathrm{d}x$$

for every $\phi \in H^{\frac{1}{2}}(\Gamma_s)$. By the Cauchy-Schwarz inequality we have

$$\begin{aligned} \left|\left\langle \frac{\partial u}{\partial \nu}\bigg|_{\Gamma_s}, \phi \right\rangle\right| &\le \|\Delta u\|_{L^2(\Omega_f)}\|\ell\phi\|_{L^2(\Omega_f)} + \|\nabla u\|_{L^2(\Omega_f)}\|\nabla(\ell\phi)\|_{L^2(\Omega_f)} \\ &\le \sqrt{2}\|\ell\|_{\mathcal{L}(H^{\frac{1}{2}}(\partial\Omega_f), H^1(\Omega_f))}\|u\|_{\mathcal{W}(\Omega_f)}\|\phi\|_{H^{\frac{1}{2}}(\Gamma_s)}. \end{aligned}$$

Consequently, $\frac{\partial u}{\partial \nu}|_{\Gamma_s} \in H^{-\frac{1}{2}}(\Gamma_s)$ and the operator $u \mapsto \frac{\partial u}{\partial \nu}|_{\Gamma_s}$ is a bounded linear operator from the graph space $\mathcal{W}(\Omega_f)$ into $H^{-\frac{1}{2}}(\Gamma_s)$. If $u \in H^2(\Omega_f)$ then this definition coincides with the usual first order trace of u on Γ_s. Indeed, this can be seen immediately from Green's identity. According to the definition it follows easily that

$$\int_{\Omega} (\Delta u)\varphi \,\mathrm{d}x = \left\langle \frac{\partial u}{\partial \nu}\bigg|_{\Gamma_s}, \varphi \right\rangle - \int_{\Omega_f} \nabla u \cdot \nabla\varphi \,\mathrm{d}x$$

for every $u \in \mathcal{W}(\Omega_f)$ and $\varphi \in H^1(\Omega_f)$ with $\varphi = 0$ on Γ_f. In the succeeding sections, the expression Γ_s will be removed in the notation for the trace for convenience.

2.2. **Semigroup formulation.** Keeping track of the memory terms, we introduce the following auxiliary state variables

$$y(t,\theta,x) = \frac{\partial u}{\partial \nu}(t - \theta\tau_f, x), \quad z(t,\theta,x) = w_t(t - \theta\tau_s, x)$$

for $(t,\theta,x) \in (0,\infty) \times (0,1) \times \Gamma_s$ corresponding to the delay terms in the stress of the fluid and velocity of the structure on the interface. Notice that y satisfies the following uncoupled transport system on the bounded interval $(0,1)$ with parameter $x \in \Gamma_s$

$$\begin{cases} \tau_f y_t(t,\theta,x) + y_\theta(t,\theta,x) = 0, & \text{in } (0,\infty)\times(0,1)\times\Gamma_s, \\ y(t,0,x) = \dfrac{\partial u}{\partial \nu}(t,x), & \text{on } (0,\infty)\times\Gamma_s, \\ y(0,\theta,x) = \tilde{y}_0(\theta,x), & \text{on } (0,1)\times\Gamma_s, \end{cases} \tag{4}$$

where $\tilde{y}_0(\theta,x) = y_0(-\theta\tau_f, x)$. On the other hand, z satisfies a similar transport system with parameter

$$\begin{cases} \tau_s z_t(t,\theta,x) + z_\theta(t,\theta,x) = 0, & \text{in } (0,\infty)\times(0,1)\times\Gamma_s, \\ z(t,0,x) = w_t(t,x), & \text{on } (0,\infty)\times\Gamma_s, \\ z(0,\theta,x) = \tilde{z}_0(\theta,x) & \text{on } (0,1)\times\Gamma_s, \end{cases} \tag{5}$$

where $\tilde{z}_0(\theta,x) = z_0(-\theta\tau_s, x)$.

We recast (1)–(3) as a first order system in the state variables (u,w,v,y,z), where $v = w_t$. Here, the wave equation is formulated as a first order system in terms of the displacement and velocity. Consider the following Hilbert space as our state space

$$X = L^2(\Omega_f)\times H^1(\Omega_s)\times L^2(\Omega_s)\times L^2_\theta(L^2(\Gamma_s))\times L^2_\theta(L^2(\Gamma_s)).$$

where $L^2_\theta(L^2(\Gamma_s)) = L^2(0,1;L^2(\Gamma_s))$. Due to different factors in the delay and feedback terms, this space will be equipped with a weighted inner product. For $(u_1,w_1,v_1,y_1,z_1),(u_2,w_2,v_2,y_2,z_2) \in X$ define

$$\begin{aligned} &\langle (u_1,w_1,v_1,y_1,z_1),(u_2,w_2,v_2,y_2,z_2)\rangle_{X,\mathbf{a}} \\ &= a_1\int_{\Omega_f} u_1\cdot u_2\,\mathrm{d}x + a_2\int_{\Omega_s}(w_1\cdot w_2 + \nabla w_1\cdot\nabla w_2 + v_1\cdot v_2)\,\mathrm{d}x \\ &\quad + a_3\tau_f\int_0^1\int_{\Gamma_s} y_1\cdot y_2\,\mathrm{d}x\,\mathrm{d}\theta + a_4\tau_s\int_0^1\int_{\Gamma_s} z_1\cdot z_2\,\mathrm{d}x\,\mathrm{d}\theta \end{aligned}$$

where $\mathbf{a} = (a_1,a_2,a_3,a_4)$ is a quadruple consisting of positive constants that will be specified below. Here, the dot represents either the inner product in $\mathbb{C}^d$ or $\mathbb{C}^{d\times d}$ where it is applicable. The norm associated with this inner product will be denoted by $\|\cdot\|_{X,\mathbf{a}}$.

Let $H^1_{\Gamma_f}(\Omega_f) = \{u \in H^1(\Omega_f) : u = 0 \text{ on } \Gamma_f\}$. Define the linear operator $A : D(A) \subset X \to X$ with domain

$$\begin{aligned} D(A) = \{&(u,w,v,y,z)\in X : u \in \mathcal{W}(\Omega_f)\cap H^1_{\Gamma_f}(\Omega_f),\ w\in\mathcal{W}(\Omega_s), \\ &v\in H^1(\Omega_s),\ y,z\in H^1(0,1;L^2(\Gamma_s)),\ u - \gamma\frac{\partial u}{\partial\nu} = \alpha z(1) \text{ on } \Gamma_s, \\ &\frac{\partial w}{\partial\nu} + \delta v = \beta y(1) \text{ on } \Gamma_s,\ y(0) = \frac{\partial u}{\partial \nu},\ z(0) = v \text{ on } \Gamma_s\}, \end{aligned}$$

as follows

$$A(u,w,v,y,z) = (\Delta u, v, \Delta w - w, -\tau_f^{-1}\partial_\theta y, -\tau_s^{-1}\partial_\theta z).$$

Recall that $H^1(0,1;L^2(\Gamma_s)) \subset C([0,1],L^2(\Gamma_s))$, see for instance [17, p. 286]. From the above discussion, we already know that the traces $\frac{\partial u}{\partial\nu}$ and $\frac{\partial w}{\partial\nu}$ both exist as elements in $H^{-\frac{1}{2}}(\Gamma_s)$. According to the definition of $D(A)$, we have $\frac{\partial u}{\partial\nu} = \frac{1}{\gamma}(u - \alpha z(1)) \in L^2(\Gamma_s)$ since $\gamma > 0$. This means that the condition $y(0) = \frac{\partial u}{\partial\nu}$ is meaningful under the assumption that $y \in H^1(0,1;L^2(\Gamma_s))$. Likewise, we also have $\frac{\partial w}{\partial\nu} = \beta y(1) - \delta v \in L^2(\Gamma_s)$.

System (1)–(3) can now be rewritten as a first order evolution equation on X as

$$\begin{cases} \dot{Y}(t) = AY(t), & t > 0, \\ Y(0) = Y_0, \end{cases} \tag{6}$$

with state $Y = (u, w, v, y, z)$ and initial data $Y_0 = (u_0, w_0, w_1, \tilde{y}_0, \tilde{z}_0) \in X$.

Theorem 2.1. *Let $\alpha, \beta, \gamma, \delta > 0$. Suppose that there exists $\mathbf{a} = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4_+$ such that the quadratic form $Q_{\mathbf{a}}$ defined by*

$$Q_{\mathbf{a}}(u, v, y, z) = (2a_1\gamma - a_3)u^2 + (2a_2\delta - a_4)v^2 + a_3y^2 + a_4z^2 + 2a_1\alpha uz + 2a_2\beta vy \tag{7}$$

is nonnegative definite. Then the operator A generates a strongly continuous semigroup of contractions on X. In particular, for every $Y_0 \in X$ the Cauchy problem (6) admits a unique mild solution $Y \in C([0,\infty), X)$ such that $\|Y(t)\|_{X,\mathbf{a}} \le \|Y_0\|_{X,\mathbf{a}}$ for every $t \ge 0$. Moreover, we have

$$u \in L^2(0, \infty; H^1_{\Gamma_f}(\Omega_f)). \tag{8}$$

If $\gamma\delta > \alpha\beta$ then for every $T > 0$ it holds that

$$y, z \in L^\infty(0, 1; L^2((0, T) \times \Gamma_s)). \tag{9}$$

A sufficient condition for the nonnegativity of the quadratic form $Q_{\mathbf{a}}$ is $\alpha\beta \le \gamma\delta$. The proof of this remark is provided in the Appendix.

Proof of Theorem 2.1. We shall apply the Lumer-Phillips theorem in reflexive Banach spaces, see [16, Corollary III.3.20]. The first step is to check the dissipativity of A. For this, take an arbitrary element $Y = (u, w, v, y, z) \in D(A)$. Employing Green's identity with respect to the fluid domain and using the boundary conditions $u = 0$ on Γ_f and $u = \gamma\frac{\partial u}{\partial \nu} + \alpha z(1)$ on Γ_s we have

$$\mathrm{Re}\int_{\Omega_f} \Delta u \cdot u \,\mathrm{d}x = -\int_{\Omega_f} |\nabla u|^2 \,\mathrm{d}x - \gamma\int_{\Gamma_s} \left|\frac{\partial u}{\partial \nu}\right|^2 \mathrm{d}x - \alpha \mathrm{Re}\int_{\Gamma_s} \frac{\partial u}{\partial \nu} \cdot z(1)\,\mathrm{d}x. \tag{10}$$

The negative sign on the boundary terms is due to the convention that ν on Γ_s is inward to the fluid domain Ω_f. Likewise, using Green's identity with respect to the structural domain and the boundary condition $\frac{\partial w}{\partial \nu} = \beta y(1) - \delta v$ give us

$$\mathrm{Re}\int_{\Omega_s} v \cdot w + \nabla v \cdot \nabla w + (\Delta w - w) \cdot v \,\mathrm{d}x = -\delta\int_{\Gamma_s} |v|^2 \,\mathrm{d}x + \beta\mathrm{Re}\int_{\Gamma_s} y(1) \cdot v \,\mathrm{d}x. \tag{11}$$

For the delay variable y we integrate by parts, take the real part and apply the condition $y(0) = \frac{\partial u}{\partial \nu}$ to obtain

$$\mathrm{Re}\int_0^1\int_{\Gamma_s} \Big(-\frac{1}{\tau_f}\partial_\theta y\Big) \cdot y \,\mathrm{d}x\,\mathrm{d}\theta = \frac{1}{2\tau_f}\int_{\Gamma_s} \left|\frac{\partial u}{\partial \nu}\right|^2 \mathrm{d}x - \frac{1}{2\tau_f}\int_{\Gamma_s} |y(1)|^2 \,\mathrm{d}x. \tag{12}$$

In a similar way, for the delay variable z we have using $z(0) = v$

$$\mathrm{Re}\int_0^1\int_{\Gamma_s} \Big(-\frac{1}{\tau_s}\partial_\theta z\Big) \cdot z \,\mathrm{d}x\,\mathrm{d}\theta = \frac{1}{2\tau_s}\int_{\Gamma_s} |v|^2 \,\mathrm{d}x - \frac{1}{2\tau_s}\int_{\Gamma_s} |z(1)|^2 \,\mathrm{d}x. \tag{13}$$

Multiplying (10)–(12) by constants a_1, a_2, a_3, a_4, respectively, and then using the Cauchy-Schwarz inequality to the last terms on the right hand sides of (10) and

(11), it is not difficult to see that we have

$$\operatorname{Re}\langle AY, Y\rangle_{X,\mathbf{a}} \le -a_1 \int_{\Omega_f} |\nabla u|^2 \,\mathrm{d}x - \frac{1}{2} Q_{\mathbf{a}}\Big(\Big\|\frac{\partial u}{\partial \nu}\Big\|_{L^2(\Gamma_s)}, \|v\|_{L^2(\Gamma_s)}, -\|y(1)\|_{L^2(\Gamma_s)}, -\|z(1)\|_{L^2(\Gamma_s)} \Big), \tag{14}$$

where $Q_{\mathbf{a}}$ is the quadratic form defined by (7). Consequently, it follows from (14) and the nonnegativity of $Q_{\mathbf{a}}$ that A is dissipative.

The next step is to prove that 0 lies in the resolvent set $\rho(A)$ of A. This implies that A has a nonempty resolvent, and thus A is closed according to [38, Remark 2.2.4]. The fact that the resolvent set is open implies that $\lambda \in \rho(A)$ for some $\lambda > 0$, which proves the range condition $R(\lambda I - A) = X$ in the Lumer-Phillips Theorem. Let $Y^* = (u^*, w^*, v^*, y^*, z^*) \in X$ be given. Note that the equation $AY = Y^*$ for some $Y = (u, w, v, y, z) \in D(A)$ is equivalent to the following system where $v = w^*$,

$$y(\theta) = \frac{\partial u}{\partial \nu} - \tau_f \int_0^\theta y^*(\vartheta)\,\mathrm{d}\vartheta, \tag{15}$$

$$z(\theta) = w^* - \tau_s \int_0^\theta z^*(\vartheta)\,\mathrm{d}\vartheta, \tag{16}$$

u satisfies an elliptic problem on Ω_f with mixed Dirichlet-Robin boundary conditions

$$\begin{cases} \Delta u = u^*, & \text{in } \Omega_f, \\ u = 0, & \text{on } \Gamma_f, \\ u - \gamma \dfrac{\partial u}{\partial \nu} = \alpha z(1), & \text{on } \Gamma_s, \end{cases} \tag{17}$$

and w satisfies an elliptic problem on Ω_s with Neumann boundary condition on Γ_s

$$\begin{cases} \Delta w - w = v^*, & \text{in } \Omega_s, \\ \dfrac{\partial w}{\partial \nu} = \beta y(1) - \delta w^*, & \text{on } \Gamma_s. \end{cases} \tag{18}$$

The boundary conditions on Γ_s for (17) and (18) can be written, according to (16) and (15), respectively as follows

$$\frac{\partial u}{\partial \nu} = \frac{1}{\gamma}\left(u - \alpha w^* + \alpha \tau_s \int_0^1 z^*(\theta)\,\mathrm{d}\theta \right), \tag{19}$$

$$\frac{\partial w}{\partial \nu} = \beta \frac{\partial u}{\partial \nu} - \beta \tau_f \int_0^1 y^*(\theta)\,\mathrm{d}\theta - \delta w^*. \tag{20}$$

By elliptic theory, the boundary value problem (17) admits a solution $u \in H^1_{\Gamma_f}(\Omega_f)$ and since $u^* \in L^2(\Omega_f)$ we deduce that $u \in \mathcal{W}(\Omega_f)$. Moreover, the boundary condition (19) implies that $\frac{\partial u}{\partial \nu} \in L^2(\Gamma_s)$ and thus $y \in H^1(0,1; L^2(\Gamma_s))$ where y is defined by (15). We can see immediately from (16) that $z \in H^1(0,1; L^2(\Gamma_s))$. On the other hand, the boundary value problem (18) has a unique solution $w \in H^1(\Omega_s)$ and since $v^* + w \in L^2(\Omega_s)$ we have $w \in \mathcal{W}(\Omega_s)$. Therefore there exists $Y \in D(A)$ such that $AY = Y^*$.

We show that there is a constant $C > 0$ independent of Y and Y^* such that

$$\|Y\|_{X,\mathbf{a}} \le C \|Y^*\|_{X,\mathbf{a}} \tag{21}$$

and this will prove that $0 \in \rho(A)$. Using a standard elliptic estimate, the trace theorem, Hölder's inequality and Fubini's theorem to the term involving z^*, we obtain from (17) and (19) that

$$\|u\|_{H^1(\Omega_f)} \;\leq\; C(\|u^*\|_{L^2(\Omega_f)} + \|w^*\|_{H^1(\Omega_s)} + \|z^*\|_{L^2_\theta(L^2(\Gamma_s))}). \tag{22}$$

By (19), the trace theorem once more and (22) we get

$$\Big\|\frac{\partial u}{\partial \nu}\Big\|_{L^2(\Gamma_s)} \leq C(\|u^*\|_{L^2(\Omega_f)} + \|w^*\|_{H^1(\Omega_s)} + \|z^*\|_{L^2_\theta(L^2(\Gamma_s))}). \tag{23}$$

Using (23) we can estimate the norm of y as follows

$$\|y\|_{L^2_\theta(L^2(\Gamma_s))} \leq C(\|u^*\|_{L^2(\Omega_f)} + \|w^*\|_{H^1(\Omega_s)} + \|z^*\|_{L^2_\theta(L^2(\Gamma_s))} + \|y^*\|_{L^2_\theta(L^2(\Gamma_s))}). \tag{24}$$

On the other hand, it is not difficult to see from (16) that

$$\|z\|_{L^2_\theta(L^2(\Gamma_s))} \leq C(\|w^*\|_{H^1(\Omega_s)} + \|z^*\|_{L^2_\theta(L^2(\Gamma_s))}). \tag{25}$$

Finally, from the equation $v = w^*$, a basic elliptic estimate applied to the boundary value problem (18) and (20) we have the following estimates

$$\|v\|_{L^2(\Omega_s)} \leq \|w^*\|_{H^1(\Omega_s)}, \tag{26}$$

$$\|w\|_{H^1(\Omega_s)} \leq C\Big(\|v^*\|_{L^2(\Omega_s)} + \|w^*\|_{H^1(\Omega_s)} + \|y^*\|_{L^2_\theta(L^2(\Gamma_s))} + \Big\|\frac{\partial u}{\partial \nu}\Big\|_{L^2(\Gamma_s)}\Big). \tag{27}$$

Combining (22)–(27) proves (21). This completes the proof that A is the generator of a strongly continuous semigroup of contractions on X.

For data $Y_0 \in D(A)$ we have $Y(t) \in D(A)$ for all $t > 0$ and the component u of the corresponding semigroup solution $Y(t) = e^{tA}Y_0$ satisfies $u = 0$ on $(0,\infty) \times \Gamma_f$ and according to (14) we have

$$\int_0^T \int_{\Omega_f} |\nabla u(t)|^2 \,\mathrm{d}x\,\mathrm{d}t \leq -\frac{1}{a_1}\int_0^T \mathrm{Re}\langle \dot{Y}(t), Y(t)\rangle_{X,\mathbf{a}}\,\mathrm{d}t \leq \frac{1}{a_1}\|Y_0\|^2_{X,\mathbf{a}}$$

for every $T > 0$. This estimate implies that $u \in L^2(0,\infty; H^1_{\Gamma_f}(\Omega_f))$, hence (8). The corresponding result for data in X follows immediately from the density of $D(A)$ in X. Finally, (9) is a consequence of the assumption that $\alpha\beta < \gamma\delta$, which implies that $Q_{\mathbf{a}}$ is positive definite (see Theorem 7.1 below) and the fact that they satisfy the one-dimensional transport equations (4) and (5) with parameter $x \in \Gamma_s$, respectively. Such regularity can be proved by following the arguments as in [36]. □

Next we prove the regularity of solutions with initial data in $D(A^2)$. The following theorem will be utilized in establishing the stability of (1)–(3) in Section 4.

Theorem 2.2. *Let $Y_0 \in D(A^2)$ and $(u,w,v,y,z) = e^{tA}Y_0$ be the semigroup solution of (6). For every $T > \tau_s$ it holds that*

$$u \in C^1([0,T], L^2(\Omega_f)) \cap L^2(\tau_s, T; H^2(\Omega_f))$$

and for every $T > \tau_f + \tau_s$ we have

$$w \in C^2([0,T], H^1(\Omega_s)) \cap L^2(\tau_f + \tau_s, T; H^2(\Omega_s)).$$

Proof. For $Y_0 \in D(A)$ it follows that $u \in C^1([0,T], L^2(\Omega_f)) \cap C([0,T], H^1_{\Gamma_f}(\Omega_f))$. This is a consequence of the fact that $e^{tA}Y_0 \in C^1([0,T], X) \cap C([0,T], D(A))$, where $D(A)$ is equipped with the graph norm.

For data $Y_0 \in D(A^2)$ we have $e^{tA}Y_0 \in C^1([0,T],D(A))$ and as a result $v_t = \Delta w - w \in C([0,T],H^1(\Omega_s))$. The fact that $w \in C^2([0,T],H^1(\Omega_s))$ follows from the equation $w_t = v$ and $w(0) \in H^1(\Omega_s)$. Therefore, by trace theory we have

$$\frac{\partial u}{\partial \nu} = \frac{1}{\gamma}\left(u - \alpha w_t(\cdot - \tau_s)\right) \in L^2(\tau_s, T; H^{\frac{1}{2}}(\Gamma_s))$$

for every $T > \tau_s$, and hence it follows that $u \in L^2(\tau_s, T; H^2(\Omega_f))$ by elliptic regularity. On the other hand, we have $\Delta w = v_t + w \in C([0,T],H^1(\Omega_s))$ and

$$\frac{\partial w}{\partial \nu} = \beta \frac{\partial u}{\partial \nu}(\cdot - \tau_f) - \delta w_t \in L^2(\tau_f + \tau_s, T; H^{\frac{1}{2}}(\Gamma_s))$$

for every $T > \tau_f + \tau_s$, and therefore $w \in L^2(\tau_f + \tau_s, T; H^2(\Omega_s))$ by applying elliptic regularity once more. □

3. **Spectral properties and asymptotic stability.**

In the absence of delays and feedback controls in the coupled system (1), the corresponding semigroup generator lacks the compactness of its resolvents. However, the projections of the resolvents onto the product of the state spaces corresponding to the fluid and structure velocities are compact, see [3] for the details. The goal of this section is to show that the spectrum of the generator A corresponding to the problem (1)–(3) consists of eigenvalues only. Our strategy is to rewrite the resolvent equation for a given fixed element of X in variational form and then apply a generalized Lax-Milgram argument.

The method we employ has been utilized for wave equations with viscoelastic surface [14] and for a fluid-structure interaction model with delay in the interior feedback control for the structural component [34]. The corresponding results in these works do not consider the whole spectrum and the analysis tackles only the part of spectrum neglecting the negative real axis. This is due to the fact that the boundary condition on the interface for the equation corresponding to the fluid is of Dirichlet type. This essential boundary condition entails to incorporate the compatibility condition in the definition of the space for the variational formulation. In this direction, one needs to formulate the variational form in terms of the velocities of the fluid and the structure. For our problem, due to the Neumann-type feedback on Γ_s we basically have boundary value problems that have either Neumann or Robin boundary conditions on the interface. As a result of these natural boundary conditions, we can formulate the variational equation in terms of the velocity of the fluid and the displacement of the structure, and from this we can cover the whole spectrum of the generator in the analysis. The first step is to rewrite the resolvent equation in variational form.

Lemma 3.1. *Let $\lambda \in \mathbb{C}$ and $Y^* = (u^*, w^*, v^*, y^*, z^*) \in X$ be fixed. There exists $Y = (u, w, v, y, z) \in D(A)$ satisfying the equation*

$$(\lambda I - A)Y = Y^* \tag{28}$$

if and only if there exists a pair $(u, w) \in H^1_{\Gamma_f}(\Omega_f) \times H^1(\Omega_s)$ that satisfies the variational equation

$$a_\lambda((u, w), (\varphi, \psi)) = F_{\lambda, Y^*}(\varphi, \psi) \tag{29}$$

for every $(\varphi, \psi) \in H^1_{\Gamma_f}(\Omega_f) \times H^1(\Omega_s)$, where $a_\lambda : [H^1_{\Gamma_f}(\Omega_f) \times H^1(\Omega_s)]^2 \to \mathbb{C}$ is the continuous sesquilinear form defined by

$$\begin{aligned} a_\lambda((u,w),(\varphi,\psi)) \;=\; & \lambda \int_{\Omega_f} u\cdot\varphi \,\mathrm{d}x + \int_{\Omega_f} \nabla u\cdot\nabla\varphi\,\mathrm{d}x + (\lambda^2+1)\int_{\Omega_s} w\cdot\psi\,\mathrm{d}x \\ & + \int_{\Omega_s} \nabla w\cdot\nabla\psi\,\mathrm{d}x + \frac{1}{\gamma}\int_{\Gamma_s} u\cdot(\varphi - \beta e^{-\overline{\lambda}\tau_f}\psi)\,\mathrm{d}x \\ & + \lambda\int_{\Gamma_s} w\cdot\Big[\Big(\frac{\alpha\beta}{\gamma}e^{-\overline{\lambda}(\tau_f+\tau_s)}+\delta\Big)\psi - \frac{\alpha}{\gamma}e^{-\overline{\lambda}\tau_s}\varphi\Big]\mathrm{d}x \end{aligned} \tag{30}$$

and $F_{\lambda,Y^*} : H^1_{\Gamma_f}(\Omega_f)\times H^1(\Omega_s)\to\mathbb{C}$ *is the continuous anti-linear form defined by*

$$\begin{aligned} F_{\lambda,Y^*}(\varphi,\psi) \;=\; & \int_{\Omega_f} u^*\cdot\varphi\,\mathrm{d}x + \int_{\Omega_s} v^*\cdot\psi\,\mathrm{d}x \\ & + \int_{\Gamma_s} w^*\cdot\Big[\Big(\frac{\alpha\beta}{\gamma}e^{-\overline{\lambda}(\tau_f+\tau_s)}+\delta\Big)\psi - \frac{\alpha}{\gamma}e^{-\overline{\lambda}\tau_s}\varphi\Big]\mathrm{d}x \\ & + \lambda\int_{\Omega_s} w^*\cdot\psi\,\mathrm{d}x + \beta\tau_f e^{-\lambda\tau_f}\int_0^1\int_{\Gamma_s} e^{\lambda\tau_f\theta}y^*(\theta)\cdot\psi\,\mathrm{d}x\,\mathrm{d}\theta \\ & + \frac{\alpha\tau_s}{\gamma}e^{-\lambda\tau_s}\int_0^1\int_{\Gamma_s} e^{\lambda\tau_s\theta}z^*(\theta)\cdot(\varphi-\beta e^{-\overline{\lambda}\tau_f}\psi)\,\mathrm{d}x\,\mathrm{d}\theta. \end{aligned} \tag{31}$$

Proof. First, let us suppose that the equation (28) holds. Notice that (28) is equivalent to a system where u is the solution of the boundary value problem

$$\begin{cases} \lambda u - \Delta u = u^*, & \text{in } \Omega_f, \\ u = 0, & \text{on } \Gamma_f, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{1}{\gamma}u - \dfrac{\alpha}{\gamma}z(1), & \text{on } \Gamma_s, \end{cases} \tag{32}$$

v and w satisfies the system

$$\begin{cases} \lambda w - v = w^*, & \text{in } \Omega_s, \\ \lambda v - \Delta w + w = v^*, & \text{in } \Omega_s, \\ \dfrac{\partial w}{\partial \nu} = \beta y(1) - \delta v, & \text{on } \Gamma_s, \end{cases} \tag{33}$$

while the delay variables y and z satisfy respectively the following ordinary differential equations with parameter

$$\begin{cases} \lambda\tau_f y(\theta) + y_\theta(\theta) = \tau_f y^*(\theta), & \text{on } (0,1)\times\Gamma_s, \\ y(0) = \dfrac{\partial u}{\partial\nu}, & \text{on } \Gamma_s, \end{cases} \tag{34}$$

$$\begin{cases} \lambda\tau_s z(\theta) + z_\theta(\theta) = \tau_s z^*(\theta), & \text{on } (0,1)\times\Gamma_s, \\ z(0) = v, & \text{on } \Gamma_s. \end{cases} \tag{35}$$

Applying the variation of parameters formula to (34) and (35) and using the equation $v = \lambda w - w^*$ we obtain

$$y(\theta) = e^{-\lambda\tau_f\theta}\frac{\partial u}{\partial\nu} + \tau_f e^{-\lambda\tau_f\theta}\int_0^\theta e^{\lambda\tau_f\vartheta}y^*(\vartheta)\,\mathrm{d}\vartheta, \tag{36}$$

$$z(\theta) = e^{-\lambda\tau_s\theta}(\lambda w - w^*) + \tau_s e^{-\lambda\tau_s\theta}\int_0^\theta e^{\lambda\tau_s\vartheta}z^*(\vartheta)\,\mathrm{d}\vartheta. \tag{37}$$

From (37), the boundary condition for u on Γ_s in (32) becomes

$$\frac{\partial u}{\partial \nu} = \frac{1}{\gamma}u - \frac{\alpha}{\gamma}e^{-\lambda\tau_s}(\lambda w - w^*) - \frac{\alpha\tau_s}{\gamma}e^{-\lambda\tau_s}\int_0^1 e^{\lambda\tau_s\theta}z^*(\theta)\,\mathrm{d}\theta. \tag{38}$$

Consequently, from (36) and (38) the boundary condition for w can be written as

$$\begin{aligned}\frac{\partial w}{\partial \nu} \;=\;& \frac{\beta}{\gamma}e^{-\lambda\tau_f}u - \Big(\frac{\alpha\beta}{\gamma}e^{-\lambda(\tau_s+\tau_f)} + \delta\Big)(\lambda w - w^*) \qquad (39)\\ &-\frac{\alpha\beta\tau_s}{\gamma}e^{-\lambda(\tau_s+\tau_f)}\int_0^1 e^{\lambda\tau_s\theta}z^*(\theta)\,\mathrm{d}\theta + \beta\tau_f e^{-\lambda\tau_f}\int_0^1 e^{\lambda\tau_f\theta}y^*(\theta)\,\mathrm{d}\theta.\end{aligned}$$

Let $\varphi \in H^1_{\Gamma_f}(\Omega_f)$. Taking the inner product in $L^2(\Omega_f)$ of the first equation in (32) with φ and then applying Green's identity we have

$$\int_{\Omega_f} u^*\cdot\varphi\,\mathrm{d}x = \lambda\int_{\Omega_f} u\cdot\varphi\,\mathrm{d}x + \int_{\Omega_f}\nabla u\cdot\nabla\varphi\,\mathrm{d}x + \int_{\Gamma_s}\frac{\partial u}{\partial\nu}\cdot\varphi\,\mathrm{d}x. \tag{40}$$

Using (38), the boundary integral in (40) can be written as

$$\begin{aligned}\int_{\Gamma_s}\frac{\partial u}{\partial\nu}\cdot\varphi\,\mathrm{d}x \;=\;& \frac{1}{\gamma}\int_{\Gamma_s}u\cdot\varphi\,\mathrm{d}x - \frac{\alpha\lambda}{\gamma}e^{-\lambda\tau_s}\int_{\Gamma_s}w\cdot\varphi\,\mathrm{d}x + \frac{\alpha}{\gamma}e^{-\lambda\tau_s}\int_{\Gamma_s}w^*\cdot\varphi\,\mathrm{d}x\\ &-\frac{\alpha\tau_s}{\gamma}e^{-\lambda\tau_s}\int_0^1\int_{\Gamma_s}e^{\lambda\tau_s\theta}z^*(\theta)\cdot\varphi\,\mathrm{d}x\,\mathrm{d}\theta. \qquad (41)\end{aligned}$$

With regards to the elliptic equation for w in (33), we take the inner product in $L^2(\Omega_s)$ with $\psi \in H^1(\Omega_s)$, use $v = \lambda w - w^*$ and apply Green's identity to obtain

$$\begin{aligned}\int_{\Omega_s}v^*\cdot\psi\,\mathrm{d}x \;=\;& (\lambda^2+1)\int_{\Omega_s}w\cdot\psi\,\mathrm{d}x + \int_{\Omega_s}\nabla w\cdot\nabla\psi\,\mathrm{d}x\\ &-\int_{\Gamma_s}\frac{\partial w}{\partial\nu}\cdot\psi\,\mathrm{d}x - \lambda\int_{\Omega_s}w^*\cdot\psi\,\mathrm{d}x. \qquad (42)\end{aligned}$$

From (39) the boundary term in the above equation can be expressed as

$$\begin{aligned}-\int_{\Gamma_s}\frac{\partial w}{\partial\nu}\cdot\psi\,\mathrm{d}x \;=\;& -\frac{\beta}{\gamma}e^{-\lambda\tau_f}\int_{\Gamma_s}u\cdot\psi\,\mathrm{d}x + \lambda\Big(\frac{\alpha\beta}{\gamma}e^{-\lambda(\tau_s+\tau_f)} + \delta\Big)\int_{\Gamma_s}w\cdot\psi\,\mathrm{d}x\\ &-\Big(\frac{\alpha\beta}{\gamma}e^{-\lambda(\tau_s+\tau_f)} + \delta\Big)\int_{\Gamma_s}w^*\cdot\psi\,\mathrm{d}x\\ &+\frac{\alpha\beta\tau_s}{\gamma}e^{-\lambda(\tau_s+\tau_f)}\int_0^1\int_{\Gamma_s}e^{\lambda\tau_s\theta}z^*(\theta)\cdot\psi\,\mathrm{d}x\,\mathrm{d}\theta \qquad (43)\\ &-\beta\tau_f e^{-\lambda\tau_f}\int_0^1\int_{\Gamma_s}e^{\lambda\tau_f\theta}y^*(\theta)\cdot\psi\,\mathrm{d}x\,\mathrm{d}\theta.\end{aligned}$$

Taking the sum of (40) and (42), substituting the boundary terms according to (41) and (43), and then rearranging the terms yield the variational equation (29).

Conversely, suppose that there exists $(u,w) \in H^1_{\Gamma_f}(\Omega_f)\times H^1(\Omega_s)$ such that (29) holds for every $(\varphi,\psi) \in H^1_{\Gamma_f}(\Omega_f)\times H^1(\Omega_s)$. We define $v = \lambda w - w^*$ and z by (37). By definition we have $v \in H^1(\Omega_s)$, $z \in H^1(0,1;L^2(\Gamma_s))$ and z satisfies the differential equation (34). By taking $\psi = 0$ in (29) and using the definition of z, we can see that u is the weak solution of the boundary value problem (32). Thus, in particular we have $\frac{\partial u}{\partial\nu} \in L^2(\Gamma_s)$.

Now define y according to (36), which satisfies (34) and $y \in H^1(0,1;L^2(\Gamma_s))$. Taking $\varphi = 0$ in (29) and then using (38) and the definition of z, we can see that

w satisfies the boundary value problem in (33). These observations imply that $Y = (u, w, v, y, z) \in D(A)$ and it satisfies equation (28). □

To characterize the spectrum of A, we need the following combination of the Lax-Milgram Lemma and the Fredholm alternative. For the proof we refer the reader to [14] or [35].

Lemma 3.2 (Lax-Milgram-Fredholm). *Let H_1 and H_0 be Hilbert spaces such that the embedding $H_1 \subset H_0$ is compact and dense. Suppose that $a_1 : H_1 \times H_1 \to \mathbb{C}$ and $a_2 : H_0 \times H_0 \to \mathbb{C}$ are two bounded sesquilinear forms such that a_1 is H_1-coercive and $F : H_1 \to \mathbb{C}$ is a continuous conjugate linear form. The variational equation*

$$a_1(u, v) + a_2(u, v) = F(v), \qquad \text{for every } v \in H_1,$$

has either a unique solution $u \in H_1$ for all $F \in H_1'$ or has a nontrivial solution for $F = 0$.

In the following theorem, we will use the weighted trace inequality: for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\|f\|_{L^2(\partial\Omega)} \leq \varepsilon \|\nabla f\|_{L^2(\Omega)} + C_\varepsilon \|f\|_{L^2(\Omega)}, \quad \text{for all } f \in H^1(\Omega), \tag{44}$$

where Ω is a sufficiently smooth bounded domain in $\mathbb{R}^d$. This inequality follows from the moment trace inequality

$$\|f\|_{L^2(\partial\Omega)} \leq C \|f\|_{L^2(\Omega)}^{\frac{1}{2}} \|f\|_{H^1(\Omega)}^{\frac{1}{2}}$$

for some constant $C > 0$ and Young's inequality $|ab| \leq \varepsilon a^2 + C_\varepsilon b^2$ for some $C_\varepsilon > 0$ and for every $\varepsilon > 0$ and $a, b \in \mathbb{R}$. We denote by $\sigma_p(A)$ the point spectrum of a closed linear operator A.

Theorem 3.3. *Suppose that $\gamma\delta \geq \alpha\beta$. Then $\sigma(A) = \sigma_p(A)$, that is, the spectrum of A consists of only eigenvalues.*

Proof. Let $\lambda \in \mathbb{C}$ and denote by $\tilde{a}_\lambda$ the sesquilinear form corresponding to the boundary terms in a_λ, that is, $\tilde{a}_\lambda : [H^1_{\Gamma_f}(\Omega_f) \times H^1(\Omega_s)]^2 \to \mathbb{C}$ is given by

$$\begin{aligned} \tilde{a}_\lambda((u, w), (\varphi, \psi)) \;=\; & \frac{1}{\gamma} \int_{\Gamma_s} u \cdot \varphi \, \mathrm{d}x - \frac{\alpha\lambda}{\gamma} e^{-\lambda\tau_s} \int_{\Gamma_s} w \cdot \varphi \, \mathrm{d}x \\ & - \frac{\beta}{\gamma} e^{-\lambda\tau_f} \int_{\Gamma_s} u \cdot \psi \, \mathrm{d}x + \lambda \Big(\frac{\alpha\beta}{\gamma} e^{-\lambda(\tau_f + \tau_s)} + \delta \Big) \int_{\Gamma_s} w \cdot \psi \, \mathrm{d}x. \end{aligned} \tag{45}$$

Taking $\varphi = u$ and $\psi = w$ and then applying the Cauchy-Schwarz inequality we obtain

$$|\tilde{a}_\lambda((u, w), (u, w))| \leq k_1(\lambda) \int_{\Gamma_s} |u|^2 \, \mathrm{d}x + k_2(\lambda) \int_{\Gamma_s} |w|^2 \, \mathrm{d}x \tag{46}$$

where the positive constants $k_1(\lambda)$ and $k_2(\lambda)$ are given by

$$\begin{aligned} k_1(\lambda) &= \frac{\alpha|\lambda|}{2\gamma} e^{-\operatorname{Re}\lambda\tau_s} + \frac{\beta}{2\gamma} e^{-\operatorname{Re}\lambda\tau_f} + \frac{1}{\gamma} \\ k_2(\lambda) &= \frac{\alpha|\lambda|}{2\gamma} e^{-\operatorname{Re}\lambda\tau_s} + \frac{\beta}{2\gamma} e^{-\operatorname{Re}\lambda\tau_f} + |\lambda| \Big(\frac{\alpha\beta}{\gamma} e^{-\operatorname{Re}\lambda(\tau_f + \tau_s)} + \delta \Big). \end{aligned}$$

Choose the constant $\varepsilon(\lambda) > 0$ small enough so that $1 - k_i(\lambda)\varepsilon(\lambda) > 0$ for $i = 1, 2$. Let us split the sesquilinear form a_λ into $a_\lambda = a_{\lambda,1} + a_{\lambda,2}$ where $a_{\lambda,1} : [H^1_{\Gamma_f}(\Omega_f) \times$

$H^1(\Omega_s)]^2 \to \mathbb{C}$ is the sesquilinear form defined by

$$a_{\lambda,1}((u,w),(\varphi,\psi)) = \int_{\Omega_f} \nabla u \cdot \nabla \varphi \,\mathrm{d}x + \int_{\Omega_s} \nabla w \cdot \nabla \psi \,\mathrm{d}x$$

$$+ C_{\varepsilon(\lambda)} k_1(\lambda) \int_{\Omega_f} u \cdot \varphi \,\mathrm{d}x + (1 + C_{\varepsilon(\lambda)} k_2(\lambda)) \int_{\Omega_s} w \cdot \psi \,\mathrm{d}x + \tilde{a}_\lambda((u,w),(\varphi,\psi))$$

while $a_{\lambda,2} : [L^2(\Omega_f) \times L^2(\Omega_s)]^2 \to \mathbb{C}$ is the sesquilinear form given by

$$a_{\lambda,2}((u,w),(\varphi,\psi)) = (\lambda - C_{\varepsilon(\lambda)} k_1(\lambda)) \int_{\Omega_f} u \cdot \varphi \,\mathrm{d}x + (\lambda^2 - C_{\varepsilon(\lambda)} k_2(\lambda)) \int_{\Omega_s} w \cdot \psi \,\mathrm{d}x.$$

Here $C_{\varepsilon(\lambda)} > 0$ is the constant in (44) corresponding to $\varepsilon(\lambda) > 0$.

One can immediately see that $a_{\lambda,1}$ and $a_{\lambda,2}$ are bounded. Now, we show that $a_{\lambda,1}$ is coercive. Utilizing (44), (46) and recalling the choice of $\varepsilon(\lambda)$, we can see that

$$\begin{aligned} |a_{\lambda,1}((u,w),(u,w))| \;&\geq\; (1 - k_1(\lambda)\varepsilon(\lambda)) \int_{\Omega_f} |\nabla u|^2 \,\mathrm{d}x + \int_{\Omega_s} |w|^2 \,\mathrm{d}x \\ &\quad + (1 - k_2(\lambda)\varepsilon(\lambda)) \int_{\Omega_s} |\nabla w|^2 \,\mathrm{d}x \\ &\geq\; C(\lambda)(\|u\|^2_{H^1_{\Gamma_f}(\Omega_f)} + \|w\|^2_{H^1(\Omega_s)}) \end{aligned}$$

where $C(\lambda) = \min\{1 - k_1(\lambda)\varepsilon(\lambda), 1 - k_2(\lambda)\varepsilon(\lambda)\} > 0$. From the compactness of the embedding $[L^2(\Omega_f) \times L^2(\Omega_s)]^2 \subset [H^1_{\Gamma_f}(\Omega_f) \times H^1(\Omega_s)]^2$ and Lemma 3.2 we either have λ in the resolvent set of A or in its point spectrum. This is equivalent to the conclusion of the lemma. □

Now we prove that A does not have purely imaginary eigenvalues.

Theorem 3.4. *If $\gamma\delta \geq \alpha\beta$ then $\sigma(A) \cap i\mathbb{R} = \emptyset$.*

Proof. We already know that $0 \in \rho(A)$. Let r be a nonzero real number and $Y := (u, w, v, y, z) \in D(A)$ be such that $AY = irY$. This is equivalent to the system (32)–(35) with $Y^* = 0$ and $\lambda = ir$. From the dissipativity inequality (14) and the fact that $Q_{\mathbf{a}}$ is nonnegative definite we have

$$a_1 \int_{\Omega_f} |\nabla u|^2 \,\mathrm{d}x \leq -\mathrm{Re}\langle AY, Y\rangle_{X,\mathbf{a}} = 0.$$

Thus u is constant and from the boundary condition on Γ_f it follows that u must be zero in Ω_f. From (36) we obtain $y = 0$ in $(0,1) \times \Gamma_s$. The equation for w in (33) turns into

$$\begin{cases} -\Delta w + (1 - r^2)w = 0, & \text{in } \Omega_s, \\ \dfrac{\partial w}{\partial \nu} + i\delta r w = 0, & \text{on } \Gamma_s, \end{cases} \tag{47}$$

since $v = irw$. Multiplying (47) by w, integrating over Ω_s and then using Green's identity we have

$$\int_{\Omega_s} |\nabla w|^2 \,\mathrm{d}x + (1 - r^2) \int_{\Omega_s} |w|^2 \,\mathrm{d}x + i\delta r \int_{\Gamma_s} |w|^2 \,\mathrm{d}x = 0.$$

From the imaginary part in the above equation, we can see that $w = 0$ on Γ_s and consequently $\frac{\partial w}{\partial \nu} = 0$ according to the boundary condition in (47). By elliptic regularity we have $w \in H^2(\Omega_s) \cap H^1_0(\Omega_s)$ and therefore $w = 0$ in Ω_s from the unique continuation theorem for elliptic operators, see [38, Corollary 15.2.2] for example.

Thus $v = irw = 0$ in Ω_s and from (37) we have $z = 0$ in $(0,1) \times \Gamma_s$. Therefore $Y = 0$ so that ir is not an eigenvalue of A and from Lemma 3.3 it follows that ir lies in the resolvent set of A. Thus, the imaginary axis lies in the resolvent set of A. □

Applying the Tauberian-type theorems in [1, 31, 16], from Theorem 3.3 and Theorem 3.4 the following asymptotic stability immediately follows.

Theorem 3.5. *Under the condition $\gamma\delta \geq \alpha\beta$, we have $\|e^{tA}Y_0\|_{X,\mathbf{a}} \to 0$ in X as $t \to \infty$ for every $Y_0 \in X$.*

In the succeeding sections we improve this theorem by providing explicit decay rates under additional conditions on the parameters α, β, γ and δ.

4. Uniform exponential stability for the case $\alpha\beta < \gamma\delta$.

The goal of this section is to prove the exponential decay of the energy for the solutions of the system (1)–(3) for the case $\alpha\beta < \gamma\delta$. From inequality (14), it can be seen that the dissipation is due to the diffusion of the fluid, the normal stress of the fluid component and velocity of the solid on the interface. The latter boundary dissipation for the wave equation is enough to obtain exponential decay, and this can be achieved through multipliers. We follow the methodology presented in [6] for the current problem.

First, we recall the following energy identities for the wave operator $\Box = \partial_{tt} - \Delta$. These are obtained by using the multipliers $\varphi \operatorname{div} \eta$ and $\eta \cdot \nabla\varphi$, respectively. We refer to [21, 22, 23, 24] for their proofs in the scalar version and to [6] in the vector version.

Proposition 1. *Suppose that $T > s$ and $\eta \in [C^2(\overline{\Omega})]^d$ is a vector field. Then for every $\varphi \in H^2(s,T;L^2(\Omega)) \cap H^1(s,T;H^1(\Omega)) \cap L^2(s,T;H^2(\Omega))$ we have*

$$\begin{aligned}
&\int_s^T \int_\Omega (|\varphi_t|^2 - |\nabla\varphi|^2) \operatorname{div}\eta \,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_s^T \int_\Omega (\Box\varphi) \cdot (\varphi \operatorname{div}\eta) - \varphi \cdot (\nabla(\operatorname{div}\eta) \cdot \nabla\varphi) \,\mathrm{d}x\,\mathrm{d}t \qquad (48)\\
&- \int_s^T \int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu} \cdot (\varphi \operatorname{div}\eta) \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \varphi_t(T) \cdot (\varphi(T) \operatorname{div}\eta) - \varphi_t(s) \cdot (\varphi(s) \operatorname{div}\eta) \,\mathrm{d}x.
\end{aligned}$$

Also, if $J\eta$ denotes the Jacobian of η then

$$\begin{aligned}
&\int_s^T \int_\Omega (J\eta)\nabla\varphi \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_s^T \int_\Omega (\Box\varphi) \cdot (\eta \cdot \nabla\varphi) - \frac{1}{2}(|\varphi_t|^2 - |\nabla\varphi|^2) \operatorname{div}\eta \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_s^T \int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu} \cdot (\eta \cdot \nabla\varphi) + \frac{1}{2}(|\varphi_t|^2 - |\nabla\varphi|^2)\eta \cdot \nu \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_\Omega \varphi_t(T) \cdot (\eta \cdot \nabla\varphi(T)) - \varphi_t(s) \cdot (\eta \cdot \nabla\varphi(s)) \,\mathrm{d}x. \qquad (49)
\end{aligned}$$

In the above proposition, the terms $\eta \cdot \nabla\varphi$, $\nabla(\operatorname{div}\eta) \cdot \nabla\varphi$ and $(J\eta)\nabla\varphi$ are vectors with components $\eta \cdot \nabla\varphi_i$, $\nabla(\operatorname{div}\eta) \cdot \nabla\varphi_i$ and $(J\eta)\nabla\varphi_i$ for $i = 1, \ldots, d$, respectively. To estimate the term $\nabla\varphi$ on Γ_s, we separate its normal and tangential components and utilize the following trace regularity for solutions of wave equations in [22, Proposition 6.3].

Theorem 4.1. *Let $f \in L^2(0,T;L^2(\Omega))$ and $w \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ be a solution of the wave equation $\Box w = f$. If $w_t, \frac{\partial w}{\partial \nu} \in L^2(0,T;L^2(\partial\Omega))$, then for every $\sigma \in (0,T)$ we have $\frac{\partial w}{\partial \tau} \in L^2(\sigma, T-\sigma; L^2(\partial\Omega))$. Furthermore, for every $\varepsilon \in (0,\frac{1}{2}]$ there exists a constant $C_{\sigma,\varepsilon,T} > 0$ such that*

$$\int_\sigma^{T-\sigma}\int_{\partial\Omega}\left|\frac{\partial w}{\partial \tau}\right|^2 \mathrm{d}x\,\mathrm{d}t \;\le\; C_{\sigma,\varepsilon,T}\bigg(\|w\|^2_{H^{\frac{1}{2}+\varepsilon}((0,T)\times\Omega)} + \int_0^T\int_{\partial\Omega}\left|\frac{\partial w}{\partial \nu}\right|^2 \mathrm{d}x\,\mathrm{d}t + \int_\sigma^{T-\sigma}\int_\Omega |f|^2\,\mathrm{d}x\,\mathrm{d}t\bigg).$$

The estimate provided in [22] involves the $L^2(0,T;H^{\frac{1}{2}+\varepsilon}(\Omega))$-norm instead of the $H^{\frac{1}{2}+\varepsilon}((0,T)\times\Omega)$-norm. This is admitted since $H^s((0,T)\times\Omega) = L^2(0,T;H^s(\Omega)) \cap H^s(0,T;L^2(\Omega)) \subset L^2(0,T;H^s(\Omega))$ for $s \ge 0$ according to [27, Remark 2.2] and the classical extension theorems for Sobolev spaces.

For $t \ge 0$, we define the energy of the solution to (1)–(3) associated with data in $D(A)$ by

$$E(t) = \frac{1}{2}\|(u(t), w(t), v(t), y(t), z(t))\|^2_{X,\mathbf{a}}$$

and the corresponding dissipation term by

$$\begin{aligned} D(t) \;=\; &\int_{\Omega_f} |\nabla u(t)|^2\,\mathrm{d}x + \int_{\Gamma_s} \Big|\frac{\partial u}{\partial \nu}(t,x)\Big|^2 + |w_t(t,x)|^2\,\mathrm{d}x \\ &+ \int_{\Gamma_s} \Big|\frac{\partial u}{\partial \nu}(t-\tau_f,x)\Big|^2 + |w_t(t-\tau_s,x)|^2\,\mathrm{d}x. \end{aligned}$$

Lemma 4.2. *Suppose that $\gamma\delta > \alpha\beta$. Then there exist constants $c > 0$ and $C > 0$ such that the energy of the solutions of the system (1)–(3) with initial data in $D(A)$ satisfies $-cD(t) \le E'(t) \le -CD(t)$ for every $t \ge 0$.*

Proof. The given assumption implies that the quadratic form $Q_{\mathbf{a}}$ is positive definite. Hence, it follows from (14) and Theorem 7.1 that

$$E'(t) = \mathrm{Re}\langle AY(t), Y(t)\rangle_{X,\mathbf{a}} \le -CD(t)$$

for some $C > 0$, where $Y(t) = e^{tA}X_0$ and $X_0 \in D(A)$. On the other hand, using the Cauchy-Schwarz inequality and similar calculations as in the proof of (14), we have the estimate from below

$$\begin{aligned} E'(t) \ge &-a_1\int_{\Omega_f}|\nabla u(t)|^2\,\mathrm{d}x \\ &-\frac{1}{2}Q_{\mathbf{a}}\Big(\Big\|\frac{\partial u}{\partial \nu}(t)\Big\|_{L^2(\Gamma_s)}, \|w_t(t)\|_{L^2(\Gamma_s)}, \Big\|\frac{\partial u}{\partial \nu}(t-\tau_f)\Big\|_{L^2(\Gamma_s)}, \|w_t(t-\tau_s)\|_{L^2(\Gamma_s)}\Big). \end{aligned}$$

From this inequality and (118), it can be seen that there is a constant $c > 0$ such that $E'(t) \ge -cD(t)$. This completes the proof of the lemma. ∎

We are now in position to prove the exponential stability of the semigroup e^{tA}.

Theorem 4.3. *If $\gamma\delta > \alpha\beta$ then the semigroup generated by A is uniformly exponentially stable, that is, there exist $M \ge 1$ and $\sigma > 0$ such that $\|e^{tA}Y_0\|_{X,\mathbf{a}} \le Me^{-\sigma t}\|Y_0\|_{X,\mathbf{a}}$ for every $t \ge 0$ and $Y_0 \in X$.*

Proof. By the density of $D(A^2)$ in X and strong continuity of the semigroup e^{tA}, we may suppose that the initial data $Y_0 = (u_0, w_0, v_0, y_0, z_0)$ lies in $D(A^2)$. Let $Y = (u, w, v, y, z)$ be the associated solution. The regularity of the corresponding components are provided in Theorem 2.2. From Lemma 4.2, for every $T \geq s \geq 0$ it follows that

$$E(s) \leq E(T) + c\int_s^T D(t)\,\mathrm{d}t. \tag{50}$$

The goal is to prove that there exist $T > 0$ and a constant $\varrho_T \in (0,1)$ such that $E(T) \leq \varrho_T E(0)$. Then according to standard results for semigroup theory, see for instance [33], we have exponential stability. By the linearity of the problem and the fact that the coefficients are real, we may suppose without loss of generality that the states are real-valued. The corresponding result can be obtained by separating the real and imaginary parts. We divide the arguments in several steps.

Step 1. Energy estimates for y and z. Multiplying the transport equation $\tau_s z_t + z_\theta = 0$ by $2e^{-a\theta}z$ where $a > 0$ and then integrating over $(s,T)\times(0,1)\times\Gamma_s$ yields

$$\begin{aligned}&\tau_s\int_0^1\int_{\Gamma_s} e^{-a\theta}|z(T,\theta,x)|^2\,\mathrm{d}x\,\mathrm{d}\theta\,\mathrm{d}t\\&+a\int_s^T\int_0^1\int_{\Gamma_s} e^{-a\theta}|z(t,\theta,x)|^2\,\mathrm{d}x\,\mathrm{d}\theta\,\mathrm{d}t+\int_s^T\int_{\Gamma_s} e^{-a}|z(t,1,x)|^2\,\mathrm{d}x\,\mathrm{d}t\\&=\tau_s\int_0^1\int_{\Gamma_s} e^{-a\theta}|z_0(\theta,x)|^2\,\mathrm{d}x\,\mathrm{d}\theta\,\mathrm{d}t+\int_s^T\int_{\Gamma_s}|w_t(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

since $z(t,0,x) = w_t(t,x)$. Consequently, there exists a constant $C_{\tau_s,a} > 0$ such that

$$\begin{aligned}&\int_s^T\int_0^1\int_{\Gamma_s}|z(t,\theta,x)|^2\,\mathrm{d}x\,\mathrm{d}\theta\,\mathrm{d}t\\&\leq C_{\tau_s,a}\left(\int_0^1\int_{\Gamma_s}|z_0(\theta,x)|^2\,\mathrm{d}x\,\mathrm{d}\theta\,\mathrm{d}t+\int_s^T\int_{\Gamma_s}|w_t(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t\right).\end{aligned}\tag{51}$$

A similar procedure applied to the delay variable y provides us the following estimate for some constant $C_{\tau_f,a} > 0$

$$\begin{aligned}&\int_s^T\int_0^1\int_{\Gamma_s}|y(t,\theta,x)|^2\,\mathrm{d}x\,\mathrm{d}\theta\,\mathrm{d}t\\&\leq C_{\tau_f,a}\left(\int_0^1\int_{\Gamma_s}|y_0(\theta,x)|^2\,\mathrm{d}x\,\mathrm{d}\theta\,\mathrm{d}t+\int_s^T\int_{\Gamma_s}\left|\frac{\partial u}{\partial\nu}(t,x)\right|^2\mathrm{d}x\,\mathrm{d}t\right).\end{aligned}\tag{52}$$

Step 2. Energy estimate for u. By the Poincaré inequality we have

$$\int_s^T\int_{\Omega_f}|u(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t \leq C\int_s^T\int_{\Omega_f}|\nabla u(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{53}$$

Next, we shall estimate the terms in the energy corresponding to the wave component. Let $T > 2(\tau_f + \tau_s)$ and $\tau_f + \tau_s < \sigma < T - (\tau_f + \tau_s)$. In the following, we establish that there exist a constant $C_\eta > 0$ independent of T and a constant $C_{\rho,\sigma,r,\eta,T} > 0$ such that for every $r \in (\frac{1}{2}, 1]$ we have

$$\int_\sigma^{T-\sigma}\int_{\Omega_s}|w_t(t,x)|^2 + |\nabla w(t,x)|^2 + |w(t,x)|^2\,\mathrm{d}x \tag{54}$$

$$\le C_\eta(E(T-\sigma)+E(\sigma)) + C_{\rho,\sigma,r,\eta,T}\left(\|w\|^2_{H^r((0,T)\times\Omega_s)} + \int_0^T D(t)\,\mathrm{d}t\right).$$

Step 3. Energy estimates for w. We use the energy identities provided in Proposition 1, which is possible since w satisfies the necessary regularity condition according to Theorem 2.2. We choose a vector field η having a uniformly positive definite Jacobian, that is, for some some $\rho > 0$ we have

$$(J\eta)z\cdot z \ge \rho|z|^2, \quad \text{for every } z\in\mathbb{R}^d. \tag{55}$$

Let us estimate each term on the right hand side of (49) where T and s are replaced by $T-\sigma$ and σ, respectively. Since $\Box w = -w$ we have

$$\begin{aligned}&\int_\sigma^{T-\sigma}\int_{\Omega_s}(\Box w)\cdot(\eta\cdot\nabla w)\,\mathrm{d}x\,\mathrm{d}t\\ &\le \frac{\rho}{4}\int_\sigma^{T-\sigma}\int_{\Omega_s}|\nabla w|^2\,\mathrm{d}x\,\mathrm{d}t + C_{\rho,\eta}\int_\sigma^{T-\sigma}\int_{\Omega_s}|w|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{56}$$

Applying the Cauchy-Schwarz inequality to the boundary terms in (49) one obtains

$$\begin{aligned}&\int_\sigma^{T-\sigma}\int_{\Gamma_s}\frac{\partial w}{\partial\nu}\cdot(\eta\cdot\nabla w) + \frac{1}{2}(|w_t|^2-|\nabla w|^2)\eta\cdot\nu\,\mathrm{d}x\,\mathrm{d}t\\ &\le C_\eta\int_\sigma^{T-\sigma}\int_{\Gamma_s}\left(\left|\frac{\partial w}{\partial\nu}\right|^2 + |w_t|^2 + |\nabla w|^2\right)\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{57}$$

The last integral term in (49) are estimated as follows

$$\begin{aligned}&-\int_{\Omega_s} w_t(T-\sigma)\cdot(\eta\cdot\nabla w(T-\sigma)) - w_t(\sigma)\cdot(\eta\cdot\nabla w(\sigma))\,\mathrm{d}x \qquad (58)\\ &\le C_\eta\int_{\Omega_s}|w_t(T-\sigma)|^2 + |\nabla w(T-\sigma)|^2 + |w_t(\sigma)|^2 + |\nabla w(\sigma)|^2\,\mathrm{d}x.\end{aligned}$$

The remaining term in (49) is estimated by using (48) and Young's inequality to obtain

$$\begin{aligned}\left|\int_\sigma^{T-\sigma}\int_{\Omega_s}(|w_t|^2-|\nabla w|^2)\operatorname{div}\eta\,\mathrm{d}x\,\mathrm{d}t\right| &\le \frac{\rho}{4}\int_\sigma^{T-\sigma}\int_{\Omega_s}|\nabla w|^2\,\mathrm{d}x\,\mathrm{d}t\\ &+ C_{\rho,\eta}\int_\sigma^{T-\sigma}\int_{\Omega_s}|w|^2\,\mathrm{d}x\,\mathrm{d}t + C_{\rho,\eta}\int_\sigma^{T-\sigma}\int_{\Gamma_s}\left|\frac{\partial w}{\partial\nu}\right|^2\mathrm{d}x\,\mathrm{d}t\\ &+ C_\eta\int_{\Omega_s}|w_t(T-\sigma)|^2 + |w(T-\sigma)|^2 + |w_t(\sigma)|^2 + |w(\sigma)|^2\,\mathrm{d}x.\end{aligned} \tag{59}$$

Using the estimates (56)–(59) in (49) with $\varphi = w$ we have, after rearranging terms and using (55),

$$\begin{aligned}&\frac{\rho}{4}\int_\sigma^{T-\sigma}\int_{\Omega_s}|\nabla w|^2 + |w|^2\,\mathrm{d}x\,\mathrm{d}t \le C_\eta(E(T-\sigma)+E(\sigma)) \qquad (60)\\ &+ C_{\rho,\eta}\int_\sigma^{T-\sigma}\int_{\Gamma_s}\left(\left|\frac{\partial w}{\partial\nu}\right|^2 + |w_t|^2 + |\nabla w|^2\right)\mathrm{d}x\,\mathrm{d}t + C_{\rho,\eta}\int_\sigma^{T-\sigma}\int_{\Omega_s}|w|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

It remains to estimate the velocity term w_t, and this is the place where we use the energy equation (48). Let us choose the vector field η so that $\operatorname{div}\eta = 1$, e.g. the radial vector field $\eta(x) = x/d$. We estimate the right hand side of (48) by applying the Cauchy-Schwarz inequality and then using estimate (60) with $\varphi = w$ to get

$$\int_\sigma^{T-\sigma}\int_{\Omega_s} |w_t|^2 \,\mathrm{d}x\,\mathrm{d}t \;\le\; C_\eta(E(T-\sigma) + E(\sigma)) \tag{61}$$
$$+ C_{\rho,\eta}\int_\sigma^{T-\sigma}\int_{\Gamma_s}\left(\left|\frac{\partial w}{\partial \nu}\right|^2 + |w_t|^2 + |\nabla w|^2\right)\mathrm{d}x\,\mathrm{d}t + C_{\rho,\eta}\int_\sigma^{T-\sigma}\int_{\Omega_s}|w|^2\,\mathrm{d}x\,\mathrm{d}t.$$

Taking the sum of (60) and (61), using the decomposition $|\nabla w|^2 = |\frac{\partial w}{\partial\nu}|^2 + |\frac{\partial w}{\partial \tau}|^2$ on Γ_s, the tangential trace estimate in Theorem 4.1, and the boundary condition $\frac{\partial w}{\partial \nu}(t) = \beta\frac{\partial u}{\partial\nu}(t-\tau_f) - \delta w_t(t)$ on Γ_s we obtain the desired estimate (55) where $r = \frac{1}{2} + \varepsilon$.

Step 4. From (51)–(55) and the fact that the energy is decreasing we have

$$(T-2\sigma)E(T-\sigma) \;\le\; \int_\sigma^{T-\sigma} E(t)\,\mathrm{d}t \tag{62}$$
$$\le C_\eta(E(T-\sigma)+E(0)) + C_T\bigg(\|w\|^2_{H^r((0,T)\times\Omega_s)} + \int_0^T D(t)\,\mathrm{d}t\bigg).$$

Replacing T and s by $T-\sigma$ and 0 in (50), respectively, we can estimate $E(0)$ from above in terms of $E(T-\sigma)$ and the dissipation term D. Using this in (62) we obtain

$$(T-2\sigma-2C_\eta)E(T-\sigma) \le C_T\bigg(\|w\|^2_{H^r((0,T)\times\Omega_s)} + \int_0^T D(t)\,\mathrm{d}t\bigg) \tag{63}$$

for some constant C_η independent of T. If we take $\sigma > \tau_f + \tau_s$ and $T > 2\sigma + C_\eta$ and then use the fact that $E(T) \le E(T-\sigma)$ in (63) we obtain the energy estimate

$$E(T) \le C_T\bigg(\|w\|^2_{H^r((0,T)\times\Omega_s)} + \int_0^T D(t)\,\mathrm{d}t\bigg) \tag{64}$$

where $r \in (\frac{1}{2}, 1]$.

Step 5. Absorption of lower order term. In this step, we prove that the *lower order term* on the right hand side of (64) can be absorbed by the dissipation term. More precisely, we show that there exists $C_T > 0$ such that

$$\|w\|^2_{H^r((0,T)\times\Omega_s)} \le C_T\int_0^T D(t)\,\mathrm{d}t. \tag{65}$$

This is done via a compactness-uniquess argument as in [6]. Suppose on the contrary that for each integer $n > 0$ there is a data $Y_{0n} = (u_{0n}, w_{0n}, v_{0n}, y_{0n}, z_{0n}) \in D(A)$ with

$$\|w_n\|^2_{H^r((0,T)\times\Omega_s)} > n\int_0^T D_n(t)\,\mathrm{d}t, \tag{66}$$

where D_n is the dissipation term associated with the solution $Y_n = (u_n, w_n, v_n, y_n, z_n)$ with initial data $Y_{0n} = (u_{0n}, w_{0n}, v_{0n}, y_{0n}, z_{0n})$. By rescaling the initial data, we may suppose without loss of generality that for each n we have

$$\|w_n\|_{H^r((0,T)\times\Omega_s)} = 1. \tag{67}$$

From (66) and (67) it follows that

$$\begin{cases} u_n \to 0, & \text{strongly in } L^2(0,T;H^1_{\Gamma_f}(\Omega_f)),\\ w_{nt} \to 0 \ \text{ and } \ w_{nt}(\cdot - \tau_s) \to 0, & \text{strongly in } L^2(0,T;L^2(\Gamma_s)),\\ \dfrac{\partial u_n}{\partial \nu} \to 0 \ \text{ and } \ \dfrac{\partial u_n}{\partial \nu}(\cdot - \tau_f) \to 0, & \text{strongly in } L^2(0,T;L^2(\Gamma_s)). \end{cases} \tag{68}$$

In particular, the Neumann boundary condition for w_n yields

$$\frac{\partial w_n}{\partial \nu} \to 0, \quad \text{strongly in } L^2(0,T;L^2(\Gamma_s)). \tag{69}$$

According to (64), (66) and (67), $Y_n(T)$ is uniformly bounded in X, and consequently $Y_n(0) = Y_{0n}$ is uniformly bounded in X according to (50) with $s = 0$. Thus, up to a subsequence, we have $Y_{n0} \to Y_0$ weakly in X for some $Y_0 = (u_0, w_0, v_0, y_0, z_0) \in X$. By the uniform boundedness of the adjoint semigroup on compact intervals, it follows that

$$\begin{cases} u_n \to u, & \text{weakly-star in } L^\infty(0,T;L^2(\Omega_f)),\\ w_n \to w, & \text{weakly-star in } L^\infty(0,T;H^1(\Omega_s)),\\ w_{nt} \to w_t, & \text{weakly-star in } L^\infty(0,T;L^2(\Omega_s)), \end{cases} \tag{70}$$

where $Y(t) = (u(t), w(t), v(t), y(t), z(t)) = e^{tA}Y_0$. In particular, from the last two parts in (70) we can see that w_n is uniformly bounded in $H^1((0,T)\times\Omega_s)$. By compactness, there is a subsequence which we denote by the same indices such that $w_n \to w$ strongly in $H^r((0,T)\times\Omega_s)$ for $r \in (\frac{1}{2}, 1)$, and by passing to the limit in (67) the limit satisfies

$$\|w\|_{H^r((0,T)\times\Omega_s)} = 1. \tag{71}$$

From (68) and (70) it follows that $u = 0$.

We show that $v = w_t$ is the very weak solution of the following wave equation with overdetermined boundary conditions

$$\begin{cases} v_{tt} - \Delta v + v = 0, & \text{in } (0,T)\times\Omega_s,\\ \dfrac{\partial v}{\partial \nu} = 0, \quad v = 0, & \text{on } (0,T)\times\Gamma_s,\\ v(0) = w_1 \in L^2(\Omega_s), & v_t(0) = R_s w_0 \in H^1(\Omega_s)', \end{cases} \tag{72}$$

where R_s is defined by $\langle R_s w_0, \varphi\rangle_{H^1(\Omega_s)'\times H^1(\Omega_s)} = (w_0, \varphi)_{H^1(\Omega_s)}$ for $w_0, \varphi \in H^1(\Omega_s)$. Given $f \in L^2((0,T)\times\Omega_s)$, let $\varphi \in C^1([0,T], L^2(\Omega_s)) \cap C([0,T], H^1(\Omega_s))$ be the weak solution of

$$\begin{cases} \varphi_{tt} - \Delta\varphi + \varphi = f, & \text{in } (0,T)\times\Omega_s,\\ \dfrac{\partial \varphi}{\partial \nu} = 0, & \text{on } (0,T)\times\Gamma_s,\\ \varphi(T) = \varphi_t(T) = 0, & \text{in } \Omega_s. \end{cases}$$

Integrating by parts in time and space we infer that

$$\begin{aligned} 0 &= \int_0^T \int_{\Omega_s} (w_{ntt} - \Delta w_n + w_n)\cdot \varphi_t \,\mathrm{d}x\,\mathrm{d}t\\ &= -\int_{\Omega_s} v_{0n}\cdot\varphi_t(0)\,\mathrm{d}x - \int_0^T\int_{\Gamma_s} \frac{\partial w_n}{\partial \nu}\cdot\varphi_t\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_s} w_{0n}\cdot\varphi(0)\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&- \int_0^T \int_{\Omega_s} w_{nt} \cdot \varphi_{tt} - \nabla w_n \cdot \nabla \varphi_t + w_{nt} \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t \\
= \; &- \int_{\Omega_s} v_{0n} \cdot \varphi_t(0)\,\mathrm{d}x - \langle R_s w_{0n}, \varphi(0)\rangle_{H^1(\Omega_s)' \times H^1(\Omega_s)} - \int_0^T \int_{\Gamma_s} \frac{\partial w_n}{\partial \nu} \cdot \varphi_t \,\mathrm{d}x\,\mathrm{d}t \\
&- \int_0^T \int_{\Omega_s} w_{nt} \cdot f \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Passing to the limit $n \to \infty$ and using (68)–(70), $w_{0n} \to w_0$ weakly in $H^1(\Omega_s)$ and $v_{0n} \to v_0$ weakly in $L^2(\Omega_s)$ we have, for every $f \in L^2((0,T) \times \Omega_s)$,

$$
\int_0^T \int_{\Omega_s} v \cdot f \,\mathrm{d}x\,\mathrm{d}t = - \int_{\Omega_s} v_0 \cdot \varphi_t(0)\,\mathrm{d}x - \langle R_s w_0, \varphi(0)\rangle_{H^1(\Omega_s)' \times H^1(\Omega_s)}. \tag{73}
$$

In a similar fashion it can be shown that for every $f \in L^2((0,T) \times \Omega_s)$ we have

$$
\int_0^T \int_{\Omega_s} v \cdot f \,\mathrm{d}x\,\mathrm{d}t = - \int_{\Omega_s} v_0 \cdot \psi_t(0)\,\mathrm{d}x - \langle R_s w_0, \psi(0)\rangle_{H^1(\Omega_s)' \times H^1(\Omega_s)}, \tag{74}
$$

where $\psi \in C^1([0,T], L^2(\Omega_s)) \cap C([0,T], H_0^1(\Omega_s))$ is the weak solution of

$$
\begin{cases}
\psi_{tt} - \Delta \psi + \psi = f, & \text{in } (0,T) \times \Omega_s, \\
\psi = 0, & \text{on } (0,T) \times \Gamma_s, \\
\psi(T) = \psi_t(T) = 0, & \text{in } \Omega_s.
\end{cases}
$$

From (73) and (74) it follows that v is indeed the very weak solution of (72).

For sufficiently large $T > 0$, the solution of (72) is identically zero according to the Holmgren-uniqueness principle, see [25, Chapter I, Theorem 8.2]. Therefore w is constant with respect to t and it is a weak solution of the over-determined elliptic problem

$$
\begin{cases}
-\Delta w + w = 0, & \text{in } \Omega_s, \\
\dfrac{\partial w}{\partial \nu} = 0, \; w = 0, & \text{on } \Gamma_s.
\end{cases} \tag{75}
$$

Applying the unique-continuation condition for elliptic operators we infer that $w = 0$, contradicting (71). This completes the proof of (65).

Step 6. Let us finish the proof of the theorem. First, we obtain immediately from (64) and (65) that

$$
E(T) \le C_T \int_0^T D(t)\,\mathrm{d}t.
$$

Moreover, according to Lemma 4.2 we have

$$
\int_0^T D(t)\,\mathrm{d}t \le \frac{1}{C}(E(0) - E(T)).
$$

Combining the last two estimates leads us to $E(T) \le \varrho_T E(0)$ where $\varrho_T = C_T C^{-1}/(1 + C_T C^{-1}) \in (0,1)$. This proves the desired inequality, which in turn implies the exponential stability of the semigroup generated by A. □

5. **Rational stability under the critical case $\gamma\delta = \alpha\beta$.** In the critical case $\gamma\delta = \alpha\beta$, the dissipation induced by the feedback controls is cancelled due to the delay in the interaction. However, we can see that the energy still dissipates through the diffusion of the fluid component. In this section, we show that this dissipative property leads to a rational decay rate for the interaction model, however, we are only able to prove this under an additional geometric condition. This geometrical condition has been utilized in several works and we refer to [20] in the context of fluid-structure interaction models.

Our method relies on a resolvent-based approach, which has been used in fluid-structure models without delay in [2, 5]. The method we shall employ here uses the ideas and techniques given in [5], where instead of the Dirichlet map we use the Robin map. The success of these methods is based on the following abstract result by Borichev and Tomilov [11].

Theorem 5.1. *Let A be a generator of a bounded semigroup on a Hilbert space X such that $i\mathbb{R} \subset \rho(A)$ and let $\alpha > 0$. Then there exist $r_0 > 0$ and $C(r_0) > 0$ such that*

$$\|(irI - A)^{-1}\|_X \le C_{r_0}|r|^{\sigma} \quad \text{for every } r \in \mathbb{R} \text{ with } |r| \ge r_0,$$

where $\sigma \in (0,\infty)$, if and only if there exist $t_0 > 0$ and $C_{t_0} > 0$ such that

$$\|e^{tA}X_0\|_X \le C_{t_0} t^{-\frac{1}{\sigma}}\|X_0\|_{D(A)} \quad \text{for every } X_0 \in D(A) \text{ and } t > t_0.$$

To prove rational stability, we need the static version of the energy identity (49) in Proposition 1, compare [5]. For convenience, we state it in the following proposition.

Proposition 2. *Let Ω be a smooth bounded domain and suppose that $\psi \in H^2(\Omega)$ is a vector-valued function satisfying the elliptic equation*

$$-\Delta\psi + s\psi = f$$

where $f \in L^2(\Omega)$ and $s \in \mathbb{R}$. If $\eta \in [C^2(\overline{\Omega})]^d$ is a vector field then we have

$$\begin{aligned}
&\int_{\Omega} (J\eta)\nabla\psi \cdot \nabla\psi \,\mathrm{d}x + \frac{1}{2}\int_{\partial\Omega} |\nabla\psi|^2 \eta\cdot\nu \,\mathrm{d}x \\
&= \int_{\partial\Omega} \frac{\partial\psi}{\partial\nu}\cdot(\eta\cdot\nabla\psi)\,\mathrm{d}x - \frac{s}{2}\int_{\partial\Omega} |\psi|^2\eta\cdot\nu\,\mathrm{d}x + \frac{1}{2}\int_{\partial\Omega}\frac{\partial\psi}{\partial\nu}\cdot(\psi \operatorname{div}\eta)\,\mathrm{d}x \\
&- \frac{1}{2}\int_{\Omega}\nabla\psi\cdot(\psi\cdot\nabla(\operatorname{div}\eta))\,\mathrm{d}x + \int_{\Omega} f\cdot(\eta\cdot\nabla\psi)\,\mathrm{d}x. \qquad (76)
\end{aligned}$$

We are now ready to state and prove the main result of this section.

Theorem 5.2. *Suppose that $\alpha\beta = \gamma\delta$. Assume that there exists a vector field $\eta \in [C^2(\overline{\Omega}_s)]^d$ and $\eta_0 > 0$ such that $\eta\cdot\nu \ge \eta_0$ on Γ_s and $J\eta(x)$ is uniformly positive definite, that is, there exists $\rho > 0$ such that $[J\eta(x)]z\cdot z \ge \rho|z|^2$ for every $z \in \mathbb{R}^d$ and $x \in \overline{\Omega}_s$. Then there exist $t_0 > 0$ and $C = C(t_0) > 0$ such that*

$$\|e^{tA}Y_0\|_X \le Ct^{-\frac{1}{3}}\|Y_0\|_{D(A)}, \qquad t > t_0,$$

for every $Y_0 \in D(A)$.

Proof. We will establish the following inequality

$$\|Y\|_{X,\mathbf{a}} \le C|r|^3\|Y^*\|_{X,\mathbf{a}} \qquad (77)$$

for every real number r with $|r| \ge r_0$ for some $r_0 > 1$ and $C = C(r_0) > 0$, where $Y = (u,v,w,y,z) \in D(A)$ and $(irI - A)Y = Y^* = (u^*, w^*, v^*, y^*, z^*)$. The latter equation is equivalent to system (32)–(35) where $\lambda = ir$. We note that to prove (77), it is enough to prove that

$$\|Y\|_{X,\mathbf{a}}^2 \le C|r|^3(\|\nabla u\|_{L^2(\Omega_f)}^2 + \|Y^*\|_{X,\mathbf{a}}^2) \tag{78}$$

for $|r| \ge r_0$. Indeed, since $|\langle Y^*, Y\rangle_{X,\mathbf{a}}| \ge -\mathrm{Re}\langle irY - AY, Y\rangle_{X,\mathbf{a}} \ge C\|\nabla u\|_{L^2(\Omega_f)}^2$ for some constant $C > 0$, (78) implies that

$$\|Y\|_{X,\mathbf{a}}^2 \le C|r|^3(|\langle Y^*, Y\rangle_{X,\mathbf{a}}| + \|Y^*\|_{X,\mathbf{a}}^2).$$

Applying Young's inequality, we have

$$C|r|^3|\langle Y^*, Y\rangle_{X,\mathbf{a}}| \le \frac{1}{2}\|Y\|_{X,\mathbf{a}}^2 + C|r|^6\|Y^*\|_{X,\mathbf{a}}^2.$$

The last two inequalities, together with the assumption that $r_0 > 1$, imply (77) after taking square roots. Therefore, in the following we will show (78). This will be done in several steps.

Step 1. An auxiliary variable. Define the Robin map $\mathcal{R}g = \varphi$ as follows

$$\begin{cases} \Delta\varphi = 0, & \text{in } \Omega_s, \\ \dfrac{\partial\varphi}{\partial\nu} + \varphi = g, & \text{on } \Gamma_s. \end{cases}$$

We infer from the regularity theory for elliptic operators in [26, Chapter 2] that the linear operator $\mathcal{R}$ satisfies $\mathcal{R} \in \mathcal{L}(H^\sigma(\Gamma_s), H^{\sigma+\frac{3}{2}}(\Omega_s))$ for every $\sigma \in \mathbb{R}$, assuming that Ω_s is sufficiently smooth. Let $\varphi = \mathcal{R}(\frac{\partial w}{\partial\nu} + w)$. The function $\psi = w - \varphi$ satisfies the boundary value problem

$$\begin{cases} -\Delta\psi + (1 - r^2)\psi = (r^2 - 1)\varphi + irw^* + v^*, & \text{in } \Omega_s, \\ \dfrac{\partial\psi}{\partial\nu} + \psi = 0, & \text{on } \Gamma_s. \end{cases} \tag{79}$$

Notice that $\psi \in H^2(\Omega_s)$. The right hand side of the first equation in (79) will be denoted by ψ^*.

Step 2. We prove that there is a constant $C > 0$ such that for every $|r| \ge \sqrt{6}$,

$$\begin{aligned} r^2\|\psi\|_{L^2(\Omega_s)}^2 \;\le\; & C(\|\nabla\psi\|_{L^2(\Omega_s)}^2 + \|\psi\|_{L^2(\Gamma_s)}^2 + r^2\|\varphi\|_{L^2(\Omega_s)}^2 \\ & + \|w^*\|_{L^2(\Omega_s)}^2 + \|v^*\|_{L^2(\Omega_s)}^2). \end{aligned} \tag{80}$$

Multiplying the first equation in (79) by $\overline{\psi}$, integrating over Ω_f and then using the generalized Green's identity we obtain

$$(r^2 - 1)\int_{\Omega_s} |\psi|^2 \,\mathrm{d}x = \int_{\Gamma_s} |\psi|^2 \,\mathrm{d}x + \int_{\Omega_s} |\nabla\psi|^2 \,\mathrm{d}x - \int_{\Omega_s} \psi^* \cdot \psi \,\mathrm{d}x. \tag{81}$$

For $|r| \ge 1/\sqrt{2}$ we have $|r^2 - 1| \le r^2$, and upon using this together with the elementary inequality $|ab| \le a^2 + \frac{1}{4}b^2$ we obtain from (79) that

$$\begin{aligned} \left|\int_{\Omega_s} \psi^* \cdot \psi \,\mathrm{d}x\right| \le & \int_{\Omega_s} \frac{|r^2 - 1|}{|r|}|\varphi||r||\psi| \,\mathrm{d}x + \int_{\Omega_s} |w^*||r||\psi| \,\mathrm{d}x + \int_{\Omega_s} |v^*||\psi| \,\mathrm{d}x \\ \le r^2 \int_{\Omega_s} |\varphi|^2 \,\mathrm{d}x + & \int_{\Omega_s} |w^*|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\Omega_s} |v^*|^2 \,\mathrm{d}x + \frac{1}{2}(r^2 + 1)\int_{\Omega_s} |\psi|^2 \,\mathrm{d}x. \end{aligned} \tag{82}$$

Plugging (82) in (81) yields the following estimate for some $C > 0$

$$\frac{1}{2}(r^2 - 3)\int_{\Omega_s} |\psi|^2 \,\mathrm{d}x \le C\left(\int_{\Omega_s} |\nabla\psi|^2 + |w^*|^2 + |v^*|^2 \,\mathrm{d}x + \int_{\Gamma_s} |\psi|^2 \,\mathrm{d}x\right) + r^2 \int_{\Omega_s} |\varphi|^2 \,\mathrm{d}x.$$

The assumption $|r| \geq \sqrt{6}$ implies that $(r^2-3)/2 \geq r^2/4$ and applying this to the above inequality we obtain (80).

Step 3. The next step is to show that for every $\varepsilon > 0$ there exist constants $C > 0$ and $C_\varepsilon > 0$ such that

$$\begin{aligned}\|\nabla\psi\|^2_{L^2(\Omega_s)} \;\leq\; & C_\varepsilon(r^2\|\psi\|^2_{L^2(\Gamma_s)} + r^2\|\varphi\|^2_{H^1(\Omega_s)} + r^2\|w^*\|^2_{L^2(\Omega_s)} + \|v^*\|^2_{L^2(\Omega_s)}) \\ & + (\varepsilon r^2 + C)\|\psi\|^2_{L^2(\Omega_s)} \end{aligned} \tag{83}$$

for every $|r| \geq 1$. This is the place where we apply the energy identity in Proposition 2 where $s = 1 - r^2$, $f = \psi^*$ and with the vector-field η in the statement of the theorem.

We estimate the terms on the right-hand side of (76). From the boundary condition $\frac{\partial\psi}{\partial\nu} = -\psi$ we obtain

$$\left|\frac{1}{2}\int_{\Gamma_s} \frac{\partial\psi}{\partial\nu}\cdot(\psi\,\mathrm{div}\,\eta)\,\mathrm{d}x\right| \;\leq\; C_\eta\int_{\Gamma_s}|\psi|^2\,\mathrm{d}x \tag{84}$$

$$\left|\frac{(1-r^2)}{2}\int_{\partial\Omega}|\psi|^2\eta\cdot\nu\,\mathrm{d}x\right| \;\leq\; C_\eta r^2\int_{\Gamma_s}|\psi|^2\,\mathrm{d}x. \tag{85}$$

In (85) we used the inequality $|r^2-1| \leq r^2$ which is valid for $|r| \geq \frac{1}{\sqrt{2}}$. On the other hand, using Young's inequality we derive the following estimates

$$\left|\int_{\Gamma_s}\frac{\partial\psi}{\partial\nu}\cdot(\eta\cdot\nabla\psi)\,\mathrm{d}x\right| \;\leq\; \frac{\eta_0}{4}\int_{\Gamma_s}|\nabla\psi|^2\,\mathrm{d}x + C_{\eta_0,\eta}\int_{\Gamma_s}|\psi|^2\,\mathrm{d}x \tag{86}$$

$$\left|\int_{\Omega_s}\nabla\psi\cdot(\psi\cdot\nabla(\mathrm{div}\,\eta))\,\mathrm{d}x\right| \;\leq\; \frac{\rho}{4}\int_{\Omega_s}|\nabla\psi|^2\,\mathrm{d}x + C_{\eta,\rho}\int_{\Omega_s}|\psi|^2\,\mathrm{d}x. \tag{87}$$

In the succeeding analysis, we estimate the last term in (76) where $f = \psi^* = (r^2-1)\varphi + irw^* + v^*$. First, we have

$$\left|\int_\Omega(irw^*+v^*)\cdot(\eta\cdot\nabla\psi)\,\mathrm{d}x\right| \leq C_{\eta,\rho}\int_{\Omega_s} r^2|w^*|^2 + |v^*|^2\,\mathrm{d}x + \frac{\rho}{4}\int_{\Omega_s}|\nabla\psi|^2\,\mathrm{d}x. \tag{88}$$

According to the divergence theorem it can be seen that

$$\begin{aligned}&\int_{\Omega_s} r^2\varphi\cdot(\eta\cdot\nabla\psi)\,\mathrm{d}x \\ &= \int_{\Gamma_s}(r\varphi)\cdot(r\psi)\eta\cdot\nu\,\mathrm{d}x - \int_{\Omega_s}(r\varphi)\cdot((r\psi)\,\mathrm{div}\,\eta)\,\mathrm{d}x - \int_{\Omega_s}(r\nabla\varphi)\cdot(r\psi\eta)\,\mathrm{d}x.\end{aligned}$$

Using this equation and then invoking Young's inequality we obtain the estimate

$$\begin{aligned}&\left|\int_{\Omega_s}(r^2-1)\varphi\cdot(\eta\cdot\nabla\psi)\,\mathrm{d}x\right| \\ &\leq C_{\eta,\rho,\varepsilon}\int_{\Gamma_s}|r\psi|^2\,\mathrm{d}x + C_{\eta,\rho,\varepsilon}\int_{\Omega_s}|r\varphi|^2 + |r\nabla\varphi|^2\,\mathrm{d}x + \frac{\rho\varepsilon}{2}r^2\int_{\Omega_s}|\psi|^2\,\mathrm{d}x.\end{aligned} \tag{89}$$

The terms on the left hand side of (76) can be estimated below by

$$\begin{aligned}&\int_\Omega(J\eta)\nabla\psi\cdot\nabla\psi\,\mathrm{d}x + \frac{1}{2}\int_{\Omega_s}|\nabla\psi|^2\eta\cdot\nu\,\mathrm{d}x \\ &\geq \rho\int_{\Omega_s}|\nabla\psi|^2\,\mathrm{d}x + \frac{\eta_0}{2}\int_{\Gamma_s}|\nabla\psi|\,\mathrm{d}x.\end{aligned} \tag{90}$$

Using the estimates (84)–(90) in (76) and the assumption $|r| \geq 1$ we obtain (83) with $C = C_{\eta,\rho}$ and $C_\varepsilon = C_{\eta,\rho,\varepsilon}$ after multiplying by $2/\rho$.

Step 4. The next step is to estimate the first two terms on the right hand side of (83). We show that there exists $C > 0$ such that for every $|r| \geq 1$ there holds

$$\left\|\frac{\partial w}{\partial \nu}\right\|^2_{H^{-\frac{1}{2}}(\Gamma_s)} + r^2\|w\|^2_{H^{-\frac{1}{2}}(\Gamma_s)} \leq C|r|(\|\nabla u\|^2_{L^2(\Omega_f)} + \|Y^*\|^2_{X,\mathbf{a}}) \tag{91}$$

and as a consequence for every $\varepsilon_1 > 0$ there exists $C_{\varepsilon_1} > 0$ such that for every $|r| \geq 1$

$$r^2\|\psi\|^2_{L^2(\Gamma_s)} + r^2\|\varphi\|^2_{H^1(\Omega_s)} \leq \varepsilon_1\|\psi\|^2_{H^1(\Omega_s)} + C_{\varepsilon_1}|r|^3(\|\nabla u\|^2_{L^2(\Omega_f)} + \|Y^*\|^2_{X,\mathbf{a}}). \tag{92}$$

First we prove (91). Rewrite the boundary conditions for u and w on Γ_s given in (38) and (39) as follows

$$irw = \frac{e^{ir\tau_s}}{\alpha}\left(u - \gamma\frac{\partial u}{\partial \nu}\right) + \tau_s\int_0^1 e^{ir\tau_s\theta}z^*(\theta)\,\mathrm{d}\theta + w^* \tag{93}$$

$$\frac{\partial w}{\partial \nu} = -\delta irw + \beta e^{-ir\tau_f}\left(\frac{\partial u}{\partial \nu} + \tau_f\int_0^1 e^{-ir\tau_f\theta}y^*(\theta)\,\mathrm{d}\theta\right). \tag{94}$$

Following [5], we define $\tilde{u} \in H^1(\Omega_f)$ to be the solution of the following elliptic problem

$$\Delta\tilde{u} = \Delta u + u^* \text{ in } \Omega_f, \qquad \tilde{u} = 0 \text{ on } \partial\Omega_f,$$

which, by a standard elliptic estimate, satisfies the inequality

$$\|\nabla\tilde{u}\|_{L^2(\Omega_f)} \leq C\|\Delta u + u^*\|_{H^{-1}(\Omega_f)} \leq C(\|\nabla u\|_{L^2(\Omega_f)} + \|u^*\|_{L^2(\Omega_f)}). \tag{95}$$

Since $iru = \Delta\tilde{u} = \mathrm{div}(\nabla\tilde{u})$ we have $|r|\|u\|_{H^{-1}(\Omega_f)} = \|\Delta\tilde{u}\|_{H^{-1}(\Omega_f)} \leq \|\nabla\tilde{u}\|_{L^2(\Omega_f)}$. From this estimate and (95), together with interpolation, the Poincaré and Cauchy-Schwarz inequalities, we have

$$|r|\|u\|^2_{L^2(\Omega_f)} \leq |r|\|u\|_{H^{-1}(\Omega_f)}\|u\|_{H^1(\Omega_f)} \leq C(\|\nabla u\|^2_{L^2(\Omega_f)} + \|u^*\|^2_{L^2(\Omega_f)}). \tag{96}$$

By the continuity of the first-order trace operator (recall Section 1.1), the equation $\Delta u = iru - u^*$, the Poincaré inequality and (96) we obtain for $|r| \geq 1$

$$\begin{aligned}\left\|\frac{\partial u}{\partial \nu}\right\|^2_{H^{-\frac{1}{2}}(\Gamma_s)} &\leq C(\|u\|^2_{H^1(\Omega_f)} + \|\Delta u\|^2_{L^2(\Omega_f)}) \\ &\leq C|r|(\|\nabla u\|^2_{L^2(\Omega_f)} + \|u^*\|^2_{L^2(\Omega_f)}).\end{aligned} \tag{97}$$

Using the embedding $L^2(\Gamma_s) \subset H^{-\frac{1}{2}}(\Gamma_s)$, trace theory and Poincaré inequality to majorize the norm of u in $L^2(\Gamma_s)$ by the norm of its gradient in $L^2(\Omega_f)$ in (93)

$$r^2\|w\|^2_{H^{-\frac{1}{2}}(\Gamma_s)} \leq C|r|(\|\nabla u\|^2_{L^2(\Omega_f)} + \|Y^*\|^2_{X,\mathbf{a}}). \tag{98}$$

Employing the inequalities (97) and (98) in (94) we get

$$\left\|\frac{\partial w}{\partial \nu}\right\|^2_{H^{-\frac{1}{2}}(\Gamma_s)} \leq C|r|(\|\nabla u\|^2_{L^2(\Omega_f)} + \|Y^*\|^2_{X,\mathbf{a}}). \tag{99}$$

The sum of (98) and (99) gives us (91), and this in turn implies the estimate

$$\|\varphi\|^2_{H^1(\Omega_s)} \leq C\left\|\frac{\partial w}{\partial \nu} + w\right\|^2_{H^{-\frac{1}{2}}(\Gamma_s)} \leq C|r|(\|\nabla u\|^2_{L^2(\Omega_s)} + \|Y^*\|^2_{X,\mathbf{a}}) \tag{100}$$

by the continuity of the Robin map $\mathcal{R}$. To estimate the norm of $r\psi$ in $L^2(\Gamma_s)$, we rewrite this norm, using the equation $\psi = w - \varphi$, as follows

$$r^2\|\psi\|^2_{L^2(\Gamma_s)} = \langle r^2 w, \psi\rangle - \int_{\Gamma_s} r\psi \cdot r\varphi \,\mathrm{d}x, \tag{101}$$

where the brackets denote the duality pairing between $H^{-\frac{1}{2}}(\Gamma_s)$ and $H^{\frac{1}{2}}(\Gamma_s)$. The first term on the right hand side of (101) can be estimated above by

$$|\langle r^2 w, \psi\rangle| \le r^2\|w\|_{H^{-\frac{1}{2}}(\Gamma_s)}\|\psi\|_{H^{\frac{1}{2}}(\Gamma_s)} \le C_{\varepsilon_1} r^4\|w\|^2_{H^{-\frac{1}{2}}(\Gamma_s)} + \varepsilon_1\|\psi\|^2_{H^1(\Omega_s)}. \tag{102}$$

The second term of the said equation will be simply estimated as follows

$$\left|\int_{\Gamma_s} r\psi\cdot r\varphi\,\mathrm{d}x\right| \le \frac{1}{2}\int_{\Gamma_s} r^2|\psi|^2\,\mathrm{d}x + \frac{1}{2}\int_{\Gamma_s} r^2|\varphi|^2\,\mathrm{d}x. \tag{103}$$

Using (102) and (103) in (101), taking the sum of the resulting estimate with (100) and finally using (98), we obtain (92) by virtue of the assumption that $|r| \ge 1$.

Step 5. Now we combine the estimates provided in the previous steps to prove that there exist $r_0 > \sqrt{6}$ and a constant $C > 0$ such that for $|r| \ge r_0$ there holds

$$r^2\|\psi\|^2_{L^2(\Omega_s)} + \|\nabla\psi\|^2_{L^2(\Omega_s)} \le C|r|^3(\|\nabla u\|^2_{L^2(\Omega_f)} + \|Y^*\|^2_{X,\mathbf{a}}). \tag{104}$$

From (80), (83) and (92), it follows that if $|r| \ge \sqrt{6}$ then

$$\begin{aligned} r^2\|\psi\|^2_{L^2(\Omega_s)} + \|\nabla\psi\|^2_{L^2(\Omega_s)} \ \le\ & C_\varepsilon\varepsilon_1\|\psi\|^2_{H^1(\Omega_s)} + (C\varepsilon r^2 + C)\|\psi\|^2_{L^2(\Omega_s)} \\ & + C_{\varepsilon,\varepsilon_1}|r|^3(\|\nabla u\|^2_{L^2(\Omega_f)} + \|Y^*\|^2_{X,\mathbf{a}}) \end{aligned} \tag{105}$$

where C is independent of r and ε. We choose the constants r_0, ε and ε_1 so that the following inequalities are satisfied

$$(1 - C\varepsilon)r^2 - C - C_\varepsilon\varepsilon_1 \ge \frac{r^2}{4} \quad \text{and} \quad 1 - C_\varepsilon\varepsilon_1 \ge \frac{1}{2} \tag{106}$$

whenever $|r| \ge r_0$. For example, we take $\varepsilon > 0$ small enough so that $1 - C\varepsilon \ge 1/2$, and then choose ε_1 sufficiently small so that the second inequality in (106) is satisfied. The first inequality in (106) is satisfied for every $|r| \ge r_0$ if we choose r_0 sufficiently large, for example one may take $r_0 = \max(\sqrt{6}, 2\sqrt{C + C_\varepsilon\varepsilon_1})$. Rearranging the terms in (105) and then using (106) yield (104).

Step 6. In this intermediate step, we will estimate the L^2-norm in Γ_s of the trace $\frac{\partial u}{\partial\nu}$. In fact, we will prove that if $|r| \ge 1$ then

$$\left\|\frac{\partial u}{\partial\nu}\right\|^2_{L^2(\Gamma_s)} \le C(|r|^3\|\nabla u\|^2_{L^2(\Omega_f)} + \|w\|^2_{H^1(\Omega_s)} + |r|^3\|Y^*\|^2_{X,\mathbf{a}}). \tag{107}$$

From the boundary condition (93) one can derive the following equation

$$\gamma\int_{\Gamma_s}\left|\frac{\partial u}{\partial\nu}\right|^2\mathrm{d}x = \int_{\Gamma_s}\frac{\partial u}{\partial\nu}\cdot\Big(u - \alpha e^{-ir\tau_s}(irw - w^*) - \alpha e^{-ir\tau_s}\tau_s\int_0^1 e^{ir\tau_s\theta}z^*(\theta)\,\mathrm{d}\theta\Big)\mathrm{d}x.$$

According to Young's inequality and trace theory applied to w^*, and Fubini's theorem and Hölder's inequality applied to z^* we have

$$\begin{aligned} &\left|\int_{\Gamma_s}\frac{\partial u}{\partial\nu}\cdot\Big(\alpha e^{-ir\tau_s}w^* - \alpha e^{-ir\tau_s}\tau_s\int_0^1 e^{ir\tau_s\theta}z^*(\theta)\,\mathrm{d}\theta\Big)\mathrm{d}x\right| \\ &\le \frac{\gamma}{2}\int_{\Gamma_s}\left|\frac{\partial u}{\partial\nu}\right|^2\mathrm{d}x + C_\gamma\int_{\Omega_s}|w^*|^2 + |\nabla w^*|^2\,\mathrm{d}x + C_\gamma\int_0^1\int_{\Gamma_s}|z^*(\theta)|^2\,\mathrm{d}x\,\mathrm{d}\theta. \end{aligned} \tag{108}$$

Estimate (97) and the trace theorem imply that

$$\left|\int_{\Gamma_s} \frac{\partial u}{\partial \nu}\cdot u \,\mathrm{d}x\right| \le \left\|\frac{\partial u}{\partial \nu}\right\|_{H^{-\frac{1}{2}}(\Gamma_s)}\|u\|_{H^{\frac{1}{2}}(\Gamma_s)} \le C|r|^{\frac{1}{2}}(\|\nabla u\|^2_{L^2(\Omega_f)} + \|Y^*\|^2_{X,\mathbf{a}}) \quad (109)$$

and similarly, now using the assumption $|r| \ge 1$,

$$\begin{aligned}\left|\int_{\Gamma_s} \frac{\partial u}{\partial \nu}\cdot \alpha e^{-ir\tau_s} irw \,\mathrm{d}x\right| &\le C\Big(|r|^2\left\|\frac{\partial u}{\partial \nu}\right\|^2_{H^{-\frac{1}{2}}(\Gamma_s)} + \|w\|^2_{H^{\frac{1}{2}}(\Gamma_s)}\Big)\\ &\le C(|r|^3\|\nabla u\|^2_{L^2(\Omega_f)} + \|w\|^2_{H^1(\Omega_s)} + |r|^3\|Y^*\|^2_{X,\mathbf{a}}). \quad (110)\end{aligned}$$

The estimates (108)–(110) give us the claim (107).

Step 7. Finally, we prove (78). From the equation $w = \psi + \varphi$ and the estimates (100) and (104) it can be seen that

$$\|w\|^2_{H^1(\Omega_s)} \le C|r|^3(\|\nabla u\|^2_{L^2(\Omega_f)} + \|Y^*\|^2_{X,\mathbf{a}}). \quad (111)$$

Again, using (100) and (104) once more, now with the equation $v = irw - w^* = ir(\psi+\varphi) - w^*$, we likewise have

$$\|v\|^2_{L^2(\Omega_s)} \le C|r|^3(\|\nabla u\|^2_{L^2(\Omega_f)} + \|Y^*\|^2_{X,\mathbf{a}}). \quad (112)$$

By trace theory, Poincaré inequality and the fact that $r_0 > 1$ we obtain

$$\|u\|^2_{L^2(\Omega_f)} + \|u\|^2_{L^2(\Gamma_s)} \le C|r|^3\|\nabla u\|^2_{L^2(\Omega_f)}. \quad (113)$$

Now, using (36) where $\lambda = ir$, and then applying the estimate (107) we obtain

$$\begin{aligned}\|y\|^2_{L^2_\theta(L^2(\Gamma_s))} &\le C\Bigg(\left\|\frac{\partial u}{\partial \nu}\right\|^2_{L^2(\Gamma_s)} + \|y^*\|^2_{L^2_\theta(L^2(\Gamma_s))}\Bigg)\\ &\le C|r|^3(\|\nabla u\|^2_{L^2(\Omega_f)} + \|Y^*\|^2_{X,\mathbf{a}}). \quad (114)\end{aligned}$$

On the other hand, replacing $irw - w^*$ in (93) by v the estimate

$$\|v\|^2_{L^2(\Gamma_s)} \le C\Bigg(\|u\|^2_{L^2(\Gamma_s)} + \left\|\frac{\partial u}{\partial \nu}\right\|^2_{L^2(\Gamma_s)} + \|z^*\|^2_{L^2_\theta(L^2(\Gamma_s))}\Bigg)$$

holds. This inequality, together with (107), (111), (113) and (37), implies the following

$$\|z\|^2_{L^2_\theta(L^2(\Gamma_s))} \le C|r|^3(\|\nabla u\|^2_{L^2(\Omega_s)} + \|Y^*\|^2_{X,\mathbf{a}}). \quad (115)$$

Inequalities (111)–(115) give us (78). The proof of the theorem is now completed by applying Theorem 5.1. □

6. Further remarks.

One may also consider possible delays in the feedback controls as has been done in [12, 13] for the one-dimensional wave equation and in [32] for the multidimensional case. The feedbacks in (2) will be replaced by

$$\begin{cases} F(t,x) = \gamma_1 \dfrac{\partial u}{\partial \nu}(t,x) + \gamma_2 \dfrac{\partial u}{\partial \nu}(t - \tilde{\tau}_f, x), & (t,x) \in (0,\infty)\times\Gamma_s,\\ G(t,x) = \delta_1 w_t(t,x) + \delta_2 w_t(t-\tilde{\tau}_s, x), & (t,x)\in(0,\infty)\times\Gamma_s,\end{cases} \quad (116)$$

for some delay parameters $\tilde{\tau}_f > 0$ and $\tilde{\tau}_s > 0$ and coefficients $\gamma_1 > 0$, $\delta_1 > 0$, and $\gamma_2, \delta_2 \in \mathbb{R}$. Using the methodologies presented in Section 4, it can be shown that the corresponding system is exponentially stable provided that $\gamma_1 > |\gamma_2|$, $\delta_1 > |\delta_2|$ and $(\gamma_1 - |\gamma_2|)(\delta_1 - |\delta_2|) > \alpha\beta$. This follows from the more general version of Theorem 7.1 given in [37]. On the other hand, it is rationally stable for $\gamma_1 > |\gamma_2|$, $\delta_1 > |\delta_2|$ and $(\gamma_1 - |\gamma_2|)(\delta_1 - |\delta_2|) = \alpha\beta$ by using the same techniques given in Section 5.

Finally, we note that the above results can be adapted when there is no delay, that is, $\tau_f = \tilde{\tau}_f = \tau_s = \tilde{\tau}_s = 0$.

7. Appendix.

Consider the quadratic form $Q_{\mathbf{a}} : \mathbb{R}^4 \to \mathbb{R}$ defined by

$$Q_{\mathbf{a}}(u, v, y, z) = (2a_1\gamma - a_3)u^2 + (2a_2\delta - a_4)v^2 + a_3y^2 + a_4z^2 + 2a_1\alpha uz + 2a_2\beta vy \quad (117)$$

where $\mathbf{a} = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4$. The quadratic form $Q_{\mathbf{a}}$ clearly satisfies the estimate

$$|Q_{\mathbf{a}}(u, v, y, z)| \leq R(|u|^2 + |v|^2 + |y|^2 + |z|^2), \quad \text{for every } (u, v, y, z) \in \mathbb{R}^4, \quad (118)$$

where $R > 0$ is a constant depending only on $\mathbf{a}$ and $(\alpha, \beta, \gamma, \delta)$. Under suitable assumptions on the parameters $(\alpha, \beta, \gamma, \delta)$, this form will be positive-definite, that is, there exists a constant $\rho > 0$ such that

$$Q_{\mathbf{a}}(u, v, y, z) \geq \rho(|u|^2 + |v|^2 + |y|^2 + |z|^2), \quad \text{for every } (u, v, y, z) \in \mathbb{R}^4,$$

or nonnegative definite, that is, $Q_{\mathbf{a}}(u, v, y, z) \geq 0$ for every $(u, v, y, z) \in \mathbb{R}^4$. This is the content of the following theorem.

Theorem 7.1. *Suppose that $\alpha, \beta, \gamma, \delta > 0$ satisfies $\alpha\beta < \gamma\delta$. Then the quadratic form $Q_{\mathbf{a}}$ is positive definite for some $\mathbf{a} \in \mathbb{R}^4_+$. If $\alpha\beta = \gamma\delta$ then $Q_{\mathbf{a}}$ is nonnegative definite for some $\mathbf{a} \in \mathbb{R}^4_+$.*

Proof. The positive definiteness of the form $Q_{\mathbf{a}}$ for some $\mathbf{a} \in \mathbb{R}^4_+$ under the condition $\alpha\beta < \gamma\delta$ has been already established in [37]. Thus, we only need to prove the second part. Following [37], we rewrite the quadratic form as follows

$$\begin{aligned} Q_{\mathbf{a}}(u, v, y, z) \; = \; & \left[-\frac{\alpha^2}{a_4}\left(a_1 - \frac{a_4\gamma}{\alpha^2}\right)^2 + \frac{a_4\gamma^2}{\alpha^2} - a_3 \right] u^2 + a_4\left(z + \frac{a_1\alpha}{a_4}u\right)^2 \\ & + \left[-\frac{\beta^2}{a_3}\left(a_2 - \frac{a_3\delta}{\beta^2}\right)^2 + \frac{a_3\delta^2}{\beta^2} - a_4 \right] v^2 + a_3\left(y + \frac{a_2\beta}{a_3}v\right)^2. \end{aligned}$$

If we choose the positive constants a_3 and a_4 to satisfy $a_4/a_3 = \alpha^2/\gamma^2$ and then take $a_1 = a_4\gamma/\alpha^2$, then the coefficient for u^2 in the above equation will become zero. On the other hand, by taking $a_2 = a_3\delta/\beta^2$ and using the fact that $a_3/a_4 = \gamma^2/\alpha^2 = \beta^2/\delta^2$, we can see that the coefficient of v^2 also vanishes. Therefore, with these choices for the constants a_1, a_2, a_3, a_4, the quadratic form is nonnegative definite provided that $\alpha\beta = \gamma\delta$. □

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