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On Existence of Solutions of a Class of Volterra Integral Equations

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Abstract: In this paper, we discuss sufficient conditions to guarantee the existence and uniqueness of a class of Volterra integral equations of the second kind. Our results are based on the resolvent and contraction mapping methods. Some classical existence and uniqueness theorems are given as corollaries.

Keywords: Contraction mapping theorem, Volterra integral equations, spectral radius

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1 Introduction

There are many results in establishing the existence and uniqueness of solutions of the Volterra integral equation of the second kind and various methods were developed to obtain these results. Knowing that there is only one solution does not content mathematicians to further study these integral equations. Of course, one obvious question is that what are the exact solutions for these integral equations? A lot of techniques in finding explicit solutions of these equations were made and in the event of obtaining an exact solution is impossible, approximations of the exact solution were also studied. For a more detailed discussion, we refer the readers to [2], [5], [6], and [7].

In this paper, we will consider a more general integral equation of the type

$$f(x) = \lambda \int_a^x K(x,y)F(x,y,f(y))\,\mathrm{d}y + \phi(x), \tag{1}$$

which is called a **Volterra integral equation of the second kind**. The said integral equation arises in several problems in mathematical physics (see [3]). We give sufficient conditions for the kernel K and the function F such that given a function ϕ in a certain space, for example the space of continuous function or the L^p spaces, we can find a unique function f belonging to the same class and satisfies (1). In the literature, there are results regarding the existence and uniqueness of solutions of integral equations of the type

$$f(x) = \lambda \int_a^x K(x,y)G(y,f(y))\,\mathrm{d}y + \phi(x). \tag{2}$$

We can see that if the variable x can be absorbed by $K(x,y)$, then Equation (1) can be reduced to Equation (2). To be exact, if $F(x,y,f(y)) = g(x)G(y,f(y))$, then we can let our kernel to be $K_0(x,y) = K(x,y)g(x)$. But there are integral equations for which the variable x cannot be absorbed by the kernel $K(x,y)$, for instance, the integral equation

$$f(x) = \int_0^1 (x^2+y^2)\cos(xf(y))\,\mathrm{d}y + 1.$$

Also, Gori and Santi [1] provided a method for solving (1) numerically in the case where the kernel K is of the convolution type $K(x-y)$ and $K(t)$ is continuous at $t>0$ and integrable at $t=0$. Their method is based on quasi-interpolatory splines. In this work, we establish the existence and uniqueness of solutions of the integral equation (1) in the space of continuous functions and the Lebesgue L^p spaces, for $p>1$.

2 Main Results

2.1 Continuous Kernels

First, we consider the case where the kernel is continuous. If the function F is continuous and satisfies a certain Lipschitz condition, then we are guaranteed that there is a unique solution to the Volterra integral equation (1). This is the content of the following theorem.

Theorem 2.1. *Let $F : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that it satisfies a Lipschitz condition of the form*

$$|F(x,y,z_1) - F(x,y,z_2)| \leq M(x,y)|z_1 - z_2|$$

for all $x, y \in [a,b]$ and $z_1, z_2 \in \mathbb{R}$ where

$$0 < \sup_{a \leq x,y \leq b} |M(x,y)| < \infty.$$

Assume that $K : [a,b] \times [a,b] \to \mathbb{R}$ and $\phi : [a,b] \to \mathbb{R}$ are continuous and $\lambda \in \mathbb{R}$. Then there exists a unique function $f \in C([a,b],\mathbb{R})$ such that f satisfies (1).

Proof. If either $K = 0$ or $\lambda = 0$, then we may take $f = \phi$. Now assume that neither K nor λ is zero. For convenience, we let $M_0 = \sup_{a \leq x,y \leq b} |M(x,y)|$ and $K_0 = \max_{a \leq x,y \leq b} |K(x,y)|$. By assumption M_0 and K_0 are positive. Define a mapping $T : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ by

$$(Tu)(x) = \lambda \int_a^x K(x,y)F(x,y,u(y))\,\mathrm{d}y + \phi(x).$$

The continuity of the kernel K and the function F implies that the integral in Tu is continuous on $[a,b]$, and since ϕ is continuous on $[a,b]$ it follows that $Tu \in C([a,b],\mathbb{R})$. Therefore T is well-defined. For $u, v \in C([a,b],\mathbb{R})$ we have, by induction,

$$|(T^n u)(x) - (T^n v)(x)| \leq \frac{(|\lambda| K_0 M_0 (x-a))^n \|u - v\|_\infty}{n!}. \tag{3}$$

for all $n \in \mathbb{N}$. Taking the supremum of (3) we obtain

$$\|T^n u - T^n v\|_\infty \leq \frac{(|\lambda| K_0 M_0 (b-a))^n}{n!} \|u - v\|_\infty.$$

From the inequality $\sqrt{n} \leq \sqrt[n]{n!}$ we have $1/\sqrt[n]{n!} \to 0$ as $n \to \infty$. Hence, there exists a positive integer N such that

$$\frac{1}{\sqrt[n]{n!}} < \frac{1}{|\lambda| K_0 M_0 (b-a)}.$$

whenever $n \geq N$. In particular,

$$\frac{(|\lambda| K_0 M_0 (b-a))^N}{N!} < 1$$

so that T^N is a contraction mapping. Therefore by the Contraction Mapping Theorem, there exists a unique $f \in C([a,b],\mathbb{R})$ such that $Tf = f$. This completes the proof of the theorem. □

Since the function $F : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ defined by $F(x,y,z) = z$ clearly satisfies the Lipschitz condition given in the previous theorem, we have the following classical result.

Corollary 2.2. *Assume that* $K : [a,b] \times [a,b] \to \mathbb{R}$ *and* $\phi : [a,b] \to \mathbb{R}$ *are continuous and* $\lambda \in \mathbb{R}$. *Then there exists a unique function* $f \in C([a,b], \mathbb{R})$ *such that* f *satisfies the Volterra integral equation of the second kind*

$$f(x) = \lambda \int_a^x K(x,y) f(x)\, \mathrm{d}y + \phi(x)$$

for all $x \in [a,b]$.

2.2 Measurable and Symmetric Kernels

Let I be an interval. If $F : I \times I \times \mathbb{C} \to \mathbb{C}$, $u : I \to \mathbb{C}$ and $x \in I$ we define the function $H(F,x,u;\cdot) : I \to \mathbb{C}$ by

$$H(F,x,u;y) = F(x,y,u(y))$$

for all $y \in I$.

Definition 2.3. Let $M > 0$. A Lebesgue measurable function $F : I \times I \times \mathbb{C} \to \mathbb{C}$, where I is an interval, is said to be of **class** M if it satisfies the following conditions.

(a) $|H(F,x,u;y)| \leq M|u(y)|$ for all $x, y \in I$ and $u \in C(I,\mathbb{C})$ (or $L^p(I)$).

(b) If $x \in I$ then $H(F,x,\alpha u + v;\cdot) = \alpha H(F,x,u;\cdot) + H(F,x,v;\cdot)$ for all $u, v \in C(I,\mathbb{C})$ (or $L^p(I)$) and for all scalar α.

For kernels which are not necessarily continuous, we have the following theorems.

Theorem 2.4. *Let* $-\infty \leq a < b \leq \infty, 1 < p < \infty$ *and* $\lambda \in \mathbb{C}$. *Suppose* $F : (a,b) \times (a,b) \times \mathbb{C} \to \mathbb{C}$ *is of class* M *for some* $M > 0$. *Let* $K : (a,b) \times (a,b) \to \mathbb{C}$ *be Lebesgue measurable and define* $G : (a,b) \to \mathbb{C}$ *by*

$$G(x) = \int_a^x |K(x,y)|^{p/(p-1)}\, \mathrm{d}y.$$

If $G \in L^{p-1}(a,b)$ *and* $H(F,x,u;\cdot) \in L^p(a,b)$ *for all* $x \in (a,b)$ *and* $u \in L^p(a,b)$, *then given* $\phi \in L^p(a,b)$, *there exists a unique* $f \in L^p(a,b)$ *such that* f *satisfies the Volterra integral equation of the second kind (1).*

Proof. If $\lambda = 0$ we can take $f = \phi$. Now let us consider the case where λ is nonzero. Since $H(F, x, u; \cdot) \in L^p(a, b)$ Hölders inequality implies that

$$\left(\int_a^x |K(x,y)F(x,y,u(y))| \, \mathrm{d}y \right)^p \leq M^p [G(x)]^{p-1} \int_a^x |u(y)|^p \, \mathrm{d}y, \tag{4}$$

whenever $u \in L^p(a, b)$ and $x \in (a, b)$.

Define an operator $T : L^p(a, b) \to L^p(a, b)$ by

$$(Tu)(x) = \int_a^x K(x,y)F(x,y,u(y)) \, \mathrm{d}y.$$

Given $u \in L^p(a, b)$, it follows from (4) that

$$\begin{aligned} \|Tu\|_p^p &= \int_a^b \left| \int_a^x K(x,y)F(x,y,u(y)) \, \mathrm{d}y \right|^p \mathrm{d}x \\ &\leq M^p \|u\|_p^p \|G\|_{p-1}^{p-1}. \end{aligned}$$

Thus

$$\|Tu\|_p \leq \left(M \|G\|_{p-1}^{(p-1)/p} \right) \|u\|_p < \infty,$$

because $G \in L^{p-1}(a, b)$, so that T is well-defined and $T \in \mathfrak{B}(L^p(a, b))$. Also, T is a linear operator since

$$\begin{aligned} (T(\alpha u + v))(x) &= \int_a^x K(x,y) H(F, x, \alpha u + v; y) \, \mathrm{d}y \\ &= \int_a^x K(x,y) [\alpha H(F, x, u; y) + H(F, x, v; y)] \, \mathrm{d}y \\ &= (\alpha T(u) + T(v))(x) \end{aligned}$$

whenever $\alpha \in \mathbb{C}$ and $u, v \in L^p(a, b)$.

Fix an element $u \in L^p(a, b)$ and define $v_n : (a, b) \to \mathbb{R}$ by

$$v_n(x) = \int_a^x |(T^n u)(y)|^p \, \mathrm{d}y$$

for nonnegative integers n. By replacing u by $T^n u$ in (4), we get

$$\begin{aligned} v_{n+1}(x) &= \int_a^x \left| \int_a^s K(s,y)F(s,y,(T^n u)(y)) \, \mathrm{d}y \right|^p \mathrm{d}s \\ &\leq M^p \int_a^x [G(s)]^{p-1} \left(\int_a^s |(T^n u)(y)|^p \, \mathrm{d}y \right) \mathrm{d}s \\ &= M^p \int_a^x [G(s)]^{p-1} v_n(s) \, \mathrm{d}s \end{aligned} \tag{5}$$

for all $n \geq 0$. Our next step is to prove the following estimate for the function v_n

$$v_n(x) \leq \frac{M^{np}}{(n-1)!} \int_a^x \left(\int_y^x [G(s)]^{p-1} \, \mathrm{d}s \right)^{n-1} [G(y)]^{p-1} v_0(y) \, \mathrm{d}y \tag{6}$$

for all $n \in \mathbb{N}$. We proceed by mathematical induction. Indeed, one can see that the basis step can be verified by letting $n = 0$ in (5). Assuming that (6) holds for $n = k$ we have, in virtue of (5)

$$v_{k+1}(x) \quad \leq \quad \frac{M^{(k+1)p}}{(k-1)!} \int_a^x \int_a^t \xi(t,y) [G(y)]^{p-1} v_0(y) \, \mathrm{d}y \, \mathrm{d}t$$

where

$$\xi(t,y) = \left(\int_y^t [G(s)]^{p-1} \, \mathrm{d}s \right)^{k-1} [G(t)]^{p-1}.$$

Let us consider the integral

$$\int_a^x \int_a^t \xi(t,y) [G(y)]^{p-1} v_0(y) \, \mathrm{d}y \, \mathrm{d}t.$$

Changing the order of integration gives us

$$\int_a^x \int_a^t \xi(t,y) [G(y)]^{p-1} v_0(y) \, \mathrm{d}y \, \mathrm{d}t = \int_a^x \int_y^x \xi(t,y) [G(y)]^{p-1} v_0(y) \, \mathrm{d}t \, \mathrm{d}y.$$

Consequently,

$$v_{k+1}(x) \quad \leq \quad \frac{M^{(k+1)p}}{(k-1)!} \int_a^x \left(\int_y^x \xi(t,y) \, \mathrm{d}t \right) [G(y)]^{p-1} v_0(y) \, \mathrm{d}y. \tag{7}$$

The first fundamental theorem of the calculus gives us

$$\int_y^x \xi(t,y) \, \mathrm{d}t \quad = \quad \frac{1}{k} \left(\int_y^x [G(s)]^{p-1} \, \mathrm{d}s \right)^k.$$

Using this in (7) gives us

$$v_{k+1}(x) \quad \leq \quad \frac{M^{(k+1)p}}{k!} \int_a^x \left(\int_y^x [G(s)]^{p-1} \, \mathrm{d}s \right)^k [G(y)]^{p-1} v_0(y) \, \mathrm{d}y.$$

This completes the proof of the estimate (6).

Again, note that

$$\frac{\mathrm{d}}{\mathrm{d}y} \int_y^x [G(s)]^{p-1} \, \mathrm{d}s = -[G(y)]^{p-1}.$$

and so

$$\int_a^x \left(\int_y^x [G(s)]^{p-1}\,\mathrm{d}s \right)^{n-1} [G(y)]^{p-1}\,\mathrm{d}y \quad = \quad \frac{1}{n}\left(\int_a^x [G(s)]^{p-1}\,\mathrm{d}s \right)^{n}. \tag{8}$$

Since $v_0(y) \le v_0(b)$, by (6) and (8) we get

$$\begin{aligned} v_n(x) \quad &\le \quad \frac{M^{np}}{(n-1)!} v_0(b) \int_a^x \left(\int_y^x [G(s)]^{p-1}\,\mathrm{d}s \right)^{n-1} [G(y)]^{p-1}\,\mathrm{d}y \\ &\le \quad \frac{M^{np}}{n!} v_0(b) \left(\int_a^x [G(s)]^{p-1}\,\mathrm{d}s \right)^{n} \\ &\le \quad \frac{M^{np}}{n!} \|G\|_{p-1}^{n(p-1)} v_0(b). \end{aligned}$$

Because $v_0(b) = \|u\|_p^p$ we have

$$\|T^n u\|_p = v_n(b)^{1/p} \le \left(\frac{M^{np}\|G\|_{p-1}^{n(p-1)}}{n!} \right)^{1/p} \|u\|_p.$$

Therefore, it follows that

$$\|T^n\|^{1/n} \le \left(\frac{M^{p}\|G\|_{p-1}^{p-1}}{\sqrt[n]{n!}} \right)^{1/p} \|u\|_p^{1/n},$$

and this estimate implies that $\|T^n\|^{1/n} \to 0$ as $n \to \infty$. Hence, the spectral radius of T is 0. The boundedness of the closed operator T implies that its spectrum is nonempty and so $\sigma(T) = \{0\}$ and $\rho(T) = \mathbb{C} \setminus \{0\}$. Thus $\lambda^{-1} \in \rho(T)$ for each nonzero λ and since $\lambda^{-1}\phi \in L^p(a,b)$ we have

$$f = (\lambda^{-1}I - T)^{-1}(\lambda^{-1}\phi) \in L^p(a,b).$$

From this, we get $(\lambda^{-1}I - T)f = \lambda^{-1}\phi$ and so $f - \lambda T f = \phi$. Therefore f is the solution of the integral equation. From the above discussion, it is also clear that such f is unique. This completes the proof of the theorem. □

From the proof of the previous theorem, we have the following limit (see Theorem 1.6.8 of Miklavčič [4])

$$\lim_{n\to\infty} \left\| f - \sum_{k=0}^{n} T^k \lambda^k \phi \right\|_p = 0.$$

Moreover, since $F(x,y,z) = z$ is clearly a continuous function of class $M = 1$, we have the following corollary.

Corollary 2.5. *Let* $-\infty \leq a < b \leq \infty, 1 < p < \infty$ *and* $\lambda \in \mathbb{C}$. *Let* $K : (a,b) \times (a,b) \times \mathbb{C}$ *be Lebesgue measurable and define* $G : (a,b) \to \mathbb{C}$ *by*

$$G(x) = \int_a^x |K(x,y)|^{p/(p-1)} \,\mathrm{d}y.$$

If $G \in L^{p-1}(a,b)$, *then given* $\phi \in L^p(a,b)$, *there exists a unique* $f \in L^p(a,b)$ *such that* f *satisfies the Volterra integral equation of the second kind*

$$f(x) = \lambda \int_a^x K(x,y) f(y) \,\mathrm{d}y + \phi(x).$$

A function $K : [a,b] \times [a,b] \to \mathbb{C}$ is said to be **symmetric** if $K(x,y) = K(y,x)$ for all $x, y \in [a,b]$. For symmetric kernels, we have the following existence and uniqueness theorem.

Theorem 2.6. *Let* $-\infty < a < b < \infty$ *and* $\lambda \in \mathbb{C}$. *Assume that* $K : [a,b] \times [a,b] \to \mathbb{C}$ *is Lebesgue measurable, bounded, and symmetric, and* $F : [a,b] \times [a,b] \times \mathbb{C} \to \mathbb{C}$ *is a continuous function of class* M *for some* $M > 0$. *If* $K(x,\cdot) \in L^1(a,b)$ *for all* $x \in [a,b]$, *then given* $\phi \in C([a,b],\mathbb{C})$, *there exists a unique* $f \in C([a,b],\mathbb{C})$ *such that* f *satisfies the Volterra integral equation (1)*

Proof. Let $G : [a,b] \to \mathbb{R}$ be defined by

$$G(x) = \sup_{a \leq y \leq b} |K(x,y)|.$$

The boundedness of the kernel K implies that $\|G\|_\infty < \infty$. Define a linear operator $T : C([a,b],\mathbb{C}) \to C([a,b],\mathbb{C})$ by

$$(Tu)(x) = \int_a^x K(x,y) F(x,y,u(y)) \,\mathrm{d}y.$$

One can easily check that $\|Tu\|_\infty \leq M(b-a)\|G\|_\infty \|u\|_\infty$, and from this we have $T \in \mathfrak{B}(C([a,b],\mathbb{C}))$.

For each nonnegative integer n let $v_n : [a,b] \to \mathbb{R}$ be the increasing function given by

$$v_n(x) = \sup_{a \leq y \leq x} |(T^n u)(y)|.$$

Then for each nonnegative integer n,

$$v_{n+1}(x) \quad \leq \quad M \sup_{a \leq y \leq x} \int_a^y |K(y,s)||(T^n u)(s)| \,\mathrm{d}s.$$

For $y \le x$, $\|K(y,\cdot)T^n u\|_{L^1(a,y)} \le \|K(y,\cdot)T^n u\|_{L^1(a,x)}$ and so

$$v_{n+1}(x) \le M \int_a^x \left(\sup_{a \le y \le x} |K(y,s)| \right) |(T^n u)(s)| \,\mathrm{d}s.$$

The symmetry of the kernel and the fact that $|(T^n u)(s)| \le v_n(s)$ we obtain

$$v_{n+1}(x) \le M \int_a^x G(s) v_n(s) \,\mathrm{d}s.$$

Using a similar argument as in the proof of Theorem 2.4 we obtain

$$v_n(x) \le \frac{M^n}{(n-1)!} \int_a^x \left(\int_y^x G(s) \,\mathrm{d}s \right)^{n-1} G(y) v_0(y) \,\mathrm{d}y$$

for all $n \ge 1, x \in [a,b]$. Because $v_0(y) \le v_0(b) = \|u\|_\infty$ we have

$$v_n(x) \le \frac{M^n}{(n-1)!} \|u\|_\infty \int_a^x \left(\int_y^x G(s) \,\mathrm{d}s \right)^{n-1} G(y) \,\mathrm{d}y.$$

From the inequality

$$\int_a^x \left(\int_y^x G(s) \,\mathrm{d}s \right)^{n-1} G(y) \,\mathrm{d}y \quad \le \quad \frac{1}{n} \|G\|_\infty^n (x-a)^n$$

we obtain

$$v_n(x) \le \frac{M^n (b-a)^n \|G\|_\infty^n}{n!} \|u\|_\infty$$

for all $x \in [a,b]$. The equality $v_n(b) = \|T^n u\|_\infty$ gives us

$$\|T^n u\|_\infty \le \frac{M^n (b-a)^n \|G\|_\infty^n}{n!} \|u\|_\infty.$$

Therefore

$$\|T^n\|^{1/n} \le \frac{M(b-a)\|G\|_\infty}{\sqrt[n]{n!}}$$

and this estimate yields $\|T^n\|^{1/n} \to 0$ as $n \to \infty$. Thus $\rho(T) = \mathbb{C} \setminus \{0\}$ and the unique solution to the nonhomogeneous Volterra integral equation is given by $f = (\lambda^{-1} I - T)^{-1} (\lambda^{-1} \phi) \in C([a,b], \mathbb{C})$. $\square$

Corollary 2.7. *Let $-\infty < a < b < \infty$ and $\lambda \in \mathbb{C}$. Assume that $K : [a,b] \times [a,b] \to \mathbb{C}$ is Lebesgue measurable, bounded, and symmetric. If $K(x,\cdot) \in L^1(a,b)$ for all $x \in [a,b]$, then given $\phi \in C([a,b],\mathbb{C})$, there exists a unique $f \in C([a,b],\mathbb{C})$ such that*

$$f(x) = \lambda \int_a^x K(x,y) \, \langle T(f(y)), h_0 \rangle_H \,\mathrm{d}y + \phi(x),$$

where $T : \mathbb{C} \to H$ is a bounded linear operator, H is a Hilbert space and $h_0 \in H$.

Proof. If $T = 0$ or $h_0 = 0$, then the integral in the conclusion of the corollary is zero and hence we may take $f = \phi$. Now, assume that T is not the zero operator and h_0 is a nonzero vector in H. From the previous theorem, it remains to show that the function $F : [a, b] \times [a, b] \times \mathbb{C} \to \mathbb{C}$ defined by $F(x, y, z) = \langle T(z), h_0 \rangle_H$ is continuous and of class M for some $M > 0$. If $\alpha \in \mathbb{C}$ and $u, v \in C([a, b], \mathbb{C})$ then using the linearity of the operator T and the linearity of the inner product in the first argument, we have $H(F, x, \alpha u + v; y) = \alpha H(F, x, u; y) + H(F, x, v; y)$ for all $y \in [a, b]$. Furthermore, using the Cauchy-Schwartz Inequality we have

$$\begin{aligned} |H(F, x, \alpha u + v; y)| &= |\langle T(u(y)), h_0 \rangle_H| \\ &\leq \|T\|_{\mathfrak{B}(\mathbb{C},H)} \|h_0\|_H |u(y)|. \end{aligned}$$

Since T and h_0 are both nonzero, then it follows that $\|T\|_{\mathfrak{B}(\mathbb{C},H)} > 0$ and $\|h_0\|_H > 0$. Note that the measurability of F follows from its continuity. Therefore it follows that F is a function of class $M = \|T\|_{\mathfrak{B}(\mathbb{C},H)} \|h_0\|_H > 0$, provided that it is continuous.

It remains to show that F is indeed continuous. To prove this, let $\epsilon > 0$. Choose $\delta = \epsilon / M$. If $|(x, y, z) - (\tilde{x}, \tilde{y}, \tilde{z})| < \delta$ then $|z - \tilde{z}| < \delta$. Hence

$$|\langle T(z), h_0 \rangle_H - \langle T(\tilde{z}), h_0 \rangle_H| \leq \|T\|_{\mathfrak{B}(\mathbb{C},H)} \|h_0\|_H |z - \tilde{z}| < \delta M = \epsilon.$$

Thus F is continuous and this establishes our result. □

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