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SOME PROPERTIES OF A SEQUENCE OF INVERSION NUMBERS

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Abstract. *In this paper, we consider a certain sequence of inversion numbers. We show that this sequence is a polynomial sequence and find its leading term. Using this, a characterization of the Hankel and inverse binomial transforms of these inversion numbers will be given and each of these transforms, together with an appropriate sequence, forms a basis for the space of real sequences having compact support. Also, with the aid of the generating function of the inversion numbers we will give a formula for a certain type of complex integral.*

1 Introduction

Let σ be a permutation of the set $\{1, 2, \ldots, n\}$. The pair $(\sigma(i), \sigma(j))$ is called an **inversion** of σ if $i < j$ and $\sigma(i) > \sigma(j)$. Inversions are used in sorting algorithms and have applications in computational molecular biology (see [2]). Denote $N(\sigma)$ to be the total number of inversions of the permutation σ. Then $N(\sigma)$ is the smallest number of factors such that the permutation σ can be written as a product of simple transpositions [1].

For each nonnegative integer k, we let $I_n(k) = |\{\sigma \in \mathbf{S}_n : N(\sigma) = k\}|$, where $\mathbf{S}_n$ is the symmetric group of degree n. That is, $I_n(k)$ is the total number of permutations of the set $\{1, 2, \ldots, n\}$ having k inversions. Then $I_n(k) = 0$ for all $k > \binom{n}{2}$ and $I_n(k) > 0$ for all $0 \leq k \leq \binom{n}{2}$. The number $I_n(k)$ is called an **inversion number**. Finding the value of $I_n(k)$ is a classic area of combinatorics. Margolius, Louchard and Prodinger give asymptotic formulas of a certain sequence of inversion numbers, the sequence $\{I_{n+k}(n) : n \geq 0\}$, where k is a fixed positive integer [4, 6]. The results of Louchard and Prodinger are based on the saddle point method. In a recent paper [5], the authors consider another sequence of inversion numbers, the sequence $\{I_{n+k}(k) : n \geq 0\}$, where $k \geq 1$ is fixed. Interestingly, these sequences are polynomial sequences as we can see later.

The inversion numbers have the following recursive formula

$$I_1(0) = I_2(0) = I_2(1) = 1$$

and

$$I_n(k) = \sum_{i=\max\{0,k-n+1\}}^{\min\{k,\binom{n-1}{2}\}} I_{n-1}(i), \quad n \geq 3. \tag{1}$$

This formula was obtained using a specific partition of the symmetric group. For $n > 1$, this can be simplified into

$$I_n(k) = \begin{cases} 1, & \text{if } k = 0; \\ I_n(k-1) + I_{n-1}(k), & \text{if } 1 \leq k \leq n-1; \\ I_n(k-1) + I_{n-1}(k) - I_{n-1}(k-n), & \text{if } n \leq k \leq \binom{n-1}{2}; \\ I_n(k-1) - I_{n-1}(k-n), & \text{if } \binom{n-1}{2} < k \leq \binom{n}{2}. \end{cases} \tag{2}$$

For more details about these recursive formulas, we refer the reader to [5].

In Section 2, we give a complete proof showing that the sequence $\{I_{n+k}(k) : n \geq 0\}$ is a polynomial sequence and that the leading term of this polynomial sequence is $(k!)^{-1}$. Further, we compare the monotonicity of the two sequences $\{I_{n+l}(n) : n \geq 0\}$ and $\{I_{n+k}(k) : n \geq 0\}$, where k and l are fixed positive integers. Section 3 relates a specific type of an integral of a complex valued function to the inversion numbers. Finally, we characterize the Hankel and inverse binomial transforms of $\{I_{n+k}(k) : n \geq 0\}$ in Section 4.

2 Characterizations of a sequence of inversion numbers

In the following lemma, we consider the sum $\sum_{j=1}^{n} j^{h-1}$. As we can see later, this sum is closely related to the sequence $\{I_{n+k}(k) : n \geq 0\}$.

Lemma 1. *For each positive integer h let $P_h(n) = \sum_{j=1}^{n} j^{h-1}$. Then $P_h(n)$ is a polynomial of the variable n of degree h and*

$$\lim_{n\to\infty} \frac{P_h(n)}{n^h} = \frac{1}{h}.$$

Proof. We prove the lemma by strong induction. Is is easy to see that the conclusion holds if $h = 1$. Now, assume that $P_l(n)$ is a polynomial of degree l for all $1 \leq l \leq h$. Using the Binomial Theorem, we get

$$\begin{aligned}
\sum_{j=1}^{n}[j^{h+1} - (j-1)^{h+1}] &= \sum_{j=1}^{n}\left[j^{h+1} - \sum_{l=0}^{h+1}(-1)^l\binom{h+1}{l}j^{h-l+1}\right] \\
&= \sum_{l=1}^{h+1}(-1)^{l+1}\binom{h+1}{l}\left(\sum_{j=1}^{n} j^{h-l+1}\right) \\
&= \sum_{l=1}^{h+1}(-1)^{l+1}\binom{h+1}{l}P_{h-l+2}(n) \\
&= (h+1)P_{h+1}(n) + \sum_{l=0}^{h-1}(-1)^{l+1}\binom{h+1}{l+2}P_{h-l}(n).
\end{aligned}$$

But

$$\sum_{j=1}^{n}[j^{h+1} - (j-1)^{h+1}] = n^{h+1},$$

and so

$$P_{h+1}(n) = \frac{n^{h+1}}{h+1} + Q(n), \tag{3}$$

where

$$Q(n) = \frac{1}{h+1}\sum_{l=0}^{h-1}(-1)^l\binom{h+1}{l+2}P_{h-l}(n). \tag{4}$$

Using Equation (4) and the induction hypothesis, we can see that $Q(n)$ is a polynomial of degree h. Thus, from Equation (3), $P_{h+1}(n)$ is a polynomial of degree $h+1$. Further, since $Q(n)/n^{h+1} \to 0$ as $n \to \infty$ we have $P_{h+1}(n)/n^{h+1} \to 1/(h+1)$ as $n \to \infty$. $\square$

Lemma 2. *Let k be a fixed positive integer and n be a nonnegative integer. Then*

$$I_{n+k}(k) = I_k(k) + \sum_{j=1}^{n} I_{j+k}(k-1).$$

Proof. The above formula is clear if $n = 0$, so let us assume that $n \geq 1$. Note that $1 \leq k \leq (n+k-i)-1$ for all $i = 0, 1, \ldots, n-1$. Using this and the recursive formula (2) we have

$$\begin{aligned}
I_{n+k}(k) &= I_{n+k}(k-1) + I_{n+k-1}(k) \\
&= I_{n+k}(k-1) + I_{n+k-1}(k-1) + I_{n+k-2}(k) \\
&= I_{n+k}(k-1) + I_{n+k-1}(k-1) + \cdots + I_{k+1}(k-1) + I_k(k) \\
&= \sum_{j=1}^{n} I_{j+k}(k-1) + I_k(k).
\end{aligned}$$

This completes the proof of the lemma. $\square$

Theorem 3. *If $k \geq 1$, then the sequence $\{I_{n+k}(k) : n \geq 0\}$ is a polynomial sequence of degree k and*

$$\lim_{n\to\infty} \frac{I_n(k)}{n^k} = \frac{1}{k!}.$$

Moreover, the leading term of $I_{n+k}(k)$ is $1/k!$.

Proof. Since $I_{n+1}(1) = n$ for all $n \geq 0$, the theorem trivially holds if $k = 1$. Assume that $I_{n+k}(k) = \sum_{i=0}^{k} a_{ki} n^i$, where $a_{kk} \neq 0$ in order for $I_{n+k}(k)$ to have degree k. Following [5] and using Lemma 2 we have

$$\begin{aligned} I_{n+k+1}(k+1) &= I_{k+1}(k+1) + \sum_{j=1}^{n} I_{j+1+k}(k) \\ &= C_{k+1} + \sum_{j=1}^{n}\sum_{i=0}^{k} a_{ki}(j+1)^i \\ &= C_{k+1} + \sum_{j=1}^{n}\sum_{i=0}^{k} a_{ki}\left(\sum_{h=0}^{i}\binom{i}{h} j^h\right) \\ &= C_{k+1} + \sum_{i=0}^{k}\sum_{h=0}^{i}\binom{i}{h} a_{ki} P_{h+1}(n), \end{aligned}$$

where $C_{k+1} = I_{k+1}(k+1)$. Using Lemma 1 it follows that $I_{n+k+1}(k+1)$ is a polynomial of degree $k+1$. Moreover, observe that

$$I_n(k+1) = C_{k+1} + \sum_{i=0}^{k}\sum_{h=0}^{i}\binom{i}{h} a_{ki} P_{h+1}(n-k-1),$$

for all $n \geq k+1$. Notice that $\lim_{n\to\infty} I_n(1)/n = 1$. Assume that $\lim_{n\to\infty} I_n(k)/n^k = 1/k!$, and so $a_{kk} = 1/k!$. From Lemma 1 we obtain

$$\lim_{n\to\infty} \frac{P_{h+1}(n-k-1)}{n^{k+1}} = \begin{cases} 0, & \text{if } 0 \leq h \leq k-1; \\ \dfrac{1}{k+1}, & \text{if } h = k. \end{cases}$$

Hence

$$\lim_{n\to\infty} \frac{I_n(k+1)}{n^{k+1}} = \lim_{n\to\infty} \frac{a_{kk} P_{k+1}(n-k-1)}{n^{k+1}} = \frac{a_{kk}}{k+1} = \frac{1}{(k+1)!}.$$

The 'moreover' part follows immediately. This establishes the theorem. $\square$

Using Lemma 2 and Faulhaber's formulas we have

$$\begin{aligned} I_{n+1}(1) &= n, \\ I_{n+2}(2) &= n(n+3)/2, \\ I_{n+3}(3) &= (n+3)(n^2+6n+2)/6, \\ I_{n+4}(4) &= (n+4)(n+5)(n^2+9n+6)/24, \\ I_{n+5}(5) &= (n+4)(n+11)(n^3+15n^2+66n+60)/120, \\ I_{n+6}(6) &= (n+5)(n+6)(n^4+34n^3+401n^2+1844n+2160)/720, \\ I_{n+7}(7) &= (n^7+63n^6+1645n^5+22995n^4+184534n^3+841302n^2+1983540n+1809360)/5040. \end{aligned}$$

Suppose that $I_{n+k}(k) = \sum_{i=0}^{k} a_{ki} n^i$. It can be shown that the constant term of the polynomial $P_{h+1}(n)$, where $h \geq 0$, is zero. Thus, we can write $P_{h+1}(n) = \sum_{j=1}^{h+1} p_{h+1,j} n^j$. From the proof of Theorem 3 we have

$$I_{n+k+1}(k+1) = I_{k+1}(k+1) + \sum_{i=0}^{k}\sum_{h=0}^{i}\sum_{j=1}^{h+1}\binom{i}{h} a_{ki} p_{h+1,j} n^j.$$

Therefore, if $I_{n+k+1}(k+1) = \sum_{i=0}^{k+1} a_{k+1,i} n^i$, then the coefficients of $I_{n+k+1}(k+1)$ is related to the coefficients of $I_{n+k}(k)$ and $P_{h+1}(n)$ and we have

$$a_{k+1,l} = \begin{cases} I_{k+1}(k+1), & \text{if } l = 0; \\ \displaystyle\sum_{i=l-1}^{k} \sum_{h=l-1}^{i} \binom{i}{h} a_{ki} p_{h+1,l}, & \text{if } 1 \leq l \leq k; \\ \dfrac{1}{(k+1)!}, & \text{if } l = k+1. \end{cases}$$

As a consequence of the previous theorem we have the following corollary.

Corollary 4. *For each real number x we have* $\displaystyle\sum_{j=0}^{\infty} \lim_{n\to\infty} I_n(j) \left(\frac{x}{n}\right)^j = e^x$.

From Euler's pentagonal number theorem we have

$$Q(z) = \prod_{j=1}^{\infty}(1 - z^j) = \sum_{i\in\mathbb{Z}} (-1)^i z^{i(3i-1)/2}.$$

Set $q_0 = Q(1/2), q_1 = Q'(1/2)$ and $q_2 = Q''(1/2)/2$.

Corollary 5. *For each $k, l \geq 1$,*

$$\frac{I_{n+l}(n)}{I_{n+k}(k)} = \frac{2^{2n+l-1}k!}{\sqrt{\pi}n^{k+1/2}} \left(q_0 - \frac{8q_0 l^2 + 2(q_1 - q_0)l + q_2 - 2q_1 + (1 + 8k! a_{k,k-1})q_0}{8n} + O(n^{-2}) \right).$$

Proof. If $k \geq 2$ then

$$\begin{aligned} I_{n+k}(k) &= \frac{n^k}{k!} \left(1 + \frac{a_{k,k-1}k!}{n} + \frac{1}{n^2} \sum_{i=0}^{k-2} \frac{a_{ki}k!}{n^{k-i-2}} \right) \\ &= \frac{n^k}{k!} \left(1 + \frac{a_{k,k-1}k!}{n} + O(n^{-2}) \right). \end{aligned}$$

If $k = 1$ then we have the same result. Combining this with the result of Louchard and Prodinger, which is

$$I_{n+l}(n) = \frac{2^{2n+l-1}}{\sqrt{\pi n}} \left(q_0 - \frac{8q_0 l^2 + 2(q_1 - q_0)l + q_2 - 2q_1 + q_0}{8n} + O(n^{-2}) \right),$$

we obtain the desired asymptotic formula. □

Let k and l be two fixed positive integers. We can see that after a sufficiently large number of terms, the sequence $\{I_{n+l}(n) : n \geq 0\}$ increases faster than the sequence $\{I_{n+k}(k) : n \geq 0\}$. Indeed, from Corollary 5

$$\lim_{n\to\infty} \frac{I_{n+k}(k)}{I_{n+l}(n)} = 0.$$

3 Inversion numbers and integrals

We will use Equation (1) to prove algebraically that the generating function of the sequence $\{I_n(k) : k = 0, 1, \ldots, \binom{n}{2}\}$ is

$$\Phi_n(x) = \sum_{k=0}^{\binom{n}{2}} I_n(k)x^k = \prod_{k=1}^{n} \sum_{i=0}^{k-1} x^i. \tag{5}$$

It can be easily verified that Equation (5) holds if $n = 1, 2$. Suppose $n \geq 3$. Then

$$\begin{aligned}
\Phi_{n-1}(x) \sum_{j=0}^{n-1} x^j &= \left(\sum_{i=0}^{\binom{n-1}{2}} I_{n-1}(i) x^i \right) \left(\sum_{j=0}^{n-1} x^j \right) \\
&= \sum_{k=0}^{\binom{n}{2}} \left(\sum_{i+j=k} I_{n-1}(i) \right) x^k \\
&= \sum_{k=0}^{\binom{n}{2}} \left(\sum_{i=\max\{0,k-n+1\}}^{\min\{k,\binom{n-1}{2}\}} I_{n-1}(i) \right) x^k.
\end{aligned}$$

From this, we get $\Phi_{n-1}(x) \sum_{j=0}^{n-1} x^j = \Phi_n(x)$. Using this and an induction argument proves (5).

For each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, where each α_i is a nonnegative integer, we define $\alpha! = \alpha_1! \cdots \alpha_n!$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The following lemma is the generalized Leibniz's rule for differentiation.

Lemma 6. *If $f_1, \ldots, f_n$ are analytic complex valued functions in an open set $U \subset \mathbb{C}$, then*

$$\frac{d^m}{dz^m} \prod_{j=1}^{n} f_j(z) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \prod_{j=1}^{n} f_j^{(\alpha_j)}(z)$$

for all $m \in \mathbb{N}$ and for all $z \in U$.

Proof. We prove the lemma by induction on m. Notice that the lemma is clear if $m = 1$. Suppose that the lemma holds for $m = k$. Now we show that the lemma is true for $m = k + 1$. Using the induction hypothesis we get

$$\begin{aligned}
\frac{d^{k+1}}{dz^{k+1}} \prod_{j=1}^{n} f_j(z) &= \frac{d}{dz} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \prod_{j=1}^{n} f_j^{(\alpha_j)}(z) \\
&= \sum_{|\alpha|=k} \sum_{|\beta|=1} \frac{k!}{\alpha!} \prod_{j=1}^{n} f_j^{(\alpha_j+\beta_j)}(z).
\end{aligned}$$

If $\beta_j = 1$ then

$$\frac{(k+1)!}{(\alpha+\beta)!} = \frac{k+1}{\alpha_j + \beta_j} \cdot \frac{k!}{\alpha!}.$$

Now, let $\gamma = \alpha + \beta$. Then $|\gamma| = |\alpha| + |\beta| = k + 1$ and

$$\begin{aligned}
\frac{d^{k+1}}{dz^{k+1}} \prod_{j=1}^{n} f_j(z) &= \sum_{|\gamma|=k+1} \sum_{j=1}^{n} \frac{\gamma_j}{k+1} \cdot \frac{(k+1)!}{\gamma!} \prod_{j=1}^{n} f_j^{(\gamma_j)}(z) \\
&= \sum_{|\gamma|=k+1} \frac{(k+1)!}{\gamma!} \prod_{j=1}^{n} f_j^{(\gamma_j)}(z).
\end{aligned}$$

This completes the proof of the lemma. $\square$

Theorem 7. *Let $f : D \subset \mathbb{C} \to \mathbb{C}$ and suppose that f is analytic in an open set $U \subset D$. If $m \in \mathbb{N}$ and $\mathcal{C}$ is a closed simple contour lying inside U and z_0 is any point interior to $\mathcal{C}$, then for all positive integer k we have*

$$\int_{\mathcal{C}} \frac{1}{(z-z_0)^{k+1}} \left(\prod_{l=1}^{m} \sum_{i=0}^{l-1} [f(z)]^i \right) dz = \frac{2\pi i}{k!} \sum_{1 \leq j \leq \binom{m}{2}} I_m(j) M_j(z_0), \tag{6}$$

where

$$M_j(z_0) = \sum_{|(\alpha_1,\ldots,\alpha_j)|=k} \frac{k!}{\alpha_1! \cdots \alpha_j!} f^{(\alpha_1)}(z_0) \cdots f^{(\alpha_j)}(z_0).$$

Proof. Letting $x = f(z)$ in Equation (5), dividing by $(z - z_0)^{k+1}$ and then integrating we get

$$\int_{\mathcal{C}} \frac{1}{(z - z_0)^{k+1}} \left(\prod_{l=1}^{m} \sum_{i=0}^{l-1} [f(z)]^i \right) dz = \sum_{0 \le j \le \binom{m}{2}} I_m(j) \int_{\mathcal{C}} \frac{[f(z)]^j}{(z - z_0)^{k+1}}\, dz. \tag{7}$$

By Cauchy's integral formula,

$$\int_{\mathcal{C}} \frac{[f(z)]^j}{(z - z_0)^{k+1}}\, dz = \begin{cases} 0, & \text{if } j = 0; \\ \dfrac{2\pi i}{k!} \cdot \dfrac{d^k([f(z_0)]^j)}{dz^k}, & \text{if } j \ge 1. \end{cases}$$

Using the generalized Leibniz's rule for differentiation we get

$$\frac{d^k([f(z_0)]^j)}{dz^k} = \sum_{|(\alpha_1, \ldots, \alpha_j)| = k} \frac{k!}{\alpha_1! \cdots \alpha_j!} f^{(\alpha_1)}(z_0) \cdots f^{(\alpha_j)}(z_0), \tag{8}$$

for all $j \ge 1$. Hence, Equation (6) follows from Equations (7) and (8). □

If we let $z_0 = 0$, $f(z) = z$ and $m = n + k$, we have the following corollary.

Corollary 8. *Let k be a fixed positive integer. Then for each nonnegative integer n we have*

$$\int_{\mathcal{C}} \frac{(1+z)(1+z+z^2)\cdots(1+z+\cdots+z^{n+k-1})}{z^{k+1}}\, dz = 2\pi i I_{n+k}(k),$$

where $\mathcal{C}$ is any simple closed contour containing the origin.

Example 9. Using the previous corollary we have

$$\begin{aligned} \int_{\mathcal{C}} \frac{(1+z)(1+z+z^2)\cdots(1+z+\cdots+z^n)}{z^2}\, dz &= 2n\pi i, \\ \int_{\mathcal{C}} \frac{(1+z)(1+z+z^2)\cdots(1+z+\cdots+z^{n+1})}{z^3}\, dz &= (n^2+3n)\pi i, \\ \int_{\mathcal{C}} \frac{(1+z)(1+z+z^2)\cdots(1+z+\cdots+z^{n+2})}{z^4}\, dz &= \frac{(n^3+9n^2+20n+6)\pi i}{3}, \end{aligned}$$

for all $n \ge 0$, where $\mathcal{C}$ is any closed contour containing the origin.

4 Hankel and inverse binomial transforms

Let $A = \{a_n\}_{n=0}^{\infty}$ be a sequence. The **inverse binomial transform** of the sequence A is the sequence denoted by $B^{-1}(A) = \{b_n\}_{n=1}^{\infty}$ where b_n is defined by the formula

$$b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k$$

for all $n \ge 0$. Let $H = [h_{ij}]_{i,j \in \mathbb{N}}$, where $h_{ij} = a_{i+j-2}$. Thus

$$H = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The **Hankel matrix** H_n of order n of the sequence A is defined to be the $(n+1) \times (n+1)$ upper left submatrix of H, that is, $H_n = [h_{ij}]_{1 \le i,j \le n+1}$. Let h_n denote the determinant of the Hankel matrix H_n of order n. The sequence $H(A) = \{h_n\}_{n=0}^{\infty}$ is called the **Hankel transform** of the sequence A.

Some properties of the Hankel transform are discussed in [3] and [7]. Further, Spivey and Steil [7] proved that the Hankel transform is invariant under falling k-binomial transform and since the inverse binomial transform is just a special type of a falling k-binomial transform, where $k = -1$, it follows that the Hankel transform is also invariant under inverse binomial transform. (For more details, we refer the reader to the work of Spivey and Steil [7].) Hence we have the following theorem.

Theorem 10. *If $A = \{a_n\}_{n=0}^{\infty}$ is a sequence, then $H(B^{-1}(A)) = H(A)$.*

Given a sequence $A = \{a_k\}_{k=0}^{\infty}$, the **support** of A is defined by $\operatorname{supp}(A) = \{k : a_k \neq 0\}$. The set of all real sequences having a finite support is denoted by c_{00}. Note that c_{00} is a vector space over $\mathbb{R}$ under the usual componentwise addition and scalar multiplication.

The next two theorems characterize the inverse binomial transform and the Hankel transform of the sequence $\{I_{n+k}(k) : n \geq 0\}$.

Theorem 11. *For each $k \geq 1$, let $A_k = \{I_{n+k}(k) : n \geq 0\}$ and $I_{n+k}(k) = \sum_{i=0}^{k} a_{ki} n^i$. Then $B^{-1}(A_k) = \{b_m\}_{m=0}^{\infty} \in c_{00}$ and $b_m = m! \sum_{i=m}^{k} a_{ki} S(i, m)$, for all $1 \leq m \leq k$, where $S(i, m)$ is a Stirling number of the second kind, and $b_m = 0$ for all $m > k$.*

Proof. Let $m \geq 1$. Using the definition, we have

$$b_m \quad = \quad a_{k0} \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} + \sum_{i=1}^{k} a_{ki} \left(\sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} n^i \right).$$

Note that we have

$$\sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} = 0$$

and

$$S(i, m) = \frac{1}{m!} \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} n^i,$$

for all $1 \leq m \leq i$. Define Δ_x by $\Delta_x = x \frac{d}{dx}$. Then for $1 \leq i < m$

$$\Delta_x^i (x-1)^m = \sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} n^i x^n.$$

Since $\Delta_x (x-1)^m = mx(x-1)^{m-1}$ then $(x-1)^{m-1}$ divides $\Delta_x (1-x)^m$. Suppose that $1 \leq i < m-1$ and $(x-1)^{m-i}$ divides $\Delta_x^i (x-1)^m$. Thus $\Delta_x^i (x-1)^m = (x-1)^{m-i} g_i(x)$ for some polynomial $g_i(x)$. Applying Δ_x once more, we get

$$\Delta_x^{i+1} (x-1)^m = x \frac{d[(x-1)^{m-i} g_i(x)]}{dx} = x(m-i)(x-1)^{m-i-1} g_i(x) + x(x-1)^{m-i} g_i'(x).$$

Hence $(x-1)^{m-(i+1)}$ divides $\Delta_x^{i+1}(x-1)^m$. This shows that for all $m > i \geq 1$, we can find a polynomial $g_i(x)$ satisfying $\Delta_x^i (x-1)^m = (x-1)^{m-i} g_i(x)$. If we let $x = 1$ we get

$$\sum_{n=0}^{m} (-1)^{m-n} \binom{m}{n} n^i = \Delta_x^i (x-1)^m \big|_{x=1} = 0$$

for all $m > i$. From these, we have

$$b_m = m! \sum_{i=m}^{k} a_{ki} S(i, m)$$

for all $1 \leq m \leq k$ and $b_m = 0$ for all $m > k$. Therefore $\{b_m\}_{m=0}^{\infty} \in c_{00}$. $\square$

Now, $b_k = k! a_{kk} S(k, k) = 1$. Therefore, the last nonzero term of $B^{-1}(A_k)$ is 1. Let $A_0 = \{I_n(0) : n \geq 1\}$. Then $A_0 = \{1, 1, \ldots\}$ and $B^{-1}(A_0) = \{1, 0, 0, \ldots\}$. From these it follows that $\{B^{-1}(A_k)\}_{k=0}^{\infty}$ forms a basis for c_{00}.

Example 12. Using the above theorem, we have

$$\begin{aligned}
B^{-1}(A_1) &= \{0, 1, 0, 0, 0, 0, 0, \ldots\}, \\
B^{-1}(A_2) &= \{0, 2, 1, 0, 0, 0, 0, \ldots\}, \\
B^{-1}(A_3) &= \{1, 5, 4, 1, 0, 0, 0, \ldots\}, \\
B^{-1}(A_4) &= \{5, 15, 14, 6, 1, 0, \ldots\}.
\end{aligned}$$

Theorem 13. *For each positive integer k, $H(A_k) \in c_{00}$. Furthermore, $\{H(A_k)\}_{k=0}^{\infty}$ forms a basis for c_{00}.*

Proof. First, note that $H(A_0) = H(B^{-1}(A_0)) = B^{-1}(A_0)$. Suppose $k \geq 1$. Let $B^{-1}(A_k) = \{b_m\}_{m=0}^{\infty}$ and $H(B^{-1}(A)) = \{h_n\}_{n=0}^{\infty}$. From the previous theorem, we have $b_m = 0$ for all $m > k$. Hence the $(m+1)$st row of the mth order Hankel matrix H_m has only zero entries for all $m > k$. Consequently, the determinant of of H_m is zero for all $m > k$. Therefore $h_m = 0$ for all $m > k$. Since $H(A) = H(B^{-1}(A))$, it follows that the Hankel transform of A_k lies in c_{00}. Consider the Hankel matrix of $B^{-1}(A_k)$ of order k. Then we have $h_{ij} = 0$ for all $i + j > k + 2$ and $h_{ij} = 1$ for all $i + j = k + 2$. It follows that $h_k = -1$ if $k \equiv 1, 2 \pmod 4$ and $h_k = 1$ if $k \equiv 0, 3 \pmod 4$. Therefore the last nonzero term of the Hankel transform of A_k is either -1 or 1. Consequently, $\{H(A_k)\}_{k=0}^{\infty}$ is a basis for c_{00}. □

Example 14. From Example 12, we get the following

$$\begin{aligned}
H(A_1) &= \{0, -1, 0, 0, 0, 0, 0, 0, 0, 0, \ldots\},\\
H(A_2) &= \{0, -4, -1, 0, 0, 0, 0, 0, 0, \ldots\},\\
H(A_3) &= \{1, -21, -25, 1, 0, 0, 0, 0, \ldots\},\\
H(A_4) &= \{5, -155, -559, 155, 1, 0, \ldots\}.
\end{aligned}$$

In general, one can similarly prove the following theorem.

Theorem 15. *For each $k \in \mathbb{N}$ let $p_k(n)$ be a polynomial in the variable n such that* $\deg p_k(n) = k$. *Let $A_k = \{p_k(n) : n \in \mathbb{N}\}$. Then $B^{-1}(A_k), H(A_k) \in c_{00}$. Furthermore $\{B^{-1}(A_k)\}_{k=0}^{\infty}$ or $\{H(A_k)\}_{k=0}^{\infty}$ forms a basis for c_{00}.*

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