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Optimal Control for the Stationary Navier–Stokes Equation Involving the Pressure, Stress and Their Pointwise Evaluations

OPTIMAL CONTROL FOR THE STATIONARY NAVIER–STOKES EQUATION INVOLVING THE PRESSURE, STRESS AND THEIR POINTWISE EVALUATIONS

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Abstract.
We consider optimal control problems for the two-dimensional stationary Navier–Stokes equation with cost functionals involving the pressure, stress, diffusion, and convection. Likewise, we study observations for the velocity, pressure, and stress, concentrated on a finite collection of points located either in the domain or on the boundary. Such observations are known to produce PDEs with measure data for the corresponding adjoint equation. Depending on the nature of the objective cost functional, the control set will be either the space of square-integrable functions without constraints, with constraints in Lebesgue spaces, or with weak derivatives. We prove generalized Green's theorems for the weak and very weak solutions that involve the trace and normal stress on the boundary. Finally, the local optimality systems for such control problems will be derived and the regularity of the optimal states will be established.

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1. Introduction

Consider an open, connected, and bounded domain $\Omega \subset \mathbb{R}^2$ with sufficiently smooth boundary Γ. We study optimal control problems of the form

$$\min_{(\mathbf{y},p,\mathbf{u})\in \mathbf{W}^{2,2}(\Omega)\times W^{1,2}(\Omega)\times \mathbf{L}^2(\Omega)} \mathcal{J}_0(\mathbf{y},p,\mathbf{u}) := J_0(\mathbf{y},p) + \frac{\rho}{2}\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \tag{1.1}$$

where for a given $\mathbf{u} \in \mathbf{L}^2(\Omega)$, the pair $(\mathbf{y},p) \in \mathbf{W}^{2,2}(\Omega) \times W^{1,2}(\Omega)$ is a solution to the two-dimensional stationary Navier–Stokes equation

$$\left[\begin{array}{l} -\nu\Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{u} \ \text{ in } \Omega, \\ \operatorname{div}\mathbf{y} = 0 \ \text{ in } \Omega, \quad \mathbf{y} = \mathbf{0} \ \text{ on } \Gamma, \quad \displaystyle\int_\Omega p\,\mathrm{d}x = 0. \end{array}\right. \tag{1.2}$$

The functions $\mathbf{y} : \Omega \to \mathbb{R}^2$ and $p : \Omega \to \mathbb{R}$ in (1.2) represent the velocity and pressure, respectively, and $\mathbf{u} : \Omega \to \mathbb{R}^2$ acts as a control distributed over the domain Ω. The detailed discussion on the notation and definition of function spaces as well as the review of the existence and regularity of solutions and the corresponding a priori estimates will be given in the succeeding section. We refer to Section 3 for the concrete definitions of the cost functionals.

We shall also consider (1.1) with control constraints, that is, we replace $\mathbf{L}^2(\Omega)$ with the control space

$$\mathbf{U}_{\text{ad}} := \{\mathbf{u} \in \mathbf{L}^2(\Omega) : \mathbf{a} \le \mathbf{u} \le \mathbf{b} \text{ a.e. } \Omega\}$$

for suitable $\mathbf{a},\mathbf{b} : \Omega \to \mathbb{R}^2$, taken as a closed subspace of $\mathbf{L}^2(\Omega)$. The relation $\le$ on $\mathbb{R}^2$ is to be understood componentwise. The control space $\mathbf{U}_{\text{ad}}$ with $\mathbf{a},\mathbf{b} \in \mathbf{L}^s(\Omega)$ where $2 < s < \infty$ will be appropriate in dealing with functionals that involve point evaluations of the pressure or the normal stress on the boundary. In this case, it holds that $\mathbf{U}_{\rm ad} \subset \mathbf{L}^s(\Omega)$ and it can be shown from regularity theory for the Stokes equation that $(\mathbf{y}, p) \in \mathbf{W}^{2,s}(\Omega) \times W^{1,s}(\Omega)$ (see Subsection 2.2 for the details). With this, aside from (1.1), we also study minimization problems of the form

$$\min_{(\mathbf{y},p,\mathbf{u})\in \mathbf{W}^{2,s}(\Omega)\times W^{1,s}(\Omega)\times \mathbf{U}_{\rm ad}} \mathcal{J}_1(\mathbf{y}, p, \mathbf{u}) := J_1(\mathbf{y}, p) + \frac{\rho}{2}\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \tag{1.3}$$

subject to (1.2).

Alternatively, if one wishes to avoid control constraints, then one can utilize controls that are weakly differentiable. More precisely, for the control space $\mathbf{W}^{1,2}(\Omega)$, we also consider optimal control problems of the form

$$\min_{(\mathbf{y},p,\mathbf{u})\in \mathbf{W}^{3,2}(\Omega)\times W^{2,2}(\Omega)\times \mathbf{W}^{1,2}(\Omega)} \mathcal{J}_2(\mathbf{y}, p, \mathbf{u}) := J_2(\mathbf{y}, p) + \frac{\rho}{2}\|\mathbf{u}\|^2_{\mathbf{W}^{1,2}(\Omega)} \tag{1.4}$$

subject to (1.2). The optimal control problem (1.4) can be viewed as a regularization of (1.1) if we equip $\mathbf{W}^{1,2}(\Omega)$ with the norm $\|\mathbf{u}\|_{\mathbf{W}^{1,2}(\Omega)} = (\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \varepsilon\|\nabla\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)^2})^{1/2}$, where $\varepsilon > 0$ is a given small parameter. Here, we shall take $\varepsilon = 1$ for simplicity as we are not interested on the asymptotic behavior of the solutions when $\varepsilon \to 0$. Specific and precise definitions of (1.1), (1.3), and (1.4) will be given in Section 3.

Note that $p(x)$, for $x \in \bar{\Omega}$, is well-defined as soon as $p \in W^{1,s}(\Omega)$ or $p \in W^{2,2}(\Omega)$, thanks to the Sobolev embeddings $W^{1,s}(\Omega) \subset C(\bar{\Omega})$ for $2 < s < \infty$ and $W^{2,2}(\Omega) \subset C(\bar{\Omega})$. Hence, (1.3) and (1.4) are suitable for problems with point observations of the pressure.

The cost functional J_k will include derivatives of the velocity, such as the gradient, Laplacian, or convection within the domain, the derivative normal to the boundary, or point evaluations within the domain or on the boundary. With respect to the pressure, we also investigate the interior, boundary, or point tracking problems. For optimal control problems of the Navier–Stokes equation, the pressure is typically not considered in the vast existing literature, particularly for boundary and point observations of the pressure. The analysis of optimal control problems involving the pressure is the main contribution of the current work.

The Cauchy stress

$$\mathbf{T}(\mathbf{y}, p) := -\nu\nabla\mathbf{y} + p\mathbf{I}$$

within the domain or its associated conormal derivative

$$\mathbf{T}(\mathbf{y}, p)\mathbf{n} = -\nu\partial_{\mathbf{n}}\mathbf{y} + p\mathbf{n}$$

on the boundary, where $\mathbf{n}$ is the unit normal outward on Γ with respect to Ω, will be examined in this manuscript as well. In order for the observations of the states on the boundary or at points in the domain to be well-defined, suitable regularity is needed, hence there is a need to consider smooth controls. As mentioned above, smoothness can be achieved by either imposing control constraints or by allowing weak differentiability of the controls.

Pointwise tracking leads to the investigation of linear PDEs with measure data. For example, optimality systems and finite element approximations for linear elliptic, semilinear elliptic, and the Stokes equations have been examined in [8], [2], and [6, 7], respectively. When considering point-evaluations of the pressure or the normal Cauchy stress tensor at points on the boundary in the cost functional, we must take a space larger than the space of regular Borel measures. One framework in the study of PDEs with measure data involves reformulating the problem in suitable negative-indexed Sobolev spaces and resorting to duality arguments.

In general, the adjoint equation to the optimal control problems we consider here will be a Stokes-type equation having a non-homogenous source, divergence, or boundary conditions. In particular, for problems with measure data, the notion of very weak solutions can be employed. Such solutions for the Stokes, Oseen, and Navier–Stokes equations have been thoroughly investigated and extensively studied over the past decades (see, for instance, [5, 4, 14, 16, 18, 21, 22], and the references therein).

In this work, we deal with the optimal control problems (1.1), (1.3) and (1.4) having the specific structures given in Section 3, all subject to (1.2), locally at regular solutions. Roughly speaking, regular solutions are those solutions that yield topological isomorphisms for the linearized operators, and consequently, leading to uniqueness of solutions in a certain neighborhood. In this way, a control-to-state mapping can be defined locally. This framework was introduced in [10] to study Borel measure controls to the stationary Navier–Stokes equation (1.2). We adapt the method presented in the latter paper to the linearized problem around an optimal regular solution, wherein we demonstrate the existence of two mutually exclusive (Fredholm) alternatives: the associated operator is either an isomorphism or has a nontrivial kernel. Earlier works that utilized the notion of regular or non-singular solutions for the steady-state Navier–Stokes equation can be found in [17], where such concept was used as a framework for numerical approximations, and in [11] as an application to optimal control problems with distributed control. We emphasize here that our results are valid provided that appropriate smallness conditions on the data are satisfied, namely, the smallness of the Reynolds' number or the norm of the controls. Such conditions will guarantee the existence and uniqueness of the so-called regular solutions.

The structure of the paper is as follows: Section 2 deals with the well-posedness of the state, linearized state, and adjoint equations. Moreover, we prove that weak and very weak solutions in L^p-spaces for $1 < p < \infty$ to the Stokes equation possess a well-defined normal Cauchy stress on the boundary, extending the results in the Hilbertian case presented in [21, 22]. Section 3 focuses on the optimality systems and the regularity of the optimal solutions of the localized problems around regular solutions.

2. Analysis of state, linearized state and adjoint equations

2.1. Function spaces.

Let $1 \le s \le \infty$ and $s' = \frac{s}{s-1}$ be the Hölder conjugate when $1 < s < \infty$. The Lebesgue spaces and Sobolev spaces will be denoted by $L^s(\Omega)$ and $W^{r,s}(\Omega)$, respectively [1]. For the vector case, we set $\mathbf{L}^s(\Omega) := L^s(\Omega) \times L^s(\Omega)$ and $\mathbf{W}^{r,s}(\Omega) := W^{r,s}(\Omega) \times W^{r,s}(\Omega)$. The set of all elements in $\mathbf{W}^{1,s}(\Omega)$ that vanish on the boundary in the sense of traces will be denoted by $\mathbf{W}^{1,s}_0(\Omega)$. We let $\mathbf{X}^{2,s}(\Omega) := \mathbf{W}^{1,s}_0(\Omega) \cap \mathbf{W}^{2,s}(\Omega)$,

$$\widehat{L}^s(\Omega) := \left\{ p \in L^s(\Omega) : \int_\Omega p \,\mathrm{d}x = 0 \right\},$$

and $\widehat{W}^{1,s}(\Omega) := W^{1,s}(\Omega) \cap \widehat{L}^s(\Omega)$.

With regard to divergence-free vector fields, we use the following notations:

$$\mathbf{L}^s_\sigma(\Omega) := \{\mathbf{y} \in \mathbf{L}^s(\Omega) : \operatorname{div} \mathbf{y} = 0 \text{ in } \Omega, \ \mathbf{y} \cdot \mathbf{n} = 0 \text{ on } \Gamma\},$$
$$\mathbf{V}^{1,s}(\Omega) := \{\mathbf{y} \in \mathbf{W}^{1,s}_0(\Omega) : \operatorname{div} \mathbf{y} = \mathbf{0} \text{ in } \Omega\},$$
$$\mathbf{V}^{2,s}(\Omega) := \{\mathbf{y} \in \mathbf{X}^{2,s}(\Omega) : \operatorname{div} \mathbf{y} = \mathbf{0} \text{ in } \Omega\}.$$

In the definition of $\mathbf{L}^s_\sigma(\Omega)$, the equation $\mathbf{y} \cdot \mathbf{n} = 0$ on Γ is taken in the sense of $W^{-\frac{1}{s},s}(\Gamma)$, see [23, Lemma I.1.2.2] for instance. We denote the associated norms by

$$\|\mathbf{u}\|_{\mathbf{L}^s_\sigma(\Omega)} := \|\mathbf{u}\|_{\mathbf{L}^s(\Omega)},$$
$$\|\mathbf{v}\|_{\mathbf{V}^{1,s}(\Omega)} := \|\nabla \mathbf{v}\|_{\mathbf{L}^s(\Omega)^2},$$
$$\|\mathbf{w}\|_{\mathbf{V}^{2,s}(\Omega)} := \|\Delta \mathbf{w}\|_{\mathbf{L}^s(\Omega)},$$

for $\mathbf{u} \in \mathbf{L}^s_\sigma(\Omega)$, $\mathbf{v} \in \mathbf{V}^{1,s}(\Omega)$, and $\mathbf{w} \in \mathbf{V}^{2,s}(\Omega)$.

The dual spaces will be denoted with a negative order, that is, we shall write $\mathbf{X}^{-2,s}(\Omega) := \mathbf{X}^{2,s'}(\Omega)'$, $\mathbf{W}^{-1,s}(\Omega) := \mathbf{W}^{1,s'}_0(\Omega)'$, and $\mathbf{V}^{-k,s}(\Omega) := \mathbf{V}^{k,s'}(\Omega)'$ for $k = 1, 2$. For the spaces $W^{k,s'}(\Omega)$ and $\widehat{W}^{k,s'}(\Omega)$, the respective dual spaces will be denoted by the usual notation $W^{k,s'}(\Omega)'$ and $\widehat{W}^{k,s'}(\Omega)'$. Similarly, we set $\mathbf{W}^{-\sigma,s}(\Gamma) := \mathbf{W}^{\sigma,s'}(\Gamma)'$ for $\sigma > 0$. We also set

$$\mathbf{L}^s_{\mathrm{div}}(\Omega) := \{\mathbf{u} \in \mathbf{L}^s(\Omega) : \operatorname{div} \mathbf{u} \in L^s(\Omega)\}$$

endowed with the graph norm $\|\mathbf{u}\|_{\mathbf{L}^s_{\mathrm{div}}(\Omega)} := \|\mathbf{u}\|_{\mathbf{L}^s(\Omega)} + \|\operatorname{div} \mathbf{u}\|_{L^s(\Omega)}$.

The space of bounded linear operators from a Banach space X into a Banach space Y is denoted by $\mathcal{L}(X, Y)$ and $\mathcal{L}(X) := \mathcal{L}(X, X)$. The collection of isomorphisms in $\mathcal{L}(X, Y)$ will be written as $\mathcal{L}_{\mathrm{iso}}(X, Y)$. Here, by an isomorphism we mean a topological one, that is, the bounded linear operator is invertible, hence, has a bounded inverse according to the open mapping theorem.

Recall that the well-posedness for the Stokes equation with non-homogeneous divergence and boundary conditions requires compatibility conditions. However, the data in the observations may not satisfy such conditions, hence, there is a need to achieve compatibility. In this direction, let $t, r \in \mathbb{R}$ and $1 < s < \infty$. To have a single definition on the spaces associated with the compatibility conditions, we introduce the notation

$$F^{t,s}(\Omega) := \begin{cases} W^{t,s}(\Omega) & \text{if } t \geq 0, \\ W^{-t,s'}(\Omega)' & \text{if } t < 0. \end{cases} \tag{2.1}$$

Same definition will be applied to $\widehat{F}^{t,s}(\Omega)$ where we replace W with $\widehat{W}$, and to $\mathbf{F}^{r,s}(\Gamma)$ where we replace t with r, W with $\mathbf{W}$, and Ω with Γ.

Given $g \in F^{t,s}(\Omega)$ and $\mathbf{h} \in \mathbf{F}^{r,s}(\Gamma)$, we set

$$\langle g, 1\rangle_\Omega := \begin{cases} \displaystyle\int_\Omega g \,\mathrm{d}x & \text{if } t \geq 0, \\ \langle g, 1\rangle_{W^{-t,s'}(\Omega)', W^{-t,s'}(\Omega)} & \text{if } t < 0, \end{cases}$$

$$\langle \mathbf{h}, \mathbf{n}\rangle_\Gamma := \begin{cases} \displaystyle\int_\Gamma \mathbf{h} \cdot \mathbf{n} \,\mathrm{d}s & \text{if } r \geq 0, \\ \langle \mathbf{h}, \mathbf{n}\rangle_{W^{-r,s'}(\Gamma)', W^{-r,s'}(\Gamma)} & \text{if } r < 0. \end{cases}$$

Then, we consider the function space

$$\mathbf{Z}^{t,r,s}(\Omega,\Gamma) := \left\{(g,\mathbf{h}) \in F^{t,s}(\Omega)\times \mathbf{F}^{r,s}(\Gamma) : \langle g,1\rangle_\Omega + \langle \mathbf{h},\mathbf{n}\rangle_\Gamma = 0\right\}$$

taken as a closed subspace of $F^{t,s}(\Omega)\times \mathbf{F}^{r,s}(\Gamma)$. The dual space of $\mathbf{Z}^{t,r,s}(\Omega,\Gamma)$ with respect to the pivot space $\mathbf{Z}^{0,0,2}(\Omega,\Gamma)$ is given by $\mathbf{Z}^{t,r,s}(\Omega,\Gamma)' = \mathbf{Z}^{-t,-r,s'}(\Omega,\Gamma)$.

Define the bounded linear operator $\Pi : F^{t,s}(\Omega)\times \mathbf{F}^{r,s}(\Gamma) \to F^{t,s}(\Omega)$ by

$$\Pi(g,\mathbf{h}) := g - \frac{1}{|\Omega|}(\langle g,1\rangle_\Omega + \langle \mathbf{h},\mathbf{n}\rangle_\Gamma).$$

Then, it follows that

$$(\Pi(g,\mathbf{h}),\mathbf{h}) \in \mathbf{Z}^{t,r,s}(\Omega,\Gamma) \quad \forall (g,\mathbf{h}) \in F^{t,s}(\Omega)\times \mathbf{F}^{r,s}(\Gamma). \tag{2.2}$$

Likewise, we introduce the bounded linear operators $\Lambda : F^{t,s}(\Omega) \to \widehat{F}^{t,s}(\Omega)$ and $\Sigma : \mathbf{F}^{r,s}(\Gamma) \to \mathbb{R}$ by

$$\Lambda g := \Pi(g,\mathbf{0}) = g - \frac{1}{|\Omega|}\langle g,1\rangle_\Omega,$$
$$\Sigma\mathbf{h} := \Pi(0,\mathbf{h}) = -\frac{1}{|\Omega|}\langle \mathbf{h},\mathbf{n}\rangle_\Gamma.$$

By definition, it is clear that $\Lambda g \in \widehat{F}^{t,s}(\Omega)$ and $(\Sigma\mathbf{h},\mathbf{h}) \in \mathbf{Z}^{t,r,s}(\Omega,\Gamma)$ for every $g \in F^{t,s}(\Omega)$ and $\mathbf{h} \in \mathbf{F}^{r,s}(\Gamma)$.

2.2. Strong solutions to the state equation. In this subsection, we recall briefly the well-posedness theory and the a priori estimates for the solutions of the stationary Navier–Stokes equation (1.2) for a given control $\mathbf{u}$. In what follows, c or with a subscript will denote a generic positive constant independent on the state and control variables, unless stated otherwise.

Suppose that $\mathbf{u} \in \mathbf{W}^{-1,2}(\Omega)$. Then, (1.2) admits a weak solution $(\mathbf{y},p) \in \mathbf{V}^{1,2}(\Omega)\times \widehat{L}^2(\Omega)$ satisfying the a priori estimates

$$\|\mathbf{y}\|_{\mathbf{V}^{1,2}(\Omega)} \le \frac{1}{\nu}\|\mathbf{u}\|_{\mathbf{W}^{-1,2}(\Omega)}, \tag{2.3}$$
$$\|p\|_{\widehat{L}^2(\Omega)} \le c\left(\|\mathbf{u}\|_{\mathbf{W}^{-1,2}(\Omega)} + \frac{1}{\nu^2}\|\mathbf{u}\|^2_{\mathbf{W}^{-1,2}(\Omega)}\right),$$

for some constant $c > 0$ independent of $\mathbf{y}$, p, $\mathbf{u}$, and ν, see [15, Theorems IX.3.1].

If $\mathbf{u} \in \mathbf{L}^2(\Omega)$, then $(\mathbf{y},p) \in \mathbf{V}^{2,2}(\Omega)\times \widehat{W}^{1,2}(\Omega)$. Let us derive a priori estimates where the dependence on the viscosity coefficient ν is explicitly stated. For this, we rewrite (1.2) as follows:

$$\left[\begin{array}{l} -\nu\Delta\mathbf{y} + \nabla p = \mathbf{u} - (\mathbf{y}\cdot\nabla)\mathbf{y} \ \text{ in } \Omega, \\ \operatorname{div}\mathbf{y} = 0 \ \text{ in } \Omega, \quad \mathbf{y} = \mathbf{0} \ \text{ on } \Gamma, \quad \displaystyle\int_\Omega p\,\mathrm{d}x = 0. \end{array}\right. \tag{2.4}$$

Invoking the a priori estimate for the Stokes equation to (2.4), see [23, Theorem III.1.5.3] for instance, we obtain

$$\|\mathbf{y}\|_{\mathbf{V}^{2,2}(\Omega)} + \frac{1}{\nu}\|p\|_{\widehat{W}^{1,2}(\Omega)} \le \frac{c}{\nu}(\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} + \|(\mathbf{y}\cdot\nabla)\mathbf{y}\|_{\mathbf{L}^2(\Omega)}), \tag{2.5}$$

where $c > 0$ denotes a generic constant independent of ν. Using the Hölder and Ladyzhenkaya inequalities, the continuity of the embedding $\mathbf{V}^{1,2}(\Omega) \subset \mathbf{L}^4(\Omega)$, and Young inequality, we get

$$\begin{aligned}\|(\mathbf{y}\cdot\nabla)\mathbf{y}\|_{\mathbf{L}^2(\Omega)} &\le \|\mathbf{y}\|_{\mathbf{L}^4(\Omega)}\|\nabla\mathbf{y}\|_{\mathbf{L}^4(\Omega)^2}\\ &\le c\|\mathbf{y}\|^{3/2}_{\mathbf{V}^{1,2}(\Omega)}\|\mathbf{y}\|^{1/2}_{\mathbf{V}^{2,2}(\Omega)}\\ &\le \frac{c}{2\nu}\|\mathbf{y}\|^3_{\mathbf{V}^{1,2}(\Omega)} + \frac{\nu}{2c}\|\mathbf{y}\|_{\mathbf{V}^{2,2}(\Omega)}.\end{aligned}$$

Plugging the last estimate in (2.5), applying (2.3), and using the continuity of $\mathbf{L}^2(\Omega) \subset \mathbf{W}^{-1,2}(\Omega)$, we deduce that

$$\|\mathbf{y}\|_{\mathbf{V}^{2,2}(\Omega)} + \frac{1}{\nu}\|p\|_{\widehat{W}^{1,2}(\Omega)} \le \frac{c}{\nu}\left(\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} + \frac{1}{\nu^4}\|\mathbf{u}\|^3_{\mathbf{L}^2(\Omega)}\right). \tag{2.6}$$

Assume that $\mathbf{a}, \mathbf{b} \in \mathbf{L}^s(\Omega)$ where $2 < s < \infty$. If $\mathbf{u} \in \mathbf{U}_{\mathrm{ad}}$, then $\mathbf{u} \in \mathbf{L}^s(\Omega)$ and it holds that

$$\|\mathbf{u}\|_{\mathbf{L}^s(\Omega)} \le c_s(\|\mathbf{a}\|_{\mathbf{L}^s(\Omega)} + \|\mathbf{b}\|_{\mathbf{L}^s(\Omega)}) \tag{2.7}$$

for some (generic) constant $c_s > 0$, thanks to the inequalities $|\mathbf{u}| \le \max\{|\mathbf{a}|, |\mathbf{b}|\} \le |\mathbf{a}| + |\mathbf{b}|$ almost everywhere in Ω. Thus, we have

$$\|(\mathbf{y}\cdot\nabla)\mathbf{y}\|_{\mathbf{L}^s(\Omega)} \le \|\mathbf{y}\|_{\mathbf{L}^\infty(\Omega)}\|\nabla\mathbf{y}\|_{\mathbf{L}^s(\Omega)^2} \le c\|\mathbf{y}\|^2_{\mathbf{V}^{2,2}(\Omega)}, \tag{2.8}$$

thanks to the continuity of $\mathbf{V}^{2,2}(\Omega) \subset \mathbf{L}^\infty(\Omega) \cap \mathbf{W}^{1,s}(\Omega)$ due to the Sobolev embedding theorem. Regularity theory for the Stokes equation [3, 12] in general Lebesgue spaces, (2.6), and (2.8) lead to $(\mathbf{y}, p) \in \mathbf{V}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega)$ and

$$\begin{aligned}\|\mathbf{y}\|_{\mathbf{V}^{2,s}(\Omega)} + \frac{1}{\nu}\|p\|_{\widehat{W}^{1,s}(\Omega)} &\le \frac{c}{\nu}(\|\mathbf{u}\|_{\mathbf{L}^s(\Omega)} + \|(\mathbf{y}\cdot\nabla)\mathbf{y}\|_{\mathbf{L}^s(\Omega)})\\ &\le \frac{c}{\nu}\left(\|\mathbf{u}\|_{\mathbf{L}^s(\Omega)} + \frac{1}{\nu^2}\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{\nu^{10}}\|\mathbf{u}\|^6_{\mathbf{L}^2(\Omega)}\right).\end{aligned} \tag{2.9}$$

Without explicit dependence on ν, this a priori estimate can also be derived by applying the well-known result for the stationary Navier–Stokes equation, see [15, Theorem IX.5.2].

According to Sobolev embedding theorem [13, Section 5.6.3], $\mathbf{V}^{2,s}(\Omega) \subset \mathbf{X}^{2,s}(\Omega) \subset \mathbf{C}_0(\Omega) \cap \mathbf{C}^{1,1-\frac{2}{s}}(\bar{\Omega})$ and $\widehat{W}^{1,s}(\Omega) \subset C^{0,1-\frac{2}{s}}(\bar{\Omega})$ continuously. Therefore, we deduce from (2.9) that

$$\begin{aligned}&\|\mathbf{y}\|_{\mathbf{C}_0(\Omega)\cap\mathbf{C}^{1,1-\frac{2}{s}}(\bar{\Omega})} + \frac{1}{\nu}\|p\|_{C^{0,1-\frac{2}{s}}(\bar{\Omega})}\\ &\quad\le \frac{c}{\nu}\left(\|\mathbf{u}\|_{\mathbf{L}^s(\Omega)} + \frac{1}{\nu^2}\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{\nu^{10}}\|\mathbf{u}\|^6_{\mathbf{L}^2(\Omega)}\right).\end{aligned}$$

In particular, since $\mathbf{C}^{1,1-\frac{2}{s}}(\bar{\Omega}) \subset \mathbf{C}^1(\bar{\Omega})$ and $C^{0,1-\frac{2}{s}}(\bar{\Omega}) \subset C(\bar{\Omega})$ continuously, the following a priori estimate holds:

$$\|\mathbf{y}\|_{\mathbf{C}_0(\Omega)\cap\mathbf{C}^1(\bar{\Omega})} + \frac{1}{\nu}\|p\|_{C(\bar{\Omega})} \le \frac{c}{\nu}\left(\|\mathbf{u}\|_{\mathbf{L}^s(\Omega)} + \frac{1}{\nu^2}\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{\nu^{10}}\|\mathbf{u}\|^6_{\mathbf{L}^2(\Omega)}\right).$$

Suppose that we have a control $\mathbf{u} \in \mathbf{W}^{1,2}(\Omega)$. Then, by the Hölder's inequality and the continuous embedding $\mathbf{V}^{2,2}(\Omega) \subset \mathbf{L}^\infty(\Omega) \cap \mathbf{W}^{1,4}(\Omega)$, we get

$$\begin{aligned}\|\nabla((\mathbf{y}\cdot\nabla)\mathbf{y})\|_{\mathbf{L}^2(\Omega)^2} &\le c(\|\nabla\mathbf{y}\|^2_{\mathbf{L}^4(\Omega)^2} + \|\mathbf{y}\|_{\mathbf{L}^\infty(\Omega)}\|\nabla^2\mathbf{y}\|_{\mathbf{L}^2(\Omega)^{2\times 2}}) \\ &\le c\|\mathbf{y}\|^2_{\mathbf{V}^{2,2}(\Omega)}. \end{aligned} \tag{2.10}$$

Invoking the regularity theory for the Stokes equation [9, Theorem IV.5.8], (2.8), and (2.10), we obtain

$$\begin{aligned}\|\mathbf{y}\|_{\mathbf{V}^{3,2}(\Omega)} + \frac{1}{\nu}\|p\|_{\widehat{W}^{2,2}(\Omega)} &\le \frac{c}{\nu}(\|\mathbf{u}\|_{\mathbf{W}^{1,2}(\Omega)} + \|(\mathbf{y}\cdot\nabla)\mathbf{y}\|_{\mathbf{W}^{1,2}(\Omega)}) \\ &\le \frac{c}{\nu}\left(\|\mathbf{u}\|_{\mathbf{W}^{1,2}(\Omega)} + \frac{1}{\nu^2}\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{\nu^{10}}\|\mathbf{u}\|^6_{\mathbf{L}^2(\Omega)}\right).\end{aligned}$$

Consequently, by the continuity of the embeddings $\mathbf{V}^{3,2}(\Omega) \subset \mathbf{C}_0(\Omega) \cap \mathbf{C}^1(\bar{\Omega})$ and $\widehat{W}^{2,2}(\Omega) \subset C(\bar{\Omega})$, it holds that

$$\|\mathbf{y}\|_{\mathbf{C}_0(\Omega)\cap\mathbf{C}^1(\bar{\Omega})} + \frac{1}{\nu}\|p\|_{C(\bar{\Omega})} \le \frac{c}{\nu}\left(\|\mathbf{u}\|_{\mathbf{W}^{1,2}(\Omega)} + \frac{1}{\nu^2}\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{\nu^{10}}\|\mathbf{u}\|^6_{\mathbf{L}^2(\Omega)}\right).$$

Take note that the results for the cases where the controls belong to $\mathbf{U}_{\text{ad}}$ or $\mathbf{W}^{1,2}(\Omega)$ yield that point-evaluations involving the pressure and the gradient of the velocity are well-defined. As a result, observations involving such quantities can be studied.

2.3. Well-Posedness of the Linearized and Adjoint Systems.

In this subsection, we analyze the linearized version of (1.2) and the associated dual problem for this linearized system. Specifically, we consider the linearized system with non-homogenous divergence and boundary conditions and apply duality arguments to derive the associated dual problem having similar non-homogeneities.

The linearized problem around a reference state $\mathbf{y}$ is given by

$$\left[\begin{array}{l} -\nu\Delta\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{w} + \nabla\pi = \mathbf{r} \ \text{ in } \Omega, \\ -\operatorname{div}\mathbf{w} = q \ \text{ in } \Omega, \quad \mathbf{w} = \mathbf{z} \ \text{ on } \Gamma, \quad \displaystyle\int_\Omega \pi\,\mathrm{d}x = 0, \end{array}\right. \tag{2.11}$$

where $\mathbf{r}$, q, and $\mathbf{z}$ are suitable data with the compatibility condition

$$\int_\Omega q\,\mathrm{d}x + \int_\Gamma \mathbf{z}\cdot\mathbf{n}\,\mathrm{d}s = 0. \tag{2.12}$$

In the case of less regular q or $\mathbf{z}$, these integrals should be replaced by duality pairings. Note that the compatibility condition (2.12) is a consequence of the second and third equations in (2.11) and the divergence theorem.

The dual problem corresponding to (2.11) is given by

$$\left[\begin{array}{l} -\nu\Delta\mathbf{v} + (\nabla\mathbf{y})^\top\mathbf{v} - (\mathbf{y}\cdot\nabla)\mathbf{v} + \nabla\sigma = \mathbf{f} \ \text{ in } \Omega, \\ -\operatorname{div}\mathbf{v} = g \ \text{ in } \Omega, \quad \mathbf{v} = \mathbf{h} \ \text{ on } \Gamma, \quad \displaystyle\int_\Omega \sigma\,\mathrm{d}x = 0, \end{array}\right. \tag{2.13}$$

for appropriate $\mathbf{f}$, g, and $\mathbf{h}$ with the compatibility condition

$$\int_\Omega g\,\mathrm{d}x + \int_\Gamma \mathbf{h}\cdot\mathbf{n}\,\mathrm{d}s = 0. \tag{2.14}$$

For smooth enough solutions $(\mathbf{w},\pi)$ for (2.11) and $(\mathbf{v},\sigma)$ for (2.13), integration by parts leads to the following equation:

$$\begin{aligned}&\int_\Omega \mathbf{f}\cdot\mathbf{w}\,\mathrm{d}x + \int_\Omega g\pi\,\mathrm{d}x + \int_\Gamma \mathbf{h}\cdot\mathbf{T}(\mathbf{w},\pi)\mathbf{n}\,\mathrm{d}s\\ &\quad= \int_\Omega \mathbf{v}\cdot\mathbf{r}\,\mathrm{d}x + \int_\Omega \sigma q\,\mathrm{d}x + \int_\Gamma \mathbf{T}(\mathbf{v},\sigma)\mathbf{n}\cdot\mathbf{z}\,\mathrm{d}s. \qquad (2.15)\end{aligned}$$

In particular, when $\mathbf{z} = \mathbf{0}$, (2.16) reduces to

$$\int_\Omega \mathbf{f}\cdot\mathbf{w}\,\mathrm{d}x + \int_\Omega g\pi\,\mathrm{d}x + \int_\Gamma \mathbf{h}\cdot\mathbf{T}(\mathbf{w},\pi)\mathbf{n}\,\mathrm{d}s = \int_\Omega \mathbf{v}\cdot\mathbf{r}\,\mathrm{d}x + \int_\Omega \sigma q\,\mathrm{d}x. \qquad (2.16)$$

This equation motivates the definition of very weak solutions to (2.13) with rough data (see Definition 2.11 below) and the conversion of the required compatibility of the data arising from the cost functional.

Sufficient regularity assumptions for the validity of the equation (2.15) are as follows: $\mathbf{y}\in\mathbf{V}^{2,2}(\Omega)$, $\mathbf{r}\in\mathbf{L}^s(\Omega)$, $q\in W^{1,s}(\Omega)$, $\mathbf{z}\in\mathbf{W}^{2-\frac{1}{s},s}(\Gamma)$, $\mathbf{f}\in\mathbf{L}^{s'}(\Omega)$, $g\in W^{1,s'}(\Omega)$, and $\mathbf{h}\in\mathbf{W}^{2-\frac{1}{s'},s'}(\Gamma)$, where $1<s<\infty$. Consequently, we have $\mathbf{w}\in\mathbf{W}^{2,s}(\Omega)$, $\pi\in\widehat{W}^{1,s}(\Omega)$, $\mathbf{v}\in\mathbf{W}^{2,s'}(\Omega)$, and $\sigma\in\widehat{W}^{1,s'}(\Omega)$, and hence, $\mathbf{T}(\mathbf{w},\pi)\mathbf{n}\in\mathbf{W}^{1-\frac{1}{s},s}(\Gamma)$, and $\mathbf{T}(\mathbf{v},\sigma)\mathbf{n}\in\mathbf{W}^{1-\frac{1}{s'},s'}(\Gamma)$. For less regular $\mathbf{f}$, g, or $\mathbf{h}$, the integrals appearing on the left-hand side of (2.15) will be replaced by duality pairings. In particular, (2.15) will be generalized in Theorem 2.14 and Theorem 2.15.

Given a fixed $\mathbf{y}\in\mathbf{V}^{2,2}(\Omega)$, consider the bounded linear operator

$$\mathfrak{A}_\mathbf{y} : \mathbf{W}^{1,s}_0(\Omega)\times\widehat{L}^s(\Omega)\to\mathbf{W}^{-1,s}(\Omega)\times\widehat{L}^s(\Omega)$$

defined by

$$\begin{aligned}\mathfrak{A}_\mathbf{y}(\mathbf{w},\pi) &:= (A_\mathbf{y}(\mathbf{w},\pi), -\operatorname{div}\mathbf{w}),\\ A_\mathbf{y}(\mathbf{w},\pi) &:= -\nu\Delta\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{w} + \nabla\pi.\end{aligned}$$

Note that for $(\mathbf{w},\pi)\in\mathbf{W}^{1,s}_0(\Omega)\times\widehat{L}^s(\Omega)$ and $\mathbf{v}\in\mathbf{W}^{1,s'}_0(\Omega)$,

$$\begin{aligned}&\langle A_\mathbf{y}(\mathbf{w},\pi),\mathbf{v}\rangle_{\mathbf{W}^{-1,s}(\Omega),\mathbf{W}^{1,s'}_0(\Omega)}\\ &\quad:= \int_\Omega \nu\nabla\mathbf{w}:\nabla\mathbf{v} + (\mathbf{w}\cdot\nabla)\mathbf{y}\cdot\mathbf{v} + (\mathbf{y}\cdot\nabla)\mathbf{w}\cdot\mathbf{v} - \pi\operatorname{div}\mathbf{v}\,\mathrm{d}x. \qquad (2.17)\end{aligned}$$

Similarly, define the bounded linear operator

$$\mathfrak{A}^\star_\mathbf{y} : \mathbf{W}^{1,s'}_0(\Omega)\times\widehat{L}^{s'}(\Omega)\to\mathbf{W}^{-1,s'}(\Omega)\times\widehat{L}^{s'}(\Omega)$$

according to

$$\begin{aligned}\mathfrak{A}^\star_\mathbf{y}(\mathbf{v},\sigma) &:= (A^\star_\mathbf{y}(\mathbf{v},\sigma), -\operatorname{div}\mathbf{v}),\\ A^\star_\mathbf{y}(\mathbf{v},\sigma) &:= -\nu\Delta\mathbf{v} + (\nabla\mathbf{y})^\top\mathbf{v} - (\mathbf{y}\cdot\nabla)\mathbf{v} + \nabla\sigma,\end{aligned}$$

where, for $(\mathbf{v},\sigma)\in\mathbf{W}^{1,s'}_0(\Omega)\times\widehat{L}^{s'}(\Omega)$ and $\mathbf{w}\in\mathbf{W}^{1,s}_0(\Omega)$,

$$\begin{aligned}&\langle A^\star_\mathbf{y}(\mathbf{v},\sigma),\mathbf{w}\rangle_{\mathbf{W}^{-1,s'}(\Omega),\mathbf{W}^{1,s}_0(\Omega)}\\ &\quad:= \int_\Omega \nu\nabla\mathbf{w}:\nabla\mathbf{v} + (\nabla\mathbf{y})^\top\mathbf{v}\cdot\mathbf{w} - (\mathbf{y}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} - \sigma\operatorname{div}\mathbf{w}\,\mathrm{d}x. \qquad (2.18)\end{aligned}$$

The operators $\mathfrak{A}_{\mathbf{y}}$ and $\mathfrak{A}_{\mathbf{y}}^{\star}$ are dual to each other in the sense that

$$\begin{aligned}&\langle \mathfrak{A}_{\mathbf{y}}(\mathbf{w},\pi),(\mathbf{v},\sigma)\rangle_{\mathbf{W}^{-1,s}(\Omega)\times\widehat{L}^{s}(\Omega),\mathbf{W}_0^{1,s'}(\Omega)\times\widehat{L}^{s'}(\Omega)}\\&\quad=\langle(\mathbf{w},\pi),\mathfrak{A}_{\mathbf{y}}^{\star}(\mathbf{v},\sigma)\rangle_{\mathbf{W}^{-1,s'}(\Omega)\times\widehat{L}^{s'}(\Omega),\mathbf{W}_0^{1,s}(\Omega)\times\widehat{L}^{s}(\Omega)}\end{aligned}\tag{2.19}$$

for every $(\mathbf{w},\pi)\in\mathbf{W}_0^{1,s}(\Omega)\times\widehat{L}^{s}(\Omega)$ and $(\mathbf{v},\sigma)\in\mathbf{W}_0^{1,s'}(\Omega)\times\widehat{L}^{s'}(\Omega)$. This can be established easily from the above definitions.

Definition 2.1. *A solution $\mathbf{y}\in\mathbf{V}^{2,2}(\Omega)$ to (1.2) with associated pressure $p\in\widehat{W}^{1,2}(\Omega)$ corresponding to a source $\mathbf{u}\in\mathbf{L}^2(\Omega)$ is called regular if*

$$A_{\mathbf{y}}\in\mathcal{L}_{\mathrm{iso}}(\mathbf{V}^{1,2}(\Omega)\times\widehat{L}^{2}(\Omega),\mathbf{W}^{-1,2}(\Omega)).$$

To deal with optimal control problems having point-evaluations of the velocity as observations, it is necessary to extend this definition so that $A_{\mathbf{y}}\in\mathcal{L}_{\mathrm{iso}}(\mathbf{V}^{1,r}(\Omega)\times\widehat{L}^{r}(\Omega),\mathbf{W}^{-1,r}(\Omega))$ with $r\neq 2$. We will prove this in Lemma 2.7. Moreover, to treat observations that are point-evaluations of the pressure or the normal stress on the boundary, we need to study the case of non-homogeneous divergence and Dirichlet boundary data. This will be done in Lemma 2.10 and Corollary 2.13.

The following lemma tells us that there are two alternatives for the operator $A_{\mathbf{y}}\in\mathcal{L}(\mathbf{V}^{1,2}(\Omega)\times\widehat{L}^{2}(\Omega),\mathbf{W}^{-1,2}(\Omega))$, compare with [15, Lemma IX.2.2].

Proposition 2.2. *Given $\mathbf{y}\in\mathbf{V}^{2,2}(\Omega)$, either $A_{\mathbf{y}}\in\mathcal{L}_{\mathrm{iso}}(\mathbf{V}^{1,2}(\Omega)\times\widehat{L}^{2}(\Omega),\mathbf{W}^{-1,2}(\Omega))$ or $A_{\mathbf{y}}$ has a nontrivial kernel.*

Proof. First, let us note that by the triangle and Hölder inequalities

$$\begin{aligned}\|(\mathbf{w}\cdot\nabla)\mathbf{y}+(\mathbf{y}\cdot\nabla)\mathbf{w}\|_{\mathbf{L}^2(\Omega)}&\le\|\mathbf{w}\|_{\mathbf{L}^4(\Omega)}\|\nabla\mathbf{y}\|_{\mathbf{L}^4(\Omega)^2}+\|\mathbf{y}\|_{\mathbf{L}^\infty(\Omega)}\|\nabla\mathbf{w}\|_{\mathbf{L}^2(\Omega)^2}\\&\le c\|\mathbf{y}\|_{\mathbf{V}^{2,2}(\Omega)}\|\mathbf{w}\|_{\mathbf{V}^{1,2}(\Omega)}\end{aligned}\tag{2.20}$$

thanks to the continuous embeddings $\mathbf{V}^{2,2}(\Omega)\subset\mathbf{L}^\infty(\Omega)\cap\mathbf{W}^{1,4}(\Omega)$ and $\mathbf{V}^{1,2}(\Omega)\subset\mathbf{L}^4(\Omega)$.

Consider the bounded linear operator $\widehat{A}_{\mathbf{y}}:\mathbf{V}^{1,2}(\Omega)\to\mathbf{V}^{-1,2}(\Omega)$ defined by

$$\widehat{A}_{\mathbf{y}}\mathbf{w}:=-\nu\Delta\mathbf{w}+I[(\mathbf{w}\cdot\nabla)\mathbf{y}+(\mathbf{y}\cdot\nabla)\mathbf{w}],$$

where I denotes the canonical embedding from $\mathbf{L}^2(\Omega)$ into $\mathbf{V}^{-1,2}(\Omega)$. Let us decompose this operator as $\widehat{A}_{\mathbf{y}}=:C_{\mathbf{y}}+K_{\mathbf{y}}$, where

$$C_{\mathbf{y}}\mathbf{w}:=-\nu\Delta\mathbf{w}+I[(\mathbf{y}\cdot\nabla)\mathbf{w}],\quad K_{\mathbf{y}}\mathbf{w}:=I[(\mathbf{w}\cdot\nabla)\mathbf{y}].$$

Since $\langle C_{\mathbf{y}}\mathbf{w},\mathbf{w}\rangle_{\mathbf{V}^{-1,2}(\Omega),\mathbf{V}^{1,2}(\Omega)}=\nu\|\mathbf{w}\|_{\mathbf{V}^{1,2}(\Omega)}$, it follows that

$$C_{\mathbf{y}}\in\mathcal{L}_{\mathrm{iso}}(\mathbf{V}^{1,2}(\Omega),\mathbf{V}^{-1,2}(\Omega))$$

from the Lax–Milgram Lemma.

On the other hand, for every $\mathbf{w}\in\mathbf{L}^2_\sigma(\Omega)$ and $\mathbf{v}\in\mathbf{V}^{1,2}(\Omega)$

$$\begin{aligned}|\langle K_{\mathbf{y}}\mathbf{w},\mathbf{v}\rangle_{\mathbf{V}^{-1,2}(\Omega),\mathbf{V}^{1,2}(\Omega)}|&\le\|\mathbf{w}\|_{\mathbf{L}^2(\Omega)}\|\nabla\mathbf{y}\|_{\mathbf{L}^4(\Omega)^2}\|\mathbf{v}\|_{\mathbf{L}^4(\Omega)}\\&\le c\|\mathbf{w}\|_{\mathbf{L}^2_\sigma(\Omega)}\|\mathbf{y}\|_{\mathbf{V}^{2,2}(\Omega)}\|\mathbf{v}\|_{\mathbf{V}^{1,2}(\Omega)}.\end{aligned}$$

Thus, $K_{\mathbf{y}}\in\mathcal{L}(\mathbf{L}^2_\sigma(\Omega),\mathbf{V}^{-1,2}(\Omega))$, and as result, $K_{\mathbf{y}}\in\mathcal{L}(\mathbf{V}^{1,2}(\Omega),\mathbf{V}^{-1,2}(\Omega))$ is compact due to the compactness of $\mathbf{V}^{1,2}(\Omega)\subset\mathbf{L}^2_\sigma(\Omega)$.

For $\mathbf{w} \in \mathbf{V}^{1,2}(\Omega)$ and $\mathbf{r} \in \mathbf{V}^{-1,2}(\Omega)$, the equation $\widehat{A}_{\mathbf{y}}\mathbf{w} = \mathbf{r}$ in $\mathbf{V}^{-1,2}(\Omega)$ is equivalent to $(1 + C_{\mathbf{y}}^{-1}K_{\mathbf{y}})\mathbf{w} = C_{\mathbf{y}}^{-1}\mathbf{r}$ in $\mathbf{V}^{1,2}(\Omega)$. Since $C_{\mathbf{y}}^{-1}K_{\mathbf{y}} \in \mathcal{L}(\mathbf{V}^{1,2}(\Omega))$ is compact, we deduce from the Fredholm alternative that -1 is either in the resolvent set or in the point spectrum of $C_{\mathbf{y}}^{-1}K_{\mathbf{y}}$. In the first alternative, we obtain that $\widehat{A}_{\mathbf{y}} \in \mathcal{L}_{\text{iso}}(\mathbf{V}^{1,2}(\Omega), \mathbf{V}^{-1,2}(\Omega))$. Thanks to the Banach inverse theorem, if $\widehat{A}_{\mathbf{y}}\mathbf{w} = \mathbf{r}$, then

$$\|\mathbf{w}\|_{\mathbf{V}^{1,2}(\Omega)} \le c\|\mathbf{r}\|_{\mathbf{V}^{-1,2}(\Omega)}, \quad c := \|\widehat{A}_{\mathbf{y}}^{-1}\|_{\mathcal{L}(\mathbf{V}^{-1,2}(\Omega),\mathbf{V}^{1,2}(\Omega))}.$$

The second alternative implies that $\widehat{A}_{\mathbf{y}}$ has a nontrivial kernel.

Given $\mathbf{r} \in \mathbf{W}^{-1,2}(\Omega) \subset \mathbf{V}^{-1,2}(\Omega)$, the preceding paragraph yields either $\widehat{A}_{\mathbf{y}}\mathbf{w} = \mathbf{r}$ for a unique $\mathbf{w} \in \mathbf{V}^{1,2}(\Omega)$ and

$$\|\mathbf{w}\|_{\mathbf{V}^{1,2}(\Omega)} \le c\|\mathbf{r}\|_{\mathbf{W}^{-1,2}(\Omega)},$$

or $\widehat{A}_{\mathbf{y}}\mathbf{w} = \mathbf{0}$ for some $\mathbf{0} \neq \mathbf{w} \in \mathbf{V}^{1,2}(\Omega)$. In the first case, we obtain from de Rham's theorem the existence of a unique $\pi \in \widehat{L}^2(\Omega)$ such that $A_{\mathbf{y}}(\mathbf{w}, \pi) = \mathbf{r}$ and

$$\|\pi\|_{\widehat{L}^2(\Omega)} \le c\|\nabla\pi\|_{\mathbf{W}^{-1,2}(\Omega)} \le c(\|\mathbf{r}\|_{\mathbf{W}^{-1,2}(\Omega)} + \|\mathbf{w}\|_{\mathbf{V}^{1,2}(\Omega)}) \le c\|\mathbf{r}\|_{\mathbf{W}^{-1,2}(\Omega)}.$$

Thus, we have

$$\|\mathbf{w}\|_{\mathbf{V}^{1,2}(\Omega)} + \|\pi\|_{\widehat{L}^2(\Omega)} \le c\|\mathbf{r}\|_{\mathbf{W}^{-1,2}(\Omega)}. \tag{2.21}$$

It follows that $A_{\mathbf{y}}$ is surjective. It is also injective, since $A_{\mathbf{y}}(\mathbf{w}, \pi) = (\mathbf{0}, 0)$ implies $\widehat{A}_{\mathbf{y}}\mathbf{w} = \mathbf{0}$ so that $\mathbf{w} = \mathbf{0}$ by injectivity of $\widehat{A}_{\mathbf{y}}$, and hence $\nabla\pi = \mathbf{0}$, which leads to $\pi = 0$ in $\widehat{L}^2(\Omega)$. The second case obviously implies that $(\mathbf{w}, 0) \neq (\mathbf{0}, 0)$ lies in the kernel of $A_{\mathbf{y}}$. □

Corollary 2.3. *Let $\mathbf{y} \in \mathbf{V}^{2,2}(\Omega)$. If for every $\mathbf{0} \neq \mathbf{w} \in \mathbf{V}^{1,2}(\Omega)$ we have*

$$\int_\Omega \nu|\nabla\mathbf{w}|^2 + (\mathbf{w}\cdot\nabla)\mathbf{y}\cdot\mathbf{w}\,\mathrm{d}x \neq 0,$$

then $A_{\mathbf{y}} \in \mathcal{L}_{\text{iso}}(\mathbf{V}^{1,2}(\Omega) \times \widehat{L}^2(\Omega), \mathbf{W}^{-1,2}(\Omega))$.

Proof. If the kernel of $A_{\mathbf{y}}$ contains a nontrivial element, say $(\mathbf{w}, \pi) \in \mathbf{V}^{1,2}(\Omega) \times \widehat{L}^2(\Omega)$, then we have $\mathbf{w} \neq \mathbf{0}$ and

$$0 = \langle A_{\mathbf{y}}(\mathbf{w}, \pi), \mathbf{w}\rangle_{\mathbf{W}^{-1,2}(\Omega),\mathbf{W}_0^{1,2}(\Omega)} = \int_\Omega \nu|\nabla\mathbf{w}|^2 + (\mathbf{w}\cdot\nabla)\mathbf{y}\cdot\mathbf{w}\,\mathrm{d}x.$$

This is a contradiction to the given hypothesis. Hence, $A_{\mathbf{y}}$ must be an isomorphism due to Proposition 2.2. □

A sufficient condition for the existence of regular points is given in the following theorem (see also [10, Definition 2.7]). In particular, this result tells us that if the viscosity coefficient is sufficiently large, then it is guaranteed that there is at least one regular point of (1.2). At this point, there is no need to specify the tracking parts J_k, for $k = 0, 1, 2$, of the cost functionals $\mathcal{J}_k$ as the non-negativity of J_k will suffice for the validity of the following theorem. Non-negativity of J_k is satisfied by all functionals defined in Section 3. Unlike in the introduction, we now include the last three equations in (1.2) for the function spaces for $\mathbf{y}$ and p.

Theorem 2.4. *Let $(\mathbf{y}, p, \mathbf{u}) \in \mathbf{V}^{2,2}(\Omega) \times \widehat{W}^{1,2}(\Omega) \times \mathbf{L}^2(\Omega)$ be such that $(\mathbf{y}, p)$ is a solution of (1.2) with control $\mathbf{u}$ and $\mathcal{J}_0(\mathbf{y}, p, \mathbf{u}) \le \mathcal{J}_0(\mathbf{0}, 0, \mathbf{0})$, where $\mathcal{J}_0 : \mathbf{V}^{2,2}(\Omega) \times \widehat{W}^{1,2}(\Omega) \times \mathbf{L}^2(\Omega) \to \mathbb{R}$ is of the form*

$$\mathcal{J}_0(\mathbf{y}, p, \mathbf{u}) := J_0(\mathbf{y}, p) + \frac{\rho}{2}\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)}$$

with $J_0 : \mathbf{V}^{2,2}(\Omega) \times \widehat{W}^{1,2}(\Omega) \to \mathbb{R}$ a non-negative function. Then, there exists $\nu_0 > 0$ such that if $\nu > \nu_0$, then $\mathbf{y}$ is a regular solution to (1.2). Analogous results hold for the case of cost functionals $\mathcal{J}_1 : \mathbf{V}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega) \times \mathbf{L}^s(\Omega) \to \mathbb{R}$, with $2 < s < \infty$, of the form

$$\mathcal{J}_1(\mathbf{y}, p, \mathbf{u}) := J_1(\mathbf{y}, p) + \frac{\rho}{2}\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)}$$

for some non-negative $J_1 : \mathbf{V}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega) \to \mathbb{R}$, and $\mathcal{J}_2 : (\mathbf{V}^{2,2}(\Omega) \cap \mathbf{W}^{3,2}(\Omega)) \times \widehat{W}^{2,2}(\Omega) \times \mathbf{W}^{1,2}(\Omega) \to \mathbb{R}$ of the form

$$\mathcal{J}_2(\mathbf{y}, p, \mathbf{u}) := J_2(\mathbf{y}, p) + \frac{\rho}{2}\|\mathbf{u}\|^2_{\mathbf{W}^{1,2}(\Omega)}$$

for some non-negative $J_2 : (\mathbf{V}^{2,2}(\Omega) \cap \mathbf{W}^{3,2}(\Omega)) \times \widehat{W}^{2,2}(\Omega) \to \mathbb{R}$.

Proof. Let $\mathcal{J}^{(0)} := \mathcal{J}_0(\mathbf{0}, 0, \mathbf{0}) = J_0(\mathbf{0}, 0)$ and $\mathbf{0} \neq \mathbf{w} \in \mathbf{V}^{1,2}(\Omega)$. Notice that $(\mathbf{0}, 0)$ is a feasible point of (1.1). From the non-negativity of J_0, we have $\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \le 2\mathcal{J}_0(\mathbf{y}, p, \mathbf{u})/\rho \le 2\mathcal{J}^{(0)}/\rho$, and using this inequality in (2.3) we obtain

$$\|\mathbf{y}\|_{\mathbf{V}^{1,2}(\Omega)} \le \nu^{-1} c_0 (2\mathcal{J}^{(0)}/\rho)^{1/2}$$

where $c_0 > 0$ denotes the operator norm of the embedding $\mathbf{L}^2(\Omega) \subset \mathbf{W}^{-1,2}(\Omega)$.

Note that

$$\begin{aligned}\left|\int_\Omega (\mathbf{w} \cdot \nabla)\mathbf{y} \cdot \mathbf{w} \,\mathrm{d}x\right| &\le c\|\mathbf{w}\|^2_{\mathbf{L}^4(\Omega)}\|\nabla \mathbf{y}\|_{\mathbf{L}^2(\Omega)^2} \\ &\le c c_1^2 \nu^{-1} c_0 (2\mathcal{J}^{(0)}/\rho)^{1/2}\|\mathbf{w}\|^2_{\mathbf{V}^{1,2}(\Omega)},\end{aligned}$$

where $c_1 > 0$ is the operator norm of $\mathbf{V}^{1,2}(\Omega) \subset \mathbf{L}^4(\Omega)$. By taking

$$\nu_0 > c c_1^2 \nu^{-1} c_0 (2\mathcal{J}^{(0)}/\rho)^{1/2} \ge 0,$$

we deduce that for $\nu > \nu_0$ we have

$$\int_\Omega \nu|\nabla \mathbf{w}|^2 + (\mathbf{w} \cdot \nabla)\mathbf{y} \cdot \mathbf{w} \,\mathrm{d}x \ge (\nu - \nu_0)\int_\Omega |\nabla \mathbf{w}|^2 \,\mathrm{d}x > 0.$$

Using Corollary 2.3, we see that $\mathbf{y}$ is a regular solution of (1.2).

For the last statement of the theorem, note that the proof for the case of $\mathcal{J}_1$ is the same but with $\mathcal{J}^{(0)}$ replaced by $\mathcal{J}^{(1)} := \mathcal{J}_1(\mathbf{0}, 0, \mathbf{0}) = J_1(\mathbf{0}, 0)$. On the other hand, that of $\mathcal{J}_2$ can be established by applying the embedding $\mathbf{W}^{1,2}(\Omega) \subset \mathbf{L}^2(\Omega)$ and $\mathcal{J}^{(0)}$ replaced by $\mathcal{J}^{(2)} := \mathcal{J}_2(\mathbf{0}, 0, \mathbf{0}) = J_2(\mathbf{0}, 0)$. □

For a given fix $\nu > 0$, it is possible to prove that a solution $\mathbf{y}$ of (1.2) is regular if one imposes smallness condition on the size of the control. For example, if $\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} < \delta$, where $\delta > 0$ is small enough so that $\nu^2 - c c_1^2 c_0 \delta > 0$, then following the same argument as above we can verify that $\mathbf{y}$ is regular.

The results of Theorem 2.4 also hold if we replace $J_k(\mathbf{y},p)$ and $\mathcal{J}_k(\mathbf{y},p,\mathbf{u})$ for $k=0,1,2$ with $J_k(\mathbf{y})$ and $\mathcal{J}_k(\mathbf{y},\mathbf{u})$, or with $J_k(p)$ and $\mathcal{J}_k(p,\mathbf{u})$, respectively, with appropriate modifications on the spaces where these functionals are defined. Such functionals are suitable when one has to deal with either the velocity or the pressure.

The following lemma provides an equivalent definition of regular points.

Lemma 2.5. *Let $\mathbf{y}\in\mathbf{V}^{2,2}(\Omega)$. Then, $A_{\mathbf{y}}\in\mathcal{L}_{\rm iso}(\mathbf{V}^{1,2}(\Omega)\times\widehat{L}^2(\Omega),\mathbf{W}^{-1,2}(\Omega))$ if and only if $\mathfrak{A}_{\mathbf{y}}\in\mathcal{L}_{\rm iso}(\mathbf{W}^{1,2}_0(\Omega)\times\widehat{L}^2(\Omega),\mathbf{W}^{-1,2}(\Omega)\times\widehat{L}^2(\Omega))$.*

Proof. If $\mathfrak{A}_{\mathbf{y}}:\mathbf{W}^{1,2}_0(\Omega)\times\widehat{L}^2(\Omega)\to\mathbf{W}^{-1,2}(\Omega)\times\widehat{L}^2(\Omega)$ is an isomorphism, then given $\mathbf{r}\in\mathbf{W}^{-1,2}(\Omega)$ there exists a unique $(\mathbf{w},\pi)\in\mathbf{W}^{1,2}_0(\Omega)\times\widehat{L}^2(\Omega)$ such that $\mathfrak{A}_{\mathbf{y}}(\mathbf{w},\pi)=(\mathbf{r},0)$ and (2.21) holds. Hence, $\mathbf{w}\in\mathbf{V}^{1,2}(\Omega)$ and $A_{\mathbf{y}}(\mathbf{w},\pi)=\mathbf{r}$. Therefore, $A_{\mathbf{y}}$ is an isomorphism.

Conversely, suppose that $(\mathbf{r},q)\in\mathbf{W}^{-1,2}(\Omega)\times\widehat{L}^2(\Omega)$. Let $(\mathbf{w}_1,\pi_1)\in\mathbf{W}^{1,2}_0(\Omega)\times\widehat{L}^2(\Omega)$ be the weak solution of the Stokes problem

$$-\nu\Delta\mathbf{w}_1+\nabla\pi_1=\mathbf{r}\ \text{ in }\Omega,\quad -\operatorname{div}\mathbf{w}_1=q\ \text{ in }\Omega,\quad \mathbf{w}_1=\mathbf{0}\ \text{ on }\Gamma.$$

Then, we have

$$\|\mathbf{w}_1\|_{\mathbf{W}^{1,2}_0(\Omega)}+\|\pi_1\|_{\widehat{L}^2(\Omega)}\le c(\|\mathbf{r}\|_{\mathbf{W}^{-1,2}(\Omega)}+\|q\|_{\widehat{L}^2(\Omega)}).\tag{2.22}$$

If $A_{\mathbf{y}}:\mathbf{V}^{1,2}(\Omega)\times\widehat{L}^2(\Omega)\to\mathbf{W}^{-1,2}(\Omega)$ is an isomorphism, then there exists a unique $(\mathbf{w}_2,\pi_2)\in\mathbf{V}^{1,2}(\Omega)\times\widehat{L}^2(\Omega)$ such that $A_{\mathbf{y}}(\mathbf{w}_2,\pi_2)=-(\mathbf{w}_1\cdot\nabla)\mathbf{y}-(\mathbf{y}\cdot\nabla)\mathbf{w}_1$ and

$$\begin{aligned}\|\mathbf{w}_2\|_{\mathbf{W}^{1,2}_0(\Omega)}+\|\pi_2\|_{\widehat{L}^2(\Omega)}&\le c\|-(\mathbf{w}_1\cdot\nabla)\mathbf{y}-(\mathbf{y}\cdot\nabla)\mathbf{w}_1\|_{\mathbf{W}^{-1,2}(\Omega)}\\&\le c\|\mathbf{y}\|_{\mathbf{V}^{2,2}(\Omega)}(\|\mathbf{r}\|_{\mathbf{W}^{-1,2}(\Omega)}+\|q\|_{\widehat{L}^2(\Omega)}),\end{aligned}\tag{2.23}$$

where we used (2.22) in the second inequality.

Setting $(\mathbf{w},\pi):=(\mathbf{w}_1+\mathbf{w}_2,\pi_1+\pi_2)\in\mathbf{W}^{1,2}_0(\Omega)\times\widehat{L}^2(\Omega)$, we have $\mathfrak{A}_{\mathbf{y}}(\mathbf{w},\pi)=(\mathbf{r},q)$ and from (2.22) and (2.23), one has

$$\|\mathbf{w}\|_{\mathbf{W}^{1,2}_0(\Omega)}+\|\pi\|_{\widehat{L}^2(\Omega)}\le c(1+\|\mathbf{y}\|_{\mathbf{V}^{2,2}(\Omega)})(\|\mathbf{r}\|_{\mathbf{W}^{-1,2}(\Omega)}+\|q\|_{\widehat{L}^2(\Omega)}).$$

Injectivity of $\mathfrak{A}_{\mathbf{y}}$ follows from that of $A_{\mathbf{y}}$. Hence, $\mathfrak{A}_{\mathbf{y}}$ is an isomorphism. □

In the following lemma, we establish that $\mathfrak{A}_{\mathbf{y}}$ and $\mathfrak{A}^\star_{\mathbf{y}}$ are isomorphisms in general Lebesgue spaces. Here, we adapt and generalize the proof provided in [10, Theorem 2.9].

Lemma 2.6. *Let $\mathbf{y}\in\mathbf{V}^{2,2}(\Omega)$ be a regular solution of (1.2) and $1<s<\infty$. Then, $\mathfrak{A}_{\mathbf{y}},\mathfrak{A}^\star_{\mathbf{y}}\in\mathcal{L}_{\rm iso}(\mathbf{W}^{1,s}_0(\Omega)\times\widehat{L}^s(\Omega),\mathbf{W}^{-1,s}(\Omega)\times\widehat{L}^s(\Omega))$.*

Proof. First, we consider the case $s=2$ for the operator $\mathfrak{A}^\star_{\mathbf{y}}$. Let $(\mathbf{r},q)\in\mathbf{W}^{-1,2}(\Omega)\times\widehat{L}^2(\Omega)$. Then, for some $(\mathbf{w},\pi)\in\mathbf{W}^{1,2}_0(\Omega)\times\widehat{L}^2(\Omega)$ we have $\mathfrak{A}_{\mathbf{y}}(\mathbf{w},\pi)=(\mathbf{r},q)$ by Lemma 2.5 and the assumption that $\mathbf{y}$ is regular. Moreover, for each $(\mathbf{v},\sigma)\in\mathbf{W}^{1,2}_0(\Omega)\times\widehat{L}^2(\Omega)$, it holds that

$$\begin{aligned}&|\langle(\mathbf{r},q),(\mathbf{v},\sigma)\rangle_{\mathbf{W}^{-1,2}(\Omega)\times\widehat{L}^2(\Omega),\mathbf{W}^{1,2}_0(\Omega)\times\widehat{L}^2(\Omega)}|\\&\quad\le c\|\mathfrak{A}^\star_{\mathbf{y}}(\mathbf{v},\sigma)\|_{\mathbf{W}^{-1,2}(\Omega)\times\widehat{L}^2(\Omega)}(\|\mathbf{w}\|_{\mathbf{W}^{1,2}_0(\Omega)}+\|\pi\|_{\widehat{L}^2(\Omega)})\\&\quad\le c\|\mathfrak{A}^\star_{\mathbf{y}}(\mathbf{v},\sigma)\|_{\mathbf{W}^{-1,2}(\Omega)\times\widehat{L}^2(\Omega)}(\|\mathbf{r}\|_{\mathbf{W}^{-1,2}(\Omega)}+\|q\|_{\widehat{L}^2(\Omega)}).\end{aligned}$$

By duality, this estimate implies that $\mathfrak{A}_{\mathbf{y}}^\star$ is injective and it has a closed range. Suppose that $\mathfrak{A}_{\mathbf{y}}^\star$ does not have a dense range so that

$$\langle \mathfrak{A}_{\mathbf{y}}^\star(\mathbf{v},\sigma),(\mathbf{w},\pi)\rangle_{\mathbf{W}^{-1,2}(\Omega)\times\widehat{L}^2(\Omega),\mathbf{W}_0^{1,2}(\Omega)\times\widehat{L}^2(\Omega)} = 0$$

for some nonzero $(\mathbf{w},\pi)\in\mathbf{W}_0^{1,2}(\Omega)\times\widehat{L}^2(\Omega)$ and for all $(\mathbf{v},\sigma)\in\mathbf{W}_0^{1,2}(\Omega)\times\widehat{L}^2(\Omega)$. Then, $\mathfrak{A}_{\mathbf{y}}(\mathbf{w},\pi) = (\mathbf{0},0)$ so that $(\mathbf{w},\pi) = (\mathbf{0},0)$ since $\mathfrak{A}_{\mathbf{y}}$ is injective, which is a contradiction to the fact that $\mathfrak{A}_{\mathbf{y}}$ is an isomorphism. Thus, $\mathfrak{A}_{\mathbf{y}}^\star$ must have a dense and closed range, and consequently, $\mathfrak{A}_{\mathbf{y}}^\star$ is surjective.

Suppose $2 < s < \infty$. From the continuous embedding $\mathbf{W}^{-1,s}(\Omega)\times\widehat{L}^s(\Omega)\subset\mathbf{W}^{-1,2}(\Omega)\times\widehat{L}^2(\Omega)$, given $(\mathbf{f},g)\in\mathbf{W}^{-1,s}(\Omega)\times\widehat{L}^s(\Omega)$, the result of the previous paragraph implies that there exists only one $(\mathbf{v},\sigma)\in\mathbf{W}^{-1,2}(\Omega)\times\widehat{L}^2(\Omega)$ such that $\mathfrak{A}_{\mathbf{y}}^\star(\mathbf{v},\sigma) = (\mathbf{f},g)$, and moreover, we have the priori estimate

$$\begin{aligned}\|\mathbf{v}\|_{\mathbf{W}_0^{1,2}(\Omega)} + \|\sigma\|_{\widehat{L}^2(\Omega)} &\le c(\|\mathbf{f}\|_{\mathbf{W}^{-1,2}(\Omega)} + \|g\|_{\widehat{L}^2(\Omega)})\\ &\le c(\|\mathbf{f}\|_{\mathbf{W}^{-1,s}(\Omega)} + \|g\|_{\widehat{L}^s(\Omega)}).\end{aligned}$$

As before, it can be deduced that $(\nabla\mathbf{y})^\top\mathbf{v} - (\mathbf{y}\cdot\nabla)\mathbf{v}\in\mathbf{L}^2(\Omega)\subset\mathbf{W}^{-1,s}(\Omega)$ and

$$\begin{aligned}\|(\nabla\mathbf{y})^\top\mathbf{v} - (\mathbf{y}\cdot\nabla)\mathbf{v}\|_{\mathbf{W}^{-1,s}(\Omega)} &\le c\|(\nabla\mathbf{y})^\top\mathbf{v} - (\mathbf{y}\cdot\nabla)\mathbf{v}\|_{\mathbf{L}^2(\Omega)}\\ &\le c\|\mathbf{y}\|_{\mathbf{V}^{2,2}(\Omega)}\|\mathbf{v}\|_{\mathbf{W}_0^{1,2}(\Omega)}.\end{aligned}$$

Hence, the L^s-regularity theory for the Stokes equation leads to $(\mathbf{v},\sigma)\in\mathbf{W}_0^{1,s}(\Omega)\times\widehat{L}^s(\Omega)$ and

$$\|\mathbf{v}\|_{\mathbf{W}_0^{1,s}(\Omega)} + \|\sigma\|_{\widehat{L}^s(\Omega)} \le c(1+\|\mathbf{y}\|_{\mathbf{V}^{2,2}(\Omega)})(\|\mathbf{f}\|_{\mathbf{W}^{-1,s}(\Omega)} + \|g\|_{\widehat{L}^s(\Omega)}).$$

This completes the proof of $\mathfrak{A}_{\mathbf{y}}^\star$ for $2\le s<\infty$. For the operator $\mathfrak{A}_{\mathbf{y}}$, the case $s=2$ follows from Lemma 2.5, while the case $2<s<\infty$ can be handled in a similar manner as that of $\mathfrak{A}_{\mathbf{y}}^\star$.

Now, assume that $1 < s < 2$ and proceed by a density argument. Given $(\mathbf{r},q)\in\mathbf{W}^{-1,s}(\Omega)\times\widehat{L}^s(\Omega)$, there is a sequence $(\mathbf{r}_n,q_n)\in\mathbf{W}^{-1,2}(\Omega)\times\widehat{L}^2(\Omega)$ such that $(\mathbf{r}_n,q_n)\to(\mathbf{r},q)$ in $\mathbf{W}^{-1,s}(\Omega)\times\widehat{L}^s(\Omega)$ since $\mathbf{W}^{-1,2}(\Omega)\times\widehat{L}^2(\Omega)$ is dense in $\mathbf{W}^{-1,s}(\Omega)\times\widehat{L}^s(\Omega)$. For each n, we have $\mathfrak{A}_{\mathbf{y}}(\mathbf{w}_n,\pi_n) = (\mathbf{r}_n,q_n)$ for some $(\mathbf{w}_n,\pi_n)\in\mathbf{W}_0^{1,2}(\Omega)\times\widehat{L}^2(\Omega)$. Let $(\mathbf{f},g)\in\mathbf{W}^{-1,s'}(\Omega)\times\widehat{L}^{s'}(\Omega)$, where $2<s'<\infty$. The above discussion shows that $\mathfrak{A}_{\mathbf{y}}^\star(\mathbf{v},\sigma) = (\mathbf{f},g)$ for some $(\mathbf{v},\sigma)\in\mathbf{W}_0^{1,s'}(\Omega)\times\widehat{L}^{s'}(\Omega)$ and

$$\begin{aligned}&|\langle(\mathbf{f},g),(\mathbf{w}_n,\pi_n)\rangle_{\mathbf{W}^{-1,s'}(\Omega)\times\widehat{L}^{s'}(\Omega),\mathbf{W}_0^{1,s}(\Omega)\times\widehat{L}^s(\Omega)}|\\ &\quad= |\langle(\mathbf{r}_n,q_n),(\mathbf{v},\sigma)\rangle_{\mathbf{W}^{-1,s}(\Omega)\times\widehat{L}^s(\Omega),\mathbf{W}_0^{1,s'}(\Omega)\times\widehat{L}^{s'}(\Omega)}|\\ &\quad\le \|(\mathbf{r}_n,q_n)\|_{\mathbf{W}^{-1,s}(\Omega)\times\widehat{L}^s(\Omega)}\|(\mathbf{v},\sigma)\|_{\mathbf{W}_0^{1,s'}(\Omega)\times\widehat{L}^{s'}(\Omega)}\\ &\quad\le c\|(\mathbf{r}_n,q_n)\|_{\mathbf{W}^{-1,s}(\Omega)\times\widehat{L}^s(\Omega)}\|(\mathbf{f},g)\|_{\mathbf{W}^{-1,s'}(\Omega)\times\widehat{L}^{s'}(\Omega)}.\end{aligned}$$

By duality, we obtain up to a subsequence that $(\mathbf{w}_n,\pi_n)\rightharpoonup(\mathbf{w},\pi)$ weakly in $\mathbf{W}_0^{1,s}(\Omega)\times\widehat{L}^s(\Omega)$ for some $(\mathbf{w},\pi)\in\mathbf{W}_0^{1,s}(\Omega)\times\widehat{L}^s(\Omega)$. Passing $n\to\infty$ in the variational form of $\mathfrak{A}_{\mathbf{y}}(\mathbf{w}_n,\pi_n) = (\mathbf{r}_n,q_n)$, we deduce that $\mathfrak{A}_{\mathbf{y}}(\mathbf{w},\pi) = (\mathbf{r},q)$, showing that $\mathfrak{A}_{\mathbf{y}}$ maps $\mathbf{W}_0^{1,s}(\Omega)\times\widehat{L}^s(\Omega)$ onto $\mathbf{W}^{-1,s}(\Omega)\times\widehat{L}^s(\Omega)$. The fact that that this map is injective follows from the surjectivity of $\mathfrak{A}_{\mathbf{y}}^\star:\mathbf{W}_0^{1,s'}(\Omega)\times\widehat{L}^{s'}(\Omega)\to\mathbf{W}^{-1,s'}(\Omega)\times\widehat{L}^{s'}(\Omega)$. Therefore, $\mathfrak{A}_{\mathbf{y}}\in\mathcal{L}_{\text{iso}}(\mathbf{W}_0^{1,s}(\Omega)\times\widehat{L}^s(\Omega),\mathbf{W}^{-1,s}(\Omega)\times\widehat{L}^s(\Omega))$. Analogously, it can be shown with the same arguments that $\mathfrak{A}_{\mathbf{y}}^\star \in \mathcal{L}_{\rm iso}(\mathbf{W}_0^{1,s}(\Omega)\times \widehat{L}^s(\Omega), \mathbf{W}^{-1,s}(\Omega)\times\widehat{L}^s(\Omega))$. $\square$

Lemma 2.7. *Let $1 < s < \infty$. If $\mathbf{y} \in \mathbf{V}^{2,2}(\Omega)$ is a regular solution to (1.2), then*

$$\mathfrak{A}_{\mathbf{y}}, \mathfrak{A}_{\mathbf{y}}^\star \in \mathcal{L}_{\rm iso}(\mathbf{X}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega), \mathbf{L}^s(\Omega) \times \widehat{W}^{1,s}(\Omega)).$$

In particular, we have $A_{\mathbf{y}}, A_{\mathbf{y}}^\star \in \mathcal{L}_{\rm iso}(\mathbf{V}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega), \mathbf{L}^s(\Omega))$.

Proof. We only deal with $\mathfrak{A}_{\mathbf{y}}$ since the case of $\mathfrak{A}_{\mathbf{y}}^\star$ is entirely similar. Given $(\mathbf{r}, q) \in \mathbf{L}^s(\Omega) \times \widehat{W}^{1,s}(\Omega) \subset \mathbf{W}^{-1,s}(\Omega) \times \widehat{L}^s(\Omega)$, there exists $(\mathbf{w}, \pi) \in \mathbf{W}_0^{1,s}(\Omega) \times \widehat{L}^s(\Omega)$ for which $\mathfrak{A}_{\mathbf{y}}(\mathbf{w}, \pi) = (\mathbf{r}, q)$ and according to Lemma 2.6, we have

$$\begin{aligned} \|\mathbf{w}\|_{\mathbf{W}_0^{1,s}(\Omega)} + \|\pi\|_{\widehat{L}^s(\Omega)} &\le c_{\mathbf{y}}(\|\mathbf{r}\|_{\mathbf{W}^{-1,s}(\Omega)} + \|q\|_{\widehat{L}^s(\Omega)}) \\ &\le c_{\mathbf{y}}(\|\mathbf{r}\|_{\mathbf{L}^s(\Omega)} + \|q\|_{\widehat{W}^{1,s}(\Omega)}) \end{aligned} \tag{2.24}$$

where $c_{\mathbf{y}} := c(\|\mathbf{y}\|_{\mathbf{V}^{2,2}(\Omega)}) > 0$ and $c : [0, \infty) \to (0, \infty)$ denotes a generic continuous monotone increasing function.

If $s = 2$, then it follows from the standard regularity theory for Stokes equation, (2.20) with $\|\mathbf{w}\|_{\mathbf{V}^{1,2}(\Omega)}$ replaced by $\|\mathbf{w}\|_{\mathbf{W}_0^{1,2}(\Omega)}$, and $\mathfrak{A}_{\mathbf{y}} \in \mathcal{L}_{\rm iso}(\mathbf{W}_0^{1,2}(\Omega) \times \widehat{L}^2(\Omega), \mathbf{W}^{-1,2}(\Omega) \times \widehat{L}^2(\Omega))$ by Lemma 2.5 that

$$\|\mathbf{w}\|_{\mathbf{X}^{2,2}(\Omega)} + \|\pi\|_{\widehat{W}^{1,2}(\Omega)} \le c_{\mathbf{y}}(\|\mathbf{r}\|_{\mathbf{L}^2(\Omega)} + \|q\|_{\widehat{W}^{1,2}(\Omega)}).$$

For $1 < s < 2$, we have $(\mathbf{w}\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla\mathbf{w}) \in \mathbf{L}^s(\Omega)$ since

$$\begin{aligned} \|(\mathbf{w}\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla\mathbf{w})\|_{\mathbf{L}^s(\Omega)} &\le \|\mathbf{w}\|_{\mathbf{L}^{2s/(2-s)}(\Omega)}\|\nabla\mathbf{y}\|_{\mathbf{L}^2(\Omega)^2} + \|\mathbf{y}\|_{\mathbf{L}^\infty(\Omega)}\|\nabla\mathbf{w}\|_{\mathbf{L}^s(\Omega)^2} \\ &\le c\|\mathbf{y}\|_{\mathbf{V}^{2,2}(\Omega)}\|\mathbf{w}\|_{\mathbf{W}_0^{1,s}(\Omega)} \end{aligned}$$

due to $\mathbf{W}_0^{1,s}(\Omega) \subset \mathbf{L}^{2s/(2-s)}(\Omega)$ and $\mathbf{V}^{2,2}(\Omega) \subset \mathbf{L}^\infty(\Omega)$.

If $2 < s < \infty$, we also have $(\mathbf{w}\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla\mathbf{w}) \in \mathbf{L}^s(\Omega)$ since

$$\begin{aligned} \|(\mathbf{w}\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla\mathbf{w})\|_{\mathbf{L}^s(\Omega)} &\le \|\mathbf{w}\|_{\mathbf{L}^\infty(\Omega)}\|\nabla\mathbf{y}\|_{\mathbf{L}^s(\Omega)^2} + \|\mathbf{y}\|_{\mathbf{L}^\infty(\Omega)}\|\nabla\mathbf{w}\|_{\mathbf{L}^s(\Omega)^2} \\ &\le c\|\mathbf{y}\|_{\mathbf{V}^{2,2}(\Omega)}\|\mathbf{w}\|_{\mathbf{W}_0^{1,s}(\Omega)} \end{aligned}$$

in virtue of the continuity of $\mathbf{W}_0^{1,s}(\Omega) \subset \mathbf{L}^\infty(\Omega)$ and $\mathbf{V}^{2,2}(\Omega) \subset \mathbf{W}_0^{1,s}(\Omega)$. Using the fact that $\mathfrak{A}_{\mathbf{y}} \in \mathcal{L}_{\rm iso}(\mathbf{W}_0^{1,s}(\Omega) \times \widehat{L}^s(\Omega), \mathbf{W}^{-1,s}(\Omega) \times \widehat{L}^s(\Omega))$, (2.24), and the L^s-regularity theory for the Stokes equation, we have

$$\|\mathbf{w}\|_{\mathbf{X}^{2,s}(\Omega)} + \|\pi\|_{\widehat{W}^{1,s}(\Omega)} \le c_{\mathbf{y}}(\|\mathbf{r}\|_{\mathbf{L}^s(\Omega)} + \|q\|_{\widehat{W}^{1,s}(\Omega)}).$$

Therefore, we obtain that $\mathfrak{A}_{\mathbf{y}} \in \mathcal{L}_{\rm iso}(\mathbf{X}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega), \mathbf{L}^s(\Omega) \times \widehat{W}^{1,s}(\Omega))$, and as a result, $A_{\mathbf{y}} \in \mathcal{L}_{\rm iso}(\mathbf{V}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega), \mathbf{L}^s(\Omega))$. $\square$

For each $\varrho > 0$ and $1 < s < \infty$, we denote the open ball in $\mathbf{V}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega)$ centered at $(\mathbf{y}, p) \in \mathbf{V}^{2,s}(\Omega)\times\widehat{W}^{1,s}(\Omega)$ by $B_{\varrho,s}(\mathbf{y}, p)$. Similarly, if $(\mathbf{y}, p) \in (\mathbf{V}^{2,2}(\Omega)\cap \mathbf{W}^{3,2}(\Omega))\times\widehat{W}^{2,2}(\Omega)$, then $B_\varrho(\mathbf{y}, p)$ is the open ball in $(\mathbf{V}^{2,2}(\Omega)\cap\mathbf{W}^{3,2}(\Omega))\times\widehat{W}^{2,2}(\Omega)$.

Theorem 2.8. *Let $2 \leq s < \infty$ and $\mathbf{y}^* \in \mathbf{V}^{2,s}(\Omega)$ be a regular solution to (1.2) corresponding to $\mathbf{u}^* \in \mathbf{L}^s(\Omega)$ with an associated pressure $p^* \in \widehat{W}^{1,s}(\Omega)$. Then, there exists $\varrho > 0$, an open, bounded, and convex set $\mathcal{U}_s(\mathbf{u}^*) \subset \mathbf{L}^s(\Omega)$ containing $\mathbf{u}^*$, and a C^∞-map $\mathcal{S}_s : \mathcal{U}_s(\mathbf{u}^*) \to B_{\varrho,s}(\mathbf{y}^*, p^*)$ such that for each $\mathbf{u} \in \mathcal{U}_s(\mathbf{u}^*)$, the pair $(\mathbf{y}, p) = \mathcal{S}_s(\mathbf{u}) \in \mathbf{V}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega)$ is the unique solution of (1.2) and $\mathcal{S}_s'(\mathbf{u}) \in \mathcal{L}_{\text{iso}}(\mathbf{L}^s(\Omega), \mathbf{V}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega))$ for every $\mathbf{u} \in \mathcal{U}_s(\mathbf{u}^*)$. Moreover, $(\mathbf{w}, \pi) := \mathcal{S}_s'(\mathbf{u})\mathbf{r}$ for $\mathbf{u} \in \mathcal{U}_s(\mathbf{u}^*)$ and $\mathbf{r} \in \mathbf{L}^s(\Omega)$ if and only if $A_{\mathbf{y}}(\mathbf{w}, \pi) = \mathbf{r}$, that is,*

$$\left[\begin{array}{l} -\nu\Delta\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{w} + \nabla\pi = \mathbf{r} \quad \text{in } \Omega, \\ -\operatorname{div}\mathbf{w} = 0 \quad \text{in } \Omega, \quad \mathbf{w} = \mathbf{0} \quad \text{on } \Gamma, \quad \displaystyle\int_\Omega \pi \,\mathrm{d}x = 0. \end{array}\right. \tag{2.25}$$

Proof. We follow the proof of [10, Theorem 2.10] by using the implicit function theorem. Define the nonlinear operator $\mathcal{T}_s : \mathbf{V}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega) \to \mathbf{L}^s(\Omega) \times \widehat{W}^{1,s}(\Omega)$ according to

$$\mathcal{T}_s(\mathbf{y}, p, \mathbf{u}) := -\nu\Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p - \mathbf{u}.$$

Clearly, $\mathcal{T}_s$ is a C^∞-mapping and $\mathcal{T}_s(\mathbf{y}^*, p^*, \mathbf{u}^*) = 0$. By Lemma 2.7

$$\frac{\partial\mathcal{T}_s}{\partial(\mathbf{y}, p)}(\mathbf{y}^*, p^*, \mathbf{u}^*) = A_{\mathbf{y}^*} \in \mathcal{L}_{\text{iso}}(\mathbf{V}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega), \mathbf{L}^s(\Omega))$$

due to the assumption that $\mathbf{y}^*$ is regular. Therefore, the conclusions of the theorem follow from the implicit function theorem for Banach spaces [26, Section 4.7] and (2.25) can be obtained via implicit differentiation. □

The analog of the previous theorem for controls in the Sobolev space $\mathbf{W}^{1,2}(\Omega)$ is given below.

Theorem 2.9. *Let $(\mathbf{y}^*, p^*) \in (\mathbf{V}^{2,2}(\Omega) \cap \mathbf{W}^{3,2}(\Omega)) \times \widehat{W}^{2,2}(\Omega)$ be a solution to (1.2) with control $\mathbf{u}^* \in \mathbf{W}^{1,2}(\Omega)$ such that $\mathbf{y}^*$ is regular. Then, there is $\varrho > 0$, an open, bounded, and convex set $\mathcal{V}(\mathbf{u}^*) \subset \mathbf{W}^{1,2}(\Omega)$ containing $\mathbf{u}^*$, and a C^∞-map $\mathcal{R} : \mathcal{V}(\mathbf{u}^*) \to B_\varrho(\mathbf{y}^*, p^*)$ so that for each $\mathbf{u} \in \mathcal{V}(\mathbf{u}^*)$, the unique solution of (1.2) is given by $(\mathbf{y}, p) = \mathcal{R}(\mathbf{u}) \in (\mathbf{V}^{2,2}(\Omega) \cap \mathbf{W}^{3,2}(\Omega)) \times \widehat{W}^{2,2}(\Omega)$.*

Proof. The proof is the same as that of the previous theorem, but now recognizing the fact that $A_{\mathbf{y}^*} \in \mathcal{L}_{\text{iso}}((\mathbf{V}^{2,2}(\Omega) \cap \mathbf{W}^{3,2}(\Omega)) \times \widehat{W}^{2,2}(\Omega), \mathbf{W}^{1,2}(\Omega))$ whenever $\mathbf{y}^*$ is regular. Indeed, suppose $\mathbf{u} \in \mathbf{W}^{1,2}(\Omega)$. Due to $A_{\mathbf{y}^*} \in \mathcal{L}_{\text{iso}}(\mathbf{V}^{2,2}(\Omega) \times \widehat{W}^{1,2}(\Omega), \mathbf{L}^2(\Omega))$ by regularity of $\mathbf{y}^*$, there exists a unique $(\mathbf{w}, \pi) \in \mathbf{V}^{2,2}(\Omega) \times \widehat{W}^{1,2}(\Omega)$ such that $A_{\mathbf{y}^*}(\mathbf{w}, \pi) = \mathbf{u}$. The latter equation is equivalent to

$$\left[\begin{array}{l} -\nu\Delta\mathbf{w} + \nabla\pi = \mathbf{u} - (\mathbf{y}^*\cdot\nabla)\mathbf{w} - (\mathbf{w}\cdot\nabla)\mathbf{y}^* \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{w} = 0 \quad \text{in } \Omega, \quad \mathbf{w} = \mathbf{0} \quad \text{on } \Gamma, \quad \displaystyle\int_\Omega \pi \,\mathrm{d}x = 0. \end{array}\right.$$

Moreover, we have

$$\|\mathbf{w}\|_{\mathbf{V}^{2,2}(\Omega)} + \|\pi\|_{\widehat{W}^{1,2}(\Omega)} \leq c\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)}. \tag{2.26}$$

As in (2.10), one can show that $(\mathbf{y}^*\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{y}^* \in \mathbf{W}^{1,2}(\Omega)$ and

$$\|(\mathbf{y}^*\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{y}^*\|_{\mathbf{W}^{1,2}(\Omega)} \leq c\|\mathbf{y}^*\|_{\mathbf{V}^{2,2}(\Omega)}\|\mathbf{w}\|_{\mathbf{V}^{2,2}(\Omega)}. \tag{2.27}$$

Applying the regularity theory for the Stokes equation, (2.26), and (2.27), we deduce that $(\mathbf{w},\pi) \in (\mathbf{V}^{2,2}(\Omega)\cap \mathbf{W}^{3,2}(\Omega)) \times \widehat{W}^{2,2}(\Omega)$ and

$$\|\mathbf{w}\|_{\mathbf{V}^{2,2}(\Omega)\cap\mathbf{W}^{3,2}(\Omega)} + \|\pi\|_{\widehat{W}^{2,2}(\Omega)} \le c(\|\mathbf{y}^*\|_{\mathbf{V}^{2,2}(\Omega)})\|\mathbf{u}\|_{\mathbf{W}^{1,2}(\Omega)}.$$

Therefore, $A_{\mathbf{y}^*} : (\mathbf{V}^{2,2}(\Omega)\cap\mathbf{W}^{3,2}(\Omega))\times\widehat{W}^{2,2}(\Omega) \to \mathbf{W}^{1,2}(\Omega)$ is an isomorphism. □

Now, we include the traces of the linearized and adjoint velocities in the definition of the operators discussed above. In this direction, we define the bounded linear operators

$$\mathfrak{B}_{\mathbf{y}}, \mathfrak{D}_{\mathbf{y}} : \mathbf{W}^{1,s}(\Omega)\times\widehat{L}^s(\Omega) \to \mathbf{W}^{-1,s}(\Omega)\times\mathbf{Z}^{0,1-\frac{1}{s},s}(\Omega,\Gamma)$$

according to

$$\mathfrak{B}_{\mathbf{y}}(\mathbf{w},\pi) := (\mathfrak{A}_{\mathbf{y}}(\mathbf{w},\pi), \mathbf{w}|_\Gamma), \quad \mathfrak{D}_{\mathbf{y}}(\mathbf{v},\sigma) := (\mathfrak{A}^\star_{\mathbf{y}}(\mathbf{v},\sigma), \mathbf{v}|_\Gamma),$$

for $(\mathbf{w},\pi), (\mathbf{v},\sigma) \in \mathbf{W}^{1,s}(\Omega)\times\widehat{L}^s(\Omega)$. Here, $\mathfrak{A}_{\mathbf{y}}(\mathbf{w},\pi)$ and $\mathfrak{A}^\star_{\mathbf{y}}(\mathbf{v},\sigma)$ are defined as in (2.17) and (2.18), respectively. These maps are well-defined in virtue of the trace and Gauss divergence theorems:

$$\int_\Omega (-\operatorname{div}\mathbf{w})\,\mathrm{d}x + \int_\Gamma \mathbf{w}|_\Gamma\cdot\mathbf{n}\,\mathrm{d}s = 0 = \int_\Omega (-\operatorname{div}\mathbf{v})\,\mathrm{d}x + \int_\Gamma \mathbf{v}|_\Gamma\cdot\mathbf{n}\,\mathrm{d}s.$$

Hence, $(-\operatorname{div}\mathbf{w}, \mathbf{w}|_\Gamma), (-\operatorname{div}\mathbf{v}, \mathbf{v}|_\Gamma) \in \mathbf{Z}^{0,1-\frac{1}{s},s}(\Omega,\Gamma)$.

The linear maps $\mathfrak{B}_{\mathbf{y}}$ and $\mathfrak{D}_{\mathbf{y}}$ are isomorphisms provided that $\mathbf{y}$ is a regular solution according to the succeeding lemma. For this, we define the Stokes operator

$$\mathfrak{S} : \mathbf{W}^{2,s}(\Omega)\times\widehat{W}^{1,s}(\Omega) \to \mathbf{L}^s(\Omega)\times\mathbf{Z}^{1,2-\frac{1}{s},s}(\Omega,\Gamma)$$

according to

$$\mathfrak{S}(\mathbf{w},\pi) := (-\nu\Delta\mathbf{w} + \nabla\pi, -\operatorname{div}\mathbf{w}, \mathbf{w}|_\Gamma). \tag{2.28}$$

It is well-known that

$$\mathfrak{S} \in \mathcal{L}_{\mathrm{iso}}(\mathbf{W}^{k,s}(\Omega)\times\widehat{F}^{k-1,s}(\Omega), \mathbf{Y}^{k-2,s}(\Omega)\times\mathbf{Z}^{k-1,k-\frac{1}{s},s}(\Omega,\Gamma)) \tag{2.29}$$

for each non-negative integer k, where $\widehat{F}^{k-1,s}(\Omega)$ is defined in a similar way as in (2.1) with W replaced by $\widehat{W}$ and k replaced by $k-1$, and if Ω is bounded $C^{\max\{2,k\}}$-domain, where $\mathbf{Y}^{-2,s}(\Omega) := \mathbf{X}^{-2,s}(\Omega)$, $\mathbf{Y}^{-1,s}(\Omega) := \mathbf{W}^{-1,s}(\Omega)$, $\mathbf{Y}^{0,s}(\Omega) := \mathbf{L}^s(\Omega)$, and $\mathbf{Y}^{k,s}(\Omega) := \mathbf{W}^{k,s}(\Omega)$ when $k > 0$ is an integer. For this, see for instance [15, Theorem IV.6.1] for $k \ge 2$, [24, Proposition 2.3] for $k \ge 1$, and [18, Theorem 7] for $k = 0$.

For the operator $\mathfrak{S}_0$ given by

$$\mathfrak{S}_0(\mathbf{w},\pi) := -\nu\Delta\mathbf{w} + \nabla\pi, \tag{2.30}$$

we have

$$\mathfrak{S}_0 \in \mathcal{L}_{\mathrm{iso}}(\mathbf{V}^{k,s}(\Omega)\times\widehat{F}^{k-1,s}(\Omega), \mathbf{Y}^{k-2,s}(\Omega)). \tag{2.31}$$

Lemma 2.10. *If $\mathbf{y} \in \mathbf{V}^{2,2}(\Omega)$ is a regular solution to (1.2), then for every $1 < s < \infty$ we have*

$$\mathfrak{B}_{\mathbf{y}}, \mathfrak{D}_{\mathbf{y}} \in \mathcal{L}_{\mathrm{iso}}(\mathbf{W}^{1,s}(\Omega)\times\widehat{L}^s(\Omega), \mathbf{W}^{-1,s}(\Omega)\times\mathbf{Z}^{0,1-\frac{1}{s},s}(\Omega,\Gamma))$$

$$\cap\, \mathcal{L}_{\text{iso}}(\mathbf{W}^{2,s}(\Omega)\times \widehat{W}^{1,s}(\Omega), \mathbf{L}^s(\Omega)\times \mathbf{Z}^{1,2-\frac{1}{s},s}(\Omega,\Gamma)).$$

Proof. Let $(\mathbf{r},q,\mathbf{z})\in \mathbf{W}^{-1,s}(\Omega)\times \mathbf{Z}^{0,1-\frac{1}{s},s}(\Omega,\Gamma)$. By (2.29) with $k=1$, there exists $(\mathbf{w}_1,\pi_1)\in \mathbf{W}^{1,s}(\Omega)\times \widehat{L}^s(\Omega)$ such that $\mathfrak{S}(\mathbf{w}_1,\pi_1)=(\mathbf{r},q,\mathbf{z})$ and satisfying the a priori estimate

$$\|\mathbf{w}_1\|_{\mathbf{W}^{1,s}(\Omega)}+\|\pi_1\|_{\widehat{L}^s(\Omega)}\le c_{\mathbf{y}}\|(\mathbf{r},q,\mathbf{z})\|_{\mathbf{W}^{-1,s}(\Omega)\times\mathbf{Z}^{0,1-\frac{1}{s},s}(\Omega,\Gamma)}. \tag{2.32}$$

Since $(\mathbf{w}_1\cdot\nabla)\mathbf{y}+(\mathbf{y}\cdot\nabla)\mathbf{w}_1\in \mathbf{L}^s(\Omega)$, we obtain from Lemma 2.7 that $\mathfrak{A}_{\mathbf{y}}(\mathbf{w}_2,\pi_2)=(-(\mathbf{w}_1\cdot\nabla)\mathbf{y}-(\mathbf{y}\cdot\nabla)\mathbf{w}_1,0)$ for some $(\mathbf{w}_2,\pi_2)\in \mathbf{V}^{2,s}(\Omega)\times\widehat{W}^{1,s}(\Omega)$ and

$$\|\mathbf{w}_2\|_{\mathbf{V}^{2,s}(\Omega)}+\|\pi_2\|_{\widehat{W}^{1,s}(\Omega)}\le c_{\mathbf{y}}\|\mathbf{w}_1\|_{\mathbf{W}^{1,s}(\Omega)}. \tag{2.33}$$

If $(\mathbf{w},\pi):=(\mathbf{w}_1+\mathbf{w}_2,\pi_1+\pi_2)\in \mathbf{W}^{1,s}(\Omega)\times\widehat{L}^s(\Omega)$, then $\mathfrak{B}_{\mathbf{y}}(\mathbf{w},\pi)=(\mathbf{r},q,\mathbf{z})$, and by (2.32), (2.33), and the triangle inequality, one has

$$\|\mathbf{w}\|_{\mathbf{W}^{1,s}(\Omega)}+\|\pi\|_{\widehat{L}^s(\Omega)}\le c_{\mathbf{y}}\|(\mathbf{r},q,\mathbf{z})\|_{\mathbf{W}^{-1,s}(\Omega)\times\mathbf{Z}^{0,1-\frac{1}{s},s}(\Omega,\Gamma)}.$$

On the other hand, if $(\mathbf{r},q,\mathbf{z})\in \mathbf{L}^s(\Omega)\times\mathbf{Z}^{1,2-\frac{1}{s},s}(\Omega,\Gamma)$, then $(\mathbf{w}_1,\pi_1)\in\mathbf{W}^{2,s}(\Omega)\times \widehat{W}^{1,s}(\Omega)$ from (2.29) for $k=2$ with the a priori estimate

$$\|\mathbf{w}_1\|_{\mathbf{W}^{2,s}(\Omega)}+\|\pi_1\|_{\widehat{W}^{1,s}(\Omega)}\le c_{\mathbf{y}}\|(\mathbf{r},q,\mathbf{z})\|_{\mathbf{L}^s(\Omega)\times\mathbf{Z}^{1,2-\frac{1}{s},s}(\Omega,\Gamma)}. \tag{2.34}$$

Therefore, (2.33) and (2.34) lead to

$$\|\mathbf{w}\|_{\mathbf{W}^{2,s}(\Omega)}+\|\pi\|_{\widehat{W}^{1,s}(\Omega)}\le c_{\mathbf{y}}\|(\mathbf{r},q,\mathbf{z})\|_{\mathbf{L}^s(\Omega)\times\mathbf{Z}^{1,2-\frac{1}{s},s}(\Omega,\Gamma)}.$$

The injectivity of $\mathfrak{D}_{\mathbf{y}}$ follows from that of $\mathfrak{A}_{\mathbf{y}}$. This completes the proof for the case of the operator $\mathfrak{B}_{\mathbf{y}}$. The case of $\mathfrak{D}_{\mathbf{y}}$ is completely the same, where we use $\mathfrak{A}_{\mathbf{y}}^\star$ instead of $\mathfrak{A}_{\mathbf{y}}$. □

We now discuss the very weak formulation of the adjoint equation (2.13). The following definition is based on the duality equation (2.16). This definition extends that of the case of Hilbert spaces in [21, 22]. Moreover, in contrast to [18] for the stationary Stokes and Navier–Stokes equations, we include the pressure in the definition, hence, one must consider test functions that are not necessarily divergence-free. Although, these two formulations are equivalent by de Rham's Theorem, the definition provided below is more appropriate when studying optimal control problems with observations involving the pressure.

Definition 2.11. *Let $1<s<\infty$, $\mathbf{f}\in\mathbf{X}^{-2,s}(\Omega)$, and $(g,\mathbf{h})\in\mathbf{Z}^{-1,-\frac{1}{s},s}(\Omega,\Gamma)$. Then, $(\mathbf{v},\sigma)\in\mathbf{L}^s(\Omega)\times\widehat{W}^{1,s'}(\Omega)'$ is called a very weak solution to the adjoint problem (2.13) if the variational equation*

$$\begin{aligned}&\int_\Omega \mathbf{v}\cdot(-\nu\Delta\mathbf{w}+(\mathbf{w}\cdot\nabla)\mathbf{y}+(\mathbf{y}\cdot\nabla)\mathbf{w}+\nabla\pi)\,\mathrm{d}x\\ &\qquad-\langle\sigma,\operatorname{div}\mathbf{w}\rangle_{\widehat{W}^{1,s'}(\Omega)',\widehat{W}^{1,s'}(\Omega)}=\langle\mathbf{f},\mathbf{w}\rangle_{\mathbf{X}^{-2,s}(\Omega),\mathbf{X}^{2,s'}(\Omega)}\\ &\qquad+\langle g,\pi\rangle_{W^{1,s'}(\Omega)',W^{1,s'}(\Omega)}+\langle\mathbf{h},\mathbf{T}(\mathbf{w},\pi)\mathbf{n}\rangle_{\mathbf{W}^{-\frac{1}{s},s}(\Gamma),\mathbf{W}^{1-\frac{1}{s'},s'}(\Gamma)}\end{aligned} \tag{2.35}$$

holds for every test function $(\mathbf{w},\pi)\in\mathbf{X}^{2,s'}(\Omega)\times W^{1,s'}(\Omega)$.

Theorem 2.12. *Let $\mathbf{y} \in \mathbf{V}^{2,2}(\Omega)$ be a regular solution to (1.2). Given $1 < s < \infty$, the adjoint problem (2.13) with data $\mathbf{f} \in \mathbf{X}^{-2,s}(\Omega)$ and $(g, \mathbf{h}) \in \mathbf{Z}^{-1,-\frac{1}{s},s}(\Omega, \Gamma)$ admits a unique very weak solution $(\mathbf{v}, \sigma) \in \mathbf{L}^s(\Omega) \times \widehat{W}^{1,s'}(\Omega)'$ such that*

$$\|\mathbf{v}\|_{\mathbf{L}^s(\Omega)} + \|\sigma\|_{\widehat{W}^{1,s'}(\Omega)'} \leq c(\|\mathbf{f}\|_{\mathbf{X}^{-2,s}(\Omega)} + \|g\|_{W^{1,s'}(\Omega)'} + \|\mathbf{h}\|_{\mathbf{W}^{-\frac{1}{s},s}(\Gamma)}). \tag{2.36}$$

Proof. Take note that due to the compatibility condition for $(g, \mathbf{h})$, the variational equation (2.35) is equivalent to the one with test functions $(\mathbf{w}, \pi) \in \mathbf{X}^{2,s'}(\Omega) \times \widehat{W}^{1,s'}(\Omega)$. Consider the linear map $\ell : \mathbf{L}^{s'}(\Omega) \times \widehat{W}^{1,s'}(\Omega) \to \mathbb{R}$ given by

$$\begin{aligned}\ell(\mathbf{r}, q) := \langle \mathbf{f}, \mathbf{w}\rangle_{\mathbf{X}^{-2,s}(\Omega),\mathbf{X}^{2,s'}(\Omega)} + \langle g, \pi\rangle_{W^{1,s'}(\Omega)',W^{1,s'}(\Omega)} \\ + \langle \mathbf{h}, \mathbf{T}(\mathbf{w}, \pi)\mathbf{n}\rangle_{\mathbf{W}^{-\frac{1}{s},s}(\Gamma),\mathbf{W}^{1-\frac{1}{s'},s'}(\Gamma)}\end{aligned} \tag{2.37}$$

where, for a given $(\mathbf{r}, q) \in \mathbf{L}^{s'}(\Omega) \times \widehat{W}^{1,s'}(\Omega)$, the pair $(\mathbf{w}, \pi) \in \mathbf{X}^{2,s'}(\Omega) \times \widehat{W}^{1,s'}(\Omega)$ satisfies $\mathfrak{A}_{\mathbf{y}}(\mathbf{w}, \pi) = (\mathbf{r}, q)$. Such a pair exists due to Lemma 2.7, and furthermore,

$$\|\mathbf{w}\|_{\mathbf{X}^{2,s'}(\Omega)} + \|\pi\|_{\widehat{W}^{1,s'}(\Omega)} \leq c(\|\mathbf{r}\|_{\mathbf{L}^{s'}(\Omega)} + \|q\|_{\widehat{W}^{1,s'}(\Omega)}). \tag{2.38}$$

By standard trace theory, we have

$$\|\mathbf{T}(\mathbf{w}, \pi)\mathbf{n}\|_{\mathbf{W}^{1-\frac{1}{s'},s'}(\Gamma)} \leq c(\|\mathbf{w}\|_{\mathbf{X}^{2,s'}(\Omega)} + \|\pi\|_{\widehat{W}^{1,s'}(\Omega)}). \tag{2.39}$$

Combining the estimates (2.38) and (2.39), we get

$$\|\mathbf{w}\|_{\mathbf{X}^{2,s'}(\Omega)} + \|\pi\|_{\widehat{W}^{1,s'}(\Omega)} + \|\mathbf{T}(\mathbf{w}, \pi)\mathbf{n}\|_{\mathbf{W}^{1-\frac{1}{s'},s'}(\Gamma)} \leq c(\|\mathbf{r}\|_{\mathbf{L}^{s'}(\Omega)} + \|q\|_{\widehat{W}^{1,s'}(\Omega)}).$$

By duality, this inequality implies the existence of a pair $(\mathbf{v}, \sigma) \in \mathbf{L}^s(\Omega) \times \widehat{W}^{1,s'}(\Omega)'$ such that

$$\begin{aligned}\langle (\mathbf{v}, \sigma), (\mathbf{r}, q)\rangle_{\mathbf{L}^s(\Omega)\times\widehat{W}^{1,s'}(\Omega)',\mathbf{L}^{s'}(\Omega)\times\widehat{W}^{1,s'}(\Omega)} \\ = \ell(\mathbf{r}, q) \quad \forall(\mathbf{r}, q) \in \mathbf{L}^{s'}(\Omega) \times \widehat{W}^{1,s'}(\Omega).\end{aligned} \tag{2.40}$$

In addition, we obtain

$$\begin{aligned}\|\mathbf{v}\|_{\mathbf{L}^s(\Omega)} + \|\sigma\|_{\widehat{W}^{1,s'}(\Omega)'} &= \|\ell\|_{[\mathbf{L}^{s'}(\Omega)\times\widehat{W}^{1,s'}(\Omega)]'} \\ &\leq c(\|\mathbf{f}\|_{\mathbf{X}^{-2,s}(\Omega)} + \|g\|_{W^{1,s'}(\Omega)'} + \|\mathbf{h}\|_{\mathbf{W}^{-\frac{1}{s},s}(\Gamma)}).\end{aligned}$$

Given a test function $(\mathbf{w}, \pi) \in \mathbf{X}^{2,s'}(\Omega) \times \widehat{W}^{1,s'}(\Omega)$, we set $(\mathbf{r}, q) = \mathfrak{A}_{\mathbf{y}}(\mathbf{w}, q)$ so that from (2.37) and (2.40),

$$\begin{aligned}\langle (\mathbf{v}, \sigma), \mathfrak{A}_{\mathbf{y}}(\mathbf{w}, \pi)\rangle_{\mathbf{L}^s(\Omega)\times\widehat{W}^{1,s'}(\Omega)',\mathbf{L}^{s'}(\Omega)\times\widehat{W}^{1,s'}(\Omega)} = \langle \mathbf{f}, \mathbf{w}\rangle_{\mathbf{X}^{-2,s}(\Omega),\mathbf{X}^{2,s'}(\Omega)} \\ + \langle g, \pi\rangle_{W^{1,s'}(\Omega)',W^{1,s'}(\Omega)} + \langle \mathbf{h}, \mathbf{T}(\mathbf{w}, \pi)\mathbf{n}\rangle_{\mathbf{W}^{-\frac{1}{s},s}(\Gamma),\mathbf{W}^{1-\frac{1}{s'},s'}(\Gamma)}.\end{aligned}$$

This shows that $(\mathbf{v}, \sigma)$ is a very weak solution to (2.13) and (2.36) holds for this pair.

If $(\mathbf{v}_1, \sigma_1)$ and $(\mathbf{v}_2, \sigma_2)$ are two very weak solutions of (2.13) in $\mathbf{L}^s(\Omega) \times \widehat{W}^{1,s'}(\Omega)'$, then the difference $(\mathbf{v}, \sigma) := (\mathbf{v}_1 - \mathbf{v}_2, \sigma_1 - \sigma_2)$ satisfies

$$\langle (\mathbf{v}, \sigma), \mathfrak{A}_{\mathbf{y}}(\mathbf{w}, \pi)\rangle_{\mathbf{L}^s(\Omega)\times\widehat{W}^{1,s'}(\Omega)',\mathbf{L}^{s'}(\Omega)\times\widehat{W}^{1,s'}(\Omega)} = 0$$

for every $(\mathbf{w}, \pi) \in \mathbf{X}^{2,s'}(\Omega) \times \widehat{W}^{1,s'}(\Omega)$. From the surjectivity of $\mathfrak{A}_{\mathbf{y}} : \mathbf{X}^{2,s'}(\Omega) \times \widehat{W}^{1,s'}(\Omega) \to \mathbf{L}^{s'}(\Omega) \times \widehat{W}^{1,s'}(\Omega)$ in Lemma 2.7, we conclude that

$(\mathbf{v},\sigma) = (\mathbf{0},0)$, establishing the uniqueness of the very weak solution. □

Corollary 2.13. *If $\mathbf{y}\in \mathbf{V}^{2,2}(\Omega)$ is a regular solution to (1.2), then for every $1 < s < \infty$, the linear operators $\mathfrak{B}_{\mathbf{y}}$ and $\mathfrak{D}_{\mathbf{y}}$ admit unique extensions such that*

$$\mathfrak{B}_{\mathbf{y}}, \mathfrak{D}_{\mathbf{y}} \in \mathcal{L}_{\mathrm{iso}}(\mathbf{L}^s(\Omega)\times \widehat{W}^{1,s'}(\Omega)', \mathbf{X}^{-2,s}(\Omega)\times \mathbf{Z}^{-1,-\frac{1}{s},s}(\Omega,\Gamma)).$$

Proof. The case of $\mathfrak{D}_{\mathbf{y}}$ is precisely the result of Theorem 2.12, while the case of $\mathfrak{B}_{\mathbf{y}}$ can be handled in a similar fashion where we use the operator $\mathfrak{A}_{\mathbf{y}}^\star$ instead of $\mathfrak{A}_{\mathbf{y}}$. □

We close this section by proving general integration by parts formula and the regularity of the normal Cauchy stress on the boundary for weak and very weak solutions under certain conditions. From now on, we implicitly assume that $\mathbf{y}\in \mathbf{V}^{2,2}(\Omega)$ is a regular solution to (1.2). For analogous results, we refer to [18].

Theorem 2.14. *Let $1 < s < \infty$, $\mathbf{v}\in \mathbf{W}^{1,s}_0(\Omega)$, and $\sigma \in \widehat{L}^s(\Omega)$ be such that $A_{\mathbf{y}}^\star(\mathbf{v},\sigma)\in \mathbf{W}^{1,s'}(\Omega)'$. Then, there exists a unique $\mathbf{s}\in \mathbf{W}^{-\frac{1}{s},s}(\Gamma)$ for which the generalized Green's identity*

$$\begin{aligned}\langle A_{\mathbf{y}}^\star(\mathbf{v},\sigma),\mathbf{w}\rangle_{\mathbf{W}^{1,s'}(\Omega)',\mathbf{W}^{1,s'}(\Omega)} - \int_\Omega \pi \operatorname{div}\mathbf{v}\,\mathrm{d}x \\ = \langle A_{\mathbf{y}}(\mathbf{w},\pi),\mathbf{v}\rangle_{\mathbf{W}^{-1,s'}(\Omega),\mathbf{W}^{1,s}_0(\Omega)} - \int_\Omega \sigma\operatorname{div}\mathbf{w} + \langle \mathbf{s},\mathbf{w}|_\Gamma\rangle_{\mathbf{W}^{-\frac{1}{s},s}(\Gamma),\mathbf{W}^{1-\frac{1}{s'},s'}(\Gamma)}\end{aligned} \tag{2.41}$$

holds for every $(\mathbf{w},\pi)\in \mathbf{W}^{1,s'}(\Omega)\times \widehat{L}^{s'}(\Omega)$ such that $A_{\mathbf{y}}(\mathbf{w},\pi)\in \mathbf{W}^{-1,s'}(\Omega)$ and

$$\|\mathbf{s}\|_{\mathbf{W}^{-\frac{1}{s},s}(\Gamma)} \le c(\|A_{\mathbf{y}}^\star(\mathbf{v},\sigma)\|_{\mathbf{W}^{1,s'}(\Omega)'} + \|\operatorname{div}\mathbf{v}\|_{\widehat{L}^s(\Omega)}). \tag{2.42}$$

In addition, if $\mathbf{v}\in \mathbf{X}^{2,s}(\Omega)$ and $\sigma\in \widehat{W}^{1,s}(\Omega)$, then $\mathbf{s} = \mathbf{T}(\mathbf{v},\sigma)\mathbf{n}$ in $\mathbf{W}^{1-\frac{1}{s},s}(\Gamma)$.

Proof. Let $\sigma\in\widehat{L}^s(\Omega)$ and $\mathbf{v}\in \mathbf{W}^{1,s}_0(\Omega)$ be such that $A_{\mathbf{y}}^\star(\mathbf{v},\sigma)\in \mathbf{W}^{1,s'}(\Omega)'$. Consider the linear functional $\ell : \mathbf{W}^{-1,s'}(\Omega)\times \mathbf{Z}^{0,1-\frac{1}{s'},s'}(\Omega,\Gamma)\to\mathbb{R}$ defined by

$$\ell(\mathbf{r},q,\mathbf{z}) := \langle A_{\mathbf{y}}^\star(\mathbf{v},\sigma),\mathbf{w}\rangle_{\mathbf{W}^{1,s'}(\Omega)',\mathbf{W}^{1,s'}(\Omega)} - \int_\Omega \pi\operatorname{div}\mathbf{v}\,\mathrm{d}x,$$

where $\mathfrak{B}_{\mathbf{y}}(\mathbf{w},\pi) = (\mathbf{r},q,\mathbf{z})\in \mathbf{W}^{-1,s'}(\Omega)\times \mathbf{Z}^{0,1-\frac{1}{s'},s'}(\Omega,\Gamma)$ and $(\mathbf{w},\pi)\in \mathbf{W}^{1,s'}(\Omega)\times \widehat{L}^{s'}(\Omega)$. The existence of $(\mathbf{w},\pi)$ follows from Lemma 2.10, and moreover,

$$\|\mathbf{w}\|_{\mathbf{W}^{1,s'}(\Omega)} + \|\pi\|_{\widehat{L}^{s'}(\Omega)} \le c(\|\mathbf{r}\|_{\mathbf{W}^{-1,s'}(\Omega)} + \|(q,\mathbf{z})\|_{\mathbf{Z}^{0,1-\frac{1}{s'},s'}(\Omega,\Gamma)}).$$

Thus, we deduce that $\ell\in[\mathbf{W}^{-1,s'}(\Omega)\times \mathbf{Z}^{0,1-\frac{1}{s'},s'}(\Omega,\Gamma)]'$, and by duality there exists $(\widetilde{\mathbf{v}},\widetilde{\sigma},\widetilde{\mathbf{s}})\in \mathbf{W}^{1,s}_0(\Omega)\times \mathbf{Z}^{0,-\frac{1}{s},s}(\Omega,\Gamma)$ such that

$$\ell(\mathbf{r},q,\mathbf{z}) = \langle(\mathbf{r},q,\mathbf{z}),(\widetilde{\mathbf{v}},\widetilde{\sigma},\widetilde{\mathbf{s}})\rangle_{\mathbf{W}^{-1,s'}(\Omega)\times\mathbf{Z}^{0,1-\frac{1}{s'},s'}(\Omega,\Gamma),\mathbf{W}^{1,s}_0(\Omega)\times\mathbf{Z}^{0,-\frac{1}{s},s}(\Omega,\Gamma)}$$

for every $(\mathbf{r},q,\mathbf{z})\in \mathbf{W}^{-1,s'}(\Omega)\times \mathbf{Z}^{0,1-\frac{1}{s'},s'}(\Omega,\Gamma)$. Hence, we have

$$\begin{aligned}\langle A_{\mathbf{y}}(\mathbf{w},\pi),\widetilde{\mathbf{v}}\rangle_{\mathbf{W}^{-1,s'}(\Omega),\mathbf{W}^{1,s}_0(\Omega)} - \int_\Omega \widetilde{\sigma}\operatorname{div}\mathbf{w}\,\mathrm{d}x + \langle\widetilde{\mathbf{s}},\mathbf{w}|_\Gamma\rangle_{\mathbf{W}^{-\frac{1}{s},s}(\Gamma)\times\mathbf{W}^{1-\frac{1}{s'},s'}(\Gamma)} \\ = \langle\mathfrak{B}_{\mathbf{y}}(\mathbf{w},\pi),(\widetilde{\mathbf{v}},\widetilde{\sigma},\widetilde{\mathbf{s}})\rangle_{\mathbf{W}^{-1,s'}(\Omega)\times\mathbf{Z}^{0,1-\frac{1}{s'},s'}(\Omega,\Gamma),\mathbf{W}^{1,s}_0(\Omega)\times\mathbf{Z}^{0,-\frac{1}{s},s}(\Omega,\Gamma)}\end{aligned}$$

$$= \langle A_{\mathbf{y}}^{\star}(\mathbf{v},\sigma), \mathbf{w}\rangle_{\mathbf{W}^{1,s'}(\Omega)',\mathbf{W}^{1,s'}(\Omega)} - \int_\Omega \pi \operatorname{div} \mathbf{v}\, \mathrm{d}x \tag{2.43}$$

for every $(\mathbf{w},\pi) \in \mathbf{W}^{1,s'}(\Omega) \times \widehat{L}^{s'}(\Omega)$ with $A_{\mathbf{y}}(\mathbf{w},\pi) \in \mathbf{W}^{-1,s'}(\Omega)$. Observe that the uniqueness of the triple $(\widetilde{\mathbf{v}},\widetilde{\sigma},\widetilde{\mathbf{s}})$ follows from the surjectivity of $\mathfrak{B}_{\mathbf{y}} : \mathbf{W}^{1,s'}(\Omega) \times \widehat{L}^{s'}(\Omega) \to \mathbf{W}^{-1,s'}(\Omega) \times \mathbf{Z}^{0,1-\frac{1}{s'},s'}(\Omega,\Gamma)$. Moreover, one has the estimate

$$\|\widetilde{\mathbf{v}}\|_{\mathbf{W}_0^{1,s}(\Omega)} + \|(\widetilde{\sigma},\widetilde{\mathbf{s}})\|_{\mathbf{Z}^{0,-\frac{1}{s},s}(\Omega,\Gamma)} \le c(\|A_{\mathbf{y}}^{\star}(\mathbf{v},\sigma)\|_{\mathbf{W}^{1,s'}(\Omega)'} + \|\operatorname{div}\mathbf{v}\|_{\widehat{L}^s(\Omega)}). \tag{2.44}$$

Indeed, this follows from the estimate

$$\begin{aligned}
&|\langle(\mathbf{r},q,\mathbf{z}),(\widetilde{\mathbf{v}},\widetilde{\sigma},\widetilde{\mathbf{s}})\rangle_{\mathbf{W}^{-1,s'}(\Omega)\times\mathbf{Z}^{0,1-\frac{1}{s'},s'}(\Omega,\Gamma),\mathbf{W}_0^{1,s}(\Omega)\times\mathbf{Z}^{0,-\frac{1}{s},s}(\Omega,\Gamma)}| \\
&\quad = \left|\langle A_{\mathbf{y}}^{\star}(\mathbf{v},\sigma),\mathbf{w}\rangle_{\mathbf{W}^{1,s'}(\Omega)',\mathbf{W}^{1,s'}(\Omega)} - \int_\Omega \pi\operatorname{div}\mathbf{v}\,\mathrm{d}x\right| \\
&\quad \le \|A_{\mathbf{y}}^{\star}(\mathbf{v},\sigma)\|_{\mathbf{W}^{1,s'}(\Omega)'}\|\mathbf{w}\|_{\mathbf{W}^{1,s'}(\Omega)} + \|\operatorname{div}\mathbf{v}\|_{\widehat{L}^s(\Omega)}\|\pi\|_{\widehat{L}^{s'}(\Omega)} \\
&\quad \le c(\|A_{\mathbf{y}}^{\star}(\mathbf{v},\sigma)\|_{\mathbf{W}^{1,s'}(\Omega)'} + \|\operatorname{div}\mathbf{v}\|_{\widehat{L}^s(\Omega)})\|(\mathbf{r},q,\mathbf{z})\|_{\mathbf{W}^{-1,s'}(\Omega)\times\mathbf{Z}^{0,1-\frac{1}{s'},s'}(\Omega,\Gamma)}
\end{aligned}$$

and by invoking the definition of the dual norm.

Suppose that $(\mathbf{v},\sigma) \in \mathbf{X}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega)$ and $(\mathbf{w},\pi) \in \mathbf{W}^{1,s'}(\Omega) \times \widehat{L}^{s'}(\Omega)$ with $A_{\mathbf{y}}(\mathbf{w},\pi) \in \mathbf{W}^{-1,s'}(\Omega)$. By density, we can take a sequence such that $(\mathbf{r}_n,q_n,\mathbf{z}_n) \in \mathbf{L}^{s'}(\Omega)\times\mathbf{Z}^{1,2-\frac{1}{s'},s'}(\Omega,\Gamma)$ for each $n\in\mathbb{N}$, $\mathbf{r}_n \to A_{\mathbf{y}}(\mathbf{w},\pi)$ as $n\to\infty$ in $\mathbf{W}^{-1,s'}(\Omega)$, and $(q_n,\mathbf{z}_n) \to (-\operatorname{div}\mathbf{w},\mathbf{w}|_\Gamma)$ in $\mathbf{Z}^{0,1-\frac{1}{s'},s'}(\Omega,\Gamma)$. Let $(\mathbf{w}_n,\pi_n) \in \mathbf{W}^{2,s'}(\Omega)\times\widehat{W}^{1,s'}(\Omega)$ be such that $\mathfrak{B}_{\mathbf{y}}(\mathbf{w}_n,\pi_n) = (\mathbf{r}_n,q_n,\mathbf{z}_n)$. Then, $(\mathbf{w}_n,\pi_n)\to(\mathbf{w},\pi)$ in $\mathbf{W}^{1,s'}(\Omega)\times\widehat{L}^{s'}(\Omega)$ by continuity of $\mathfrak{B}_{\mathbf{y}}^{-1}$, and in particular, we have $\mathbf{w}_n|_\Gamma \to \mathbf{w}|_\Gamma$ in $\mathbf{W}^{1-\frac{1}{s'},s'}(\Gamma)$ by the trace theorem. Passing to the limit in the equation

$$\begin{aligned}
&\int_\Omega A_{\mathbf{y}}^{\star}(\mathbf{v},\sigma)\cdot\mathbf{w}_n\,\mathrm{d}x - \int_\Omega \pi_n\operatorname{div}\mathbf{v}\,\mathrm{d}x \\
&\quad = \int_\Omega A_{\mathbf{y}}(\mathbf{w}_n,\pi_n)\cdot\mathbf{v}\,\mathrm{d}x - \int_\Omega \sigma\operatorname{div}\mathbf{w}_n\,\mathrm{d}x + \int_\Gamma \mathbf{T}(\mathbf{v},\sigma)\mathbf{n}\cdot\mathbf{w}_n\,\mathrm{d}s \\
&\quad = \int_\Omega A_{\mathbf{y}}(\mathbf{w}_n,\pi_n)\cdot\mathbf{v}\,\mathrm{d}x - \int_\Omega [(\sigma+\kappa(\mathbf{v},\sigma)]\operatorname{div}\mathbf{w}_n\,\mathrm{d}x \\
&\qquad + \int_\Gamma [\mathbf{T}(\mathbf{v},\sigma)\mathbf{n} + \kappa(\mathbf{v},\sigma)\mathbf{n}]\cdot\mathbf{w}_n\,\mathrm{d}s
\end{aligned}$$

where

$$\kappa(\mathbf{v},\sigma) := -\frac{1}{|\Omega|+|\Gamma|}\langle\mathbf{T}(\mathbf{v},\sigma)\mathbf{n},\mathbf{n}\rangle_\Gamma, \tag{2.45}$$

we can see that (2.43) holds for the triple $(\mathbf{v},\sigma+\kappa(\mathbf{v},\sigma),\mathbf{T}(\mathbf{v},\sigma)\mathbf{n}+\kappa(\mathbf{v},\sigma)\mathbf{n}) \in \mathbf{X}^{2,s}(\Omega)\times\mathbf{Z}^{1,1-\frac{1}{s},s}(\Omega,\Gamma) \subset \mathbf{W}_0^{1,s}(\Omega)\times\mathbf{Z}^{0,-\frac{1}{s},s}(\Omega,\Gamma)$, and in virtue of uniqueness, we must have $\widetilde{\mathbf{v}} = \mathbf{v}$, $\widetilde{\sigma} = \sigma+\kappa(\mathbf{v},\sigma)$, and $\widetilde{\mathbf{s}} = \mathbf{T}(\mathbf{v},\sigma)\mathbf{n}+\kappa(\mathbf{v},\sigma)\mathbf{n}$.

Let us return to the case $(\mathbf{v},\sigma)\in\mathbf{W}_0^{1,s}(\Omega)\times\widehat{L}^s(\Omega)$ where $A_{\mathbf{y}}^{\star}(\mathbf{v},\sigma)\in\mathbf{W}^{1,s'}(\Omega)'$. Using density once again, there exists a sequence $(\mathbf{f}_n,g_n)\in\mathbf{L}^s(\Omega)\times\widehat{W}^{1,s}(\Omega)$ such that $\mathbf{f}_n \to A_{\mathbf{y}}^{\star}(\mathbf{v},\sigma)$ in $\mathbf{W}^{1,s'}(\Omega)'$ and $g_n \to -\operatorname{div}\mathbf{v}$ in $\widehat{L}^s(\Omega)$. Define $(\mathbf{v}_n,\sigma_n)\in\mathbf{X}^{2,s}(\Omega)\times\widehat{W}^{1,s}(\Omega)$ by $\mathfrak{A}_{\mathbf{y}}^{\star}(\mathbf{v}_n,\sigma_n) = (\mathbf{f}_n,g_n)$. By construction, we have $\mathfrak{A}_{\mathbf{y}}^{\star}(\mathbf{v}-\mathbf{v}_n,\sigma-\sigma_n) = (A_{\mathbf{y}}^{\star}(\mathbf{v},\sigma)-\mathbf{f}_n,-\operatorname{div}\mathbf{v}-g_n)$. Hence, $(\mathbf{v}_n,\sigma_n)\to(\mathbf{v},\sigma)$ in $\mathbf{W}_0^{1,s}(\Omega)\times\widehat{L}^s(\Omega)$ by continuity of $(\mathfrak{A}_{\mathbf{y}}^\star)^{-1}$ and the construction of $\mathbf{f}_n$ and g_n. Applying the result of the preceding paragraph to the pair $(\mathbf{v}_n, \sigma_n)$, we obtain that

$$\begin{aligned}\langle A_{\mathbf{y}}(\mathbf{w},\pi), \widetilde{\mathbf{v}}_n\rangle_{\mathbf{W}^{-1,s'}(\Omega),\mathbf{W}_0^{1,s}(\Omega)} - \int_\Omega \widetilde{\sigma}_n \operatorname{div}\mathbf{w}\,\mathrm{d}x \\ + \langle \widetilde{\mathbf{s}}_n, \mathbf{w}|_\Gamma\rangle_{\mathbf{W}^{-\frac{1}{s},s}(\Gamma)\times\mathbf{W}^{1-\frac{1}{s'},s'}(\Gamma)} = \langle \mathbf{f}_n, \mathbf{w}\rangle_{\mathbf{W}^{1,s'}(\Omega)',\mathbf{W}^{1,s'}(\Omega)} + \int_\Omega \pi g_n\,\mathrm{d}x,\end{aligned} \tag{2.46}$$

where $\widetilde{\mathbf{v}}_n := \mathbf{v}_n$, $\widetilde{\sigma}_n := \sigma_n + \kappa(\mathbf{v}_n,\sigma_n)$ and $\widetilde{\mathbf{s}}_n := \mathbf{T}(\mathbf{v}_n,\sigma_n)\mathbf{n} + \kappa(\mathbf{v}_n,\sigma_n)\mathbf{n}$.

Subtracting (2.43) and (2.46), and then using a similar argument as in (2.44), we have

$$\begin{aligned}&\|\widetilde{\mathbf{v}}_n - \widetilde{\mathbf{v}}\|_{\mathbf{W}_0^{1,s}(\Omega)} + \|(\widetilde{\sigma}_n - \widetilde{\sigma}, \widetilde{\mathbf{s}}_n - \widetilde{\mathbf{s}})\|_{\mathbf{Z}^{0,-\frac{1}{s},s}(\Omega,\Gamma)} \\ &\quad\le c(\|\mathbf{f}_n - A_{\mathbf{y}}^\star(\mathbf{v},\sigma)\|_{\mathbf{W}^{1,s'}(\Omega)'} + \|g_n - \operatorname{div}\mathbf{v}\|_{\widehat{L}^s(\Omega)}),\end{aligned}$$

Thus, $\widetilde{\mathbf{v}}_n \to \widetilde{\mathbf{v}}$ in $\mathbf{W}_0^{1,s}(\Omega)$, so that $\widetilde{\mathbf{v}} = \mathbf{v}$, and $(\widetilde{\sigma}_n, \widetilde{\mathbf{s}}_n) \to (\widetilde{\sigma}, \widetilde{\mathbf{s}})$ in $\mathbf{Z}^{0,-\frac{1}{s},s}(\Omega,\Gamma)$. This implies that $\kappa(\mathbf{v}_n,\sigma_n) = \widetilde{\sigma}_n - \sigma_n \to \widetilde{\sigma} - \sigma$ and $\langle \mathbf{T}(\mathbf{v}_n,\sigma_n)\mathbf{n},\mathbf{n}\rangle_\Gamma = \langle \widetilde{\mathbf{s}}_n - \kappa(\mathbf{v}_n,\sigma_n)\mathbf{n},\mathbf{n}\rangle_\Gamma \to \langle \widetilde{\mathbf{s}},\mathbf{n}\rangle_\Gamma - |\Gamma|(\widetilde{\sigma} - \sigma)$. From the definition of κ in (2.45) and the operator Σ, we get $\widetilde{\sigma} = \sigma + \Sigma\widetilde{\mathbf{s}}$.

Sending $n \to \infty$ in (2.46), setting $\mathbf{s} := \widetilde{\mathbf{s}} - (\Sigma\widetilde{\mathbf{s}})\mathbf{n} \in \mathbf{W}^{-\frac{1}{s},s}(\Gamma)$, and using

$$-\int_\Omega \widetilde{\sigma} \operatorname{div}\mathbf{w}\,\mathrm{d}x = -\int_\Omega \sigma \operatorname{div}\mathbf{w}\,\mathrm{d}x - \langle (\Sigma\widetilde{\mathbf{s}})\mathbf{n}, \mathbf{w}|_\Gamma\rangle_{\mathbf{W}^{-\frac{1}{s},s}(\Gamma),\mathbf{W}^{1-\frac{1}{s'},s'}(\Gamma)},$$

we obtain the desired equation (2.41). Also, (2.42) follows immediately from (2.44). Recall that $\widetilde{\mathbf{s}} = \mathbf{T}(\mathbf{v},\sigma)\mathbf{n} + \kappa(\mathbf{v},\sigma)\mathbf{n}$ when $(\mathbf{v},\sigma) \in \mathbf{X}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega)$, hence direct computation yields

$$\mathbf{s} = \mathbf{T}(\mathbf{v},\sigma)\mathbf{n} + \kappa(\mathbf{v},\sigma)\mathbf{n} + \frac{1}{|\Omega|}\langle \mathbf{T}(\mathbf{v},\sigma)\mathbf{n} + \kappa(\mathbf{v},\sigma)\mathbf{n},\mathbf{n}\rangle_\Gamma\mathbf{n} = \mathbf{T}(\mathbf{v},\sigma)\mathbf{n}.$$

Finally, suppose that $\mathbf{s}_1, \mathbf{s}_2 \in \mathbf{W}^{-\frac{1}{s},s}(\Gamma)$ satisfy (2.41). Then,

$$\langle \mathbf{s}_1 - \mathbf{s}_2, \mathbf{w}|_\Gamma\rangle_{\mathbf{W}^{-\frac{1}{s},s}(\Gamma),\mathbf{W}^{1-\frac{1}{s'},s'}(\Gamma)} = 0$$

for every $\mathbf{w} \in \mathbf{W}^{2,s'}(\Omega)$. Since the trace map $\mathbf{w} \mapsto \mathbf{w}|_\Gamma : \mathbf{W}^{2,s'}(\Omega) \to \mathbf{W}^{2-\frac{1}{s'},s'}(\Gamma)$ is surjective and $\mathbf{W}^{2-\frac{1}{s'},s'}(\Gamma)$ is dense in $\mathbf{W}^{1-\frac{1}{s'},s'}(\Gamma)$, we have $\mathbf{s}_1 - \mathbf{s}_2 = \mathbf{0}$ in $\mathbf{W}^{-\frac{1}{s},s}(\Gamma)$. This establishes the uniqueness of $\mathbf{s}$ in $\mathbf{W}^{-\frac{1}{s},s}(\Gamma)$. The proof of the theorem is now complete. □

Theorem 2.15. *Let $1 < s < \infty$, $\mathbf{v} \in \mathbf{L}^s(\Omega)$, $\sigma \in \widehat{W}^{1,s'}(\Omega)'$, $(\mathbf{f}, g, \mathbf{h}) := \mathfrak{D}_{\mathbf{y}}(\mathbf{v},\sigma)$, and assume that $\mathbf{f} \in \mathbf{W}^{2,s'}(\Omega)'$. Then, there exists a unique $\mathbf{s} \in \mathbf{W}^{-1-\frac{1}{s},s}(\Gamma)$ such that the following generalized Green's identity*

$$\begin{aligned}&\langle \mathbf{f}, \mathbf{w}\rangle_{\mathbf{W}^{2,s'}(\Omega)',\mathbf{W}^{2,s'}(\Omega)} \\ &\quad + \langle (g,\mathbf{h}), (\pi + \kappa(\mathbf{w},\pi), \mathbf{T}(\mathbf{w},\pi)\mathbf{n} + \kappa(\mathbf{w},\pi)\mathbf{n})\rangle_{\mathbf{Z}^{-1,-\frac{1}{s},s}(\Omega,\Gamma),\mathbf{Z}^{1,1-\frac{1}{s'},s'}(\Omega,\Gamma)} \\ &\quad = \int_\Omega \mathbf{v}\cdot A_{\mathbf{y}}(\mathbf{w},\pi)\,\mathrm{d}x \\ &\quad + \langle (\sigma + \kappa_0(\mathbf{s}), \mathbf{s} + \kappa_0(\mathbf{s})\mathbf{n}), (-\operatorname{div}\mathbf{w}, \mathbf{w}|_\Gamma)\rangle_{\mathbf{Z}^{-1,-1-\frac{1}{s},s}(\Omega,\Gamma),\mathbf{Z}^{1,2-\frac{1}{s'},s'}(\Omega,\Gamma)}\end{aligned} \tag{2.47}$$

holds for every $(\mathbf{w}, \pi) \in \mathbf{W}^{2,s'}(\Omega) \times \widehat{W}^{1,s'}(\Omega)$*, where*

$$\kappa_0(\mathbf{s}) := \frac{|\Omega|}{|\Omega| + |\Gamma|} \Sigma\mathbf{s}$$

and we have the estimate

$$\|\mathbf{s}\|_{\mathbf{W}^{-\frac{1}{s},s}(\Gamma)} \le c(\|\mathbf{f}\|_{\mathbf{W}^{2,s'}(\Omega)'} + \|\mathbf{v}\|_{\mathbf{L}^s(\Omega)} + \|\sigma\|_{\widehat{W}^{1,s'}(\Omega)'}). \tag{2.48}$$

In addition, if $\mathbf{v} \in \mathbf{W}^{1,s}(\Omega)$ *and* $\sigma \in \widehat{L}^s(\Omega)$*, then* $\mathbf{h} = \mathbf{v}|_\Gamma$ *in* $\mathbf{W}^{1-\frac{1}{s},s}(\Gamma)$*. If* $\mathbf{v} \in \mathbf{W}_0^{1,s}(\Omega)$*,* $\sigma \in \widehat{L}^s(\Omega)$*, and* $A_{\mathbf{y}}^\star(\mathbf{v}, \sigma) \in \mathbf{W}^{1,s'}(\Omega)'$*, then* $\mathbf{s} \in \mathbf{W}^{-\frac{1}{s},s}(\Gamma)$ *and this coincides with the one given in Theorem 2.14. Furthermore, if* $\mathbf{v} \in \mathbf{W}^{2,s}(\Omega)$ *and* $\sigma \in \widehat{W}^{1,s}(\Omega)$*, then* $\mathbf{s} = \mathbf{T}(\mathbf{v}, \sigma)\mathbf{n}$ *in* $\mathbf{W}^{1-\frac{1}{s},s}(\Gamma)$*.*

Proof. By assumption, $\mathbf{f} \in \mathbf{W}^{2,s'}(\Omega)' \subset \mathbf{X}^{-2,s}(\Omega)$. From Corollary 2.13, we immediately obtain the estimate

$$\|\mathbf{f}\|_{\mathbf{X}^{-2,s}(\Omega)} + \|(g, \mathbf{h})\|_{\mathbf{Z}^{-1,-1-\frac{1}{s},s}(\Omega,\Gamma)} \le c(\|\mathbf{v}\|_{\mathbf{L}^s(\Omega)} + \|\sigma\|_{\widehat{W}^{1,s'}(\Omega)'}). \tag{2.49}$$

In addition, if $\mathbf{v} \in \mathbf{W}^{1,s}(\Omega)$ and $\sigma \in \widehat{L}^s(\Omega)$, then $\mathbf{h} = \mathbf{v}|_\Gamma$ in $\mathbf{W}^{1-\frac{1}{s},s}(\Gamma)$ according to the definition of $\mathfrak{D}_{\mathbf{y}}$.

Following the proof provided for Theorem 2.14, we consider the bounded linear functional $\ell : \mathbf{L}^{s'}(\Omega) \times \mathbf{Z}^{1,2-\frac{1}{s'},s'}(\Omega, \Gamma)$ given by

$$\begin{aligned} \ell(\mathbf{r}, q, \mathbf{z}) &:= \langle \mathbf{f}, \mathbf{w} \rangle_{\mathbf{W}^{2,s'}(\Omega)', \mathbf{W}^{2,s'}(\Omega)} \\ &\quad + \langle (g, \mathbf{h}), (\pi + \kappa(\mathbf{w}, \pi), \mathbf{T}(\mathbf{w}, \pi)\mathbf{n} + \kappa(\mathbf{w}, \pi)\mathbf{n}) \rangle_{\mathbf{Z}^{-1,-\frac{1}{s},s}(\Omega,\Gamma), \mathbf{Z}^{1,1-\frac{1}{s'},s'}(\Omega,\Gamma)} \end{aligned} \tag{2.50}$$

where, for a given $(\mathbf{r}, q, \mathbf{z}) \in \mathbf{L}^{s'}(\Omega) \times \mathbf{Z}^{1,2-\frac{1}{s'},s'}(\Omega, \Gamma)$, the pair $(\mathbf{w}, \pi) \in \mathbf{W}^{2,s'}(\Omega) \times \widehat{W}^{1,s'}(\Omega)$ is the solution of the equation $\mathfrak{B}_{\mathbf{y}}(\mathbf{w}, \pi) = (\mathbf{r}, q, \mathbf{z})$, see Lemma 2.10. Then, there exists a unique $(\widetilde{\mathbf{v}}, \widetilde{\sigma}, \widetilde{\mathbf{s}}) \in \mathbf{L}^s(\Omega) \times \mathbf{Z}^{-1,-1-\frac{1}{s},s}(\Omega, \Gamma)$ such that

$$\begin{aligned} &\langle \mathbf{f}, \mathbf{w} \rangle_{\mathbf{W}^{2,s'}(\Omega)', \mathbf{W}^{2,s'}(\Omega)} \\ &\quad + \langle (g, \mathbf{h}), (\pi + \kappa(\mathbf{w}, \pi), \mathbf{T}(\mathbf{w}, \pi)\mathbf{n} + \kappa(\mathbf{w}, \pi)\mathbf{n}) \rangle_{\mathbf{Z}^{-1,-\frac{1}{s},s}(\Omega,\Gamma), \mathbf{Z}^{1,1-\frac{1}{s'},s'}(\Omega,\Gamma)} \\ &\quad = \int_\Omega \widetilde{\mathbf{v}} \cdot A_{\mathbf{y}}(\mathbf{w}, \pi) \, dx + \langle (\widetilde{\sigma}, \widetilde{\mathbf{s}}), (-\operatorname{div} \mathbf{w}, \mathbf{w}|_\Gamma) \rangle_{\mathbf{Z}^{-1,-1-\frac{1}{s},s}(\Omega,\Gamma), \mathbf{Z}^{1,2-\frac{1}{s'},s'}(\Omega,\Gamma)} \end{aligned}$$

for any $(\mathbf{w}, \pi) \in \mathbf{W}^{2,s'}(\Omega) \times \widehat{W}^{1,s'}(\Omega)$. Using (2.49) along with the definition of the dual norm, we have

$$\begin{aligned} &\|\widetilde{\mathbf{v}}\|_{\mathbf{L}^s(\Omega)} + \|(\widetilde{\sigma}, \widetilde{\mathbf{s}})\|_{\mathbf{Z}^{-1,-1-\frac{1}{s},s}(\Omega,\Gamma)} \\ &\qquad \le c(\|\mathbf{f}\|_{\mathbf{W}^{2,s'}(\Omega)'} + \|(g, \mathbf{h})\|_{\mathbf{Z}^{-1,-1-\frac{1}{s},s}(\Omega,\Gamma)}) \\ &\qquad \le c(\|\mathbf{f}\|_{\mathbf{W}^{2,s'}(\Omega)'} + \|\mathbf{v}\|_{\mathbf{L}^s(\Omega)} + \|\sigma\|_{\widehat{W}^{1,s'}(\Omega)'}). \end{aligned} \tag{2.51}$$

If $\mathbf{v} \in \mathbf{W}^{2,s}(\Omega)$ and $\sigma \in \widehat{W}^{1,s}(\Omega)$, then $\mathbf{h} = \mathbf{v}|_\Gamma$ in $\mathbf{W}^{2-\frac{1}{s},s}(\Gamma)$, and by applying the uniqueness of the triple $(\widetilde{\mathbf{v}}, \widetilde{\sigma}, \widetilde{\mathbf{s}})$, we have $\widetilde{\mathbf{v}} = \mathbf{v}$, $\widetilde{\sigma} = \sigma - \kappa_0(\mathbf{s})$, and $\widetilde{\mathbf{s}} = \mathbf{s} - \kappa_0(\mathbf{s})\mathbf{n}$, where $\mathbf{s} = \mathbf{T}(\mathbf{v}, \sigma)\mathbf{n}$, based on the proof of Theorem 2.14 and since $\kappa_0(\mathbf{s}) = -\kappa(\mathbf{v}, \sigma)$.

Take a sequence $(\mathbf{f}_n, g_n, \mathbf{h}_n) \in \mathbf{L}^s(\Omega) \times \mathbf{Z}^{1,2-\frac{1}{s},s}(\Omega, \Gamma)$ such that $\mathbf{f}_n \to \mathbf{f}$ in $\mathbf{W}^{2,s'}(\Omega)'$ and $(g_n, \mathbf{h}_n) \to (g, \mathbf{h})$ in $\mathbf{Z}^{-1,-\frac{1}{s},s}(\Omega, \Gamma)$. Introduce $(\mathbf{v}_n, \sigma_n) \in \mathbf{W}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega)$ by $\mathfrak{D}_{\mathbf{y}}(\mathbf{v}_n, \sigma_n) = (\mathbf{f}_n, g_n, \mathbf{h}_n)$. Then, the difference $(\mathbf{v} - \mathbf{v}_n, \sigma - \sigma_n)$ satisfies $\mathfrak{D}_{\mathbf{y}}(\mathbf{v} -$

$\mathbf{v}_n, \sigma - \sigma_n) = (\mathbf{f} - \mathbf{f}_n, g - g_n, \mathbf{h} - \mathbf{h}_n)$. Hence, we see that $(\mathbf{v}_n, \sigma_n) \to (\mathbf{v}, \sigma)$ in $\mathbf{L}^s(\Omega) \times \widehat{W}^{1,s'}(\Omega)'$ due to the continuity of $\mathfrak{D}_{\mathbf{y}}^{-1}$. Moreover,

$$\begin{aligned} &\langle \mathbf{f}_n, \mathbf{w}\rangle_{\mathbf{W}^{2,s'}(\Omega)',\mathbf{W}^{2,s'}(\Omega)} \\ &\quad + \langle (g_n, \mathbf{h}_n), (\pi + \kappa(\mathbf{w}, \pi), \mathbf{T}(\mathbf{w}, \pi)\mathbf{n} + \kappa(\mathbf{w}, \pi)\mathbf{n})\rangle_{\mathbf{Z}^{-1,-\frac{1}{s},s}(\Omega,\Gamma),\mathbf{Z}^{1,1-\frac{1}{s'},s'}(\Omega,\Gamma)} \\ &\quad = \int_\Omega \widetilde{\mathbf{v}}_n \cdot A_{\mathbf{y}}(\mathbf{w}, \pi)\, dx + \langle (\widetilde{\sigma}_n, \widetilde{\mathbf{s}}_n), (-\operatorname{div} \mathbf{w}, \mathbf{w}|_\Gamma)\rangle_{\mathbf{Z}^{-1,-1-\frac{1}{s},s}(\Omega,\Gamma),\mathbf{Z}^{1,2-\frac{1}{s'},s'}(\Omega,\Gamma)}, \end{aligned} \tag{2.52}$$

where $\widetilde{\mathbf{v}}_n = \mathbf{v}_n$, $\widetilde{\sigma}_n = \sigma_n - \kappa_0(\mathbf{s}_n)$, $\widetilde{\mathbf{s}}_n = \mathbf{s}_n - \kappa_0(\mathbf{s}_n)\mathbf{n}$, $\mathbf{h}_n = \mathbf{v}_n|_\Gamma$, and $\mathbf{s}_n = \mathbf{T}(\mathbf{v}_n, \sigma_n)\mathbf{n}$. Furthermore, thanks to (2.51), we have the estimate

$$\begin{aligned} &\|\widetilde{\mathbf{v}}_n - \widetilde{\mathbf{v}}\|_{\mathbf{L}^s(\Omega)} + \|(\widetilde{\sigma}_n - \widetilde{\sigma}, \widetilde{\mathbf{s}}_n - \widetilde{\mathbf{s}})\|_{\mathbf{Z}^{-1,-1-\frac{1}{s},s}(\Omega,\Gamma)} \\ &\qquad \le c(\|\mathbf{f}_n - \mathbf{f}\|_{\mathbf{W}^{2,s'}(\Omega)'} + \|(g_n - g, \mathbf{h}_n - \mathbf{h})\|_{\mathbf{Z}^{-1,-1-\frac{1}{s},s}(\Omega,\Gamma)}). \end{aligned}$$

Similar to the argument provided in Theorem 2.14, we deduce that $\widetilde{\mathbf{v}} = \mathbf{v}$, $\widetilde{\sigma} = \sigma + \Sigma\widetilde{\mathbf{s}}$, and $\widetilde{\mathbf{s}} = \mathbf{s} + (\Sigma\widetilde{\mathbf{s}})\mathbf{n}$ in (2.48). The last equation implies that $\Sigma\widetilde{\mathbf{s}} = \frac{|\Omega|}{|\Omega|+|\Gamma|}\Sigma\mathbf{s} = \kappa_0(\mathbf{s})$. Sending (2.52) to the limit leads to (2.47), and we obtain (2.48) from (2.51).

Suppose that $\mathbf{v} \in \mathbf{W}_0^{1,s}(\Omega)$, $\sigma \in \widehat{L}^s(\Omega)$, and $\mathbf{f} = A_{\mathbf{y}}^\star(\mathbf{v}, \sigma) \in \mathbf{W}^{1,s'}(\Omega)' \subset \mathbf{W}^{-1,s}(\Omega)$. In this case, $g = -\operatorname{div} \mathbf{v} \in \widehat{L}^s(\Omega)$ and $\mathbf{h} = 0$ from the definition of $\mathfrak{D}_{\mathbf{y}}$. Hence, (2.47) reduces to

$$\begin{aligned} &\langle A_{\mathbf{y}}^\star(\mathbf{v}, \sigma), \mathbf{w}\rangle_{\mathbf{W}^{2,s'}(\Omega)',\mathbf{W}^{2,s'}(\Omega)} - \int_\Omega \pi \operatorname{div} \mathbf{v}\, dx \\ &\quad = \int_\Omega \mathbf{v} \cdot A_{\mathbf{y}}(\mathbf{w}, \pi)\, dx - \int_\Omega \sigma \operatorname{div} \mathbf{w}\, dx + \langle \mathbf{s}, \mathbf{w}|_\Gamma\rangle_{\mathbf{W}^{-1-\frac{1}{s},s}(\Gamma),\mathbf{W}^{2-\frac{1}{s'},s'}(\Gamma)} \end{aligned}$$

for all $(\mathbf{w}, \pi) \in \mathbf{W}^{2,s'}(\Omega) \times \widehat{W}^{1,s'}(\Omega)$. Given a fixed $\mathbf{z} \in \mathbf{W}^{2-\frac{1}{s'},s'}(\Gamma)$, take $(\mathbf{w}, \pi) \in \mathbf{W}^{2,s'}(\Omega) \times \widehat{W}^{1,s'}(\Omega)$ such that $\mathfrak{B}_{\mathbf{y}}(\mathbf{w}, \pi) = (\mathbf{0}, \Sigma\mathbf{z}, \mathbf{z})$. Thus, we deduce the estimate

$$\begin{aligned} |\langle \mathbf{s}, \mathbf{z}\rangle_{\mathbf{W}^{-1-\frac{1}{s},s}(\Gamma),\mathbf{W}^{2-\frac{1}{s'},s'}(\Gamma)}| &\le c_{\mathbf{v},\sigma}(\|\mathbf{w}\|_{\mathbf{W}^{1,s'}(\Omega)} + \|\pi\|_{\widehat{L}^{s'}(\Omega)}) \\ &\le c_{\mathbf{v},\sigma}\|(\Sigma\mathbf{z}, \mathbf{z})\|_{\mathbf{Z}^{1,2-\frac{1}{s'},s'}(\Omega,\Gamma)} \le c_{\mathbf{v},\sigma}\|\mathbf{z}\|_{\mathbf{W}^{2-\frac{1}{s'},s'}(\Gamma)} \end{aligned}$$

where $c_{\mathbf{v},\sigma} = c(\|\mathbf{v}\|_{\mathbf{L}^s(\Omega)} + \|\sigma\|_{\widehat{W}^{1,s'}(\Omega)'})$. Using the density of $\mathbf{W}^{2-\frac{1}{s'},s'}(\Gamma)$ in $\mathbf{W}^{1-\frac{1}{s'},s'}(\Gamma)$, we obtain from the above estimate that $\mathbf{s}$ admits an extension, denoted by the same notation, such that $\mathbf{s} \in \mathbf{W}^{-1-\frac{1}{s},s}(\Gamma)$ and (2.41) holds for $(\mathbf{w}, \pi) \in \mathbf{W}^{2,s'}(\Omega) \times \widehat{W}^{1,s'}(\Omega)$, hence for all $(\mathbf{w}, \pi) \in \mathbf{W}^{1,s'}(\Omega) \times \widehat{L}^{s'}(\Omega)$ such that $A_{\mathbf{y}}(\mathbf{w}, \pi) \in \mathbf{W}^{-1,s'}(\Omega)$ by employing the same density argument as in the proof of Theorem 2.14. Due to uniqueness, it follows that the functional $\mathbf{s}$ agrees with the one given in Theorem 2.14.

Finally, suppose that $\mathbf{s}_1, \mathbf{s}_2 \in \mathbf{W}^{-1-\frac{1}{s},s}(\Gamma)$ satisfy (2.47). Taking the difference of (2.47) for $\mathbf{s}_1$ and $\mathbf{s}_2$ with test functions $(\mathbf{w}, \pi) \in \mathbf{W}^{2,s'}(\Omega) \times \widehat{W}^{1,s'}(\Omega)$ yields

$$\begin{aligned} &\langle (\kappa_0(\mathbf{s}_1 - \mathbf{s}_2), \mathbf{s}_1 - \mathbf{s}_2 + \kappa_0(\mathbf{s}_1 - \mathbf{s}_2)\mathbf{n}), (-\operatorname{div} \mathbf{w}, \mathbf{w}|_\Gamma)\rangle_{\mathbf{Z}^{-1,-1-\frac{1}{s},s}(\Omega,\Gamma),\mathbf{Z}^{1,2-\frac{1}{s'},s'}(\Omega,\Gamma)} \\ &= \langle \mathbf{s}_1 - \mathbf{s}_2, \mathbf{w}|_\Gamma\rangle_{\mathbf{W}^{-1-\frac{1}{s},s}(\Gamma),\mathbf{W}^{2-\frac{1}{s'},s'}(\Gamma)} = 0 \end{aligned}$$

from the divergence theorem. The surjectivity of the trace map $\mathbf{w} \mapsto \mathbf{w}|_\Gamma$: $\mathbf{W}^{2,s'}(\Omega) \to \mathbf{W}^{2-\frac{1}{s'},s'}(\Gamma)$ shows that $\mathbf{s}_1 = \mathbf{s}_2$. □

Let us compare the previous theorem with that of [18, Theorem 5]. Instead of the generalized trace of the velocity, we studied the generalized normal stress on the boundary. However, let us point out that [18, Theorem 5] is not entirely correct as the result is valid only for more regular data [19]. Note that in the above theorem, we did not prove the existence of the trace $\mathbf{v}|_\Gamma$, but instead used the extension of $\mathfrak{D}_\mathbf{y}$ to implicitly construct the generalized trace. Nevertheless, the result given above can be viewed as a generalization of [22, Theorem 9.3].

3. Optimality systems and regularity of optimal solutions

In this section, we study the optimal control problems (1.1), (1.3), and (1.4) for various cost functionals. The existence of solutions to the optimal control problems can be established by following the methods in the standard text [25], in particular, using weak sequential compactness arguments. Nonetheless, we provide the proofs for the sake of completeness and clarity.

For $k \geq 2$ integer and $1 < r < \infty$, we let

$$\mathbf{X}^{k,r}(\Omega) := \mathbf{W}_0^{1,r}(\Omega) \cap \mathbf{W}^{k,r}(\Omega), \qquad \mathbf{V}^{k,r}(\Omega) := \mathbf{V}^{1,r}(\Omega) \cap \mathbf{W}^{k,r}(\Omega).$$

In the following, if we write $(\mathbf{y}^*, p^*) \in \mathbf{V}^{k,r}(\Omega) \times \widehat{W}^{k-1,r}(\Omega)$ for some non-negative integer k, then we implicitly assumed that Ω is a bounded $C^{\max\{2,k\}}$-domain. While existence of solutions to the optimal control problems is guaranteed, the analysis on the regularity of the optimal solutions will be done locally at regular solutions as in [10], see also [11, 17]. In this way, the optimization problems with two or three variables and a PDE constraint is converted to a local problem in one variable.

We point out that even though the regularity for the state variables can be derived from the known results for the stationary Navier–Stokes equation, we provide direct proofs by invoking those from the Stokes equation and by using a simple bootstrapping argument. Indeed, note that (1.2) can be written as

$$\mathfrak{S}_0(\mathbf{y}^*, p^*) = \mathbf{u}^* - (\mathbf{y}^* \cdot \nabla)\mathbf{y}^* \tag{3.1}$$

where $\mathfrak{S}_0$ is the Stokes operator given by (2.30).

3.1. Stress and pressure tracking.

We start our discussion with the optimization problem

$$\left[\begin{array}{c} \displaystyle\min_{(\mathbf{y},p,\mathbf{u}) \in \mathbf{V}^{2,2}(\Omega) \times \widehat{W}^{1,2}(\Omega) \times \mathbf{L}^2(\Omega)} \mathcal{J}(\mathbf{y}, p, \mathbf{u}) := J(\mathbf{y}, p) + \frac{\rho}{2}\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \\ \text{subject to (1.2)} \end{array}\right. \tag{3.2}$$

with $J : \mathbf{V}^{2,2}(\Omega) \times \widehat{W}^{1,2}(\Omega) \to \mathbb{R}$ given by

$$J(\mathbf{y}, p) := \frac{1}{2}\int_\Omega \alpha|\nabla\mathbf{y} - \mathbf{Y}_\Omega|^2 + \beta|p - p_\Omega|^2 + \lambda|\mathbf{T}(\mathbf{y}, p) - \mathbf{S}_\Omega|^2 \, dx, \tag{3.3}$$

where $\alpha, \beta, \lambda \geq 0$, with $\alpha + \beta + \lambda > 0$, $\mathbf{Y}_\Omega \in \mathbf{L}^2(\Omega)^2$, $p_\Omega \in L^2(\Omega)$, and $\mathbf{S}_\Omega \in \mathbf{L}^2(\Omega)^2$.

Let us prove that the problem (3.2)–(3.3) has at least one solution. Since $\mathcal{J}$ is bounded from below, it has an infimum, say $\mathcal{J}^*$. Take a minimizing sequence $\{(\mathbf{y}_n, p_n, \mathbf{u}_n)\}_{n=1}^\infty$ in $\mathbf{V}^{2,2}(\Omega) \times \widehat{W}^{1,2}(\Omega) \times \mathbf{L}^2(\Omega)$ such that $\mathcal{J}(\mathbf{y}_n, p_n, \mathbf{u}_n) \leq \mathcal{J}(\mathbf{0}, 0, \mathbf{0})$ for every n and $\mathcal{J}(\mathbf{y}_n, p_n, \mathbf{u}_n) \to \mathcal{J}^*$. This implies that $\{\mathbf{u}_n\}_{n=1}^\infty$ is bounded in $\mathbf{L}^2(\Omega)$, and hence, $\{(\mathbf{y}_n, p_n)\}_{n=1}^\infty$ is also bounded in $\mathbf{V}^{2,2}(\Omega) \times \widehat{W}^{1,2}(\Omega)$ thanks to (2.6). Let $\mathbf{u}^* \in \mathbf{L}^2(\Omega)$ and $(\mathbf{y}^*, p^*) \in \mathbf{V}^{2,2}(\Omega) \times \widehat{W}^{1,2}(\Omega)$ be such that, by passing to a subsequence which we do not relabel for simplicity, we have $\mathbf{u}_n \rightharpoonup \mathbf{u}^*$ in $\mathbf{L}^2(\Omega)$, $\mathbf{y}_n \rightharpoonup \mathbf{y}^*$ in $\mathbf{V}^{2,2}(\Omega)$, and $p_n \rightharpoonup p^*$ in $\widehat{W}^{1,2}(\Omega)$. By the weak lower semicontinuity of the norm

$$\|\mathbf{u}^*\|^2_{\mathbf{L}^2(\Omega)} \leq \liminf_{n\to\infty} \|\mathbf{u}_n\|^2_{\mathbf{L}^2(\Omega)}. \tag{3.4}$$

From the compactness of $\mathbf{V}^{2,2}(\Omega) \subset \mathbf{W}^{1,4}(\Omega)$ and $\widehat{W}^{1,2}(\Omega) \subset \widehat{L}^2(\Omega)$, we get by extraction of another subsequence that $\mathbf{y}_n \to \mathbf{y}^*$ in $\mathbf{W}^{1,4}(\Omega)$ and $p_n \to p^*$ in $\widehat{L}^2(\Omega)$. Hence, $(\mathbf{y}_n \cdot \nabla)\mathbf{y}_n \to (\mathbf{y}^* \cdot \nabla)\mathbf{y}^*$ strongly in $\mathbf{L}^2(\Omega)$ due to

$$\begin{aligned}&\|(\mathbf{y}_n \cdot \nabla)\mathbf{y}_n - (\mathbf{y}^* \cdot \nabla)\mathbf{y}^*\|_{\mathbf{L}^2(\Omega)}\\ &\quad \leq \|\mathbf{y}_n - \mathbf{y}^*\|_{\mathbf{L}^4(\Omega)} \|\nabla \mathbf{y}_n\|_{\mathbf{L}^4(\Omega)^2} + \|\mathbf{y}^*\|_{\mathbf{L}^4(\Omega)} \|\nabla \mathbf{y}_n - \nabla \mathbf{y}^*\|_{\mathbf{L}^4(\Omega)^2} \to 0.\end{aligned}$$

Passing to the weak limit of both sides of the equation satisfied by $(\mathbf{y}_n, p_n, \mathbf{u}_n)$, we see that $(\mathbf{y}^*, p^*)$ is a solution of (1.2) with control $\mathbf{u}^*$. Since we also have $\nabla \mathbf{y}_n \to \nabla \mathbf{y}^*$ and $\mathbf{T}(\mathbf{y}_n, p_n) \to \mathbf{T}(\mathbf{y}^*, p^*)$ both in $\mathbf{L}^2(\Omega)^2$, it follows that

$$\lim_{n\to\infty} J(\mathbf{y}_n, p_n) = J(\mathbf{y}^*, p^*). \tag{3.5}$$

From (3.4) and (3.5), we obtain

$$\mathcal{J}^* \leq \mathcal{J}(\mathbf{y}^*, p^*, \mathbf{u}^*) \leq \liminf_{n\to\infty} \mathcal{J}(\mathbf{y}_n, p_n, \mathbf{u}_n) = \mathcal{J}^*. \tag{3.6}$$

Therefore, $(\mathbf{y}^*, p^*, \mathbf{u}^*)$ is a solution of (3.2)–(3.3).

Suppose that the local solution $(\mathbf{y}^*, p^*)$ corresponding to $\mathbf{u}^*$ is regular and let $\mathcal{S}_2 : \mathcal{U}_2(\mathbf{u}^*) \to B_{\varrho,2}(\mathbf{y}^*, p^*)$ be the map given in Theorem 2.8. Let $J_{\mathrm{r}} : \mathcal{U}_2(\mathbf{u}^*) \to \mathbb{R}$ be given by

$$J_{\mathrm{r}}(\mathbf{u}) := J(\mathbf{y}(\mathbf{u}), p(\mathbf{u})) \tag{3.7}$$

and introduce the reduced cost functional $\mathcal{J}_{\mathrm{r}} : \mathcal{U}_2(\mathbf{u}^*) \to \mathbb{R}$ defined by

$$\mathcal{J}_{\mathrm{r}}(\mathbf{u}) := \mathcal{J}(\mathbf{y}(\mathbf{u}), p(\mathbf{u}), \mathbf{u}) = J(\mathbf{y}(\mathbf{u}), p(\mathbf{u})) + \frac{\rho}{2}\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \tag{3.8}$$

where $(\mathbf{y}(\mathbf{u}), p(\mathbf{u})) = \mathcal{S}_2(\mathbf{u})$, that is, $(\mathbf{y}(\mathbf{u}), p(\mathbf{u}))$ is the unique solution of (1.2) corresponding to $\mathbf{u} \in \mathcal{U}_2(\mathbf{u}^*)$. Thus, $\mathbf{u}^*$ is a solution of the following local problem:

$$\min_{\mathbf{u} \in \mathcal{U}_2(\mathbf{u}^*)} \mathcal{J}_{\mathrm{r}}(\mathbf{u}). \tag{3.9}$$

Assumption 3.1. In what follows, we will assume in all localized problems that the solution to the state equation corresponding to an optimal control is regular in the sense of Definition 2.1.

Recall that Theorem 2.4 provides conditions for the existence of regular solutions. Note that the condition $\mathcal{J}(\mathbf{y}^*, p^*, \mathbf{u}^*) \leq \mathcal{J}(\mathbf{0}, 0, \mathbf{0})$ in Theorem 2.4 is always satisfied by solutions of the optimal control problems.

For simplicity, let us write $(\mathbf{y},\mathbf{p}) = (\mathbf{y}(\mathbf{u}), p(\mathbf{u}))$. Then, the derivative of J_{r} at $\mathbf{u} \in \mathcal{U}_2(\mathbf{u}^*)$ in the direction $\mathbf{r} \in \mathbf{L}^2(\Omega)$ can be expressed as

$$J_{\mathrm{r}}'(\mathbf{u})\mathbf{r} = \int_\Omega \alpha(\nabla\mathbf{y} - \mathbf{Y}_\Omega) : \nabla\mathbf{w} + \beta(p - p_\Omega)\pi + \lambda(\mathbf{T}(\mathbf{y},p) - \mathbf{S}_\Omega) : \mathbf{T}(\mathbf{w},\pi)\,\mathrm{d}x.$$

Here and for the rest of the paper, $(\mathbf{w},\pi) \in \mathbf{V}^{2,2}(\Omega) \times \widehat{W}^{1,2}(\Omega)$ is the solution of the linearized problem (2.11) with $\mathbf{y} = \mathbf{y}(\mathbf{u})$, $q = 0$, and $\mathbf{z} = \mathbf{0}$.

The integrals involving the gradient and the stress can be expressed as follows:

$$\begin{aligned}
&\int_\Omega \alpha(\nabla\mathbf{y} - \mathbf{Y}_\Omega) : \nabla\mathbf{w}\,\mathrm{d}x = \langle \alpha(\operatorname{div}\mathbf{Y}_\Omega - \Delta\mathbf{y}), \mathbf{w}\rangle_{\mathbf{W}^{-1,2}(\Omega),\mathbf{W}_0^{1,2}(\Omega)}\\
&\int_\Omega \lambda(\mathbf{T}(\mathbf{y},p) - \mathbf{S}_\Omega) : \mathbf{T}(\mathbf{w},\pi)\,\mathrm{d}x = \int_\Omega \lambda(\mathbf{T}(\mathbf{y},p) - \mathbf{S}_\Omega) : (-\nu\nabla\mathbf{w} + \pi\mathbf{I})\,\mathrm{d}x\\
&\quad = \langle \lambda\operatorname{div}\mathbf{S}_\Omega - \lambda\nu^2\Delta\mathbf{y} + \lambda\nu\nabla p, \mathbf{w}\rangle_{\mathbf{W}^{-1,2}(\Omega),\mathbf{W}_0^{1,2}(\Omega)} + \int_\Omega \lambda(2p - \operatorname{Tr}\mathbf{S}_\Omega)\pi\,\mathrm{d}x
\end{aligned}$$

since $\operatorname{div}\mathbf{T}(\mathbf{y},p) = -\nu\Delta\mathbf{y} + \nabla p$ and $\operatorname{Tr}\mathbf{T}(\mathbf{y},p) = -\nu\operatorname{div}\mathbf{y} + 2p = 2p$. Using the fact that π has zero mean over Ω, we obtain

$$J_{\mathrm{r}}'(\mathbf{u})\mathbf{r} = \langle \mathbf{f}(\mathbf{y},p), \mathbf{w}\rangle_{\mathbf{W}^{-1,2}(\Omega),\mathbf{W}_0^{1,2}(\Omega)} + \int_\Omega \pi\varLambda g(p)\,\mathrm{d}x,$$

where

$$\mathbf{f}(\mathbf{y},p) := \alpha\operatorname{div}\mathbf{Y}_\Omega + \lambda\operatorname{div}\mathbf{S}_\Omega - (\alpha + \lambda\nu^2)\Delta\mathbf{y} + \lambda\nu\nabla p, \tag{3.10}$$

$$g(p) := (\beta + 2\lambda)p - \beta p_\Omega - \lambda\operatorname{Tr}\mathbf{S}_\Omega. \tag{3.11}$$

As a consequence, the directional derivative is given by

$$\mathcal{J}_{\mathrm{r}}'(\mathbf{u})\mathbf{r} = \int_\Omega (\mathbf{v} + \rho\mathbf{u}) \cdot \mathbf{r}\,\mathrm{d}x \tag{3.12}$$

where $\mathfrak{A}_{\mathbf{y}}^\star(\mathbf{v},\sigma) = (\mathbf{f}(\mathbf{y},p), \varLambda g(p))$.

As we are in the unconstrained setting, it follows that the local optimality system corresponding to $(\mathbf{y}^*, p^*, \mathbf{u}^*)$ is given by

$$\left[\begin{array}{l} -\nu\Delta\mathbf{y}^* + (\mathbf{y}^*\cdot\nabla)\mathbf{y}^* + \nabla p^* = \mathbf{u}^* \ \text{ in } \Omega,\\ \operatorname{div}\mathbf{y}^* = 0 \ \text{ in } \Omega, \quad \mathbf{y}^* = \mathbf{0} \ \text{ on } \Gamma, \quad \langle p^*, 1\rangle_\Omega = 0, \end{array}\right. \tag{3.13}$$

$$\left[\begin{array}{l} -\nu\Delta\mathbf{v}^* + (\nabla\mathbf{y}^*)^\top\mathbf{v}^* - (\mathbf{y}^*\cdot\nabla)\mathbf{v}^* + \nabla\sigma^* = \mathbf{f}(\mathbf{y}^*, p^*) \ \text{ in } \Omega,\\ \operatorname{div}\mathbf{v}^* = -\varLambda g(p^*) \ \text{ in } \Omega, \quad \mathbf{v}^* = \mathbf{0} \ \text{ on } \Gamma, \quad \langle \sigma^*, 1\rangle_\Omega = 0, \end{array}\right. \tag{3.14}$$

$$\mathbf{v}^* + \rho\mathbf{u}^* = 0 \ \text{ in } \Omega, \tag{3.15}$$

where $\mathbf{f}(\mathbf{y}^*, p^*)$ and $g(p^*)$ are defined as in (3.10) and (3.11), respectively. In this case, the optimal control and optimal adjoint velocity has the same regularity.

To see (3.15), let us start with the fact that $\mathcal{J}_{\mathrm{r}}'(\mathbf{u}^*)(\mathbf{u} - \mathbf{u}^*) \geq 0$ for every $\mathbf{u} \in \mathcal{U}_2(\mathbf{u}^*)$. Given $\mathbf{r} \in \mathbf{L}^2(\Omega)$, choose $\varepsilon > 0$ small enough so that $\mathbf{u} := \mathbf{u}^* \pm \varepsilon\mathbf{r} \in \mathcal{U}_2(\mathbf{u}^*)$. This is possible since $\mathcal{U}_2(\mathbf{u}^*)$ is an open set that contains $\mathbf{u}^*$. With this, we have $\mathcal{J}_{\mathrm{r}}'(\mathbf{u}^*)\mathbf{r} = 0$ for every $\mathbf{r} \in \mathbf{L}^2(\Omega)$ and consequently (3.15).

For emphasis, let us present special cases of (3.14). In the case $\alpha = \lambda = 0$ in (3.3) where we only keep track of the pressure, the adjoint system (3.14) reduces to

$$\left[\begin{array}{l} -\nu\Delta\mathbf{v}^* + (\nabla\mathbf{y}^*)^\top\mathbf{v}^* - (\mathbf{y}^*\cdot\nabla)\mathbf{v}^* + \nabla\sigma^* = \mathbf{0} \ \text{ in } \Omega, \\ \operatorname{div}\mathbf{v}^* = -\beta\Lambda(p^* - p_\Omega) \ \text{ in } \Omega, \quad \mathbf{v}^* = \mathbf{0} \ \text{ on } \Gamma, \quad \langle \sigma^*, 1\rangle_\Omega = 0. \end{array}\right.$$

On the other hand, for the case where we only keep track of the Cauchy stress, that is, $\alpha = \beta = 0$, (3.14) becomes

$$\left[\begin{array}{l} -\nu\Delta\mathbf{v}^* + (\nabla\mathbf{y}^*)^\top\mathbf{v}^* - (\mathbf{y}^*\cdot\nabla)\mathbf{v}^* + \nabla\sigma^* = \lambda(\operatorname{div}\mathbf{S}_\Omega - \nu^2\Delta\mathbf{y}^* + \nu\nabla p^*) \ \text{ in } \Omega, \\ \operatorname{div}\mathbf{v}^* = -\lambda\Lambda(2p^* - \operatorname{Tr}\mathbf{S}_\Omega) \ \text{ in } \Omega, \quad \mathbf{v}^* = \mathbf{0} \ \text{ on } \Gamma, \quad \langle \sigma^*, 1\rangle_\Omega = 0. \end{array}\right.$$

Finally, keeping track of the velocity gradient only, the adjoint system with $\beta = \lambda = 0$ is given by

$$\left[\begin{array}{l} -\nu\Delta\mathbf{v}^* + (\nabla\mathbf{y}^*)^\top\mathbf{v}^* - (\mathbf{y}^*\cdot\nabla)\mathbf{v}^* + \nabla\sigma^* = \alpha(\operatorname{div}\mathbf{Y}_\Omega - \Delta\mathbf{y}^*) \ \text{ in } \Omega, \\ \operatorname{div}\mathbf{v}^* = 0 \ \text{ in } \Omega, \quad \mathbf{v}^* = \mathbf{0} \ \text{ on } \Gamma, \quad \langle \sigma^*, 1\rangle_\Omega = 0. \end{array}\right.$$

Theorem 3.2. *Let $\alpha, \beta, \lambda \geq 0$ with $\alpha + \beta + \lambda > 0$ in (3.3). If $\mathbf{Y}_\Omega, \mathbf{S}_\Omega \in \mathbf{L}^s(\Omega)^2$ and $p_\Omega \in L^s(\Omega)$ for some $2 \leq s < \infty$, then for the solution of the optimality system (3.13)–(3.15), we have*

$$(\mathbf{y}^*, p^*, \mathbf{u}^*, \mathbf{v}^*, \sigma^*) \in \mathbf{V}^{3,s}(\Omega) \times \widehat{W}^{2,s}(\Omega) \times \mathbf{W}_0^{1,s}(\Omega) \times \mathbf{W}_0^{1,s}(\Omega) \times \widehat{L}^s(\Omega). \tag{3.16}$$

If $\mathbf{Y}_\Omega, \mathbf{S}_\Omega \in \mathbf{L}^2(\Omega)^2$, $\operatorname{div}\mathbf{Y}_\Omega, \operatorname{div}\mathbf{S}_\Omega \in \mathbf{L}^r(\Omega)$, and $p_\Omega, \operatorname{Tr}\mathbf{S}_\Omega \in W^{1,r}(\Omega)$ for some $1 < r < \infty$, then

$$(\mathbf{y}^*, p^*, \mathbf{u}^*, \mathbf{v}^*, \sigma^*) \in \mathbf{V}^{4,r}(\Omega) \times \widehat{W}^{3,r}(\Omega) \times \mathbf{X}^{2,r}(\Omega) \times \mathbf{X}^{2,r}(\Omega) \times \widehat{W}^{1,r}(\Omega). \tag{3.17}$$

Moreover, if $\beta = \lambda = 0$, then $\mathbf{u}^*, \mathbf{v}^* \in \mathbf{V}^{1,s}(\Omega)$ in (3.16) and $\mathbf{u}^*, \mathbf{v}^* \in \mathbf{V}^{2,r}(\Omega)$ in (3.17).

Proof. In the first situation, (3.10) and (3.11) give us $\mathbf{f}(\mathbf{y}^*, p^*) \in \mathbf{W}^{-1,s}(\Omega)$ and $\Lambda g(p^*) \in \widehat{L}^s(\Omega)$ since $\operatorname{div}\mathbf{Y}_\Omega, \operatorname{div}\mathbf{S}_\Omega \in \mathbf{W}^{-1,s}(\Omega)$, $\Delta\mathbf{y}^*, \nabla p^* \in \mathbf{L}^2(\Omega) \subset \mathbf{W}^{-1,s}(\Omega)$, and $p^* \in \widehat{W}^{1,2}(\Omega) \subset L^s(\Omega)$. Hence, $(\mathbf{v}^*, \sigma^*) \in \mathbf{W}_0^{1,s}(\Omega) \times \widehat{L}^s(\Omega)$ and $\mathbf{u}^* \in \mathbf{W}_0^{1,s}(\Omega)$ according to Lemma 2.6 for $\mathfrak{A}^\star_{\mathbf{y}^*}$ and (3.15). As a consequence, we deduce that $(\mathbf{y}^*, p^*) \in \mathbf{V}^{3,2}(\Omega) \times \widehat{W}^{2,2}(\Omega)$ since $(\mathbf{y}^*\cdot\nabla)\mathbf{y}^* \in \mathbf{W}^{1,2}(\Omega)$, thanks to (2.31) with $(k, s) = (3, 2)$ and (3.1). This gives us $\mathbf{y}^*, \nabla\mathbf{y}^* \in \mathbf{W}^{2,2}(\Omega)$, and since $\mathbf{W}^{2,2}(\Omega)$ is a Banach algebra, we obtain that $(\mathbf{y}^*\cdot\nabla)\mathbf{y}^* \in \mathbf{W}^{2,2}(\Omega) \subset \mathbf{W}^{1,r}(\Omega)$ for every $1 < r < \infty$. Applying (2.31) with $k = 3$, we see that $(\mathbf{y}^*, p^*) \in \mathbf{V}^{3,s}(\Omega) \times \widehat{W}^{2,s}(\Omega)$. This establishes (3.16).

If the additional regularity assumptions $\operatorname{div}\mathbf{Y}_\Omega, \operatorname{div}\mathbf{S}_\Omega \in \mathbf{L}^r(\Omega)$ and $p_\Omega, \operatorname{Tr}\mathbf{S}_\Omega \in W^{1,r}(\Omega)$ for the desired states hold, then we have $\mathbf{f}(\mathbf{y}^*, p^*) \in \mathbf{L}^r(\Omega)$ and $\Lambda g(p^*) \in \widehat{W}^{1,r}(\Omega)$ since $\Delta\mathbf{y}^*, \nabla p^* \in \mathbf{W}^{1,s}(\Omega) \subset \mathbf{W}^{1,2}(\Omega) \subset \mathbf{L}^r(\Omega)$ and $p^* \in \widehat{W}^{2,s}(\Omega) \subset W^{2,2}(\Omega) \subset W^{1,r}(\Omega)$. Thus, $(\mathbf{v}^*, \sigma^*) \times \mathbf{X}^{2,r}(\Omega) \times \widehat{W}^{1,r}(\Omega)$ and $\mathbf{u}^* \in \mathbf{X}^{2,r}(\Omega)$ by Lemma 2.6 for $\mathfrak{A}^\star_{\mathbf{y}^*}$ and (3.15) once again. Observe that $(\mathbf{y}^*\cdot\nabla)\mathbf{y}^* \in \mathbf{W}^{1,r}(\Omega)$ from the above arguments, and so $\mathbf{y}^* \in \mathbf{V}^{3,r}(\Omega) \subset \mathbf{W}^{1,\infty}(\Omega)$. Thus,

$$\begin{aligned} &\|\nabla^2((\mathbf{y}^*\cdot\nabla)\mathbf{y}^*)\|_{\mathbf{L}^r(\Omega)^{2\times 2}} \\ &\quad \leq c(\|\nabla^2\mathbf{y}^*\|_{\mathbf{L}^r(\Omega)^{2\times 2}}\|\nabla\mathbf{y}^*\|_{\mathbf{L}^\infty(\Omega)^2} + \|\mathbf{y}^*\|_{\mathbf{L}^\infty(\Omega)}\|\nabla^3\mathbf{y}^*\|_{\mathbf{L}^r(\Omega)^{2\times 3}}) \end{aligned}$$

$$\leq c\|\mathbf{y}^*\|^2_{\mathbf{V}^{3,r}(\Omega)}.$$

Therefore, $(\mathbf{y}^* \cdot \nabla)\mathbf{y}^* \in \mathbf{W}^{2,r}(\Omega)$ and we get $(\mathbf{y}^*, p^*) \in \mathbf{V}^{4,r}(\Omega) \times \widehat{W}^{3,r}(\Omega)$ by (2.31) with $(k, s) = (4, r)$, and this shows (3.17).

Finally, if $\beta = \lambda = 0$, then $g(p^*) = 0$, and hence $\operatorname{div} \mathbf{u}^* = -\rho^{-1} \operatorname{div} \mathbf{v}^* = 0$. These equations lead into $\mathbf{u}^*, \mathbf{v}^* \in \mathbf{V}^{1,2}(\Omega)$ in (3.22) and $\mathbf{u}^*, \mathbf{v}^* \in \mathbf{V}^{2,r}(\Omega)$ in (3.17). $\square$

Next, let us consider the case with control constraints

$$\left[\begin{array}{c} \displaystyle\min_{(\mathbf{y},p,\mathbf{u})\in\mathbf{V}^{2,s}(\Omega)\times\widehat{W}^{1,s}(\Omega)\times\mathbf{U}_{\mathrm{ad}}} \mathcal{J}(\mathbf{y}, p, \mathbf{u}) := J(\mathbf{y}, p) + \frac{\rho}{2}\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \\ \text{subject to (1.2)} \end{array}\right. \tag{3.18}$$

with $J : \mathbf{V}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega) \to \mathbb{R}$ given by (3.3), $\mathbf{a}, \mathbf{b} \in \mathbf{L}^s(\Omega)$, and $2 \leq s < \infty$. Since $\mathbf{u} \in \mathbf{U}_{\mathrm{ad}} \subset \mathbf{L}^s(\Omega)$, we have a solution $(\mathbf{y}, p) \in \mathbf{V}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega)$ to (1.2), and so J in the current case is well-defined.

The proof of existence of solutions to (3.18) is similar to (3.2). Indeed, following the same notation as above and noting that $\mathbf{U}_{\mathrm{ad}}$ is a bounded, closed, and convex subset of $\mathbf{L}^2(\Omega)$, it is weakly compact in $\mathbf{L}^2(\Omega)$. Thus, up to a subsequence, $\mathbf{u}_n \rightharpoonup \mathbf{u}^*$ in $\mathbf{L}^2(\Omega)$ for some $\mathbf{u}^* \in \mathbf{U}_{\mathrm{ad}}$. With this, we can now proceed as in the unconstrained case to show the existence of at least one solution $(\mathbf{y}^*, p^*, \mathbf{u}^*) \in \mathbf{V}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega) \times \mathbf{U}_{\mathrm{ad}}$ to (3.18).

Let us localize the current problem. Assume that $(\mathbf{y}^*, p^*, \mathbf{u}^*)$ is a solution to (3.18) such that $(\mathbf{y}^*, p^*)$ is regular, let $\mathcal{S}_s : \mathcal{U}_s(\mathbf{u}^*) \to B_{\varrho,s}(\mathbf{y}^*, p^*)$ be the mapping provided by Theorem 2.8, and define $J_{\mathrm{r}} : \mathcal{U}_s(\mathbf{u}^*) \to \mathbb{R}$ and $\mathcal{J}_{\mathrm{r}} : \mathcal{U}_s(\mathbf{u}^*) \to \mathbb{R}$ as in (3.7) and (3.8), where $(\mathbf{y}(\mathbf{u}), p(\mathbf{u})) = \mathcal{S}_s(\mathbf{u})$. Our discussion shows that $\mathbf{u}^*$ is a solution of

$$\min_{\mathbf{u}\in\mathcal{U}_s(\mathbf{u}^*)\cap\mathbf{U}_{\mathrm{ad}}} \mathcal{J}_{\mathrm{r}}(\mathbf{u}). \tag{3.19}$$

Let $\eta > 0$ be such that $B_\eta(\mathbf{u}^*) \subset \mathcal{U}_s(\mathbf{u}^*)$, where $B_\eta(\mathbf{u}^*)$ is the open ball in $\mathbf{L}^s(\Omega)$ with center at $\mathbf{u}^*$ and radius $\eta > 0$. We consider the following localized version of (3.19):

$$\min_{\mathbf{u}\in\mathbf{U}_{\mathrm{ad}}\cap B_\eta(\mathbf{u}^*)} \mathcal{J}_{\mathrm{r}}(\mathbf{u}). \tag{3.20}$$

By local optimality of $\mathbf{u}^*$ and convexity of $\mathbf{U}_{\mathrm{ad}} \cap B_\eta(\mathbf{u}^*)$, it holds that $\mathcal{J}_{\mathrm{r}}'(\mathbf{u}^*)(\mathbf{u} - \mathbf{u}^*) \geq 0$ for every $\mathbf{u} \in \mathbf{U}_{\mathrm{ad}} \cap B_\eta(\mathbf{u}^*)$, with $\mathcal{J}_{\mathrm{r}}'$ given by (3.12). Adapting the discussion in [25, pp. 67-71] and noting that $\mathbf{u}^*$ lies in the interior of $B_\eta(\mathbf{u}^*)$ so that the constraint is not active, the following projection formula for the pointwise optimality condition can be deduced:

$$\begin{aligned} \mathbf{u}^* = \mathbf{P}_{[\mathbf{a},\mathbf{b}]}(-\rho^{-1}\mathbf{v}^*) &:= \min\{\max\{-\rho^{-1}\mathbf{v}^*, \mathbf{b}\}, \mathbf{a}\} \\ &= -\max\{\mathbf{a} - \max\{\rho^{-1}\mathbf{v}^* - \mathbf{b}, \mathbf{0}\} - \mathbf{b}, \mathbf{0}\} + \mathbf{a}. \end{aligned} \tag{3.21}$$

Then, the local optimality system for (3.20) is given by (3.13), (3.14), and (3.21). We see that the regularity of the control depends also on the constraints.

Theorem 3.3. *Suppose that $\alpha, \beta, \lambda \geq 0$ with $\alpha + \beta + \lambda > 0$ in (3.3). Let $2 \leq s < \infty$ and $\mathbf{a}, \mathbf{b} \in \mathbf{W}^{1,s}(\Omega)$. If $\mathbf{Y}_\Omega, \mathbf{S}_\Omega \in \mathbf{L}^s(\Omega)^2$ and $p_\Omega \in L^s(\Omega)$, then for the solution of* *the optimality system* (3.13), (3.14), *and* (3.21) *corresponding to* (3.20), *we have*

$$(\mathbf{y}^*, p^*, \mathbf{u}^*, \mathbf{v}^*, \sigma^*) \in \mathbf{V}^{3,s}(\Omega) \times \widehat{W}^{2,s}(\Omega) \times \mathbf{W}^{1,s}(\Omega) \times \mathbf{W}_0^{1,s}(\Omega) \times \widehat{L}^s(\Omega). \tag{3.22}$$

Moreover, if $\mathbf{Y}_\Omega, \mathbf{S}_\Omega \in \mathbf{L}^2(\Omega)^2$, $\operatorname{div} \mathbf{Y}_\Omega, \operatorname{div} \mathbf{S}_\Omega \in \mathbf{L}^r(\Omega)$, *and* $p_\Omega, \operatorname{Tr} \mathbf{S}_\Omega \in W^{1,r}(\Omega)$ *for some* $1 < r < \infty$, *then*

$$(\mathbf{v}^*, \sigma^*) \in \mathbf{X}^{2,r}(\Omega) \times \widehat{W}^{1,r}(\Omega). \tag{3.23}$$

In particular, if $\beta = \lambda = 0$, *then* $\mathbf{v}^* \in \mathbf{V}^{1,s}(\Omega)$ *and* $\mathbf{v}^* \in \mathbf{V}^{2,r}(\Omega)$ *in* (3.22) *and* (3.23), *respectively.*

Proof. The first assumption on the desired states implies that $(\mathbf{y}, p^*) \in \mathbf{V}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega)$ due to (2.31) with $k = 2$ and $\mathbf{U}_{\mathrm{ad}} \subset \mathbf{L}^s(\Omega)$. Hence, $\mathbf{f}(\mathbf{y}^*, p^*) \in \mathbf{W}^{-1,s}(\Omega)$ and $\Lambda g(p^*) \in \widehat{L}^s(\Omega)$ from (3.10) and (3.11), and we obtain the regularity of $(\mathbf{v}^*, \sigma^*)$ in (3.22) from Lemma 2.6. The classical theorem for the projection [20, Lemma A.1, p. 50], $\mathbf{a}, \mathbf{b} \in \mathbf{W}^{1,s}(\Omega)$, and (3.21) imply that $\mathbf{u}^* \in \mathbf{W}^{1,s}(\Omega)$. Hence, we obtain the regularity of $(\mathbf{y}^*, p^*)$ in (3.22) by (2.31) with $k = 3$ since $(\mathbf{y}^* \cdot \nabla)\mathbf{y}^* \in \mathbf{W}^{1,s}(\Omega)$ from the proof of Theorem 3.2.

If the additional assumption for the desired states hold, then $\mathbf{f}(\mathbf{y}^*, p^*) \in \mathbf{L}^r(\Omega)$ and $\Lambda g(p^*) \in \widehat{W}^{1,r}(\Omega)$. Thus, we get (3.23) by Lemma 2.7. The last part follows since as in Theorem 3.2, $\mathbf{v}^*$ is divergence-free when $\beta = \lambda = 0$. □

Clearly, the results of the previous theorem hold in the standard case where the controls satisfy $\mathbf{a} \le \mathbf{u} \le \mathbf{b}$ almost everywhere in Ω with constant constraints $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$.

Now, we consider the following problem with controls in a Sobolev space

$$\left[\begin{array}{c} \displaystyle\min_{(\mathbf{y},p,\mathbf{u}) \in \mathbf{V}^{3,2}(\Omega) \times \widehat{W}^{2,2}(\Omega) \times \mathbf{W}^{1,2}(\Omega)} \mathcal{J}(\mathbf{y}, p, \mathbf{u}) := J(\mathbf{y}, p) + \frac{\rho}{2}\|\mathbf{u}\|^2_{\mathbf{W}^{1,2}(\Omega)} \\ \text{subject to (1.2)} \end{array}\right. \tag{3.24}$$

and $J : \mathbf{V}^{3,2}(\Omega) \times \widehat{W}^{2,2}(\Omega) \to \mathbb{R}$ is given by (3.3). Existence of local solutions can be established as before. In fact, following the same notation as above, we have $\mathbf{u}_n \rightharpoonup \mathbf{u}^*$ in $\mathbf{W}^{1,2}(\Omega)$, and passing to another subsequence, this yields $\mathbf{u}_n \to \mathbf{u}^*$ in $\mathbf{L}^2(\Omega)$ in virtue of the compactness of $\mathbf{W}^{1,2}(\Omega) \subset \mathbf{L}^2(\Omega)$.

Let $(\mathbf{y}^*, p^*, \mathbf{u}^*)$ be a solution to (3.24) with a regular $(\mathbf{y}^*, p^*)$ and let $\mathcal{R} : \mathcal{V}(\mathbf{u}^*) \to B_\varrho(\mathbf{y}^*, p^*)$ be the C^∞-map given in Theorem 2.9. Define $J_{\mathrm{r}} : \mathcal{V}(\mathbf{u}^*) \to \mathbb{R}$ as in (3.7) with $(\mathbf{y}(\mathbf{u}), p(\mathbf{u})) = \mathcal{R}(\mathbf{u})$ and the corresponding reduced cost functional $\mathcal{J}_{\mathrm{r}} : \mathcal{V}(\mathbf{u}^*) \to \mathbb{R}$ by

$$\mathcal{J}_{\mathrm{r}}(\mathbf{u}) := \mathcal{J}(\mathbf{y}(\mathbf{u}), p(\mathbf{u}), \mathbf{u}) = J(\mathbf{y}(\mathbf{u}), p(\mathbf{u})) + \frac{\rho}{2}\|\mathbf{u}\|^2_{\mathbf{W}^{1,2}(\Omega)}. \tag{3.25}$$

Again, $\mathbf{u}^*$ is a solution to the following local optimal control problem:

$$\min_{\mathbf{u} \in \mathcal{V}(\mathbf{u}^*)} \mathcal{J}_{\mathrm{r}}(\mathbf{u}). \tag{3.26}$$

It can be shown that the directional derivative of $\mathcal{J}_{\mathrm{r}}$ at $\mathbf{u} \in \mathcal{V}(\mathbf{u}^*)$ in the direction of $\mathbf{r} \in \mathbf{W}^{1,2}(\Omega)$ is

$$\mathcal{J}_{\mathrm{r}}'(\mathbf{u})\mathbf{r} = \int_\Omega (\mathbf{v} + \rho\mathbf{u}) \cdot \mathbf{r} + \rho\nabla\mathbf{u} \cdot \nabla\mathbf{r} \, \mathrm{d}x \tag{3.27}$$

where $\mathfrak{A}_{\mathbf{y}}^{\star}(\mathbf{v},\sigma) = (\mathbf{f}(\mathbf{y},p), \Lambda g(p))$ and the components of the right-hand side are given by (3.10) and (3.11). Therefore, the optimality system for $\mathbf{u}^*$ with respect to (3.26) is given by (3.13), (3.14), and with $\mathbf{u}^* \in \mathbf{W}^{1,2}(\Omega)$ the weak solution to the homogenous Neumann boundary problem

$$-\Delta\mathbf{u}^* + \mathbf{u}^* = -\rho^{-1}\mathbf{v}^* \ \text{ in } \Omega, \quad \partial_{\mathbf{n}}\mathbf{u}^* = \mathbf{0} \ \text{ on } \Gamma. \tag{3.28}$$

By standard elliptic regularity theory, $\mathbf{u}^*$ is more regular than $\mathbf{v}^*$.

Remark 3.4. *Replacing the control space $\mathbf{W}^{1,2}(\Omega)$ by $\mathbf{W}_0^{1,2}(\Omega)$ leads to the above elliptic problem but with homogeneous Dirichlet condition $\mathbf{u}^* = \mathbf{0}$ on Γ in place of the Neumann one.*

Theorem 3.5. *Assume that $\alpha, \beta, \lambda \geq 0$ with $\alpha + \beta + \lambda > 0$ in (3.3). If $\mathbf{Y}_\Omega, \mathbf{S}_\Omega \in \mathbf{L}^s(\Omega)^2$ and $p_\Omega \in L^s(\Omega)$, where $2 \leq s < \infty$, then for the solution of the optimality system (3.13), (3.14), and (3.28) for (3.26), it holds that*

$$(\mathbf{y}^*, p^*, \mathbf{u}^*, \mathbf{v}^*, \sigma^*) \in \mathbf{V}^{5,s}(\Omega) \times \widehat{W}^{4,s}(\Omega) \times \mathbf{W}^{3,s}(\Omega) \times \mathbf{W}_0^{1,s}(\Omega) \times \widehat{L}^s(\Omega). \tag{3.29}$$

If $\mathbf{Y}_\Omega, \mathbf{S}_\Omega \in \mathbf{L}^2(\Omega)^2$, $\operatorname{div}\mathbf{Y}_\Omega, \operatorname{div}\mathbf{S}_\Omega \in \mathbf{L}^r(\Omega)$, and $p_\Omega, \operatorname{Tr}\mathbf{S}_\Omega \in W^{1,r}(\Omega)$ for some $1 < r < \infty$, then

$$(\mathbf{y}^*, p^*, \mathbf{u}^*, \mathbf{v}^*, \sigma^*) \in \mathbf{V}^{6,r}(\Omega) \times \widehat{W}^{5,r}(\Omega) \times \mathbf{W}^{4,r}(\Omega) \times \mathbf{X}^{2,r}(\Omega) \times \widehat{W}^{1,r}(\Omega). \tag{3.30}$$

Furthermore, if $\beta = \lambda = 0$, then $\mathbf{v}^* \in \mathbf{V}^{1,s}(\Omega)$ in (3.29), and $\mathbf{v}^* \in \mathbf{V}^{2,r}(\Omega)$ in (3.30), respectively.

Proof. The regularity of the adjoint variables $(\mathbf{v}^*, \sigma^*)$ in (3.29) and (3.30) follow from Theorem 3.3 since $\mathbf{W}^{1,2}(\Omega) \subset \mathbf{L}^s(\Omega)$. Hence, the results for the control $\mathbf{u}^*$ in (3.29) and (3.30) follow from (3.28) and classical elliptic regularity theory.

The same argument as in the proof of (3.17) leads to $\mathbf{y}^* \in \mathbf{V}^{4,s}(\Omega) \subset \mathbf{W}^{3,\infty}(\Omega)$. Thus, $(\mathbf{y}^* \cdot \nabla)\mathbf{y}^* \in \mathbf{W}^{3,s}(\Omega)$ since

$$\begin{aligned}\|\nabla^3((\mathbf{y}^* \cdot \nabla)\mathbf{y}^*)\|_{\mathbf{L}^s(\Omega)^{2\times 3}} &\leq c(\|\nabla\mathbf{y}^*\|_{\mathbf{L}^\infty(\Omega)^2}\|\nabla^3\mathbf{y}\|_{\mathbf{L}^\infty(\Omega)^{2\times 3}} \\ &\quad + \|\nabla^2\mathbf{y}^*\|_{\mathbf{L}^\infty(\Omega)^{2\times 2}}^2 + \|\mathbf{y}^*\|_{\mathbf{L}^\infty(\Omega)^2}\|\nabla^4\mathbf{y}^*\|_{\mathbf{L}^s(\Omega)^{2\times 4}}) \leq c\|\mathbf{y}^*\|_{\mathbf{V}^{4,s}(\Omega)}^2.\end{aligned}$$

Applying (2.31) with $k = 5$ and $\mathbf{u}^* \in \mathbf{W}^{3,s}(\Omega)$, we have $(\mathbf{y}^*, p^*) \in \mathbf{V}^{5,s}(\Omega) \times \widehat{W}^{4,s}(\Omega)$. This completes the proof of (3.29). To show (3.30) in the case of $(\mathbf{y}^*, p^*)$, first we note that $\mathbf{y}^* \in \mathbf{V}^{5,s}(\Omega) \subset \mathbf{W}^{5,2}(\Omega)$ since $s \geq 2$. As $\mathbf{W}^{4,2}(\Omega)$ is a Banach algebra, $(\mathbf{y}^* \cdot \nabla)\mathbf{y}^* \in \mathbf{W}^{4,2}(\Omega)$, and so $\mathbf{y}^* \in \mathbf{V}^{6,2}(\Omega)$ by (2.31) with $(k, s) = (5, 2)$.

Finally, using $\mathbf{V}^{6,2}(\Omega) \subset \mathbf{W}^{5,r}(\Omega) \cap \mathbf{W}^{4,\infty}(\Omega)$, one has

$$\begin{aligned}\|\nabla^4((\mathbf{y}^* \cdot \nabla)\mathbf{y}^*)\|_{\mathbf{L}^r(\Omega)^{2\times 4}} &\leq c(\|\nabla\mathbf{y}^*\|_{\mathbf{L}^\infty(\Omega)^2}\|\nabla^4\mathbf{y}^*\|_{\mathbf{L}^\infty(\Omega)^{2\times 4}} \\ &\quad + \|\nabla^2\mathbf{y}^*\|_{\mathbf{L}^\infty(\Omega)^{2\times 2}}\|\nabla^3\mathbf{y}^*\|_{\mathbf{L}^\infty(\Omega)^{2\times 3}} + \|\mathbf{y}^*\|_{\mathbf{L}^\infty(\Omega)^2}\|\nabla^5\mathbf{y}^*\|_{\mathbf{L}^r(\Omega)^{2\times 5}}) \\ &\quad \leq c\|\mathbf{y}^*\|_{\mathbf{V}^{6,2}(\Omega)}^2\end{aligned}$$

so that $(\mathbf{y}^* \cdot \nabla)\mathbf{y}^* \in \mathbf{W}^{4,r}(\Omega)$. Therefore, $(\mathbf{y}^*, p^*) \in \mathbf{V}^{6,r}(\Omega) \times \widehat{W}^{5,r}(\Omega)$ by (2.31) with $(k, s) = (6, r)$ and $\mathbf{u}^* \in \mathbf{W}^{4,r}(\Omega)$. □

Remark 3.6. *Without pressure terms in the cost functional, that is, when $\beta = \lambda = 0$, the divergence of the optimal control in the case of controls in $\mathbf{W}^{1,2}(\Omega)$ are smooth in the interior of Ω. Indeed, taking the divergence of the first equation in (3.28) and* *using* $\mathbf{v}^* \in \mathbf{L}^2_\sigma(\Omega)$, *we have* $-\Delta \operatorname{div} \mathbf{u}^* + \operatorname{div} \mathbf{u}^* = \operatorname{div} \mathbf{v}^* = 0$ *in* ω *in the sense of distributions for any smooth subset* $\omega \subset \overline{\omega} \subset \Omega$. *For instance, one can take* ω *to be an open ball whose closure lies in* Ω. *It follows that* $\operatorname{div} \mathbf{u}^* \in C^\infty(\omega)$ *by classical elliptic regularity. Since* ω *is arbitrary, we have* $\operatorname{div} \mathbf{u}^* \in C^\infty(\Omega)$.

3.2. Convection, diffusion and pressure gradient tracking. For this subsection, we study optimal control problems involving $(\mathbf{y} \cdot \nabla)\mathbf{y}$, $\Delta \mathbf{y}$, and ∇p.

First, let us consider the optimal control problem

$$\min_{(\mathbf{y},\mathbf{u}) \in \mathbf{V}^{2,2}(\Omega) \times \mathbf{L}^2(\Omega)} \mathcal{J}(\mathbf{y}, \mathbf{u}) := J(\mathbf{y}) + \frac{\rho}{2}\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \text{ subject to (1.2)} \tag{3.31}$$

where $J : \mathbf{V}^{2,2}(\Omega) \to \mathbb{R}$ is the convection-tracking functional

$$J(\mathbf{y}) := \frac{1}{2}\int_\Omega \lambda |(\mathbf{y} \cdot \nabla)\mathbf{y} - \mathbf{c}_\Omega|^2 \,\mathrm{d}x, \tag{3.32}$$

with $\mathbf{c}_\Omega \in \mathbf{L}^2(\Omega)$ and $\lambda > 0$. Following the notation in Subsection 3.1, the existence of a solution to (3.31) can be established as we have $J(\mathbf{y}_n) \to J(\mathbf{y}^*)$.

For regular solutions, we consider the localized problem (3.9) corresponding to (3.31) with $\mathcal{J}_{\mathrm{r}} : \mathcal{U}_2(\mathbf{u}^*) \to \mathbb{R}$ given by

$$\mathcal{J}_{\mathrm{r}}(\mathbf{u}) := \mathcal{J}(\mathbf{y}(\mathbf{u}), \mathbf{u}) = J_{\mathrm{r}}(\mathbf{u}) + \frac{\rho}{2}\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)}$$

where $(\mathbf{y}(\mathbf{u}), p(\mathbf{u})) = \mathcal{S}_2(\mathbf{u})$ and $J_{\mathrm{r}}(\mathbf{u}) = J(\mathbf{y}(\mathbf{u}))$. The derivative of $J_{\mathrm{r}} : \mathcal{U}_2(\mathbf{u}^*) \to \mathbb{R}$ at $\mathbf{u} \in \mathcal{U}_2(\mathbf{u}^*)$ in the direction $\mathbf{r} \in \mathbf{L}^2(\Omega)$ is given by

$$\begin{aligned} J_{\mathrm{r}}'(\mathbf{u})\mathbf{r} &= \int_\Omega \lambda((\mathbf{y} \cdot \nabla)\mathbf{y} - \mathbf{c}_\Omega) \cdot ((\mathbf{w} \cdot \nabla)\mathbf{y} + (\mathbf{y} \cdot \nabla)\mathbf{w}) \,\mathrm{d}x \\ &= \langle \mathbf{f}(\mathbf{y}), \mathbf{w} \rangle_{\mathbf{X}^{-2,2}(\Omega), \mathbf{X}^{2,2}(\Omega)} \end{aligned}$$

where we set

$$\mathbf{f}(\mathbf{y}) := \lambda(\nabla \mathbf{y})^\top((\mathbf{y} \cdot \nabla)\mathbf{y} - \mathbf{c}_\Omega) - \lambda(\mathbf{y} \cdot \nabla)^2\mathbf{y} + \lambda(\mathbf{y} \cdot \nabla)\mathbf{c}_\Omega. \tag{3.33}$$

Thus, (3.12) holds for $\mathfrak{D}_{\mathbf{y}}(\mathbf{v}, \sigma) = (\mathbf{f}(\mathbf{y}), 0, \mathbf{0})$.

The above discussion implies that the optimality system for the localized problem (3.9) corresponding to (3.31) consists of (3.13), (3.15), and the adjoint system

$$\left[\begin{aligned} &-\nu\Delta \mathbf{v}^* + (\nabla \mathbf{y}^*)^\top \mathbf{v}^* - (\mathbf{y}^* \cdot \nabla)\mathbf{v}^* + \nabla \sigma^* \\ &\qquad = \lambda[(\nabla \mathbf{y}^*)^\top((\mathbf{y}^* \cdot \nabla)\mathbf{y}^* - \mathbf{c}_\Omega) - (\mathbf{y}^* \cdot \nabla)^2\mathbf{y}^* + (\mathbf{y}^* \cdot \nabla)\mathbf{c}_\Omega] \ \text{ in } \Omega, \\ &\operatorname{div} \mathbf{v}^* = 0 \ \text{ in } \Omega, \quad \mathbf{v}^* = \mathbf{0} \ \text{ on } \Gamma, \quad \langle \sigma^*, 1 \rangle_\Omega = 0. \end{aligned}\right. \tag{3.34}$$

Similarly, one can formulate the localized versions (3.21) and (3.28) of (3.31), but now with controls in $\mathbf{U}_{\mathrm{ad}}$ and $\mathbf{W}^{1,2}(\Omega)$, respectively. We do not repeat the discussions here for brevity. For the case of $\mathbf{U}_{\mathrm{ad}}$, the optimality system is given by (3.13), (3.34), and (3.15), and for the case of $\mathbf{W}^{1,2}(\Omega)$ we have (3.13), (3.34), and (3.28).

Theorem 3.7. *Let* $\lambda > 0$ *in* (3.32) *and* $2 \le s < \infty$. *Then, we have the following properties for the localized problems:*

(i) *In the case of* $\mathbf{L}^2(\Omega)$, *if* $\mathbf{c}_\Omega \in \mathbf{L}^s(\Omega)$, *then* (3.16) *holds with* $\mathbf{u}^*, \mathbf{v}^* \in \mathbf{V}^{1,s}(\Omega)$. *If* $\mathbf{c}_\Omega \in \mathbf{W}^{1,r}(\Omega)$ *for some* $1 < r < \infty$, *then* (3.17) *is satisfied with* $\mathbf{u}^*, \mathbf{v}^* \in \mathbf{V}^{2,r}(\Omega)$.

(ii) *In the case of* $\mathbf{U}_{\rm ad}$, *if* $\mathbf{a}, \mathbf{b}, \mathbf{c}_\Omega \in \mathbf{L}^s(\Omega)$, *then* (3.22) *is satisfied with* $\mathbf{v}^* \in \mathbf{V}^{1,s}(\Omega)$. *If* $\mathbf{c}_\Omega \in \mathbf{W}^{1,r}(\Omega)$ *for some* $1 < r < \infty$, *then* (3.23) *holds.*

(iii) *In the case of* $\mathbf{W}^{1,2}(\Omega)$, *if* $\mathbf{c}_\Omega \in \mathbf{L}^s(\Omega)$, *then we have* (3.29) *with* $\mathbf{v}^* \in \mathbf{V}^{1,s}(\Omega)$. *If* $\mathbf{c}_\Omega \in \mathbf{W}^{1,r}(\Omega)$ *for some* $1 < r < \infty$, *then we have* (3.30) *with* $\mathbf{v}^* \in \mathbf{V}^{2,r}(\Omega)$.

Proof. First, since $\mathbf{y}^* \in \mathbf{V}^{2,2}(\Omega) \subset \mathbf{W}^{1,2r}(\Omega)$ for every $1 < r < \infty$, we have the following estimates

$$\begin{aligned}
&\|(\nabla \mathbf{y}^*)^\top (\mathbf{y}^* \cdot \nabla)\mathbf{y}^*\|_{\mathbf{L}^r(\Omega)} \le c\|\nabla \mathbf{y}^*\|^2_{\mathbf{L}^{2r}(\Omega)^2}\|\mathbf{y}^*\|_{\mathbf{L}^\infty(\Omega)} \le c\|\mathbf{y}^*\|^3_{\mathbf{V}^{2,2}(\Omega)}\\
&\|(\mathbf{y}^* \cdot \nabla)^2\mathbf{y}^*\|_{\mathbf{L}^2(\Omega)} \le c(\|\mathbf{y}^*\|_{\mathbf{L}^\infty(\Omega)}\|\nabla \mathbf{y}^*\|^2_{\mathbf{L}^4(\Omega)^2} + \|\mathbf{y}^*\|^2_{\mathbf{L}^\infty(\Omega)}\|\nabla^2\mathbf{y}^*\|_{\mathbf{L}^2(\Omega)^{2\times 2}})\\
&\qquad\qquad\qquad\qquad \le c\|\mathbf{y}^*\|^3_{\mathbf{V}^{2,2}(\Omega)}
\end{aligned}$$

by the Hölder's inequality. These imply that $(\nabla \mathbf{y}^*)^\top (\mathbf{y}^* \cdot \nabla)\mathbf{y}^* \in \mathbf{L}^r(\Omega)$ and $(\mathbf{y}^* \cdot \nabla)^2\mathbf{y}^* \in \mathbf{L}^2(\Omega) \subset \mathbf{W}^{-1,r}(\Omega)$ for any $1 < r < \infty$.

Let us prove (i). Assume that $\mathbf{c}_\Omega \in \mathbf{L}^s(\Omega)$. Then, $(\mathbf{y}^* \cdot \nabla)\mathbf{c}_\Omega \in \mathbf{W}^{-1,s}(\Omega)$ due to integration by parts and

$$\begin{aligned}
\int_\Omega (\mathbf{y}^* \cdot \nabla)\boldsymbol{\varphi} \cdot \mathbf{c}_\Omega \,\mathrm{d}x &\le \|\mathbf{y}^*\|_{\mathbf{L}^\infty(\Omega)}\|\nabla \boldsymbol{\varphi}\|_{\mathbf{L}^{s'}(\Omega)^2}\|\mathbf{c}_\Omega\|_{\mathbf{L}^s(\Omega)}\\
&\le c\|\mathbf{y}^*\|_{\mathbf{V}^{2,2}(\Omega)}\|\mathbf{c}_\Omega\|_{\mathbf{L}^s(\Omega)}\|\boldsymbol{\varphi}\|_{\mathbf{W}_0^{1,s'}(\Omega)}
\end{aligned}$$

for all $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,s'}(\Omega)$. Let us show that $(\nabla \mathbf{y}^*)^\top \mathbf{c}_\Omega \in \mathbf{W}^{-1,s}(\Omega)$. Indeed, if $s = 2$, then $(\nabla \mathbf{y}^*)^\top \mathbf{c}_\Omega \in \mathbf{W}^{-1,2}(\Omega)$ according to

$$\begin{aligned}
\int_\Omega (\nabla \mathbf{y}^*)^\top \mathbf{c}_\Omega \cdot \boldsymbol{\varphi} \,\mathrm{d}x &\le \|\nabla \mathbf{y}^*\|_{\mathbf{L}^4(\Omega)^2}\|\mathbf{c}_\Omega\|_{\mathbf{L}^2(\Omega)}\|\boldsymbol{\varphi}\|_{\mathbf{L}^4(\Omega)}\\
&\le c\|\mathbf{y}^*\|_{\mathbf{V}^{2,2}(\Omega)}\|\mathbf{c}_\Omega\|_{\mathbf{L}^2(\Omega)}\|\boldsymbol{\varphi}\|_{\mathbf{W}_0^{1,2}(\Omega)}
\end{aligned}$$

for all $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,2}(\Omega) \subset \mathbf{L}^4(\Omega)$. If $2 < s < \infty$, then $1 < s' < 2$ and $(\nabla \mathbf{y}^*)^\top \mathbf{c}_\Omega \in \mathbf{W}^{-1,s}(\Omega)$ since

$$\begin{aligned}
\int_\Omega (\nabla \mathbf{y}^*)^\top \mathbf{c}_\Omega \cdot \boldsymbol{\varphi} \,\mathrm{d}x &\le \|\nabla \mathbf{y}^*\|_{\mathbf{L}^2(\Omega)^2}\|\mathbf{c}_\Omega\|_{\mathbf{L}^s(\Omega)}\|\boldsymbol{\varphi}\|_{\mathbf{L}^{2s'/(2-s')}(\Omega)}\\
&\le c\|\mathbf{y}^*\|_{\mathbf{V}^{2,2}(\Omega)}\|\mathbf{c}_\Omega\|_{\mathbf{L}^s(\Omega)}\|\boldsymbol{\varphi}\|_{\mathbf{W}_0^{1,s'}(\Omega)}
\end{aligned}$$

for all $\boldsymbol{\varphi} \in \mathbf{W}_0^{1,s'}(\Omega) \subset \mathbf{L}^{2s'/(2-s')}(\Omega)$. Hence, $\mathbf{f}(\mathbf{y}^*) \in \mathbf{W}^{-1,s}(\Omega)$ from (3.33). From the proof of Theorem 3.2, we obtain (3.16) and $\mathbf{u}^*, \mathbf{v}^* \in \mathbf{V}^{1,s}(\Omega)$ since $g(p^*)$ is zero.

Let $\mathbf{c}_\Omega \in \mathbf{W}^{1,r}(\Omega)$ for some $1 < r < \infty$. Then, $(\mathbf{y}^* \cdot \nabla)\mathbf{c}_\Omega, (\nabla \mathbf{y}^*)^\top \mathbf{c}_\Omega \in \mathbf{L}^r(\Omega)$ as in the proof of Theorem 2.7. Also, we have $\mathbf{c}_\Omega \in \mathbf{L}^2(\Omega)$, and so $\mathbf{y}^* \in \mathbf{V}^{3,2}(\Omega) \subset \mathbf{W}^{2,r}(\Omega) \cap \mathbf{W}^{1,2r}(\Omega)$. Hence, $(\mathbf{y}^* \cdot \nabla)^2\mathbf{y}^* \in \mathbf{L}^r(\Omega)$ due to

$$\begin{aligned}
\|(\mathbf{y}^* \cdot \nabla)^2\mathbf{y}^*\|_{\mathbf{L}^r(\Omega)} &\le c(\|\mathbf{y}^*\|_{\mathbf{L}^\infty(\Omega)}\|\nabla \mathbf{y}^*\|^2_{\mathbf{L}^{2r}(\Omega)^2} + \|\mathbf{y}^*\|^2_{\mathbf{L}^\infty(\Omega)}\|\nabla^2\mathbf{y}^*\|_{\mathbf{L}^r(\Omega)^{2\times 2}})\\
&\le c\|\mathbf{y}^*\|^3_{\mathbf{V}^{3,2}(\Omega)}.
\end{aligned}$$

Thus, $\mathbf{f}(\mathbf{y}^*) \in \mathbf{L}^r(\Omega)$, to which we have (3.17) with $\mathbf{u}^*, \mathbf{v}^* \in \mathbf{V}^{2,r}(\Omega)$ from Theorem 3.2. This completes the proof of (i). The proofs of (ii) and (iii) are completely the same as with those given for Theorem 3.3 and Theorem 3.5, respectively, the main difference here is that $\mathbf{u}^*$ and $\mathbf{v}^*$ are divergence-free. □

Let us now consider the control problem involving the diffusion and pressure gradient. Given $\boldsymbol{\varphi} \in \mathbf{L}^2(\Omega)$ we define the generalized Laplacian $\Delta\boldsymbol{\varphi} \in \mathbf{X}^{-2,s}(\Omega)$ and the generalized divergence $\operatorname{div}\boldsymbol{\varphi} \in W^{1,s'}(\Omega)'$ as follows:

$$\langle \Delta\boldsymbol{\varphi}, \boldsymbol{\psi}\rangle_{\mathbf{X}^{-2,s}(\Omega),\mathbf{X}^{2,s'}(\Omega)} := \int_\Omega \boldsymbol{\varphi}\cdot\Delta\boldsymbol{\psi}\,\mathrm{d}x \quad \forall\boldsymbol{\psi}\in\mathbf{X}^{2,s'}(\Omega),$$
$$\langle \operatorname{div}\boldsymbol{\varphi}, \phi\rangle_{W^{1,s'}(\Omega)',W^{1,s'}(\Omega)} := -\int_\Omega \boldsymbol{\varphi}\cdot\nabla\phi\,\mathrm{d}x \quad \forall\phi\in W^{1,s'}(\Omega).$$

Consider the optimal control problem (3.2) with the cost functional

$$J(\mathbf{y},p) := \frac{1}{2}\int_\Omega \alpha|\Delta\mathbf{y}-\mathbf{y}_\Omega|^2 + \beta|\nabla p - \mathbf{q}_\Omega|^2\,\mathrm{d}x \tag{3.35}$$

where $\mathbf{y}_\Omega, \mathbf{q}_\Omega \in \mathbf{L}^2(\Omega)$. As in Subsection 3.1, one can define J_{r} and $\mathcal{J}_{\mathrm{r}}$ now for the functional (3.35). Then, we can write the derivative of J at $\mathbf{u}\in\mathcal{U}_2(\mathbf{u}^*)$ in the direction $\mathbf{r}\in\mathbf{L}^2(\Omega)$ as follows:

$$\begin{aligned} J_{\mathrm{r}}'(\mathbf{u})\mathbf{r} &= \int_\Omega \alpha(\Delta\mathbf{y}-\mathbf{y}_\Omega)\cdot\Delta\mathbf{w} + \beta(\nabla p-\mathbf{q}_\Omega)\cdot\nabla\pi\,\mathrm{d}x \\ &= \langle \mathbf{f}(\mathbf{y}),\mathbf{w}\rangle_{\mathbf{X}^{-2,2}(\Omega),\mathbf{X}^{2,2}(\Omega)} + \langle \Lambda g(p),\pi\rangle_{\widehat{W}^{1,2}(\Omega)',\widehat{W}^{1,2}(\Omega)}, \end{aligned}$$

where

$$\mathbf{f}(\mathbf{y}) := \alpha\Delta(\Delta\mathbf{y}-\mathbf{y}_\Omega), \quad g(p) := -\beta(\Delta p - \operatorname{div}\mathbf{q}_\Omega). \tag{3.36}$$

Then, (3.12) holds for $\mathfrak{D}_{\mathbf{y}}(\mathbf{v},\sigma) = (\mathbf{f}(\mathbf{y}), g(p), \mathbf{0})$.

For the localized problem (3.2) with (3.35) at a regular point, the optimality system is given by (3.13), (3.15), and $(\mathbf{v}^*,\sigma^*)$ is the very weak solution of

$$\left[\begin{array}{l} -\nu\Delta\mathbf{v}^* + (\nabla\mathbf{y}^*)^\top\mathbf{v}^* - (\mathbf{y}^*\cdot\nabla)\mathbf{v}^* + \nabla\sigma^* = \alpha\Delta(\Delta\mathbf{y}^*-\mathbf{y}_\Omega) \ \text{ in } \Omega, \\ \operatorname{div}\mathbf{v}^* = \beta\Lambda(\Delta p^* - \operatorname{div}\mathbf{q}_\Omega) \ \text{ in } \Omega, \quad \mathbf{v}^* = \mathbf{0} \ \text{ on } \Gamma, \quad \langle\sigma^*,1\rangle_\Omega = 0, \end{array}\right. \tag{3.37}$$

in the sense of Definition 2.11. In the case where the controls lie in $\mathbf{U}_{\mathrm{ad}}$ or $\mathbf{W}^{1,2}(\Omega)$, we have to replace (3.15) by (3.21) and (3.28), respectively, to obtain the optimality systems of the localized problems.

Theorem 3.8. *Let $\alpha,\beta\ge 0$ and $\alpha+\beta>0$ in (3.35). Then, the following properties hold for the localized problems:*

(i) *In the case of $\mathbf{L}^2(\Omega)$, if $\mathbf{y}_\Omega,\mathbf{q}_\Omega\in\mathbf{L}^2(\Omega)$, then*

$$(\mathbf{y}^*,p^*,\mathbf{u}^*,\mathbf{v}^*,\sigma^*) \in \mathbf{V}^{2,2}(\Omega)\times\widehat{W}^{1,2}(\Omega)\times\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega)\times\widehat{W}^{1,2}(\Omega)'. \tag{3.38}$$

(ii) *In the case of $\mathbf{U}_{\mathrm{ad}}$, where $\mathbf{a},\mathbf{b}\in\mathbf{L}^s(\Omega)$ for some $2\le s<\infty$, if $\mathbf{y}_\Omega,\mathbf{q}_\Omega\in\mathbf{L}^s(\Omega)$, then*

$$(\mathbf{y}^*,p^*,\mathbf{u}^*,\mathbf{v}^*,\sigma^*) \in \mathbf{V}^{2,s}(\Omega)\times\widehat{W}^{1,s}(\Omega)\times\mathbf{L}^s(\Omega)\times\mathbf{L}^s(\Omega)\times\widehat{W}^{1,s'}(\Omega)'. \tag{3.39}$$

(iii) *In the case of $\mathbf{W}^{1,2}(\Omega)$, if $\mathbf{y}_\Omega,\mathbf{q}_\Omega\in\mathbf{L}^s(\Omega)$ for some $2\le s<\infty$, then*

$$(\mathbf{y}^*,p^*,\mathbf{u}^*,\mathbf{v}^*,\sigma^*) \in \mathbf{V}^{4,s}(\Omega)\times\widehat{W}^{3,s}(\Omega)\times\mathbf{W}^{2,s}(\Omega)\times\mathbf{L}^s(\Omega)\times\widehat{W}^{1,s'}(\Omega)'. \tag{3.40}$$

If $\beta=0$, then $\mathbf{u}^*,\mathbf{v}^*\in\mathbf{L}^2_\sigma(\Omega)$ in (3.38), while $\mathbf{v}^*\in\mathbf{L}^s_\sigma(\Omega)$ in (3.39) and (3.40).

Proof. From (3.36) we can see that $\mathbf{f}(\mathbf{y}^*) \in \mathbf{X}^{-2,s}(\Omega)$ and $\Lambda g(p^*) \in \widehat{W}^{1,s'}(\Omega)'$ whenever $(\mathbf{y}^*, p^*) \in \mathbf{V}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega)$ and $\mathbf{y}_\Omega, \mathbf{q}_\Omega \in \mathbf{L}^s(\Omega)$, with $s = 2$ in the case of (i). Recall that $\mathfrak{D}_{\mathbf{y}^*}(\mathbf{v}^*, \sigma^*) = (\mathbf{f}(\mathbf{y}^*), g(p^*), \mathbf{0})$, and therefore, $(\mathbf{v}^*, \sigma^*) \in \mathbf{L}^s(\Omega) \times \widehat{W}^{1,s'}(\Omega)'$ according to Corollary 2.13. With these, we can follow the same lines of argument as above to deduce the regularity of $(\mathbf{y}^*, p^*, \mathbf{u}^*)$ in (3.38), (3.39), and (3.40). □

Let us prove further regularity of the optimal solution under additional compatibility conditions on the boundary for the case where the controls are in $\mathbf{W}^{1,2}(\Omega)$. While the succeeding two theorems provide better regularity on the optimal solutions, the compatibility conditions on the optimal states and the desired target may not be achieved from a practical perspective.

Theorem 3.9. *Let $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$, $\mathbf{y}_\Omega, \mathbf{q}_\Omega \in \mathbf{L}^2(\Omega)$ and consider the localized problem with the functional (3.35) and controls in $\mathbf{W}^{1,2}(\Omega)$. Suppose that $\nabla \mathbf{y}_\Omega \in \mathbf{L}^r(\Omega)^2$ and $\operatorname{div} \mathbf{q}_\Omega \in L^r(\Omega)$ for some $1 < r < \infty$. If $\Delta \mathbf{y}^*|_\Gamma = \mathbf{y}_\Omega|_\Gamma$ and $\partial_{\mathbf{n}} p^* = \mathbf{q}_\Omega \cdot \mathbf{n}$ on Γ, then (3.29) holds with s replaced by r. In addition, if $\Delta \mathbf{y}_\Omega \in \mathbf{L}^r(\Omega)$ and $\operatorname{div} \mathbf{q}_\Omega \in W^{1,r}(\Omega)$ for some $1 < r < \infty$, then we obtain (3.30).*

Proof. Let $t = \min\{2, r\}$ so that $\mathbf{y}_\Omega \in \mathbf{W}^{1,t}(\Omega)$ and $\mathbf{q}_\Omega \in \mathbf{L}^t_{\mathrm{div}}(\Omega)$. Thus, $\mathbf{y}_\Omega|_\Gamma \in \mathbf{W}^{1-\frac{1}{t},t}(\Gamma)$ and $\mathbf{q}_\Omega \cdot \mathbf{n} \in W^{-\frac{1}{t},t}(\Gamma)$. On the other hand, Theorem 3.8 (iii) with $s = 2$ provides us $\Delta \mathbf{y}^*|_\Gamma \in \mathbf{W}^{\frac{3}{2},2}(\Gamma) \subset \mathbf{W}^{1-\frac{1}{t},t}(\Gamma)$ and $\partial_{\mathbf{n}} p^* \in W^{\frac{3}{2},2}(\Gamma) \subset W^{-\frac{1}{t},t}(\Gamma)$. Therefore, the stated compatibility conditions are well-defined. Furthermore, using Green's identities, we have

$$
\begin{aligned}
J_{\mathrm{r}}'(\mathbf{u}^*)\mathbf{r} &= \int_\Omega \alpha(\Delta \mathbf{y}^* - \mathbf{y}_\Omega) \cdot \Delta \mathbf{w} + \beta(\nabla p^* - \mathbf{q}_\Omega) \cdot \nabla \pi \, \mathrm{d}x \\
&= -\int_\Omega \alpha(\nabla \Delta \mathbf{y}^* - \nabla \mathbf{y}_\Omega) : \nabla \mathbf{w} + \beta(\Delta p^* - \operatorname{div} \mathbf{q}_\Omega)\pi \, \mathrm{d}x \\
&= \langle \mathbf{f}(y^*), \mathbf{w} \rangle_{\mathbf{W}^{-1,r}(\Omega), \mathbf{W}_0^{1,r'}(\Omega)} + \int_\Omega \pi \Lambda g(p^*) \, \mathrm{d}x
\end{aligned}
$$

for every $\mathbf{r} \in \mathbf{W}^{1,2}(\Omega)$, where $\mathbf{f}(\mathbf{y}^*) := \alpha \operatorname{div}(\nabla \Delta \mathbf{y}^* - \nabla \mathbf{y}_\Omega) \in \mathbf{W}^{-1,r}(\Omega)$ and $\Lambda g(p^*) := -\beta \Lambda(\Delta p^* - \operatorname{div} \mathbf{q}_\Omega) \in \widehat{L}^r(\Omega)$. As in Theorem 3.5, this yields (3.29) with r in place of s.

For the second part, it holds that

$$
J_{\mathrm{r}}'(\mathbf{u}^*)\mathbf{r} = \int_\Omega \mathbf{f}(\mathbf{y}^*) \cdot \mathbf{w} \, \mathrm{d}x + \int_\Omega \pi \Lambda g(p^*) \, \mathrm{d}x.
$$

where $\mathbf{f}(\mathbf{y}^*) := \alpha(\Delta^2 \mathbf{y}^* - \Delta \mathbf{y}_\Omega) \in \mathbf{L}^r(\Omega)$ and $\Lambda g(p^*) \in \widehat{W}^{1,r}(\Omega)$. Hence, following Theorem 3.5, we obtain (3.30). □

In the next theorem, we relax the first compatibility condition in Theorem 3.9, however, with a new adjoint equation.

Theorem 3.10. *Let $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ in (3.35) and consider the associated localized problem with controls in $\mathbf{W}^{1,2}(\Omega)$. Assume that $\mathbf{y}_\Omega, \mathbf{q}_\Omega \in \mathbf{L}^s_{\mathrm{div}}(\Omega)$ for some*

$2 \leq s < \infty$,

$$\Delta \mathbf{y}^* \cdot \mathbf{n} = \mathbf{y}_\Omega \cdot \mathbf{n} \text{ on } \Gamma, \quad \text{and} \quad \partial_{\mathbf{n}} p^* = \mathbf{q}_\Omega \cdot \mathbf{n} \text{ on } \Gamma. \tag{3.41}$$

Then,

$$(\mathbf{y}^*, p^*, \mathbf{u}^*, \mathbf{v}^*, \sigma^*) \in \mathbf{V}^{4,s}(\Omega) \times \widehat{W}^{3,s}(\Omega) \times \mathbf{W}^{2,s}(\Omega) \times \mathbf{W}_0^{1,s}(\Omega) \times \widehat{L}^s(\Omega), \tag{3.42}$$

where $\mathbf{u}^*$ *is the solution of*

$$-\Delta \mathbf{u}^* + \mathbf{u}^* = -\frac{1}{\rho}\left[\mathbf{v}^* - \frac{\alpha}{\nu}(\Delta \mathbf{y}^* - \mathbf{y}_\Omega)\right] \text{ in } \Omega, \quad \partial_{\mathbf{n}} \mathbf{u}^* = \mathbf{0} \text{ on } \Gamma, \tag{3.43}$$

and $\mathfrak{A}^\star_{\mathbf{y}^*}(\mathbf{v}^*, \sigma^*) = (\mathbf{f}(\mathbf{y}^*), \Lambda g(p^*))$, *where*

$$\mathbf{f}(\mathbf{y}^*) := \frac{\alpha}{\nu}(\nabla \mathbf{y}^*)^\top(\Delta \mathbf{y}^* - \mathbf{y}_\Omega) - \frac{\alpha}{\nu}(\mathbf{y}^* \cdot \nabla)(\Delta \mathbf{y}^* - \mathbf{y}_\Omega) \tag{3.44}$$

$$g(p^*) := \frac{\alpha}{\nu} \operatorname{div} \mathbf{y}_\Omega - \beta(\Delta p^* - \operatorname{div} \mathbf{q}_\Omega). \tag{3.45}$$

If $\mathbf{y}_\Omega \in \mathbf{W}^{2,r}(\Omega)$, $\mathbf{q}_\Omega \in \mathbf{L}^2(\Omega)$, $\operatorname{div} \mathbf{y}_\Omega, \operatorname{div} \mathbf{q}_\Omega \in W^{1,r}(\Omega)$ *for some* $1 < r < \infty$, *and* (3.41) *is satisfied, then we obtain* (3.30).

Proof. Similar to the proof in Theorem 3.9, the compatibility conditions are meaningful in $W^{-\frac{1}{s},s}(\Gamma)$. Using the linearized equation (2.11), integrating by parts, invoking the conditions $\Delta \mathbf{y}^* \cdot \mathbf{n} = \mathbf{y}_\Omega \cdot \mathbf{n}$ and $\partial_{\mathbf{n}} p^* = \mathbf{q}_\Omega \cdot \mathbf{n}$ on Γ, and applying $\operatorname{div} \Delta \mathbf{y}^* = \Delta \operatorname{div} \mathbf{y}^* = 0$ in Ω, we obtain

$$\begin{aligned} J_{\mathrm{r}}'(\mathbf{u}^*)\mathbf{r} &= \int_\Omega \frac{\alpha}{\nu}(\Delta \mathbf{y}^* - \mathbf{y}_\Omega) \cdot ((\mathbf{y}^* \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{y}^* + \nabla \pi - \mathbf{r})\, dx \\ &\quad - \int_\Omega \beta(\Delta p^* - \operatorname{div} \mathbf{q}_\Omega)\pi \, dx \\ &= -\int_\Omega \frac{\alpha}{\nu}(\Delta \mathbf{y}^* - \mathbf{y}_\Omega) \cdot \mathbf{r}\, dx + \langle \mathbf{f}(\mathbf{y}^*), \mathbf{w}\rangle_{\mathbf{W}^{-1,s}(\Omega), \mathbf{W}_0^{1,s'}(\Omega)} + \int_\Omega \pi \Lambda g(p^*)\, dx, \end{aligned}$$

where $\mathbf{f}(\mathbf{y}^*) \in \mathbf{W}^{-1,s}(\Omega)$ and $\Lambda g(p^*) \in \widehat{L}^2(\Omega)$ are given by (3.44) and (3.45), respectively, for every $\mathbf{r} \in \mathbf{W}^{1,2}(\Omega)$.

Let us show the declared regularity of $\mathbf{f}(\mathbf{y}^*)$ and $g(p^*)$ in the previous statement. On one hand, that of $\Lambda g(p^*)$ is clear since $\operatorname{div} \mathbf{y}_\Omega, \operatorname{div} \mathbf{p}_\Omega \in L^s(\Omega) \subset L^2(\Omega)$ and $p^* \in \widehat{W}^{2,2}(\Omega)$. On the other hand, we have $(\nabla \mathbf{y}^*)^\top(\Delta \mathbf{y}^* - \mathbf{y}_\Omega) \in \mathbf{L}^s(\Omega) \subset \mathbf{W}^{-1,s}(\Omega)$ and $(\mathbf{y}^* \cdot \nabla)(\Delta \mathbf{y}^* - \mathbf{y}_\Omega) \in \mathbf{W}^{-1,s}(\Omega)$ since

$$\begin{aligned} \int_\Omega (\nabla \mathbf{y}^*)^\top(\Delta \mathbf{y}^* - \mathbf{y}_\Omega) \cdot \mathbf{w}\, dx &\leq \|\nabla \mathbf{y}^*\|_{\mathbf{L}^\infty(\Omega)^2}(\|\Delta \mathbf{y}^*\|_{\mathbf{L}^s(\Omega)} + \|\mathbf{y}_\Omega\|_{\mathbf{L}^s(\Omega)})\|\mathbf{w}\|_{\mathbf{L}^{s'}(\Omega)} \\ &\leq c\|\mathbf{y}^*\|_{\mathbf{V}^{3,2}(\Omega)}(\|\mathbf{y}^*\|_{\mathbf{V}^{3,2}(\Omega)} + \|\mathbf{y}_\Omega\|_{\mathbf{L}^s(\Omega)})\|\mathbf{w}\|_{\mathbf{L}^{s'}(\Omega)}, \\ \int_\Omega (\mathbf{y}^* \cdot \nabla)\mathbf{w} \cdot (\Delta \mathbf{y}^* - \mathbf{y}_\Omega)\, dx &\leq \|\mathbf{y}^*\|_{\mathbf{L}^\infty(\Omega)}\|\nabla \mathbf{w}\|_{\mathbf{L}^{s'}(\Omega)^2}(\|\Delta \mathbf{y}^*\|_{\mathbf{L}^s(\Omega)} + \|\mathbf{y}_\Omega\|_{\mathbf{L}^s(\Omega)}) \\ &\leq c\|\mathbf{y}^*\|_{\mathbf{V}^{3,2}(\Omega)}(\|\mathbf{y}^*\|_{\mathbf{V}^{3,2}(\Omega)} + \|\mathbf{y}_\Omega\|_{\mathbf{L}^s(\Omega)})\|\mathbf{w}\|_{\mathbf{W}_0^{1,s'}(\Omega)}, \end{aligned}$$

for every $\mathbf{w} \in \mathbf{W}_0^{1,s'}(\Omega)$, due to $\mathbf{y}^* \in \mathbf{V}^{3,2}(\Omega) \subset \mathbf{W}^{1,\infty}(\Omega) \cap \mathbf{W}^{2,s}(\Omega)$. Thus, $\mathbf{f}(\mathbf{y}^*) \in \mathbf{W}^{-1,s}(\Omega)$.

The derivative of the localized reduced cost functional J_{r} at the optimal control $\mathbf{u}^*$ in the direction of $\mathbf{r}$ can be written as

$$0 = J_{\mathrm{r}}'(\mathbf{u}^*)\mathbf{r} = \int_\Omega \left[\mathbf{v}^* - \frac{\alpha}{\nu}(\Delta\mathbf{y}^* - \mathbf{y}_\Omega)\right]\cdot\mathbf{r} + \rho\mathbf{u}^*\cdot\mathbf{r} + \rho\nabla\mathbf{u}^* : \nabla\mathbf{r}\,\mathrm{d}x$$

for every $\mathbf{r}\in\mathbf{W}^{1,2}(\Omega)$, where $\mathfrak{A}^\star_{\mathbf{y}^*}(\mathbf{v}^*,\sigma^*) = (\mathbf{f}(\mathbf{y}^*), \Lambda g(p^*))$. Hence, $\mathbf{u}^*$ is a solution to the elliptic boundary value problem (3.43). Since $\mathbf{f}(\mathbf{y}^*)\in\mathbf{W}^{-1,s}(\Omega)\subset\mathbf{W}^{-1,2}(\Omega)$ and $\Lambda g(p^*)\in\widehat{L}^2(\Omega)$, we deduce that $(\mathbf{v}^*,\sigma^*)\in\mathbf{W}^{1,2}_0(\Omega)\times\widehat{L}^2(\Omega)$ by Lemma 2.6. This implies that $\mathbf{v}^* - \frac{\alpha}{\nu}(\Delta\mathbf{y}^* - \mathbf{y}_\Omega)\in\mathbf{L}^s(\Omega)$ since $\mathbf{W}^{1,2}_0(\Omega)\subset\mathbf{L}^s(\Omega)$, and as a consequence, $\mathbf{u}^*\in\mathbf{W}^{2,s}(\Omega)$ by standard elliptic regularity theory. This regularity of the control leads to $(\mathbf{y}^*,p^*)\in\mathbf{V}^{4,s}(\Omega)\times\widehat{W}^{3,s}(\Omega)$. Hence, $\Delta p^*\in W^{1,s}(\Omega)$ and we have $\Lambda g(p^*)\in\widehat{L}^s(\Omega)$. In turn, we deduce that $(\mathbf{v}^*,\sigma^*)\in\mathbf{W}^{1,s}_0(\Omega)\times\widehat{L}^s(\Omega)$. This completes the proof (3.42).

Assume that $\mathbf{y}_\Omega\in\mathbf{W}^{1,r}(\Omega)$, $\mathbf{q}_\Omega\in\mathbf{L}^2(\Omega)$, and $\operatorname{div}\mathbf{y}_\Omega, \operatorname{div}\mathbf{q}_\Omega\in W^{1,r}(\Omega)$ for some $1<r<\infty$. Because $W^{1,r}(\Omega)\subset L^2(\Omega)$, we have $\mathbf{y}_\Omega,\mathbf{q}_\Omega\in\mathbf{L}^2_{\mathrm{div}}(\Omega)$. This means that (3.42) holds with $s=2$. Note that $(\nabla\mathbf{y}^*)^\top(\Delta\mathbf{y}^* - \mathbf{y}_\Omega)\in\mathbf{L}^r(\Omega)$ as above, and moreover, $(\mathbf{y}^*\cdot\nabla)(\Delta\mathbf{y}^* - \mathbf{y}_\Omega)\in\mathbf{L}^r(\Omega)$ since $\mathbf{y}^*\in\mathbf{V}^{4,2}(\Omega)\subset\mathbf{L}^\infty(\Omega)\cap\mathbf{W}^{3,r}(\Omega)$ and

$$\begin{aligned}\int_\Omega (\mathbf{y}^*\cdot\nabla)(\Delta\mathbf{y}^* - \mathbf{y}_\Omega)\cdot\mathbf{w}\,\mathrm{d}x &\le \|\mathbf{y}^*\|_{\mathbf{L}^\infty(\Omega)}(\|\nabla\Delta\mathbf{y}^*\|_{\mathbf{L}^r(\Omega)^2} + \|\mathbf{y}_\Omega\|_{\mathbf{L}^r(\Omega)})\|\mathbf{w}\|_{\mathbf{L}^{r'}(\Omega)}\\ &\le c\|\mathbf{y}^*\|_{\mathbf{V}^{4,2}(\Omega)}(\|\mathbf{y}^*\|_{\mathbf{V}^{4,2}(\Omega)} + \|\mathbf{y}_\Omega\|_{\mathbf{L}^r(\Omega)})\|\mathbf{w}\|_{\mathbf{L}^{r'}(\Omega)}\end{aligned}$$

for every $\mathbf{w}\in\mathbf{L}^{r'}(\Omega)$. Hence, $\mathbf{f}(\mathbf{y}^*)\in\mathbf{L}^r(\Omega)$.

Suppose that $1<r\le 2$. Then, from $p^*\in\widehat{W}^{3,2}(\Omega)\subset W^{3,r}(\Omega)$ we obtain $\Lambda g(p^*)\in\widehat{W}^{1,r}(\Omega)$. Thus, $(\mathbf{v}^*,\sigma^*)\in\mathbf{X}^{2,r}(\Omega)\times\widehat{W}^{1,r}(\Omega)$ by Lemma 2.7, which is the last two components in (3.30). By elliptic regularity, we obtain $\mathbf{u}^*\in\mathbf{W}^{3,r}(\Omega)$, so that $(\mathbf{y}^*,p^*)\in\mathbf{V}^{5,r}(\Omega)\times\widehat{W}^{4,r}(\Omega)$. As a consequence, the right-hand side of (3.43) belongs to $\mathbf{W}^{2,r}(\Omega)$. Thus, $\mathbf{u}^*\in\mathbf{W}^{4,r}(\Omega)$ and $(\mathbf{y}^*,p^*)\in\mathbf{V}^{6,r}(\Omega)\times\widehat{W}^{5,r}(\Omega)$, establishing the first three components in (3.30).

Now, assume that $2<r<\infty$. In this case, (3.30) holds for $r=2$, and in particular, $(\mathbf{y}^*,p^*)\in\mathbf{V}^{6,2}(\Omega)\times\widehat{W}^{5,2}(\Omega)\subset\mathbf{V}^{5,r}(\Omega)\times\widehat{W}^{4,r}(\Omega)$. From this, we again deduce (3.30). □

In the context of Theorem 3.10, the optimality system of the localized problem with cost functional (3.35) and controls in $\mathbf{W}^{1,2}(\Omega)$ is given by

$$\left[\begin{array}{l} -\nu\Delta\mathbf{y}^* + (\mathbf{y}^*\cdot\nabla)\mathbf{y}^* + \nabla p^* = \mathbf{u}^* \ \text{ in } \Omega,\\ \operatorname{div}\mathbf{y}^* = 0 \ \text{ in } \Omega, \quad \mathbf{y}^* = \mathbf{0} \ \text{ on } \Gamma, \quad \langle p^*, 1\rangle_\Omega = 0,\end{array}\right.$$

$$\left[\begin{array}{l} -\nu\Delta\mathbf{v}^* + (\nabla\mathbf{y}^*)^\top\mathbf{v}^* - (\mathbf{y}^*\cdot\nabla)\mathbf{v}^* + \nabla\sigma^*\\ \qquad = \frac{\alpha}{\nu}(\nabla\mathbf{y}^*)^\top(\Delta\mathbf{y}^* - \mathbf{y}_\Omega) - \frac{\alpha}{\nu}(\mathbf{y}^*\cdot\nabla)(\Delta\mathbf{y}^* - \mathbf{y}_\Omega) \ \text{ in } \Omega,\\ \operatorname{div}\mathbf{v}^* = -\Lambda\left(\frac{\alpha}{\nu}\operatorname{div}\mathbf{y}_\Omega - \beta(\Delta p^* - \operatorname{div}\mathbf{q}_\Omega)\right) \ \text{ in } \Omega,\\ \mathbf{v}^* = \mathbf{0} \ \text{ on } \Gamma, \quad \langle\sigma^*, 1\rangle_\Omega = 0,\end{array}\right.$$

$$\left[\begin{array}{l} -\Delta\mathbf{u}^* + \mathbf{u}^* = -\frac{1}{\rho}\left(\mathbf{v}^* - \frac{\alpha}{\nu}(\Delta\mathbf{y}^* - \mathbf{y}_\Omega)\right) \ \text{ in } \Omega,\\ \partial_{\mathbf{n}}\mathbf{u}^* = \mathbf{0} \ \text{ on } \Gamma.\end{array}\right.$$

3.3. Boundary normal stress and pressure tracking. For this subsection, we analyze the problems (3.2), (3.18), and (3.24) in case of the cost functional with boundary observations

$$J(\mathbf{y},p) := \frac{1}{2}\int_\Gamma \alpha|\partial_\mathbf{n}\mathbf{y}-\mathbf{y}_\Gamma|^2+\beta|p-p_\Gamma|^2+\lambda|\mathbf{T}(\mathbf{y},p)\mathbf{n}-\mathbf{s}_\Gamma|^2\,\mathrm{d}s \tag{3.46}$$

where $\alpha,\beta,\lambda\geq 0$ with $\alpha+\beta+\lambda>0$, $\mathbf{y}_\Gamma,\mathbf{s}_\Gamma\in\mathbf{L}^2(\Gamma)$, and $p_\Gamma\in L^2(\Gamma)$. Here, we only provide the main ideas and refer the reader to Subsection 3.1 for the complete details, in particular, to the localization of the current optimal control problem.

To show existence of solutions, it is enough to prove that (3.5) holds for J given by (3.46). We only give the details in the case where the control space lies in $\mathbf{L}^2(\Omega)$ as the other two cases can be dealt with a similar manner. As before, note that $\{(\mathbf{y}_n,p_n)\}_{n=1}^\infty$ is bounded in $\mathbf{V}^{2,2}(\Omega)\times\widehat{W}^{1,2}(\Omega)$ and $(\mathbf{y}_n,p_n)\rightharpoonup(\mathbf{y}^*,p^*)$ in $\mathbf{V}^{2,2}(\Omega)\times\widehat{W}^{1,2}(\Omega)$. Thanks to the Rellich–Kondrachev theorem, we have the compact embeddings $\mathbf{V}^{2,2}(\Omega)\subset\mathbf{W}^{2-\varepsilon,2}(\Omega)$ and $\widehat{W}^{1,2}(\Omega)\subset W^{1-\varepsilon,2}(\Omega)$, with $0<\varepsilon<\frac{1}{2}$. Hence, one can extract a subsequence so that $\mathbf{y}_n\to\mathbf{y}^*$ in $\mathbf{W}^{2-\varepsilon,2}(\Omega)$ and $p_n\to p^*$ in $W^{1-\varepsilon,2}(\Omega)$. The continuity of $\partial_\mathbf{n}:\mathbf{W}^{2-\varepsilon,2}(\Omega)\to\mathbf{W}^{\frac{1}{2}-\varepsilon,2}(\Gamma)$ and the trace operator $\gamma_0:W^{1-\varepsilon,2}(\Omega)\to W^{\frac{1}{2}-\varepsilon,2}(\Gamma)$ along with the continuity of the embeddings $\mathbf{W}^{\frac{1}{2}-\varepsilon,2}(\Gamma)\subset\mathbf{L}^2(\Gamma)$ and $W^{\frac{1}{2}-\varepsilon,2}(\Gamma)\subset L^2(\Gamma)$ imply that $\partial_\mathbf{n}\mathbf{y}_n\to\partial_\mathbf{n}\mathbf{y}^*$ in $\mathbf{L}^2(\Gamma)$, $p_n|_\Gamma\to p^*|_\Gamma$ in $L^2(\Gamma)$, and $\mathbf{T}(\mathbf{y}_n,p_n)\mathbf{n}\to\mathbf{T}(\mathbf{y}^*,p^*)\mathbf{n}$ in $\mathbf{L}^2(\Gamma)$. Therefore, (3.5) is satisfied for (3.46).

For the corresponding localized problems at regular points, we only discuss the case of $\mathbf{L}^2(\Omega)$ once again. Denote by $J_\mathrm{r}:\mathcal{U}_2(\mathbf{u}^*)\to\mathbb{R}$ the reduced cost functional for (3.46) constructed as in Subsection 3.1. For $\mathbf{u}\in\mathcal{U}_2(\mathbf{u}^*)$ and $\mathbf{r}\in\mathbf{L}^2(\Omega)$, we compute the directional derivative as follows:

$$\begin{aligned}J_\mathrm{r}'(\mathbf{u})\mathbf{r} &= \int_\Gamma \alpha(\partial_\mathbf{n}\mathbf{y}-\mathbf{y}_\Gamma)\cdot\partial_\mathbf{n}\mathbf{w}+\beta(p-p_\Gamma)\pi+\lambda(\mathbf{T}(\mathbf{y},p)\mathbf{n}-\mathbf{s}_\Gamma)\cdot\mathbf{T}(\mathbf{w},\pi)\mathbf{n}\,\mathrm{d}s\\ &= \int_\Gamma\left[\beta(p-p_\Gamma)+\frac{\alpha}{\nu}(\partial_\mathbf{n}\mathbf{y}-\mathbf{y}_\Gamma)\cdot\mathbf{n}\right]\pi\,\mathrm{d}s\\ &\quad+\int_\Gamma\left[\lambda(\mathbf{T}(\mathbf{y},p)\mathbf{n}-\mathbf{s}_\Gamma)-\frac{\alpha}{\nu}(\partial_\mathbf{n}\mathbf{y}-\mathbf{y}_\Gamma)\right]\cdot\mathbf{T}(\mathbf{w},\pi)\mathbf{n}\,\mathrm{d}s.\end{aligned}\tag{3.47}$$

Here, we used the fact that $\partial_\mathbf{n}\mathbf{w}=-\frac{1}{\nu}\mathbf{T}(\mathbf{w},\pi)\mathbf{n}+\frac{1}{\nu}\pi\mathbf{n}$.

In order to derive the data for the adjoint equation, we need to introduce certain operators that lift functions defined over the boundary into the domain. Given $\psi\in W^{-\frac{1}{s},s}(\Gamma)$, we define $\gamma_0^\star\psi\in W^{1,s'}(\Omega)'$ by

$$\langle\gamma_0^\star\psi,\phi\rangle_{W^{1,s'}(\Omega)',W^{1,s'}(\Omega)} := \langle\psi,\phi|_\Gamma\rangle_{W^{-\frac{1}{s},s}(\Gamma),W^{1-\frac{1}{s'},s'}(\Gamma)}\qquad\forall\phi\in W^{1,s'}(\Omega).$$

In a similar fashion, given $\boldsymbol{\psi}\in\mathbf{W}^{-\frac{1}{s},s}(\Gamma)$, we define $\boldsymbol{\gamma}_1^\star\boldsymbol{\psi}\in\mathbf{X}^{-2,s}(\Omega)$ by

$$\langle\boldsymbol{\gamma}_1^\star\boldsymbol{\psi},\boldsymbol{\varphi}\rangle_{\mathbf{X}^{-2,s}(\Omega),\mathbf{X}^{2,s'}(\Omega)} := \langle\boldsymbol{\psi},\partial_\mathbf{n}\boldsymbol{\varphi}\rangle_{\mathbf{W}^{-\frac{1}{s},s}(\Gamma),\mathbf{W}^{1-\frac{1}{s'},s'}(\Gamma)}\qquad\forall\boldsymbol{\varphi}\in\mathbf{X}^{2,s'}(\Omega).$$

Let us provide a formulation of the adjoint equation having homogeneous Dirichlet boundary condition. Other formulations leading to the same regularity of the optimal solutions will be provided below. Since $\mathbf{T}(\mathbf{w},\pi)\mathbf{n}=-\nu\partial_\mathbf{n}\mathbf{w}+\pi\mathbf{n}$ and using

$\pi \in \widehat{W}^{1,2}(\Omega)$, we obtain from (3.47) that

$$\begin{aligned} J_{\mathrm{r}}'(\mathbf{u})\mathbf{r} &= \int_\Gamma (\alpha(\partial_{\mathbf{n}}\mathbf{y} - \mathbf{y}_\Gamma) - \lambda\nu(\mathbf{T}(\mathbf{y},p)\mathbf{n} - \mathbf{s}_\Gamma)) \cdot \partial_{\mathbf{n}}\mathbf{w}\,\mathrm{d}s \\ &\quad + \int_\Gamma (\beta(p - p_\Gamma) + \lambda(\mathbf{T}(\mathbf{y},p)\mathbf{n} - \mathbf{s}_\Gamma)\cdot\mathbf{n})\pi\,\mathrm{d}s \\ &= \langle \mathbf{f}(\mathbf{y},p), \mathbf{w}\rangle_{\mathbf{X}^{-2,2}(\Omega),\mathbf{X}^{2,2}(\Omega)} + \langle \Lambda g(\mathbf{y},p), \pi\rangle_{\widehat{W}^{1,2}(\Omega)',\widehat{W}^{1,2}(\Omega)} \end{aligned}$$

where

$$\mathbf{f}(\mathbf{y},p) := \boldsymbol{\gamma}_1^\star[(\alpha+\lambda\nu^2)\partial_{\mathbf{n}}\mathbf{y} - \alpha\mathbf{y}_\Gamma - \lambda\nu(p\mathbf{n} - \mathbf{s}_\Gamma)], \tag{3.48}$$

$$g(\mathbf{y},p) := \gamma_0^\star[(\beta+\lambda)p - \beta p_\Gamma + \lambda(-\nu\partial_{\mathbf{n}}\mathbf{y}\cdot\mathbf{n} - \mathbf{s}_\Gamma\cdot\mathbf{n})], \tag{3.49}$$

so that (3.12) is satisfied with $\mathfrak{D}_{\mathbf{y}}(\mathbf{v},\sigma) = (\mathbf{f}(\mathbf{y},p), \Lambda g(\mathbf{y},p), \mathbf{0})$.

As a consequence, the optimality system for the localized problem having the cost functional (3.46) and controls in $\mathbf{L}^2(\Omega)$ is given by (3.13)–(3.15), where the right-hand sides of (3.14) are those from (3.48) and (3.49). Once again, we need to replace (3.15) by (3.21) or (3.28) in the situation where the controls belong to $\mathbf{U}_{\mathrm{ad}}$ or $\mathbf{W}^{1,2}(\Omega)$, respectively.

In the succeeding theorem, we obtain twice the integrability order of the desired states from the boundary to the interior with respect to the adjoint velocity.

Theorem 3.11. *In the case of $\mathbf{L}^2(\Omega)$, $\mathbf{y}_\Gamma, \mathbf{s}_\Gamma \in \mathbf{L}^s(\Gamma)$, and $p_\Gamma \in L^s(\Omega)$ for some $2 \le s < \infty$ in (3.46), we have*

$$(\mathbf{y}^*, p^*, \mathbf{u}^*, \mathbf{v}^*, \sigma^*) \in \mathbf{V}^{2,2s}(\Omega)\times\widehat{W}^{1,2s}(\Omega)\times\mathbf{L}^{2s}(\Omega)\times\mathbf{L}^{2s}(\Omega)\times\widehat{W}^{-1,2s}(\Omega). \tag{3.50}$$

The same result holds for the case of $\mathbf{U}_{\mathrm{ad}}$ provided $\mathbf{a},\mathbf{b} \in \mathbf{L}^{2s}(\Omega)$. Finally, in the case of $\mathbf{W}^{1,2}(\Omega)$, it holds that

$$(\mathbf{y}^*, p^*, \mathbf{u}^*) \in \mathbf{V}^{4,2s}(\Omega)\times\widehat{W}^{3,2s}(\Omega)\times\mathbf{W}^{2,2s}(\Omega). \tag{3.51}$$

Proof. We show that $\gamma_0^\star\psi \in W^{1,\frac{2s}{2s-1}}(\Omega)'$ whenever $\psi \in L^s(\Gamma)$. Indeed, if $\phi \in W^{1,\frac{2s}{2s-1}}(\Omega)$, then $\phi|_\Gamma \in W^{\frac{1}{2s},\frac{2s}{2s-1}}(\Gamma) \subset L^{\frac{s}{s-1}}(\Gamma) = L^{s'}(\Gamma)$ by the one-dimensional Sobolev embedding theorem since $\frac{1}{2s-1} < 1$. Thus, for each $\phi \in W^{1,s'}(\Omega) \subset W^{1,\frac{2s}{2s-1}}(\Omega)$, we have

$$\begin{aligned} \langle \gamma_0^\star\psi, \phi\rangle_{W^{-1,s}(\Omega),W^{1,s'}(\Omega)} &= \int_\Gamma \psi\phi|_\Gamma\,\mathrm{d}s \le \|\psi\|_{L^s(\Gamma)}\|\phi|_\Gamma\|_{L^{s'}(\Gamma)} \\ &\le c\|\psi\|_{L^s(\Gamma)}\|\phi\|_{W^{1,\frac{2s}{2s-1}}(\Omega)}. \end{aligned}$$

Since $W^{1,s'}(\Omega)$ is dense in $W^{1,\frac{2s}{2s-1}}(\Omega)$, we conclude that $\gamma_0^\star\psi \in W^{1,\frac{2s}{2s-1}}(\Omega)'$. In a similar fashion, $\boldsymbol{\gamma}_1^\star\boldsymbol{\psi} \in \mathbf{X}^{-2,2s}(\Omega)$ whenever $\boldsymbol{\psi} \in \mathbf{L}^s(\Omega)$.

From the equation $\mathfrak{D}_{\mathbf{y}^*}(\mathbf{v}^*,\sigma^*) = (\mathbf{f}(\mathbf{y}^*,p^*), \Lambda g(\mathbf{y}^*,p), \mathbf{0})$, where the components on right-hand sides are given by (3.48) and (3.49), we get $(\mathbf{v}^*,\sigma^*) \in \mathbf{L}^{2s}(\Omega)\times\widehat{W}^{-1,2s}(\Omega)$ from Corollary 2.13, and hence $\mathbf{u}^* \in \mathbf{L}^{2s}(\Omega)$ by (3.15) in the case of controls in $\mathbf{L}^2(\Omega)$. In light of (3.21), same conclusions hold for controls in $\mathbf{U}_{\mathrm{ad}}$ as long as $\mathbf{a},\mathbf{b} \in \mathbf{L}^{2s}(\Omega)$. For controls in $\mathbf{W}^{1,2}(\Omega)$, we have $\mathbf{u}^* \in \mathbf{W}^{2,2s}(\Omega)$ according to (3.28) and $\mathbf{v}^* \in \mathbf{L}^{2s}(\Omega)$, and therefore, $(\mathbf{y}^*,p^*) \in \mathbf{V}^{4,2s}(\Omega)\times\widehat{W}^{3,2s}(\Omega)$

by (2.31). □

Let us provide an alternative adjoint equation with non-homogenous boundary data. Utilizing the fact that $g(\mathbf{y},p) - \Pi(g(\mathbf{y},p),\mathbf{h}(\mathbf{y},p))$ is constant, the derivative provided in (3.47) can be written as

$$\begin{aligned}J_{\mathrm{r}}'(\mathbf{u})\mathbf{r} &= \langle g(\mathbf{y},p),\pi\rangle_{W^{1,2}(\Omega)',W^{1,2}(\Omega)} + \int_\Gamma \mathbf{h}(\mathbf{y},p)\cdot\mathbf{T}(\mathbf{w},\pi)\mathbf{n}\,\mathrm{d}s\\ &= \langle \Pi(g(\mathbf{y},p),\mathbf{h}(\mathbf{y},p)),\pi\rangle_{\widehat{W}^{1,2}(\Omega)',\widehat{W}^{1,2}(\Omega)} + \int_\Gamma \mathbf{h}(\mathbf{y},p)\cdot\mathbf{T}(\mathbf{w},\pi)\mathbf{n}\,\mathrm{d}s\end{aligned}$$

where

$$g(\mathbf{y},p) := \gamma_0^\star\left[\beta(p-p_\Gamma) + \frac{\alpha}{\nu}(\partial_{\mathbf{n}}\mathbf{y} - \mathbf{y}_\Gamma)\cdot\mathbf{n}\right], \tag{3.52}$$

$$\mathbf{h}(\mathbf{y},p) := -\left(\lambda\nu + \frac{\alpha}{\nu}\right)\partial_{\mathbf{n}}\mathbf{y} + \lambda(p\mathbf{n} - \mathbf{s}_\Gamma) + \frac{\alpha}{\nu}\mathbf{y}_\Gamma. \tag{3.53}$$

This yields (3.12) with $\mathfrak{D}_{\mathbf{y}}(\mathbf{v},\sigma) = (\mathbf{0}, \Pi(g(\mathbf{y},p),\mathbf{h}(\mathbf{y},p)),\mathbf{h}(\mathbf{y},p))$.

Note that the above formulation leads to the same regularity result as in Theorem 3.11. To see this, note that $\mathbf{W}^{\frac{1}{2s},\frac{2s}{2s-1}}(\Gamma) \subset \mathbf{L}^{s'}(\Gamma)$ implies that $\mathbf{L}^s(\Gamma) \subset \mathbf{W}^{-\frac{1}{2s},2s}(\Gamma)$ by duality. As a consequence, we obtain from (3.53) that $\mathbf{h}(\mathbf{y}^*,p^*) \in \mathbf{W}^{-\frac{1}{2s},2s}(\Gamma)$. As in the proof of the previous theorem, $g(\mathbf{y}^*,p^*) \in \widehat{W}^{1,\frac{2s}{2s-1}}(\Omega)'$. Since $(\Pi(g(\mathbf{y}^*,p^*),\mathbf{h}(\mathbf{y}^*,p^*)),\mathbf{h}(\mathbf{y}^*,p^*)) \in \mathbf{Z}^{-1,-\frac{1}{2s},2s}(\Omega,\Gamma)$, we have $(\mathbf{v}^*,\sigma^*) \in \mathbf{L}^{2s}(\Omega) \times \widehat{W}^{1,\frac{2s}{2s-1}}(\Omega)'$ thanks to Corollary 2.13.

It is also possible to have a constant-divergence formulation for the adjoint equation. Indeed, since $\pi = \pi\mathbf{n}\cdot\mathbf{n} = \mathbf{T}(\mathbf{w},\pi)\mathbf{n}\cdot\mathbf{n} + \nu\partial_{\mathbf{n}}\mathbf{w}\cdot\mathbf{n}$ we have

$$\begin{aligned}J_{\mathrm{r}}'(\mathbf{u})\mathbf{r} &= \int_\Gamma \alpha(\partial_{\mathbf{n}}\mathbf{y} - \mathbf{y}_\Gamma)\cdot\partial_{\mathbf{n}}\mathbf{w} + \beta(p-p_\Gamma)\pi + \lambda(\mathbf{T}(\mathbf{y},p)\mathbf{n} - \mathbf{s}_\Gamma)\cdot\mathbf{T}(\mathbf{w},\pi)\mathbf{n}\,\mathrm{d}s\\ &= \int_\Gamma (\alpha(\partial_{\mathbf{n}}\mathbf{y} - \mathbf{y}_\Gamma) + \beta\nu(p-p_\Gamma)\mathbf{n})\cdot\partial_{\mathbf{n}}\mathbf{w}\,\mathrm{d}s\\ &\quad + \int_\Gamma [\lambda(\mathbf{T}(\mathbf{y},p)\mathbf{n} - \mathbf{s}_\Gamma) + \beta(p-p_\Gamma)\mathbf{n}]\cdot\mathbf{T}(\mathbf{w},\pi)\mathbf{n}\,\mathrm{d}s\\ &= \langle \mathbf{f}(\mathbf{y},p),\mathbf{w}\rangle_{\mathbf{X}^{-2,2}(\Omega),\mathbf{X}^{2,2}(\Omega)} + \langle \Sigma\mathbf{h}(\mathbf{y},p),\pi\rangle_{\widehat{W}^{1,2}(\Omega)',\widehat{W}^{1,2}(\Omega)}\\ &\quad + \int_\Gamma \mathbf{h}(\mathbf{y},p)\cdot\mathbf{T}(\mathbf{w},\pi)\mathbf{n}\,\mathrm{d}s\end{aligned}$$

where

$$\mathbf{f}(\mathbf{y},p) := \boldsymbol{\gamma}_1^*[\alpha(\partial_{\mathbf{n}}\mathbf{y} - \mathbf{y}_\Gamma) + \beta\nu(p-p_\Gamma)\mathbf{n}], \tag{3.54}$$

$$\mathbf{h}(\mathbf{y},p) := -\lambda\nu\partial_{\mathbf{n}}\mathbf{y} + (\lambda+\beta)p\mathbf{n} - \lambda\mathbf{s}_\Gamma - \beta p_\Gamma\mathbf{n}. \tag{3.55}$$

Thus, we have (3.12) with

$$\mathfrak{D}_{\mathbf{y}}(\mathbf{v},\sigma) = (\mathbf{f}(\mathbf{y},p), \Sigma\mathbf{h}(\mathbf{y},p), \mathbf{h}(\mathbf{y},p)) \in \mathbf{X}^{-1,2s}(\Omega) \times \mathbf{Z}^{-1,-\frac{1}{2s},2s}(\Omega,\Gamma),$$

and this gives the same result (3.50).

We close this subsection by highlighting the adjoint systems with observations involving only the normal stress on the boundary, that is, when $\alpha = \beta = 0$ in

(3.46). On one hand, the corresponding adjoint problem with homogeneous Dirichlet condition is given by the very weak solution to

$$\left[\begin{array}{l} -\nu\Delta\mathbf{v}^* + (\nabla\mathbf{y}^*)^\top\mathbf{v}^* - (\mathbf{y}^*\cdot\nabla)\mathbf{v}^* + \nabla\sigma^* = \lambda\nu\gamma_1^\star(\nu\partial_{\mathbf{n}}\mathbf{y}^* - p^*\mathbf{n} + \mathbf{s}_\Gamma) \ \text{ in } \Omega, \\ \operatorname{div}\mathbf{v}^* = -\lambda\Lambda\gamma_0^\star(p^* - \nu\partial_{\mathbf{n}}\mathbf{y}^*\cdot\mathbf{n} - \mathbf{s}_\Gamma\cdot\mathbf{n}) \ \text{ in } \Omega, \quad \mathbf{v}^* = \mathbf{0} \ \text{ on } \Gamma, \quad \langle\sigma^*, 1\rangle_\Omega = 0. \end{array}\right.$$

This is based on (3.48) and (3.49) with $\alpha = \beta = 0$. On the other hand, if we utilize (3.52) and (3.53), then we obtain the following adjoint problem with homogeneous force term

$$\left[\begin{array}{l} -\nu\Delta\mathbf{v}^* + (\nabla\mathbf{y}^*)^\top\mathbf{v}^* - (\mathbf{y}^*\cdot\nabla)\mathbf{v}^* + \nabla\sigma^* = \mathbf{0} \ \text{ in } \Omega, \\ \operatorname{div}\mathbf{v}^* = \lambda\Sigma(\nu\partial_{\mathbf{n}}\mathbf{y}^* - p^*\mathbf{n} + \mathbf{s}_\Gamma) \ \text{ in } \Omega, \\ \mathbf{v}^* = -\lambda(\nu\partial_{\mathbf{n}}\mathbf{y}^* - p^*\mathbf{n} + \mathbf{s}_\Gamma) \ \text{ on } \Gamma, \ \langle\sigma^*, 1\rangle_\Omega = 0. \end{array}\right.$$

We can also obtain this adjoint problem with respect to (3.54) and (3.55) with $\alpha = \beta = 0$.

3.4. Pointwise velocity, stress, and pressure tracking. In this final section, we deal with cost functionals taking into account point evaluations of the velocity and the pressure. In the case of the pointwise tracking of the velocity, the result is analogous in the one provided in [8] for linear elliptic problems and in [6, 7] for the linear Stokes equation.

Consider the functional $J : \mathbf{V}^{2,2}(\Omega) \to \mathbb{R}$ with pointwise velocity observations

$$J(\mathbf{y}) := \frac{1}{2}\sum_{\xi\in\mathcal{D}} \alpha_\xi |\mathbf{y}(\xi) - \mathbf{y}_\xi|^2 \tag{3.56}$$

where $\mathcal{D}$ is a nonempty finite subset of Ω and for each $\xi \in \mathcal{D}$, $\alpha_\xi > 0$ and $\mathbf{y}_\xi \in \mathbb{R}^2$. Existence of a solution to the optimal control problem (3.31) with the cost functional (3.56) can be established as follows. Following the notation in Subsection 3.1, one can obtain $\mathbf{y}_n(\xi) \to \mathbf{y}^*(\xi)$ in $\mathbb{R}^2$ for every $\xi \in \bar{\Omega}$ thanks to the compact embedding $\mathbf{V}^{2,2}(\Omega) \subset \mathbf{C}(\bar{\Omega})$. This yields $J(\mathbf{y}_n) \to J(\mathbf{y}^*)$, and hence, the existence of a solution to (3.31). The same conclusion holds for the scenario where we have controls in $\mathbf{U}_{\rm ad}$ and $\mathbf{W}^{1,2}(\Omega)$.

To write the adjoint systems, let us introduce the following notation. For $a \in \mathbb{R}$, $\phi \in C(\bar{\Omega})$, and $\eta \in \bar{\Omega}$, we define $a\delta_\eta \in M(\bar{\Omega}) := C(\bar{\Omega})'$ by

$$\langle a\delta_\eta, \phi\rangle_{M(\bar{\Omega}),C(\bar{\Omega})} := a\phi(\eta).$$

Similarly, given $\mathbf{a} \in \mathbb{R}^2$, $\boldsymbol{\varphi} \in \mathbf{C}_0(\Omega)$, and $\xi \in \Omega$, we introduce $\mathbf{a}\otimes\delta_\xi \in \mathbf{M}_0(\Omega) := \mathbf{C}_0(\Omega)'$ by

$$\langle \mathbf{a}\otimes\delta_\xi, \boldsymbol{\varphi}\rangle_{\mathbf{M}_0(\Omega),\mathbf{C}_0(\Omega)} := \mathbf{a}\cdot\boldsymbol{\varphi}(\xi).$$

We now proceed with the local problems (3.2), (3.18), and (3.24) corresponding to (3.56). Again, avoiding repetitive arguments, we only give the main ideas as the procedure is completely the same as that with Subsection 3.1. Then, for the induced local reduced version $J_{\rm r}$ of (3.56), the action of the directional derivative can be written as

$$J_{\rm r}'(\mathbf{u})\mathbf{r} = \langle \mathbf{f}(\mathbf{y}), \mathbf{w}\rangle_{\mathbf{M}_0(\Omega),\mathbf{C}_0(\Omega)}$$

where

$$\mathbf{f}(\mathbf{y}) := \sum_{\xi\in\mathcal{D}} \alpha_\xi(\mathbf{y}(\xi) - \mathbf{y}_\xi) \otimes \boldsymbol{\delta}_\xi.$$

The above duality pairing is well-defined due to $\mathbf{w} \in \mathbf{W}_0^{2,2}(\Omega) \subset \mathbf{C}_0(\Omega)$ by the Sobolev embedding theorem. Thus, we obtain the equation (3.12) with $\mathfrak{A}_{\mathbf{y}}^\star(\mathbf{v},\sigma) = (\mathbf{f}(\mathbf{y}), 0)$. Therefore, the adjoint system in the case of (3.56) is given by the very weak solution of

$$\left[\begin{array}{l} -\nu\Delta\mathbf{v}^* + (\nabla\mathbf{y}^*)^\top\mathbf{v}^* - (\mathbf{y}^*\cdot\nabla)\mathbf{v}^* + \nabla\sigma^* = \displaystyle\sum_{\xi\in\mathcal{D}} \alpha_\xi(\mathbf{y}^*(\xi) - \mathbf{y}_\xi)\otimes\boldsymbol{\delta}_\xi \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{v}^* = 0 \quad \text{in } \Omega, \qquad \mathbf{v}^* = \mathbf{0} \quad \text{on } \Gamma, \qquad \langle\sigma^*, 1\rangle_\Omega = 0. \end{array}\right.$$

Theorem 3.12. *Let $1 < r < 2$ and consider the localized optimal control problems (3.9), (3.19), and (3.26) associated with (3.56). If the controls lie in $\mathbf{L}^2(\Omega)$, then we have*

$$(\mathbf{y}^*, p^*, \mathbf{u}^*, \mathbf{v}^*, \sigma^*) \in \mathbf{V}^{3,r}(\Omega) \times \widehat{W}^{2,r}(\Omega) \times \mathbf{V}^{1,r}(\Omega) \times \mathbf{V}^{1,r}(\Omega) \times \widehat{L}^r(\Omega). \tag{3.57}$$

If the controls lie in $\mathbf{U}_{\mathrm{ad}}$ with $\mathbf{a}, \mathbf{b} \in \mathbf{W}^{1,r}(\Omega)$, then (3.57) holds except that we have $\mathbf{u}^* \in \mathbf{W}^{1,r}(\Omega)$. Finally, in the case of controls in $\mathbf{W}^{1,2}(\Omega)$, we obtain

$$(\mathbf{y}^*, p^*, \mathbf{u}^*, \mathbf{v}^*, \sigma^*) \in \mathbf{V}^{5,r}(\Omega) \times \widehat{W}^{4,r}(\Omega) \times \mathbf{W}^{3,r}(\Omega) \times \mathbf{V}_0^{1,r}(\Omega) \times \widehat{L}^r(\Omega). \tag{3.58}$$

Proof. If $1 < r < 2$, then $r' > 2$ so that $\mathbf{W}_0^{1,r'}(\Omega) \subset \mathbf{C}_0(\Omega)$ by the Sobolev embedding theorem. Hence, $\mathbf{M}_0(\Omega) \subset \mathbf{W}^{-1,r}(\Omega)$ by duality. This implies that $\mathbf{f}(\mathbf{y}^*) \in \mathbf{W}^{-1,r}(\Omega)$, and since $\mathfrak{A}_{\mathbf{y}^*}^\star(\mathbf{v}^*,\sigma^*) = (\mathbf{f}(\mathbf{y}^*), 0)$, we have $(\mathbf{v}^*, \sigma^*) \in \mathbf{V}^{1,r}(\Omega) \times \widehat{L}^r(\Omega)$ by Lemma 2.6. Thus, $\mathbf{u}^* \in \mathbf{V}^{1,r}(\Omega)$ by (3.15) and $(\mathbf{y}^*, p^*) \in \mathbf{V}^{3,r}(\Omega) \times \widehat{W}^{2,r}(\Omega)$ by (2.31). The case where the controls lie in $\mathbf{U}_{\mathrm{ad}}$ or $\mathbf{W}^{1,2}(\Omega)$ can be handled in a similar way, but now invoking (3.21) and (3.28), respectively. □

Now, we shall take pointwise evaluations on the velocity gradient, the pressure, and the normal stress, namely,

$$\begin{aligned} J(\mathbf{y}, p) := &\frac{1}{2}\sum_{\xi\in\mathcal{E}} (\alpha_\xi|\nabla\mathbf{y}(\xi) - \mathbf{Y}_\xi|^2 + \beta_\xi|p(\xi) - p_\xi|^2 + \lambda_\xi|\mathbf{T}(\mathbf{y},p)(\xi) - \mathbf{S}_\xi|^2) \\ &+ \frac{1}{2}\sum_{\eta\in\mathcal{G}} (\zeta_\eta|\partial_{\mathbf{n}}\mathbf{y}(\eta) - \mathbf{y}_\eta|^2 + \rho_\eta|\mathbf{T}(\mathbf{y},p)(\eta)\mathbf{n}(\eta) - \mathbf{s}_\eta|^2) \end{aligned} \tag{3.59}$$

where $\mathcal{E}$ and $\mathcal{G}$ are nonempty finite subsets of $\bar{\Omega}$ and Γ, respectively, and $\alpha_\xi, \beta_\xi, \lambda_\xi, \zeta_\eta, \rho_\eta \geq 0$ for every $\xi \in \mathcal{E}$ and $\eta \in \mathcal{G}$, for which at least one of these parameters is nonzero. Also, $p_\xi \in \mathbb{R}$, $\mathbf{y}_\eta, \mathbf{s}_\eta \in \mathbb{R}^2$, and $\mathbf{Y}_\xi, \mathbf{S}_\xi \in \mathbb{R}^{2\times 2}$ are given data. Note that (3.59) allows the case where the points can be different for each of the terms appearing in the above summations.

Observe that controls in $\mathbf{L}^2(\Omega)$ are not amenable due to the limited regularity of the velocity gradient and pressure, that is, we only have $\nabla\mathbf{y} \in \mathbf{W}^{1,2}(\Omega)^2$ and $p \in \widehat{W}^{1,2}(\Omega)$ a priori. In particular, pointwise evaluations of $\nabla\mathbf{y}$ and p are not well-defined. Hence, $\mathbf{U}_{\mathrm{ad}}$ for suitable $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{W}^{1,2}(\Omega)$ will be used instead.

Now, we prove that (3.18) with the tracking functional (3.59) has a solution. Recall that $\mathbf{U}_{\rm ad} \subset \mathbf{L}^s(\Omega)$ whenever $\mathbf{a}, \mathbf{b} \in \mathbf{L}^s(\Omega)$, where $2 < s < \infty$. Let $\{(\mathbf{y}_n, p_n, \mathbf{u}_n)\}_{n=1}^\infty$ be a minimizing sequence. Then, $\{\mathbf{u}_n\}_{n=1}^\infty$ is bounded in $\mathbf{L}^s(\Omega)$, and consequently, $\{(\mathbf{y}_n, p_n)\}_{n=1}^\infty$ is bounded in $\mathbf{V}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega)$ thanks to (2.9). Therefore, passing to a subsequence, there exists $(\mathbf{y}^*, p^*, \mathbf{u}^*) \in \mathbf{V}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega) \times \mathbf{U}_{\rm ad}$ such that $(\mathbf{y}_n, p_n) \rightharpoonup (\mathbf{y}^*, p^*)$ in $\mathbf{V}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega)$ and $\mathbf{u}_n \rightharpoonup \mathbf{u}^*$ in $\mathbf{L}^s(\Omega)$. Since $2 < s < \infty$, we immediately see that the pair $(\mathbf{y}^*, p^*)$ is a solution to (1.2) with control $\mathbf{u}^*$.

By Rellich–Kondrachev theorem, we have the compact embeddings $\mathbf{V}^{2,s}(\Omega) \subset \mathbf{W}^{2-\varepsilon,s}(\Omega)$ and $\widehat{W}^{1,s}(\Omega) \subset W^{1-\varepsilon,s}(\Omega)$, where $0 < \varepsilon < (s-2)/s$. Hence, after extraction of another subsequence, $\nabla \mathbf{y}_n \to \nabla \mathbf{y}^*$ in $\mathbf{W}^{1-\varepsilon,s}(\Omega)$ and $p_n \to p^*$ in $W^{1-\varepsilon,s}(\Omega)$. The choice of ε leads to $(1-\varepsilon)s > 2$, and hence, we have the continuous embeddings $\mathbf{W}^{1-\varepsilon,s}(\Omega) \subset \mathbf{C}(\bar{\Omega})$ and $W^{1-\varepsilon,s}(\Omega) \subset C(\bar{\Omega})$. As a result, for every $\xi \in \bar{\Omega}$, we obtain $\nabla \mathbf{y}_n(\xi) \to \nabla \mathbf{y}^*(\xi)$ in $\mathbb{R}^{2\times 2}$ and $p_n(\xi) \to p^*(\xi)$ in $\mathbb{R}$. These also imply $\mathbf{T}(\mathbf{y}_n, p_n)(\xi) \to \mathbf{T}(\mathbf{y}^*, p^*)(\xi)$ in $\mathbb{R}^{2\times 2}$.

From above, we also get from the continuity of the trace operators that $\partial_{\mathbf{n}} \mathbf{y}_n \to \partial_{\mathbf{n}} \mathbf{y}^*$ and $\mathbf{T}(\mathbf{y}_n, p_n)\mathbf{n} \to \mathbf{T}(\mathbf{y}^*, p^*)\mathbf{n}$ in $\mathbf{W}^{1-\frac{1}{s}-\varepsilon,s}(\Gamma)$. Due to $(1-\frac{1}{s}-\varepsilon)s > 1$ and the one-dimensional Sobolev embedding theorem, we have $\mathbf{W}^{1-\frac{1}{s}-\varepsilon,s}(\Gamma) \subset C(\Gamma)$. Thus, $\partial_{\mathbf{n}} \mathbf{y}_n(\eta) \to \partial_{\mathbf{n}} \mathbf{y}^*(\eta)$ and $\mathbf{T}(\mathbf{y}_n, p_n)(\eta)\mathbf{n}(\eta) \to \mathbf{T}(\mathbf{y}^*, p^*)(\eta)\mathbf{n}(\eta)$ in $\mathbb{R}^2$ for every $\eta \in \Gamma$.

With the above considerations, we now see that $J(\mathbf{y}_n, p_n) \to J(\mathbf{y}^*, p^*)$, and again, this results into the existence of solutions to (3.18) with (3.59). For the case of (3.24) with (3.59), we immediately obtain existence of a solution by recognizing the continuous embedding $\mathbf{W}^{1,2}(\Omega) \subset \mathbf{L}^s(\Omega)$ for every $2 < s < \infty$.

In order to write the action of the derivative, we introduce $\mathbf{a} \otimes \boldsymbol{\delta}'_\xi, \mathbf{A} \otimes \boldsymbol{\delta}'_\xi \in \mathbf{M}_1(\bar{\Omega}) := [\mathbf{C}_0(\Omega) \cap \mathbf{C}^1(\bar{\Omega})]'$ for $\mathbf{a} \in \mathbb{R}^2$, $\mathbf{A} \in \mathbb{R}^{2\times 2}$, and $\xi \in \bar{\Omega}$ as follows:

$$\langle \mathbf{a} \otimes \boldsymbol{\delta}'_\xi, \boldsymbol{\varphi} \rangle_{\mathbf{M}_1(\bar{\Omega}), \mathbf{C}_0(\Omega)\cap \mathbf{C}^1(\bar{\Omega})} := \nabla \boldsymbol{\varphi}(\xi)\mathbf{a},$$
$$\langle \mathbf{A} \otimes \boldsymbol{\delta}'_\xi, \boldsymbol{\varphi} \rangle_{\mathbf{M}_1(\bar{\Omega}), \mathbf{C}_0(\Omega)\cap \mathbf{C}^1(\bar{\Omega})} := \nabla \boldsymbol{\varphi}(\xi) : \mathbf{A},$$

for $\boldsymbol{\varphi} \in \mathbf{C}_0(\Omega) \cap \mathbf{C}^1(\bar{\Omega})$.

Theorem 3.13. *Let $1 < r < 2$ and consider the localized problems (3.19) and (3.26) associated with (3.59). In the case of controls in $\mathbf{U}_{\rm ad}$ with $\mathbf{a}, \mathbf{b} \in \mathbf{L}^s(\Omega)$ for some $2 < s < \infty$, then for (3.59) we have*

$$(\mathbf{y}^*, p^*, \mathbf{u}^*, \mathbf{v}^*, \sigma^*) \in \mathbf{V}^{2,s}(\Omega) \times \widehat{W}^{1,s}(\Omega) \times \mathbf{L}^s(\Omega) \times \mathbf{L}^r(\Omega) \times \widehat{W}^{1,r'}(\Omega)'. \tag{3.60}$$

If we have controls in $\mathbf{W}^{1,2}(\Omega)$, then we obtain

$$(\mathbf{y}^*, p^*, \mathbf{u}^*, \mathbf{v}^*, \sigma^*) \in \mathbf{V}^{4,r}(\Omega) \times \widehat{W}^{3,r}(\Omega) \times \mathbf{W}^{2,r}(\Omega) \times \mathbf{L}^r(\Omega) \times \widehat{W}^{1,r'}(\Omega)'. \tag{3.61}$$

Proof. From the notations introduced above, the derivative of the reduced cost $J_{\rm r}$ induced by (3.59) is given by (we refer the reader back to Subsection 3.1 for the complete discussion):

$$J'_{\rm r}(\mathbf{u})\mathbf{r} = \sum_{\xi\in\mathcal{E}} \alpha_\xi(\nabla \mathbf{y}(\xi) - \mathbf{Y}_\xi) : \nabla \mathbf{w}(\xi) \tag{3.62}$$
$$= \sum_{\xi\in\mathcal{E}} [\beta_\xi(p(\xi) - p_\xi)\pi(\xi) + \lambda_\xi(\mathbf{T}(\mathbf{y}, p)(\xi) - \mathbf{S}_\xi) : \mathbf{T}(\mathbf{w}, \pi)(\xi)]$$

$$
\begin{aligned}
&+ \sum_{\eta\in\mathcal{G}} [\zeta_\eta(\partial_{\mathbf{n}}\mathbf{y}(\eta) - \mathbf{y}_\eta)\cdot \partial_{\mathbf{n}}\mathbf{w}(\eta) + \rho_\eta(\mathbf{T}(\mathbf{y},p)(\eta)\mathbf{n}(\eta) - \mathbf{s}_\eta)\cdot \mathbf{T}(\mathbf{w},\pi)(\eta)\mathbf{n}(\eta)\\
= &\langle \mathbf{f}(\mathbf{y},p), \mathbf{w}\rangle_{\mathbf{M}_1(\bar\Omega),\mathbf{C}_0(\Omega)\cap\mathbf{C}^1(\bar\Omega)} + \langle \Pi(g(\mathbf{y},p),\mathbf{h}(\mathbf{y},p)),\pi\rangle_{M(\bar\Omega),C(\bar\Omega)}\\
&+ \langle \mathbf{h}(\mathbf{y},p), \mathbf{T}(\mathbf{w},\pi)\mathbf{n}\rangle_{\mathbf{M}(\Gamma),\mathbf{C}(\Gamma)}
\end{aligned}
$$

where $\mathbf{M}(\Gamma) := \mathbf{C}(\Gamma)'$ and

$$
\begin{aligned}
\mathbf{f}(\mathbf{y},p) &:= \sum_{\xi\in\mathcal{E}} [(\alpha_\xi + \lambda_\xi\nu^2)\nabla\mathbf{y}(\xi) - \alpha_\xi\mathbf{Y}_\xi - \lambda_\xi\nu p(\xi)\mathbf{n}(\xi) + \lambda_\xi\nu\mathbf{S}_\xi]\otimes\boldsymbol{\delta}'_\xi\\
&\qquad + \sum_{\eta\in\mathcal{G}} \zeta_\eta(\partial_{\mathbf{n}}\mathbf{y}(\eta) - \mathbf{y}_\eta)\otimes[\mathbf{n}(\eta)\otimes\boldsymbol{\delta}'_\eta], && (3.63)\\
g(\mathbf{y},p) &:= \sum_{\xi\in\mathcal{E}} [(\beta_\xi + 2\lambda_\xi)p(\xi) - \beta_\xi p_\xi - \lambda_\xi \operatorname{Tr}\mathbf{S}_\xi]\delta_\xi, && (3.64)\\
\mathbf{h}(\mathbf{y},p) &:= \sum_{\eta\in\mathcal{G}} \rho_\eta(-\nu\partial_{\mathbf{n}}\mathbf{y}(\eta) + p(\eta)\mathbf{n}(\eta) - \mathbf{s}_\eta)\otimes\boldsymbol{\delta}_\eta. && (3.65)
\end{aligned}
$$

Let us show that the above duality pairings are well-defined. Note that $\mathbf{U}_{\text{ad}} \subset \mathbf{L}^s(\Omega)$ whenever $\mathbf{a},\mathbf{b}\in\mathbf{L}^s(\Omega)$, and so $\mathbf{u}^*\in\mathbf{L}^s(\Omega)$ and $(\mathbf{y}^*,p^*)\in\mathbf{V}^{2,s}(\Omega)\times\widehat{W}^{1,s}(\Omega)\subset[\mathbf{C}_0(\Omega)\cap\mathbf{C}^1(\bar\Omega)]\times C(\bar\Omega)$ since $2<s<\infty$. Similarly, $(\mathbf{w},\pi)\in[\mathbf{C}_0(\Omega)\cap\mathbf{C}^1(\bar\Omega)]\times C(\bar\Omega)$. Thus, $\mathbf{f}(\mathbf{y}^*,p^*)\in\mathbf{M}_1(\bar\Omega)\subset\mathbf{X}^{-2,r}(\Omega)$ and $g(\mathbf{y}^*,p^*)\in M(\bar\Omega)\subset\widehat{W}^{1,r'}(\Omega)'$ whenever $1<r<2$ due to $\mathbf{X}^{2,r'}(\Omega)\subset\mathbf{C}_0(\Omega)\cap\mathbf{C}^1(\bar\Omega)$ and $\widehat{W}^{1,r'}(\Omega)\subset C(\bar\Omega)$. Moreover, $\mathbf{h}(\mathbf{y}^*,p^*)\in\mathbf{M}(\Gamma)\subset\mathbf{W}^{-\frac{1}{r},r}(\Gamma)$ since $\mathbf{W}^{\frac{1}{r},r'}(\Gamma)\subset\mathbf{C}(\Gamma)$. Thus, we have

$$
\begin{aligned}
\mathfrak{D}_{\mathbf{y}^*}(\mathbf{v}^*,\sigma^*) &= (\mathbf{f}(\mathbf{y}^*,p^*), \Pi(g(\mathbf{y}^*,p^*),\mathbf{h}(\mathbf{y}^*,p^*)), \mathbf{h}(\mathbf{y}^*,p^*))\\
&\in \mathbf{X}^{-2,r}(\Omega)\times\mathbf{Z}^{-1,-\frac{1}{r},r}(\Omega,\Gamma).
\end{aligned}
$$

Therefore, by Corollary 2.13, $(\mathbf{v}^*,\sigma^*)\in\mathbf{L}^r(\Omega)\times\widehat{W}^{1,r'}(\Omega)'$ and (3.60) has been shown.

For the case of controls in $\mathbf{W}^{1,2}(\Omega)$, we have (3.60) for any $2<s<\infty$ because $\mathbf{W}^{1,2}(\Omega)\subset\mathbf{L}^s(\Omega)$. Then, the regularity of the triple $(\mathbf{y}^*,p^*,\mathbf{u}^*)$ in (3.61) follows directly from (3.28) and by elliptic regularity theory. □

As in the case of the boundary observations discussed in the previous subsection, it is also possible to formulate the adjoint equation with a homogeneous boundary condition. Indeed, we expand $\mathbf{T}(\mathbf{w},\pi)(\eta)\mathbf{n}(\eta) = -\nu\nabla\mathbf{w}(\eta)\cdot\mathbf{n}(\eta) + \pi(\eta)\mathbf{n}(\eta)$ in (3.62) to obtain

$$
J'_{\mathrm{r}}(\mathbf{u})\mathbf{r} = \langle\widetilde{\mathbf{f}}(\mathbf{y},p),\mathbf{w}\rangle_{\mathbf{M}_1(\bar\Omega),\mathbf{C}_0(\Omega)\cap\mathbf{C}^1(\bar\Omega)} + \langle \Lambda\widetilde{g}(\mathbf{y},p),\pi\rangle_{M(\bar\Omega),C(\bar\Omega)}
$$

where

$$
\begin{aligned}
\widetilde{\mathbf{f}}(\mathbf{y},p) &:= \mathbf{f}(\mathbf{y},p) - \sum_{\eta\in\mathcal{G}} \nu\rho_\eta[-\nu\partial_{\mathbf{n}}\mathbf{y}(\eta) + p(\eta)\mathbf{n}(\eta) - \mathbf{s}_\eta]\otimes[\mathbf{n}(\eta)\otimes\boldsymbol{\delta}'_\eta], && (3.66)\\
\widetilde{g}(\mathbf{y},p) &:= g(\mathbf{y},p) + \sum_{\eta\in\mathcal{G}} \nu\rho_\eta[-\nu\partial_{\mathbf{n}}\mathbf{y}(\eta)\cdot\mathbf{n}(\eta) + p(\eta) - \mathbf{s}_\eta\cdot\mathbf{n}(\eta)]\delta_\eta, && (3.67)
\end{aligned}
$$

with $\mathbf{f}(\mathbf{y},p)$ and $g(\mathbf{y},p)$ given by (3.63) and (3.64), respectively. In this case, the optimal adjoint state is the solution to

$$\mathfrak{D}_{\mathbf{y}^*}(\mathbf{v}^*,\sigma^*) = (\widetilde{\mathbf{f}}(\mathbf{y}^*,p^*), \Lambda\widetilde{g}(\mathbf{y}^*,p^*), \mathbf{0}) \in \mathbf{X}^{-2,s}(\Omega)\times \mathbf{Z}^{-1,-\frac{1}{s},s}(\Omega,\Gamma).$$

Let us highlight a simplified version of (3.59). Here, we choose the case where $\alpha_\xi = \beta_\xi = \lambda_\xi = 0$ and $\zeta_\eta = 0$ for every $\xi\in\mathcal{E}$ and $\eta\in\mathcal{G}$. This means that we have the following cost functional keeping track of the normal stress at points on the boundary:

$$J(\mathbf{y},p) = \frac{1}{2}\sum_{\eta\in\mathcal{G}}\rho_\eta|\mathbf{T}(\mathbf{y},p)(\eta)\mathbf{n}(\eta)-\mathbf{s}_\eta|^2. \tag{3.68}$$

From the proof of Theorem 3.13, we see that the adjoint problem reduces to

$$\left[\begin{array}{l} -\nu\Delta\mathbf{v}^* + (\nabla\mathbf{y}^*)^\top\mathbf{v}^* - (\mathbf{y}^*\cdot\nabla)\mathbf{v}^* + \nabla\sigma^* = \mathbf{0} \ \text{ in } \Omega,\\ \operatorname{div}\mathbf{v}^* = -\displaystyle\sum_{\eta\in\mathcal{G}}\rho_\eta\Sigma[(\mathbf{T}(\mathbf{y}^*,p^*)(\eta)\mathbf{n}(\eta)-\mathbf{s}_\eta)\otimes\boldsymbol{\delta}_\eta] \ \text{ in } \Omega,\\ \mathbf{v}^* = \displaystyle\sum_{\eta\in\mathcal{G}}\rho_\eta(\mathbf{T}(\mathbf{y}^*,p^*)(\eta)\mathbf{n}(\eta)-\mathbf{s}_\eta)\otimes\boldsymbol{\delta}_\eta \ \text{ on } \Gamma, \quad \langle\sigma^*,1\rangle_\Omega = 0. \end{array}\right.$$

On the other hand, using (3.66) and (3.67), one has the following problem with homogeneous Dirichlet data:

$$\left[\begin{array}{l} -\nu\Delta\mathbf{v}^* + (\nabla\mathbf{y}^*)^\top\mathbf{v}^* - (\mathbf{y}^*\cdot\nabla)\mathbf{v}^* + \nabla\sigma^*\\ \qquad = -\displaystyle\sum_{\eta\in\mathcal{G}}\nu\rho_\eta[\mathbf{T}(\mathbf{y}^*,p^*)(\eta)\mathbf{n}(\eta)-\mathbf{s}_\eta]\otimes[\mathbf{n}(\eta)\otimes\boldsymbol{\delta}_\eta'] \ \text{ in } \Omega,\\ \operatorname{div}\mathbf{v}^* = -\displaystyle\sum_{\eta\in\mathcal{G}}\nu\rho_\eta\Lambda\{[\mathbf{T}(\mathbf{y}^*,p^*)(\eta)\mathbf{n}(\eta)-\mathbf{s}_\eta]\cdot\mathbf{n}(\eta)\delta_\eta\} \ \text{ in } \Omega,\\ \mathbf{v}^* = \mathbf{0} \ \text{ on } \Gamma, \quad \langle\sigma^*,1\rangle_\Omega = 0. \end{array}\right.$$

We close the subsection with the following theorem wherein the control domain is disjoint from the observation points.

Theorem 3.14. *Let $\Omega_c\subset\Omega$ be open, and consider the optimal control problem*

$$\min_{(\mathbf{y},p,\mathbf{u})\in\mathbf{W}^{2,2}(\Omega)\times W^{1,2}(\Omega)\times\mathbf{L}^2(\Omega_c)} J(\mathbf{y},p) + \frac{\rho}{2}\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega_c)} \quad \text{subject to (1.2)}$$

with J given by (3.59), and $\operatorname{dist}(\Omega_c,\mathcal{E}\cup\mathcal{G}\cup\Gamma)>0$. Let $(\mathbf{y}^*,p,\mathbf{u}^*)$ be a solution such that $\mathbf{y}^*$ is regular. Then, $(\mathbf{v}^*|_{\Omega_c},\sigma^*|_{\Omega_c})\in\mathbf{W}^{2,2}(\Omega_c)\times W^{1,2}(\Omega_c)$ and $\mathbf{u}^*\in\mathbf{W}^{2,2}(\Omega_c)$.

Proof. Take a cut-off function $\varphi\in C^2(\bar\Omega)$ such that $\varphi=1$ on $\bar\Omega\setminus\Omega_0$ where $\bar\Omega_c\subset\Omega_0\subset\Omega$, $\varphi=0$ on Ω_0, and $\mathcal{E}\cup\mathcal{G}\subset\bar\Omega\setminus\Omega_0$. Note that the optimal state satisfies $(\mathbf{y}^*,p^*)\in\mathbf{X}^{2,2}(\Omega)\cap\widehat{W}^{1,2}(\Omega)\subset[\mathbf{W}^{1,r}_0(\Omega)\cap\mathbf{L}^\infty(\Omega)]\times\widehat{L}^r(\Omega)$ for every $1<r<\infty$. Setting $\mathbf{y}^*_\varphi=\varphi\mathbf{y}^*$ and $p^*_\varphi=\Lambda(\varphi p^*)$, we see from the state equation that

$$\left[\begin{array}{l} -\nu\Delta\mathbf{y}^*_\varphi + \nabla p^*_\varphi = \mathbf{r}^*_\varphi \ \text{ in } \Omega,\\ -\operatorname{div}\mathbf{y}^*_\varphi = q^*_\varphi \ \text{ in } \Omega, \quad \mathbf{y}^*_\varphi = \mathbf{0} \ \text{ on } \Gamma, \quad \displaystyle\int_\Omega p^*_\varphi\,\mathrm{d}x = 0, \end{array}\right.$$

where $\mathbf{r}^*_\varphi := -2\nu(\nabla\varphi\cdot\nabla)\mathbf{y}^* - (\nu\Delta\varphi)\mathbf{y}^* - \varphi(\mathbf{y}^*\cdot\nabla)\mathbf{y}^* + p^*\nabla\varphi\in\mathbf{L}^r(\Omega)$ and $q^*_\varphi := -\mathbf{y}^*\cdot\nabla\varphi\in\widehat{W}^{1,r}(\Omega)$. By (2.31), we have $(\mathbf{y}^*_\varphi,p^*_\varphi)\in\mathbf{X}^{2,r}(\Omega)\times\widehat{W}^{1,r}(\Omega)$. As a result,

$(\mathbf{y}^*, p^*) \in \mathbf{W}^{2,r}(\bar{\Omega} \setminus \Omega_0) \times W^{1,r}(\bar{\Omega} \setminus \Omega_0) \subset [\mathbf{C}_0(\bar{\Omega} \setminus \Omega_0) \cap \mathbf{C}^1(\bar{\Omega} \setminus \Omega_0)] \times C(\bar{\Omega} \setminus \Omega_0)$ when $r > 2$ since $\phi = 1$ on $\bar{\Omega} \setminus \Omega_0$. This implies that the cost functional (3.59) is well-defined, and existence of a solution to the above optimal control problem can be established as above.

As in the previous theorem, $(\mathbf{v}^*, \sigma^*) \in \mathbf{L}^s(\Omega) \times \widehat{W}^{1,s'}(\Omega)'$ for all $1 < s < 2$. Define $\varsigma^* := \sigma^* + \frac{1}{|\Omega|} \in W^{1,s'}(\Omega)'$, that is,

$$\langle \varsigma^*, q \rangle_{W^{1,s'}(\Omega)', W^{1,s}(\Omega)} := \langle \sigma^*, \Lambda q \rangle_{\widehat{W}^{1,s'}(\Omega)', \widehat{W}^{1,s}(\Omega)} + \frac{1}{|\Omega|} \int_\Omega q \, \mathrm{d}x \quad \forall q \in W^{1,s}(\Omega).$$

It is easy to see that $\nabla \varsigma^* = \nabla \sigma^*$ in the sense of distributions. Letting $\psi := 1 - \varphi$, $\mathbf{v}^*_\psi := \psi \mathbf{v}^*$, $\varsigma^*_\psi := \Lambda(\psi \varsigma^*)$, we get the adjoint equation

$$\left[\begin{array}{l} -\nu \Delta \mathbf{v}^*_\psi + (\nabla \mathbf{y}^*)^\top \mathbf{v}^*_\psi - (\mathbf{y}^* \cdot \nabla) \mathbf{v}^*_\psi + \nabla \varsigma^*_\psi = \mathbf{f}^*_\psi \ \text{ in } \Omega, \\ -\operatorname{div} \mathbf{v}^*_\psi = g^* \ \text{ in } \Omega, \quad \mathbf{v}^*_\psi = \mathbf{0} \ \text{ on } \Gamma. \end{array} \right. \tag{3.69}$$

where $\mathbf{f}^*_\psi := -2\nu(\nabla\psi \cdot \nabla)\mathbf{v}^* - (\nu\Delta\psi + \nabla\psi \cdot \mathbf{y}^*)\mathbf{v}^* + \varsigma^* \nabla\psi \in \mathbf{W}^{-1,s}(\Omega)$ and $g^*_\psi := -\mathbf{v}^* \cdot \nabla\psi \in \widehat{L}^s(\Omega)$. Hence, one has $(\mathbf{v}^*_\psi, \varsigma^*_\psi) \in \mathbf{W}^{1,s}_0(\Omega) \times \widehat{L}^s(\Omega)$, and so $(\mathbf{v}^*, \sigma^*) \in \mathbf{W}^{2,s}(\Omega_0) \times \widehat{W}^{1,s}(\Omega_0) \subset \mathbf{W}^{1,2}(\Omega_0) \times \widehat{L}^2(\Omega_0)$ since $\psi = 1$ in Ω_0.

Now, define $\phi \in C^2(\Omega)$ such that $\phi = 1$ on $\bar{\Omega}_c$ and $\phi = 0$ in $\bar{\Omega} \setminus \bar{\Omega}_0$. Setting $\mathbf{v}^*_\phi := \phi \mathbf{v}^*$, $\varsigma^*_\phi := \Lambda(\phi\varsigma^*)$, we deduce (3.69) with ψ replaced by ϕ. Then, $\mathbf{f}^*_\phi \in \mathbf{L}^2(\Omega)$ and $g^*_\phi \in \widehat{W}^{1,2}(\Omega)$. This yields $(\mathbf{v}^*_\phi, \varsigma^*_\phi) \in \mathbf{W}^{2,2}(\Omega) \times \widehat{W}^{1,2}(\Omega)$. Therefore, $(\mathbf{v}^*|_{\Omega_c}, \sigma^*|_{\Omega_c}) \in \mathbf{W}^{2,2}(\Omega_c) \times W^{1,2}(\Omega_c)$ since $\phi = 1$ on Ω_c, and as a consequence, $\mathbf{u}^* = -\rho^{-1} \mathbf{v}^*|_{\Omega_c} \in \mathbf{W}^{2,2}(\Omega_c)$. □

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