Department of Mathematics
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College of Science

University of the Philippines Baguio

Well-posedness of Navier–Stokes–Voigt System with Forces of Low Time-Space Regularity

WELL-POSEDNESS OF NAVIER–STOKES–VOIGT SYSTEM WITH FORCES OF LOW TIME-SPACE REGULARITY

JUNIUS WILHELM G. BUENO AND GILBERT PERALTA

Abstract.
The Navier–Stokes–Voigt equation models the motion of an incompressible non-Newtonian fluid under an abrupt external force. Existence and uniqueness of solutions of the system with forces in fractional Bessel potential spaces are established using complex interpolation theory. The linear part of the system is shown to have solution in arbitrary dimensions via semigroup theory. The contraction mapping principle is used to prove the existence and uniqueness of very weak solutions in arbitrary dimensions with small data. For dimension two, the Faedo–Galerkin method is utilized to establish the well-posedness of the nonlinear part of the system. The results for the linear and non-linear parts are combined to show well-posedness for dimension two with arbitrary initial data and source functions. Measure and measure-valued sources are covered in this framework.

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Department of Mathematics and Computer Science, University of the Philippines Baguio, Governor Pack Road, Baguio, 2600 Philippines. Email: grperalta@up.edu.ph.

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1. Introduction

Consider a time interval $I := (0, T)$ with $0 < T < \infty$ and a bounded, open, and connected domain $\Omega \subset \mathbb{R}^d$ for $d \geq 2$ with a sufficiently smooth boundary Γ. Denote the time-space domain by $Q := I \times \Omega$ and its lateral boundary by $\Sigma := I \times \Gamma$. We examine the well-posedness (in Hadamard sense) of the Navier–Stokes–Voigt equations (NSVE) given by

$$\begin{cases} \partial_t y - \kappa\partial_t\Delta y - \nu\Delta y + \operatorname{div}(y \otimes y) + \nabla\mathfrak{p} = f & \text{in } Q, \\ \operatorname{div} y = 0 \quad \text{in } Q, \qquad y = 0 \quad \text{on } \Sigma, \qquad y(0,\cdot) = y_0 \quad \text{in } \Omega, \end{cases} \tag{NSV}$$

with low time-space regular force f and homogeneous Dirichlet boundary (no-slip) condition. The unknown $y : Q \to \mathbb{R}^d$ denotes the fluid velocity vector, and $\mathfrak{p} : Q \to \mathbb{R}$ is the unknown pressure. Here, $\nu > 0$ is the kinematic viscosity coefficient, $\kappa > 0$ is the length-scale parameter for the fluid elasticity, f is a given force field, and $y_0 : \Omega \to \mathbb{R}^d$ is an initial velocity vector. Although $\operatorname{div}(y \otimes y)$ and $(y \cdot \nabla)y$ are equal for divergence-free and smooth y, the former is preferred since it allows us to consider very weak solutions.

The Navier–Stokes–Voigt equations, otherwise known as Kelvin–Voigt fluid equations (see [9, 33]), will reduce to the classical incompressible Navier–Stokes equations (NSE) when $\kappa = 0$. Because the global existence of strong solutions and the uniqueness of weak solutions of the three-dimensional NSE remains an open problem, various generalizations of the NSE have been proposed (see, for instance, [19, 20, 23] and the references therein). The NSVE was introduced by Ladyzhenskaya [26] and Oskolkov [28] as a regularization of NSE. It was Oskolkov who showed that the initial boundary value problem for the two-dimensional (2D) and three-dimensional (3D) NSVE with forces in $L_2(Q)$ has a unique weak and strong solution in the Hilbert space setting (L_2-theory); see Section 8 in [29] and [28], respectively. The additional linear term $-\kappa\partial_t\Delta y$ changes the parabolic character of the NSE and does not impose additional boundary conditions for the global well-posedness of the NSVE.

The physical properties of the NSVE, on the one hand, can be observed from its total stress tensor

$$\sigma = 2\nu\left(1 + \frac{\kappa}{\nu}\partial_t\right)\mathbb{D}(y) + (\lambda \text{div } y - \mathfrak{p})I,$$

that Oskolkov studied (see Section 8 in [29]). The coefficient $\kappa/\nu > 0$ denotes the time required to set in motion incompressible non-Newtonian fluids, called viscoelastic fluids, under abrupt forces, and

$$\mathbb{D}(y) := \frac{1}{2}[\nabla y + (\nabla y)^\top]$$

denotes the strain tensor. On the other hand, Pavlovsky [30] used the NSVE to describe the laminar flow of aqueous solutions of polymers. Aside from physical properties, the NSVE has a numerical advantage over the NSE. For instance, Cao, Lunasin, and Titi described the NSVE for small values of κ as a regularization for the numerical computations of the 3D NSE with a periodic boundary condition or a no-slip Dirichlet boundary condition (see equation 7.17 in [10]). It was observed that the NSVE behaves like a damped hyperbolic system that prevents unrealistic numerical results [6, 24]. Under appropriate decay assumptions on the forcing function and the assumption that the dual problem of the linearized steady-state system admits a positive minimal eigenvalue, both power and exponential convergence of solutions of NSVE towards steady states were established in [25].

In recent years, more properties of the NSVE have been studied. Damázio, Manholi, and Silvestre [13] extended the results of Oskolkov and proved the existence and uniqueness of weak and strong solutions of the NSVE on bounded domains in the L_q-setting for $1 < q < \infty$. Various modifications of the NSVE system were also investigated. Antontsev, Oliveira, and Khompysh [7] considered global-in-time unique weak solutions of NSVE with anisotropic viscosity, relaxation, and damping. The NSVE with position-dependent slip boundary conditions was studied by Baranovskii [9]. These boundary conditions are suitable for boundaries with a varying degree of roughness. Quadratic optimal control and time optimal control NSVE problems were been considered by Anh and Nguyet [3, 4] with controls in the Hilbert spaces $H_2^1(I; L_2(\Omega))$ and $L_2(I; L_2(\Omega))$, respectively. They also studied the optimal control problem on the NSVE with time-periodic velocity and controls in $L_{2,\text{loc}}(\mathbb{R}; L_2(\Omega))$ [5].

We describe the main contributions and techniques used in this paper. The existence and uniqueness of weak and very weak solutions of the NSVE in fractional Bessel potential spaces are established. The solutions of the linear part of (NSV) are first treated in L_p-L_q framework for $1 < p < \infty$ and $1 < q < \infty$. In order to consider solutions in the Bessel potential space, we need to extend the Stokes operator on different scales. This is done by considering fractional powers of the Stokes operator and applying its dual to itself. Using this extended Stokes operator, we form a linear Cauchy problem whose solutions, represented through the variation of parameters formula, are equivalent to the solutions of the linear part of (NSV) (see [13, 17]).

Rough non-homogeneous initial and boundary conditions are included implicitly in the non-homogeneous force field. To achieve more time regularity, we apply complex interpolation between the weak and very weak solution spaces. Following Farwig and Riechwald [17], we use the Sobolev embedding to determine conditions so that the nonlinear convection term, $\operatorname{div}(y \otimes y)$, is in the space of force fields. In this way, we can include the nonlinear term in the force field. The contraction mapping principle is then utilized together with the results in the linear part of (NSV) to show the existence and uniqueness of the solutions of (NSV) for spaces in arbitrary dimensions. Solutions on any given finite-time interval are also shown to exist for sufficiently small data.

In the 2D case, we decompose (NSV) into its linear and nonlinear parts. The nonlinear part is shown to have global-in-time solution using the Faedo–Galerkin approach. This is precisely the technique provided by Casas and Kunisch in [12]. Their method applies to the case of (NSV), the main difference is that the Voigt damping appears in both the linear and nonlinear parts. With this, maximal parabolic regularity for the linear part is no longer applicable, instead, the theory of uniformly bounded semigroups is utilized. The nonlinear part is treated using Hilbert space methods similar to those used for the Navier–Stokes–Voigt equation. We then consider a sum of Banach spaces and combine the results in the linear and nonlinear parts of (NSV) to establish the well-posedness of (NSV) in 2D with arbitrary data. Finally, we identify conditions for the well-posedness of solutions of the NSVE with measure and measure-valued forces. Such forces are desirable as these lead to optimal controls with sparse supports [11].

The asymptotic behavior of solutions to (NSV) as $t \to \infty$ will not be addressed in this paper. Likewise, the precise regularity and the corresponding a priori estimate of the associated pressure $\mathfrak{p}$ will not be discussed. In the context of solutions lying in the anisotropic spaces considered here, the analysis of these interesting problems can be quite involved. Partial results regarding the regularity of the pressure may be obtained using interpolation theory and de Rham's theorem, but these will not be pursued here.

This paper is organized as follows. The relevant function spaces and auxiliary results that will be extensively used throughout this paper are introduced in Section 2. We prove the existence and uniqueness of weak and very weak solutions to the linear part of (NSV) in Section 3. Our analysis includes cases with non-homogeneous initial and boundary conditions. Parameter values that guarantee the existence and uniqueness of very weak solutions of (NSV) with small data are determined in Section 4. Unique global-in-time solutions of (NSV) in 2D are discussed in Section 5. Here, we also identify conditions for the well-posedness of solutions of (NSV) with forces involving measures. Finally, the sensitivity of solutions of (NSV) in 2D is examined in Section 6.

2. Preliminaries

The classical function spaces based on Lebesgue spaces and their interpolations are discussed in this section. Extensions of the Stokes operator to different fractional scales are also included. We use these extensions to solve a Cauchy problem, which will be the basis for constructing the solutions of the linearized problem.

2.1. Function spaces. Throughout this paper, we let $p, q \in (1,\infty)$, $\theta \in [-1,1]$, and $s \in [-2,2]$, unless stated otherwise. Denote by $L_q(\Omega)$, $H^s_q(\Omega)$, and $B^s_{q,p}(\Omega)$ the scalar-valued (and vector-valued) Lebesgue, Bessel potential, and Besov spaces, respectively. We also write

$$(u,v)_Q := \int_Q u\cdot v \,\mathrm{d}x\,\mathrm{d}t.$$

Similar notations will be used on integrals and function spaces over Σ, I, Ω, and Γ.

The dual of a Banach space V is denoted by V^*, and $\langle\cdot,\cdot\rangle_{V^*\times V}$ is the associated duality pairing. The conjugate exponent of q is written as $q' := q/(q-1)$. Let γ_0 be the trace operator on Γ, and $\gamma_1 := \partial_{\mathrm{n}}$ be the directional derivative with respect to the outer unit normal vector n on Γ. Denote by $\mathcal{C}(\bar{I};V)$ the space of continuous functions defined on $\bar{I} = [0,T]$ whose values are in V. For each $\tau \in \bar{I}$, we define the time-evaluation operator $\delta_\tau : \mathcal{C}(\bar{I};V) \to V$ by

$$\delta_\tau y := y(\tau), \qquad y \in \mathcal{C}(\bar{I};V).$$

As in [1, 2], set

$$H^s_{q,0}(\Omega) := \begin{cases} \{u \in H^s_q(\Omega) : \gamma_0 u = 0\} & \text{if } 1/q < s \le 2,\\ \{u \in H^{1/q}_q(\Omega) :\ \mathrm{supp}(u) \subset \overline{\Omega}\} & \text{if } s = 1/q,\\ H^s_q(\Omega) & \text{if } 0 \le s < 1/q,\\ H^{-s}_{q',0}(\Omega)^* & \text{if } -2 \le s < 0, \end{cases}$$

with the support taken in the sense of distributions in the case of $s = 1/q$. Define the solenoidal Lebesgue spaces and Bessel potential spaces, respectively, as follows:

$$L_{q,\sigma}(\Omega) := \{u \in L_q(\Omega) : \mathrm{div}\, u = 0 \text{ in } \Omega,\ u\cdot \mathrm{n} = 0 \text{ on } \Gamma\}$$

and

$$H^s_{q,0,\sigma}(\Omega) := \begin{cases} H^s_{q,0}(\Omega) \cap L_{q,\sigma}(\Omega) & \text{if } 0 \le s \le 2,\\ H^{-s}_{q',0,\sigma}(\Omega)^* & \text{if } -2 \le s < 0. \end{cases}$$

Here, $u\cdot\mathrm{n}$ is taken in the sense of the Sobolev–Slobodeckii space $W^{-1/q}_q(\Gamma)$, see [21]. Note that $H^s_{q,0,\sigma}(\Omega) \subset H^r_{q,0,\sigma}(\Omega)$ for $-2 \le r \le s \le 2$. For the functions defined on the boundary, we will consider the Besov space $B^s_{q,q}(\Gamma)$, which is locally $B^s_{q,q}(\mathbb{R}^{d-1})$ after flattening Γ together with a partition of unity; see [32].

2.2. Spaces ${}_\tau\mathfrak{H}^{\theta,s}_{p,q}(Q)$. The Bessel potential space $H^s_q(\Omega)$ coincides with the Sobolev space $W^s_q(\Omega)$ for $1 < q < \infty$ and $s \in \mathbb{N}\cup\{0\}$, see Section 1.3.2 in [32]. Up to norm equivalence, we can characterize Bessel potential spaces as an interpolation space:

$$[H^{s_0}_q(\Omega), H^{s_1}_q(\Omega)]_\theta = H^{s_0(1-\theta)+s_1\theta}_q(\Omega), \tag{2.1}$$

where $[\cdot,\cdot]_\theta$ denotes the complex interpolation functor; see Section 1.6.4 in [32]. In particular, for an appropriate Banach space X and $0 < \theta < 1$, we get from Theorem 1.56 in [14] the vector-valued Bessel potential space

$$[L_p(I;X), H^1_p(I;X)]_\theta = H^\theta_p(I;X).$$

The (NSV) will be studied in the Banach space

$$\mathfrak{H}^{\theta,s}_{p,q}(Q) := \begin{cases} H^\theta_p(I; H^s_{q,0,\sigma}(\Omega)) & \text{if } \theta \in [0,1] \text{ and } s \in [-2,2], \\ \mathfrak{H}^{-\theta,-s}_{p',q'}(Q)^* & \text{if } \theta \in [-1,0) \text{ and } s \in [-2,2]. \end{cases}$$

Note that $\mathfrak{H}^{0,s}_{p,q}(Q) = L_p(I; H^s_{q,0,\sigma}(\Omega))$ and $\mathfrak{H}^{0,s}_{p,q}(Q)^* = L_{p'}(I; H^{-s}_{q',0,\sigma}(\Omega))$. By the Sobolev embedding theorem for Bessel potentials, see, for instance, Theorem 3.4 in [27], we have $\mathfrak{H}^{\theta,s}_{p,q}(Q) \subset \mathcal{C}(\bar{I}; H^s_{q,0,\sigma}(\Omega))$ for $\theta \in (1/p, 1]$ and $s \in [-2,2]$. We denote by ${}_\tau\mathfrak{H}^{\theta,s}_{p,q}(Q)$ the closure of $\{u \in \mathfrak{H}^{1,s}_{p,q}(Q) : \delta_\tau u = 0\}$ in $\mathfrak{H}^{\theta,s}_{p,q}(Q)$ for $s \in [-2,2]$, $\theta \in [0,1]$, and for $\tau = 0$ or $\tau = T$. Note that according to Remark 1.67(iii) in [14], we have

$${}_\tau\mathfrak{H}^{\theta,s}_{p,q}(Q) = \begin{cases} \{u \in \mathfrak{H}^{\theta,s}_{p,q}(Q) : \delta_\tau u = 0\} & \text{if } \theta \in (1/p, 1] \text{ and } s \in [-2,2], \\ \mathfrak{H}^{\theta,s}_{p,q}(Q) & \text{if } \theta \in [0, 1/p) \text{ and } s \in [-2,2]. \end{cases}$$

For the corresponding dual space, we set

$${}_\tau\mathfrak{H}^{\theta,s}_{p,q}(Q) := {}_\tau\mathfrak{H}^{-\theta,-s}_{p',q'}(Q)^*, \qquad \text{for } \theta \in [-1,0), s \in [-2,2].$$

By applying (2.1) and Proposition 1.69 in [14] for each $\theta \in (0,1)$, we can characterize ${}_\tau\mathfrak{H}^{\theta,s}_{p,q}(Q)$ as follows:

$${}_\tau\mathfrak{H}^{\theta,s}_{p,q}(Q) = [{}_\tau\mathfrak{H}^{0,s}_{p,q}(Q), {}_\tau\mathfrak{H}^{1,s}_{p,q}(Q)]_\theta \text{ and } {}_\tau\mathfrak{H}^{\theta-1,s}_{p,q}(Q) = [{}_\tau\mathfrak{H}^{-1,s}_{p,q}(Q), {}_\tau\mathfrak{H}^{0,s}_{p,q}(Q)]_\theta. \quad (2.2)$$

We denote by $\hookrightarrow$ and $\overset{d}{\hookrightarrow}$ the continuous and dense continuous embeddings of Banach spaces, respectively. If Ω has a sufficiently smooth boundary, then we have ${}_\tau\mathfrak{H}^{1,s}_{p,q}(Q) \hookrightarrow {}_\tau\mathfrak{H}^{0,s}_{p,q}(Q)$. In fact, by applying Theorem 1.39 in [14], we have

$${}_\tau\mathfrak{H}^{1,s}_{p,q}(Q) \overset{d}{\hookrightarrow} {}_\tau\mathfrak{H}^{\theta,s}_{p,q}(Q) \overset{d}{\hookrightarrow} {}_\tau\mathfrak{H}^{0,s}_{p,q}(Q). \quad (2.3)$$

2.3. Stokes operator and its fractional powers. The Helmholtz–Weyl decomposition is given by $L_q(\Omega) = G_q(\Omega) \oplus L_{q,\sigma}(\Omega)$, where $1 < q < \infty$ and

$$G_q(\Omega) := \{v \in L_q(\Omega) : v = \nabla u, u \in H^1_{q,\text{loc}}(\Omega)\}$$

(see Theorem III.1.2 in [21]). This implies the existence of a bounded linear operator

$$P_q : L_q(\Omega) \to L_{q,\sigma}(\Omega),$$

called the Helmholtz projector, such that $P_q^2 = P_q$ and its dual operator satisfies $P_q^* = P_{q'}$. Using the Helmholtz projector, the Stokes operator

$$A_q : \mathcal{D}(A_q) \to L_{q,\sigma}(\Omega)$$

is defined as $A_q := -P_q\Delta$. Recall that A_q is a closed bijective operator, and its domain satisfies

$$\mathcal{D}(A_q) := H^2_q(\Omega) \cap H^1_{q,0,\sigma}(\Omega) \overset{d}{\hookrightarrow} L_{q,\sigma}(\Omega).$$

Moreover, its inverse operator $A_q^{-1} : L_{q,\sigma}(\Omega) \to \mathcal{D}(A_q)$ is bounded when $\mathcal{D}(A_q)$ is equipped with the graph norm.

For $\kappa > 0$, the inverse operator $(\mathcal{I} + \kappa A_q)^{-1}$ exists as a bounded linear operator in $L_{q,\sigma}(\Omega)$, where $\mathcal{I}$ denotes the identity operator in $L_{q,\sigma}(\Omega)$ (see Corollary 1.6 in [18]). In fact, the equation

$$(\mathcal{I} + \kappa A_q)y = f,$$

for a given $f \in L_{q,\sigma}(\Omega)$, has a unique solution

$$y = (\mathcal{I} + \kappa A_q)^{-1} f \in \mathcal{D}(A_q)$$

satisfying the estimate

$$\|y\|_{\mathcal{D}(A_q)} \le C\|f\|_{L_{q,\sigma}(\Omega)},$$

for some positive constant $C = C(\kappa, q, \Omega)$ independent of y and f.

The fractional powers of the Stokes operator,

$$A_q^{s/2} : \mathcal{D}(A_q^{s/2}) \to L_{q,\sigma}(\Omega),$$

are well-defined for $0 < s < 2$ through Dunford integrals. We define A_q^0 as the identity operator in $L_{q,\sigma}(\Omega)$, and $A_q^1 := A_q$ so that $A_q^{s/2}$ is well-defined for each $s \in [0, 2]$. It is known that

$$\mathcal{D}(A_q^{s/2}) = [L_{q,\sigma}(\Omega), \mathcal{D}(A_q)]_{s/2} = H^s_{q,0,\sigma}(\Omega)$$

(see, for instance, Theorem 2 in [22]). We note that $\mathcal{D}(A_q^{s/2}) \overset{d}{\hookrightarrow} L_{q,\sigma}(\Omega)$ according to Equations 1.1 and 1.2 in [2]. Moreover, for each $s \in [0, 2]$, $A_q^{s/2}$ is bijective,

$$(A_q^{s/2})^{-1} = A_q^{-s/2} : L_{q,\sigma}(\Omega) \to \mathcal{D}(A_q^{s/2})$$

is bounded, and by duality, $(A_q^{s/2})^* = A_{q'}^{s/2}$ on $\mathcal{D}(A_{q'})$, see [16].

2.4. Operators $A_{q,s/2}$ and $\widehat{A}_{q,s/2}$. Let $\mathcal{L}(X, Y)$ denote the Banach space of all bounded linear operators from a Banach space X into a Banach space Y, equipped with the operator norm. The set of bijective bounded linear operators in $\mathcal{L}(X, Y)$ will be denoted by $\mathcal{L}_{\mathrm{is}}(X, Y)$. We use the abbreviation $\mathcal{L}(X) = \mathcal{L}(X, X)$.

The process described by Casas and Kunisch in Section 3(ii) of [12] is utilized to characterize the Stokes operator in different scales. To this end, for $s \in [0, 2]$, one can show by density arguments that the dual operator $(A_{q'}^{s/2})^*$ is an extension of $A_q^{s/2}$. The Stokes operator A_q can be extended to different scales of Bessel potential spaces by defining

$$A_{q,s/2} := (A_{q'}^{1-s/2})^* A_q^{s/2} \in \mathcal{L}_{\mathrm{is}}(H^s_{q,0,\sigma}(\Omega), H^{s-2}_{q,0,\sigma}(\Omega)).$$

This operator enjoys maximal parabolic $L_p(I; H^{s-2}_{s,0,\sigma}(\Omega))$-regularity. Thus, $-A_{q,s/2}$ generates an analytic semigroup $e^{-A_{q,s/2}t}$ for $t \ge 0$ on $H^{s-2}_{q,0,\sigma}(\Omega)$, and there exist constants $M = M(q, s, \Omega) \ge 1$ and $\beta = \beta(q, s, \Omega) > 0$ such that

$$\|e^{-A_{q,s/2}t}\|_{\mathcal{L}(H^{s-2}_{q,0,\sigma}(\Omega))} \le Me^{-\beta t}$$

(see Remark 11.3(ii) in [8]). Furthermore, note that $A_{q,s/2} = (A_{q',1-s/2})^*$ on $H^s_{q,0,\sigma}(\Omega)$. Indeed, for any $u \in H^s_{q,0,\sigma}(\Omega)$ and $v \in H^{2-s}_{q',0,\sigma}(\Omega)$, we have

$$\begin{aligned}\langle A_{q,s/2}u, v\rangle_{H^{s-2}_{q,0,\sigma}(\Omega)\times H^{2-s}_{q',0,\sigma}(\Omega)} &= \langle u, (A_q^{s/2})^* A_{q'}^{1-s/2} v\rangle_{H^s_{q,0,\sigma}(\Omega)\times H^{-s}_{q',0,\sigma}(\Omega)}\\ &= \langle u, A_{q',1-s/2}v\rangle_{H^s_{q,0,\sigma}(\Omega)\times H^{-s}_{q',0,\sigma}(\Omega)}.\end{aligned}$$

Let us now consider the operator

$$\widehat{A}_{q,s/2} := (\mathcal{I}_s + \kappa A_{q,s/2})^{-1} A_{q,s/2} \in \mathcal{L}(H^s_{q,0,\sigma}(\Omega)).$$

Here,

$$\mathcal{I}_s : H^s_{q,0,\sigma}(\Omega) \to H^s_{q,0,\sigma}(\Omega)$$

denotes the identity operator for $s \in [-2, 2]$. See the commutative diagram in Figure 1.

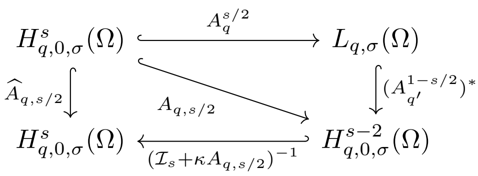

FIGURE 1. Commutative diagram for $A_{q,s/2}$ and $\widehat{A}_{q,s/2}$.

It is clear that in the domain of $A_{q,s/2}$, we have

$$\widehat{A}_{q,s/2} = \frac{1}{\kappa}\mathcal{I}_s - \frac{1}{\kappa^2}\left(\frac{1}{\kappa}\mathcal{I}_s + A_{q,s/2}\right)^{-1}.$$

It can be shown by utilizing the Hille–Yosida theorem that for each $q \in (1, \infty)$, $t \geq 0$, and a positive integer n, we have

$$\left\| \left(\frac{\mathcal{I}_s}{\kappa} + A_{q,s/2} \right)^{-n} \right\|_{\mathcal{L}(H^s_{q,0,\sigma}(\Omega))} \leq M \left(\frac{1}{\kappa} + \beta \right)^{-n} \tag{2.4}$$

(see the proof of Theorem 2.2 in [13] for $s = 2$). It follows that $\widehat{A}_{q,s/2}$ is a bounded linear operator with norm

$$\|\widehat{A}_{q,s/2}\|_{\mathcal{L}(H^s_{q,0,\sigma}(\Omega))} \leq \frac{1}{\kappa} + \frac{M}{\kappa(1+\beta\kappa)}.$$

Furthermore, the operator $-\nu\widehat{A}_{q,s/2}$ generates a uniformly continuous semigroup on $H^s_{q,0,\sigma}(\Omega)$ such that

$$e^{-\nu\widehat{A}_{q,s/2}t} = e^{\frac{-\nu}{\kappa}t} \sum_{n=0}^{\infty} \frac{(\nu t)^n}{\kappa^{2n} n!} \left(\frac{\mathcal{I}_s}{\kappa} + A_{q,s/2} \right)^{-n}.$$

Using the above definition of $e^{-\nu\widehat{A}_{q,s/2}t}$ and estimate (2.4), we have

$$\|e^{-\nu\widehat{A}_{q,s/2}t}\|_{\mathcal{L}(H^s_{q,0,\sigma}(\Omega))} \leq Me^{-\widehat{\beta}t} \qquad \forall\, t \geq 0, \tag{2.5}$$

where $\widehat{\beta} = \frac{\nu}{\kappa}\left(1 - \frac{1}{1+\beta\kappa}\right) > 0$. Therefore, the abstract Cauchy problem

$$\begin{cases} \partial_t y + \nu\widehat{A}_{q,s/2} y = F & \text{in } I, \\ \delta_0 y = y_0, \end{cases} \tag{2.6}$$

has a unique solution as stated in the following lemma. This lemma is an extension of Theorem 2.3 in [13] to Bessel potential spaces.

Lemma 2.1. *Let $p, q \in (1, \infty)$ and $s \in [0, 2]$. Given $F \in L_p(I; H^s_{q,0,\sigma}(\Omega))$ and $y_0 \in H^s_{q,0,\sigma}(\Omega)$, the Cauchy problem (2.6) has a unique solution $y \in \mathfrak{H}^{1,s}_{p,q}(Q)$ such that for some constant $C = C(\kappa, \nu, p, q, s, \Omega, T) > 0$ independent of y, y_0, and F, we have*

$$\|y\|_{\mathfrak{H}^{1,s}_{p,q}(Q)} \leq C(\|y_0\|_{H^s_{q,0,\sigma}(\Omega)} + \|F\|_{L_p(I; H^s_{q,0,\sigma}(\Omega))}). \tag{2.7}$$

Proof. One can show by direct computation that for almost all (a.a.) $t \in [0,T]$,

$$y(t) := e^{-\nu \widehat{A}_{q,s/2} t} y_0 + \int_0^t e^{-\nu \widehat{A}_{q,s/2}(t-\tau)} F(\tau)\, \mathrm{d}\tau \in H^s_{q,0,\sigma}(\Omega) \tag{2.8}$$

is the unique solution of (2.6). The estimate (2.7) follows directly from (2.8) and (2.5). In particular,

$$\begin{aligned} \|t \mapsto e^{-\nu \widehat{A}_{q,s/2} t} y_0\|_{\mathfrak{H}^{1,s}_{p,q}(Q)} &\le C_1 \|y_0\|_{H^s_{q,0,\sigma}(\Omega)}, \\ \left\| t \mapsto \int_0^t e^{-\nu \widehat{A}_{q,s/2}(t-\tau)} F(\tau)\, \mathrm{d}\tau \right\|_{\mathfrak{H}^{1,s}_{p,q}(Q)} &\le C_2 \|F\|_{\mathfrak{H}^{0,s}_{p,q}(Q)}, \end{aligned}$$

where

$$C_1 := \widehat{C}_1 \left(1 - e^{-pT\widehat{\beta}}\right)^{1/p} \text{ and } C_2 := \left[\widehat{C}_2 \left(\int_0^T (1 - e^{-p'\widehat{\beta} t})^{p-1}\, \mathrm{d}t \right)^{1/p} + 1 \right]$$

for some constants $\widehat{C}_1, \widehat{C}_2 > 0$ independent of y, y_0, and F. Hence, $y \in \mathfrak{H}^{1,s}_{p,q}(Q)$ for $p, q \in (1,\infty)$ and $s \in [0,2]$. Using (2.8) and the linearity of the system, one can prove the uniqueness of the solution in a standard manner. □

Although the evolution equation (2.6) corresponding to the operator $\nu \widehat{A}_{q,s/2}$ is not parabolic, the corresponding homogeneous system where $F = 0$ still exhibits exponential stability. Despite the operator semigroup not being analytic, it is nonetheless exponentially stable due to the viscoelastic damping. Note that such a result cannot be obtained from either classical parabolic theory nor maximal parabolic regularity.

Remark 2.2. We call y in (2.8) a *mild solution* of (2.6). Such a solution will be used to construct a weak solution of (NSV).

If we apply the transformation $t \to T - t$ and replace $(q, s/2)$ by $(q', 1 - s/2)$ in (2.6), the following result can be obtained. As in Lemma 2.3 of [17], such a formulation is crucial for the treatment of very weak solutions of (NSV).

Lemma 2.3. *Let $p, q \in (1,\infty)$ and $s \in [0,2]$. For every $G \in L_{p'}(I; H^{2-s}_{q',0,\sigma}(\Omega))$, there exists a unique $\varphi \in {}_T\mathfrak{H}^{1,2-s}_{p',q'}(Q)$ such that*

$$\begin{cases} -\partial_t \varphi + \nu \widehat{A}_{q',1-s/2} \varphi = G & \text{in } I, \\ \delta_T \varphi = 0, \end{cases} \tag{2.9}$$

which can be represented by

$$\varphi(T-t) = \int_0^t e^{-\nu \widehat{A}_{q',1-s/2}(t-\tau)} G(T-\tau)\, \mathrm{d}\tau \tag{2.10}$$

for a.a. $t \in [0,T]$. Moreover, there exists a constant $C_2 = C_2(\kappa, \nu, p, q, s, \Omega, T) > 0$ independent of φ and G such that

$$\|\varphi\|_{{}_T\mathfrak{H}^{1,2-s}_{p',q'}(Q)} \le C_2 \|G\|_{L_{p'}(I; H^{2-s}_{q',0,\sigma}(\Omega))}. \tag{2.11}$$

3. Weak and very weak solutions of linearized NSV

In this section, we present our analysis for the linear part of (NSV) with homogeneous and non-homogeneous boundary conditions. Notions of weak and very weak solutions are introduced and shown to be unique. Using complex interpolation of the solution operators corresponding to the weak and very weak formulations, our results will be extended to different scales of Bessel potential spaces.

We first consider (NSV) without the nonlinear term $\operatorname{div}(y \otimes y)$, namely,

$$\begin{cases} \partial_t y - \kappa \partial_t \Delta y - \nu \Delta y + \nabla \mathfrak{p} = f \quad \text{in } Q, \\ \operatorname{div} y = 0 \quad \text{in } Q, \qquad \gamma_0 y = 0 \quad \text{on } \Sigma, \qquad \delta_0 y = y_0 \quad \text{in } \Omega. \end{cases} \tag{3.1}$$

We derive the weak formulation of (3.1) as follows. Suppose we have a sufficiently regular solution y with an associated pressure $\mathfrak{p}$. Multiplying (3.1) by a smooth divergence-free test function φ and integrating over Ω for a.a. $t \in I$, we obtain the following equation:

$$(\partial_t y(t) - [\kappa \partial_t \Delta y(t) + \nu \Delta y(t)], \varphi(t))_\Omega + (\nabla \mathfrak{p}(t), \varphi(t))_\Omega = (f(t), \varphi(t))_\Omega. \tag{3.2}$$

Note that the time and space weak derivatives may commute (cf. Lemma 64.34 in [15]). Using Green's identity, we can write

$$\begin{aligned} ([\kappa \partial_t \Delta y(t) + \nu \Delta y(t)], \varphi(t))_\Omega = &- ([\kappa \nabla \partial_t y(t) + \nu \nabla y(t)], \nabla \varphi(t))_\Omega \\ &+ (\gamma_1 [\kappa \partial_t y(t) + \nu y(t)], \gamma_0 \varphi(t))_\Gamma, \end{aligned} \tag{3.3}$$

$$(\nabla \mathfrak{p}(t), \varphi(t))_\Omega = - (\mathfrak{p}(t), \operatorname{div} \varphi(t))_\Omega + (\mathfrak{p}(t), \mathrm{n} \cdot \gamma_0 \varphi(t))_\Gamma. \tag{3.4}$$

The boundary integral in (3.3) will vanish if we assume $\gamma_0 \varphi = 0$ on Σ. Moreover, because of the divergence-free condition on φ, (3.4) must be zero. Thus, (3.2) reduces to

$$(\partial_t y(t), \varphi(t))_\Omega - (\nabla [\kappa \partial_t y(t) + \nu y(t)], \nabla \varphi(t))_\Omega = (f(t), \varphi(t))_\Omega. \tag{3.5}$$

Integrating this equation over I leads to the following definition. Given $f \in \mathfrak{H}^{0,s-2}_{p,q}(Q)$ and $y_0 \in H^s_{q,0,\sigma}(\Omega)$, a function $y \in \mathfrak{H}^{1,s}_{p,q}(Q)$ is called a *weak solution* of (3.1) if the variational equation

$$\langle \partial_t y + \kappa A_{q,s/2} \partial_t y + \nu A_{q,s/2} y, \varphi \rangle_{\mathfrak{H}^{0,s-2}_{p,q}(Q) \times \mathfrak{H}^{0,2-s}_{p',q'}(Q)} = \langle f, \varphi \rangle_{\mathfrak{H}^{0,s-2}_{p,q}(Q) \times \mathfrak{H}^{0,2-s}_{p',q'}(Q)} \tag{3.6}$$

holds for all test functions $\varphi \in \mathfrak{H}^{0,2-s}_{p',q'}(Q)$ and the initial condition $\delta_0 y = y_0$ is satisfied in $H^s_{q,0,\sigma}(\Omega)$. This can be equivalently written in an abstract form as

$$\begin{cases} (\mathcal{I}_s + \kappa A_{q,s/2}) \partial_t y + \nu A_{q,s/2} y = f \quad \text{in } \mathfrak{H}^{0,s-2}_{p,q}(Q), \\ \delta_0 y = y_0 \quad \text{in } H^s_{q,0,\sigma}(\Omega). \end{cases} \tag{3.7}$$

Note that the initial condition is well-defined thanks to the continuous embedding $\mathfrak{H}^{1,s}_{p,q}(Q) \hookrightarrow \mathcal{C}(\bar{I}; H^s_{q,0,\sigma}(\Omega))$. The existence and uniqueness of weak solutions of (3.1) are guaranteed by the following theorem.

Theorem 3.1. *Let $p, q \in (1, \infty)$ and $s \in [0, 2]$. Given $f \in \mathfrak{H}^{0,s-2}_{p,q}(Q)$ and $y_0 \in H^s_{q,0,\sigma}(\Omega)$, then (3.1) has a unique weak solution $y \in \mathfrak{H}^{1,s}_{p,q}(Q)$. Moreover, there is a* *constant $C = C(\kappa, \nu, p, q, s, \Omega, T) > 0$ independent of y, y_0, and f such that we have the a priori estimate*

$$\|y\|_{\mathfrak{H}^{1,s}_{p,q}(Q)} \leq C \left(\|y_0\|_{H^s_{q,0,\sigma}(\Omega)} + \|f\|_{\mathfrak{H}^{0,s-2}_{p,q}(Q)} \right). \tag{3.8}$$

Proof. We can recast equation (3.7) into the form (2.6) by applying the bijective operator $(\mathcal{I}_s + \kappa A_{q,s/2})^{-1}$. Lemma 2.1 asserts that the resulting abstract form has a unique solution characterized by

$$y(t) = e^{-\nu \widehat{A}_{q,s/2} t} y_0 + \int_0^t e^{-\nu \widehat{A}_{q,s/2}(t-\tau)} (\mathcal{I}_s + \kappa A_{q,s/2})^{-1} f(\tau)\, \mathrm{d}\tau, \tag{3.9}$$

for a.a. $t \in [0, T]$. This mild solution is the unique weak solution of (3.1). Indeed, since y satisfies $(\mathcal{I}_s + \kappa A_{q,s/2})\partial_t y + \nu A_{q,s/2} y = f$ in Q, then (3.6) holds for all $\varphi \in \mathfrak{H}^{0,2-s}_{p',q'}(Q)$. Estimate (3.8) follows from (2.7) with $F := (\mathcal{I}_s + \kappa A_{q,s/2})^{-1} f$. □

Remark 3.2. Define the operator

$$\mathcal{P}^{1,s}_{p,q} : {}_0\mathfrak{H}^{1,s}_{p,q}(Q) \to {}_T\mathfrak{H}^{0,s-2}_{p,q}(Q)$$

according to

$$\begin{aligned} &\langle \mathcal{P}^{1,s}_{p,q} y, \varphi \rangle_{{}_T\mathfrak{H}^{0,s-2}_{p,q}(Q) \times {}_T\mathfrak{H}^{0,2-s}_{p',q'}(Q)} \\ &:= \langle \partial_t y + \kappa A_{q,s/2} \partial_t y + \nu A_{q,s/2} y, \varphi \rangle_{{}_T\mathfrak{H}^{0,s-2}_{p,q}(Q) \times {}_T\mathfrak{H}^{0,2-s}_{p',q'}(Q)}, \end{aligned} \tag{3.10}$$

for all $\varphi \in {}_T\mathfrak{H}^{0,2-s}_{p',q'}(Q)$. The previous theorem with $y_0 \equiv 0$ and $f \in {}_T\mathfrak{H}^{0,s-2}_{p,q}(Q)$ implies that

$$\mathcal{P}^{1,s}_{p,q} \in \mathcal{L}_{\mathrm{is}}({}_0\mathfrak{H}^{1,s}_{p,q}(Q), {}_T\mathfrak{H}^{0,s-2}_{p,q}(Q)).$$

Let us consider (3.1) with a rough non-homogeneous boundary condition:

$$\begin{cases} \partial_t y - \kappa \partial_t \Delta y - \nu \Delta y + \nabla \mathfrak{p} = f & \text{in } Q, \\ \operatorname{div} y = 0 \quad \text{in } Q, \qquad \gamma_0 y = h \quad \text{on } \Sigma, & \delta_0 y = y_0 \quad \text{in } \Omega. \end{cases} \tag{3.11}$$

A slight modification to our definition of weak solutions is needed. The same process may be followed to obtain (3.2) from (3.11). We can then apply Green's theorem to (3.3) so that

$$\begin{aligned} &([\kappa \partial_t \Delta y(t) + \nu \Delta y(t)], \varphi(t))_\Omega \\ &= -(\nabla [\kappa \partial_t y(t) + \nu y(t)], \nabla \varphi(t))_\Omega \\ &= ([\kappa \partial_t y(t) + \nu y(t)], \Delta \varphi(t))_\Omega - (\gamma_0 [\kappa \partial_t y(t) + \nu y(t)], \gamma_1 \varphi(t))_\Gamma. \end{aligned}$$

Since $h = \gamma_0 y$ on Σ, then from (3.2) together with the above equation, we obtain

$$\begin{aligned} &(\partial_t y(t), \varphi(t))_\Omega - ([\kappa \partial_t y(t) + \nu y(t)], \Delta \varphi(t))_\Omega \\ &= (f(t), \varphi(t))_\Omega - ([\kappa \partial_t h(t) + \nu h(t)], \gamma_1 \varphi(t))_\Gamma. \end{aligned} \tag{3.12}$$

For sufficiently regular h, we can use the dual of γ_1 to lift the function $\kappa \partial_t h + \nu h$ defined on Σ into the domain Q. Integrating equation (3.12) over I leads to following definition. Given $f \in \mathfrak{H}^{0,s-2}_{p,q}(Q)$ and $y_0 \in H^s_{q,0,\sigma}(\Omega)$, a function $y \in \mathfrak{H}^{1,s}_{p,q}(Q)$ is called a *weak solution* of (3.11) if

$$\langle \partial_t y + \kappa A_{q,s/2} \partial_t y + \nu A_{q,s/2} y, \varphi \rangle_{\mathfrak{H}^{0,s-2}_{p,q}(Q) \times \mathfrak{H}^{0,2-s}_{p',q'}(Q)}$$

$$= \langle f - \gamma_1^*(\kappa\partial_t + \nu)h, \varphi\rangle_{\mathfrak{H}^{0,s-2}_{p,q}(Q)\times\mathfrak{H}^{0,2-s}_{p',q'}(Q)}$$

for every test function $\varphi \in \mathfrak{H}^{0,2-s}_{p',q'}(Q)$ and $\delta_0 y = y_0$ holds in $H^s_{q,0,\sigma}(\Omega)$. In an abstract form, this is equivalent to

$$\begin{cases}(\mathcal{I}_s + \kappa A_{q,s/2})\partial_t y + \nu A_{q,s/2} y = f - \gamma_1^*(\kappa\partial_t + \nu)h & \text{in } \mathfrak{H}^{0,s}_{p,q}(Q),\\ \delta_0 y = y_0 \quad \text{in } H^s_{q,0,\sigma}(\Omega).\end{cases} \tag{3.13}$$

The following corollary is immediate from Theorem 3.1.

Corollary 3.3. *Let $p, q \in (1,\infty)$ and $s \in [0, 1/q)$. Given $f \in \mathfrak{H}^{0,s-2}_{p,q}(Q)$, $y_0 \in H^s_{q,0,\sigma}(\Omega)$, and $h \in H^1_p(I; B^{s-1/q}_{q,q}(\Gamma))$, (3.11) has a unique weak solution $y \in \mathfrak{H}^{1,s}_{p,q}(Q)$ characterized by*

$$y(t) = e^{-\nu\widehat{A}_{q,s/2}t} y_0 + \int_0^t e^{-\nu\widehat{A}_{q,s/2}(t-\tau)}(\mathcal{I}_s + \kappa A_{q,s/2})^{-1}\Big(f(\tau) - \gamma_1^*(\kappa\partial_t + \nu)h(\tau)\Big)\, \mathrm{d}\tau, \tag{3.14}$$

for a.a. $t \in [0,T]$. For some constant $C = C(\kappa, \nu, p, q, s, \Omega, \Gamma, T) > 0$ independent of y, y_0, f, and h, we have the a priori estimate

$$\|y\|_{\mathfrak{H}^{1,s}_{p,q}(Q)} \le C\left(\|y_0\|_{H^s_{q,0,\sigma}(\Omega)} + \|f\|_{\mathfrak{H}^{0,s-2}_{p,q}(Q)} + \|h\|_{H^1_p(I;B^{s-1/q}_{q,q}(\Gamma))}\right). \tag{3.15}$$

Proof. Since $s \in [0, 1/q)$, the linear operator γ_1 satisfies

$$\gamma_1 \in \mathcal{L}(\mathfrak{H}^{0,2-s}_{p',q'}(Q), L_{p'}(I; B^{1/q-s}_{q',q'}(\Gamma))).$$

Hence, for the dual operator, it holds that

$$\gamma_1^* \in \mathcal{L}(L_p(I; B^{s-1/q}_{q,q}(\Gamma)), \mathfrak{H}^{0,s-2}_{p,q}(Q)),$$

for $1 < q < \infty$, see Section 4.7.1 in [31]. Replacing f in Theorem 3.1 by $f - \gamma_1^*(\kappa\partial_t + \nu)\,h \in \mathfrak{H}^{0,s-2}_{p,q}(Q)$, we obtain a unique solution $y \in \mathfrak{H}^{1,s}_{p,q}(Q)$ satisfying the abstract form (3.13). The representation (3.14) and *a priori* estimate (3.15) follow from (3.9) and (3.8), respectively. □

Let us now characterize the very weak solution of (3.1). Such a solution, which was introduced by Amann for the NSE case, does not possess weak time derivatives [2]. We consider first the case of homogeneous initial and boundary conditions given by

$$\begin{cases}\partial_t y - \kappa\partial_t\Delta y - \nu\Delta y + \nabla\mathfrak{p} = f & \text{in } Q,\\ \operatorname{div} y = 0 \quad \text{in } Q, \qquad \gamma_0 y = 0 \quad \text{on } \Sigma, \qquad \delta_0 y = 0 & \text{in } \Omega.\end{cases} \tag{3.16}$$

Assume that we have a sufficiently regular solution y of (3.16). Multiplying (3.16) by a smooth divergence-free test function φ that is zero on the boundary Γ, integrating the resulting equation over Ω, and applying Green's identity, we obtain (3.5). Integrating by parts over I, we deduce the following equations:

$$(\partial_t y, \varphi)_Q = -(y, \partial_t\varphi)_Q + (y(T), \varphi(T))_\Omega - (y(0), \varphi(0))_\Omega,$$
$$(\nabla\partial_t y, \nabla\varphi)_Q = -(\nabla y, \nabla\partial_t\varphi)_Q + (\nabla y(T), \nabla\varphi(T))_\Omega - (\nabla y(0), \nabla\varphi(0))_\Omega.$$

Assume that $\varphi(T) = 0$ in Ω. Since $\delta_0 y = 0$ in Ω, the last two terms in each of the equations above will vanish. We then have

$$-(y, \partial_t \varphi)_Q + (\nabla y, \nabla[-\kappa \partial_t \varphi + \nu \varphi])_Q = (f, \varphi)_Q. \tag{3.17}$$

Based on the above computations, we are now in a position to introduce the notion of very weak solutions to (3.16). Given $f \in {}_T\mathfrak{H}^{-1,s-2}_{p,q}(Q)$, a function $y \in {}_0\mathfrak{H}^{0,s}_{p,q}(Q)$ is called a *very weak solution* of (3.16) if for all test functions $\varphi \in {}_T\mathfrak{H}^{1,2-s}_{p',q'}(Q)$, we have

$$\begin{aligned} &\langle y, -(\mathcal{I}_s + \kappa A_{q',1-s/2})\partial_t \varphi + \nu A_{q',1-s/2}\varphi \rangle_{{}_0\mathfrak{H}^{0,s}_{p,q}(Q) \times {}_0\mathfrak{H}^{0,-s}_{p',q'}(Q)} \\ &= \langle f, \varphi \rangle_{{}_T\mathfrak{H}^{-1,s-2}_{p,q}(Q) \times {}_T\mathfrak{H}^{1,2-s}_{p',q'}(Q)}. \end{aligned} \tag{3.18}$$

The existence and uniqueness of very weak solutions of (3.16) are shown below.

Theorem 3.4. *Let $p, q \in (1, \infty)$ and $s \in [0, 2]$. Given $f \in {}_T\mathfrak{H}^{-1,s-2}_{p,q}(Q)$, there exists a unique very weak solution $y \in {}_0\mathfrak{H}^{0,s}_{p,q}(Q)$ of (3.16). Furthermore, there is a constant $C = C(\kappa, \nu, p, q, s, \Omega, T) > 0$ independent of y and f such that*

$$\|y\|_{{}_0\mathfrak{H}^{0,s}_{p,q}(Q)} \le C\|f\|_{{}_T\mathfrak{H}^{-1,s-2}_{p,q}(Q)}. \tag{3.19}$$

Proof. Using $\mathcal{I}_s + \kappa A_{q',1-s/2} \in \mathcal{L}_{\rm is}(H^{2-s}_{q',0,\sigma}(\Omega), H^{-s}_{q',0,\sigma}(\Omega))$ together with Lemma 2.3, we see that for any $g \in \mathfrak{H}^{0,-s}_{p',q'}(Q) = {}_0\mathfrak{H}^{0,-s}_{p',q'}(Q)$, there exists a unique $\varphi_g \in {}_T\mathfrak{H}^{1,2-s}_{p',q'}(Q)$ such that

$$-\partial_t \varphi_g + \nu \widehat{A}_{q',1-s/2}\varphi_g = (\mathcal{I}_s + \kappa A_{q',1-s/2})^{-1} g,$$

and

$$\|\varphi_g\|_{{}_T\mathfrak{H}^{1,-s}_{p',q'}(Q)} \le C\|(\mathcal{I}_s + \kappa A_{q',1-s/2})^{-1} g\|_{{}_0\mathfrak{H}^{0,2-s}_{p',q'}(Q)} \le C\|g\|_{{}_0\mathfrak{H}^{0,-s}_{p',q'}(Q)}, \tag{3.20}$$

for some constant $C = C(\kappa, \nu, p, q, s, \Omega, T) > 0$ independent of g. Since φ_g depends linearly on g, we may define $y \in \mathfrak{H}^{0,s}_{p,q}(Q) = {}_0\mathfrak{H}^{0,s}_{p,q}(Q)$ by duality as follows:

$$\langle y, g \rangle_{{}_0\mathfrak{H}^{0,s}_{p,q}(Q) \times {}_0\mathfrak{H}^{0,-s}_{p',q'}(Q)} := \langle f, \varphi_g \rangle_{{}_T\mathfrak{H}^{-1,s-2}_{p,q}(Q) \times {}_T\mathfrak{H}^{1,2-s}_{p',q'}(Q)}, \tag{3.21}$$

for all $g \in {}_0\mathfrak{H}^{0,-s}_{p',q'}(Q)$. The function y is the solution that we are looking for since for any $\varphi \in {}_T\mathfrak{H}^{1,2-s}_{p',q'}(Q)$, we have $G := -(\mathcal{I}_s + \kappa A_{q',1-s/2})\partial_t \varphi + \nu A_{q',1-s/2}\varphi \in {}_0\mathfrak{H}^{0,-s}_{p',q'}(Q)$ and

$$\langle y, G \rangle_{{}_0\mathfrak{H}^{0,s}_{p,q}(Q) \times {}_0\mathfrak{H}^{0,-s}_{p',q'}(Q)} = \langle f, \varphi \rangle_{{}_T\mathfrak{H}^{-1,s-2}_{p,q}(Q) \times {}_T\mathfrak{H}^{1,2-s}_{p',q'}(Q)}.$$

The *a priori* estimate (3.19) will follow from (3.21) and (3.20), i.e., for any $g \in {}_0\mathfrak{H}^{0,-s}_{p',q'}(Q)$,

$$\begin{aligned} |\langle y, g \rangle_{{}_0\mathfrak{H}^{0,s}_{p,q}(Q) \times {}_0\mathfrak{H}^{0,-s}_{p',q'}(Q)}| &\le \|f\|_{{}_T\mathfrak{H}^{-1,s-2}_{p,q}} \|\varphi_g\|_{{}_T\mathfrak{H}^{1,2-s}_{p',q'}(Q)} \\ &\le C\|f\|_{{}_T\mathfrak{H}^{-1,s-2}_{p,q}} \|g\|_{{}_0\mathfrak{H}^{0,-s}_{p',q'}(Q)}. \end{aligned}$$

Uniqueness of very weak solutions can be shown in a standard manner thanks to the linearity of (3.16). □

Remark 3.5. Define the linear operator $\mathcal{P}^{0,s}_{p,q} : {}_0\mathfrak{H}^{0,s}_{p,q}(Q) \to {}_T\mathfrak{H}^{-1,s-2}_{p,q}(Q)$ according to the following equation:

$$\begin{aligned}&\langle \mathcal{P}^{0,s}_{p,q} y, \varphi\rangle_{{}_T\mathfrak{H}^{-1,s-2}_{p,q}(Q)\times {}_T\mathfrak{H}^{1,2-s}_{p',q'}(Q)}\\ &:= \langle y, -(\mathcal{I}_s + \kappa A_{q',1-s/2})\partial_t\varphi + \nu A_{q',1-s/2}\varphi\rangle_{{}_0\mathfrak{H}^{0,s}_{p,q}(Q)\times {}_0\mathfrak{H}^{0,-s}_{p',q'}(Q)}\end{aligned} \tag{3.22}$$

for all $\varphi \in {}_T\mathfrak{H}^{1,2-s}_{p',q'}(Q)$. Consequently, from the above theorem,

$$\mathcal{P}^{0,s}_{p,q} \in \mathcal{L}_{\text{is}}({}_0\mathfrak{H}^{0,s}_{p,q}(Q), {}_T\mathfrak{H}^{-1,s-2}_{p,q}(Q)).$$

Note that for each $y \in {}_0\mathfrak{H}^{1,s}_{p,q}(Q) \overset{d}{\hookrightarrow} {}_0\mathfrak{H}^{0,s}_{p,q}(Q)$ and for every $\varphi \in {}_T\mathfrak{H}^{1,2-s}_{p',q'}(Q) \overset{d}{\hookrightarrow} {}_T\mathfrak{H}^{0,2-s}_{p',q'}(Q)$, it can be shown that

$$\langle \mathcal{P}^{1,s}_{p,q} y, \varphi\rangle_{{}_T\mathfrak{H}^{0,s-2}_{p,q}(Q)\times {}_T\mathfrak{H}^{0,2-s}_{p',q'}(Q)} = \langle \mathcal{P}^{0,s}_{p,q} y, \varphi\rangle_{{}_T\mathfrak{H}^{-1,s-2}_{p,q}(Q)\times {}_T\mathfrak{H}^{1,2-s}_{p',q'}(Q)} \tag{3.23}$$

by duality arguments. The above dense embeddings imply that $\mathcal{P}^{0,s}_{p,q}$ is an extension of $\mathcal{P}^{1,s}_{p,q}$; see Figure 2 on the following pages. Thus, weakly differentiable (with respect to time) very weak solutions of (3.16) are precisely the weak solutions of (3.16).

For the very weak formulation of the non-homogeneous case (3.11), we continue from (3.12). Taking the integral over time and integrating by parts, we obtain

$$\begin{aligned}(\partial_t y, \varphi)_Q &= -(y, \partial_t\varphi)_Q - (y_0, \delta_0\varphi)_\Omega,\\ -(\partial_t y, \Delta\varphi)_Q + (\partial_t h, \gamma_1\varphi)_\Sigma &= (y, \Delta\partial_t\varphi)_Q + (y_0, \delta_0\Delta\varphi)_\Omega\\ &\quad - (h, \partial_t\gamma_1\varphi)_\Sigma + (h(0), \delta_0\gamma_1\varphi)_\Gamma,\end{aligned}$$

since $\delta_T\varphi = 0$ on Σ and assuming that $\delta_T\gamma_1\varphi = 0$ on Γ. Combining our observations, we get

$$\begin{aligned}(y, -\partial_t\varphi + \kappa\Delta\partial_t\varphi - \nu\Delta\varphi)_Q &= (f, \varphi)_Q - (h, [-\kappa\partial_t\gamma_1\varphi + \nu\gamma_1\varphi])_\Sigma\\ &\quad + (y_0, [\delta_0\varphi - \kappa\delta_0\Delta\varphi])_\Omega + \kappa(h(0), \delta_0\gamma_1\varphi)_\Gamma.\end{aligned}$$

For smooth solutions, when the initial y_0 has zero trace on Γ, we have the compatibility $h(0) = \gamma_0 y_0 = 0$. Hence, the last term in the above equation vanishes. If y_0 has a nonzero trace, then the situation is more complicated. We will not consider such a scenario since it is outside the scope of the current paper.

Based on the above formulation, we have the following definition of very weak solutions. Given $f \in {}_T\mathfrak{H}^{-1,s-2}_{p,q}(Q)$, $y_0 \in H^s_{q,0,\sigma}(\Omega)$, and $h \in L_p(I; B^{s-1/q}_{q,q}(\Gamma))$, we say that $y \in \mathfrak{H}^{0,s}_{p,q}(Q)$ is a *very weak solution* of (3.11), if the variational equation

$$\begin{aligned}&\langle y, -(\mathcal{I}_s + \kappa A_{q',1-s/2})\partial_t\varphi + \nu A_{q',1-s/2}\varphi\rangle_{\mathfrak{H}^{0,s}_{p,q}(Q)\times\mathfrak{H}^{0,-s}_{p',q'}(Q)}\\ &= \langle f, \varphi\rangle_{{}_T\mathfrak{H}^{-1,s-2}_{p,q}(Q)\times {}_T\mathfrak{H}^{1,2-s}_{p',q'}(Q)}\\ &\quad - \langle h, (-\kappa\partial_t + \nu)\gamma_1\varphi\rangle_{L_p(I;B^{s-1/q}_{q,q}(\Gamma))\times L_{p'}(I;B^{1/q-s}_{q',q'}(\Gamma))}\\ &\quad + \langle y_0, \delta_0(\mathcal{I}_s + \kappa A_{q',1-s/2})\varphi\rangle_{H^s_{q,0,\sigma}(\Omega)\times H^{-s}_{q',0,\sigma}(\Omega)}\end{aligned} \tag{3.24}$$

is satisfied for every test function $\varphi \in {}_T\mathfrak{H}^{1,2-s}_{p',q'}(Q)$. The following corollary is immediate from the previous theorem.

Corollary 3.6. *Let $p,q \in (1,\infty)$ and $s \in [0,1/q)$. Given $f \in {}_T\mathfrak{H}^{-1,s-2}_{p,q}(Q)$, $y_0 \in H^s_{q,0,\sigma}(\Omega)$, and $h \in L_p(I;B^{s-1/q}_{q,q}(\Gamma))$, there exists a unique very weak solution $y \in \mathfrak{H}^{0,s}_{p,q}(Q)$ of (3.11). There exists a constant $C = C(\kappa,\nu,p,q,s,\Omega,\Gamma,T) > 0$ independent of y, y_0, f, and h such that*

$$\|y\|_{\mathfrak{H}^{0,s}_{p,q}(Q)} \le C\left(\|y_0\|_{H^s_{q,0,\sigma}(\Omega)} + \|f\|_{{}_T\mathfrak{H}^{-1,s-2}_{p,q}(Q)} + \|h\|_{L_p(I;B^{s-1/q}_{q,q}(\Gamma))}\right). \tag{3.25}$$

The statements are true for any $s \in [0,2]$ if we have a homogeneous boundary condition, that is, when $h \equiv 0$.

Proof. Since ${}_T\mathfrak{H}^{1,-s}_{p',q'}(Q) \subset \mathcal{C}(\bar{I};H^{-s}_{q',0,\sigma}(\Omega))$ for $s \in [0,2]$,

$$\delta_0^* \in \mathcal{L}(H^s_{q,0,\sigma}(\Omega), {}_T\mathfrak{H}^{-1,s}_{p,q}(Q))$$

is well-defined. Consequently,

$$(\mathcal{I}_s + \kappa A_{q,s/2})\delta_0^* \in \mathcal{L}(H^s_{q,0,\sigma}(\Omega), {}_T\mathfrak{H}^{-1,s-2}_{p,q}(Q)). \tag{3.26}$$

So, (3.11) with homogeneous boundary has a unique very weak solution in the sense of (3.24) thanks to Theorem 3.4 with f replaced by $f + (\mathcal{I}_s + \kappa A_{q,s/2})\delta_0^* y_0$.

Assuming $0 \le s < 1/q$, the operator

$$(-\kappa\partial_t + \nu)\gamma_1 : {}_T\mathfrak{H}^{1,2-s}_{p',q'}(Q) \to L_{p'}(I;B^{1-s-1/q'}_{q',q'}(\Gamma))$$

is well-defined. Thus, we get

$$((-\kappa\partial_t + \nu)\gamma_1)^* \in \mathcal{L}(L_p(I;B^{s-1/q}_{q,q}(\Gamma)), {}_T\mathfrak{H}^{-1,s-2}_{p,q}(Q)). \tag{3.27}$$

We can replace f by $f - ((-\kappa\partial_t + \nu)\gamma_1)^* h + (\mathcal{I}_s + \kappa A_{q,s/2})\delta_0^* y_0 \in {}_T\mathfrak{H}^{-1,s-2}_{p,q}(Q)$ to show the existence and uniqueness of $y \in \mathfrak{H}^{0,s}_{p,q}(Q) = {}_0\mathfrak{H}^{0,s}_{p,q}(Q)$ that satisfies

$$\begin{aligned}&\langle y, -(\mathcal{I}_s + \kappa A_{q',1-s/2})\partial_t\varphi + \nu A_{q',1-s/2}\varphi\rangle_{\mathfrak{H}^{0,s}_{p,q}(Q)\times\mathfrak{H}^{0,-s}_{p',q'}(Q)}\\ &= \langle f - ((-\kappa\partial_t + \nu)\gamma_1)^* h + (\mathcal{I}_s + \kappa A_{q,s/2})\delta_0^* y_0, \varphi\rangle_{{}_T\mathfrak{H}^{-1,s-2}_{p,q}(Q)\times {}_T\mathfrak{H}^{1,2-s}_{p',q'}(Q)}.\end{aligned}$$

Therefore, y satisfies (3.24) for every $\varphi \in {}_T\mathfrak{H}^{1,2-s}_{p',q'}(Q)$. The estimate (3.25) will follow from (3.19), (3.26), and (3.27). □

We now consider complex interpolation to obtain additional regularity in time. Applying interpolation on Theorem 3.1 with a homogeneous initial condition and Theorem 3.4, we will obtain time regularity on different scales as stated in the following theorem.

Theorem 3.7. *Let $p,q \in (1,\infty)$, $s \in [0,2]$, and $\theta \in [0,1]$. Given $f \in {}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)$, then (3.16) has a unique very weak solution $y \in {}_0\mathfrak{H}^{\theta,s}_{p,q}(Q)$ in the sense that the variational equation*

$$\begin{aligned}&\langle y, -(\mathcal{I}_s + \kappa A_{q',1-s/2})\partial_t\varphi + \nu A_{q',1-s/2}\varphi\rangle_{{}_0\mathfrak{H}^{\theta,s}_{p,q}(Q)\times {}_0\mathfrak{H}^{-\theta,-s}_{p',q'}(Q)}\\ &= \langle f, \varphi\rangle_{{}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)\times {}_T\mathfrak{H}^{1-\theta,2-s}_{p',q'}(Q)}\end{aligned} \tag{3.28}$$

holds for all test functions $\varphi \in {}_T\mathfrak{H}^{1-\theta,s-2}_{p',q'}(Q)$. Furthermore, there exists a constant $C = C(\kappa,\nu,p,q,s,\theta,\Omega,T) > 0$ independent of y and f that satisfies the a priori estimate

$$\|y\|_{{}_0\mathfrak{H}^{\theta,s}_{p,q}(Q)} \le C\|f\|_{{}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)}. \tag{3.29}$$

Proof. Using (3.23), (2.2), and (2.3), and then applying Lemma 1.52 in [14], we have $\mathcal{P}_{p,q}^{\theta,s} := \mathcal{P}_{p,q}^{0,s}|_{{}_0\mathfrak{H}_{p,q}^{\theta,s}(Q)}$ with $\mathcal{P}_{p,q}^{\theta,s}({}_0\mathfrak{H}_{p,q}^{\theta,s}(Q)) = {}_T\mathfrak{H}_{p,q}^{\theta-1,s-2}(Q)$. Because $\mathcal{P}_{p,q}^{0,s}$ is an isomorphism, by complex interpolation (see Figure 2 below), we have

$$\mathcal{P}_{p,q}^{\theta,s} \in \mathcal{L}_{\text{is}}({}_0\mathfrak{H}_{p,q}^{\theta,s}(Q), {}_T\mathfrak{H}_{p,q}^{\theta-1,s-2}(Q)).$$

It follows that there exists a unique $y \in {}_0\mathfrak{H}_{p,q}^{\theta,s}(Q)$ such that

$$\langle \mathcal{P}_{p,q}^{\theta,s} y, \varphi \rangle_{{}_T\mathfrak{H}_{p,q}^{\theta-1,s-2}(Q)\times {}_T\mathfrak{H}_{p',q'}^{1-\theta,2-s}(Q)} = \langle f, \varphi \rangle_{{}_T\mathfrak{H}_{p,q}^{\theta-1,s-2}(Q)\times {}_T\mathfrak{H}_{p',q'}^{1-\theta,2-s}(Q)}$$

for all $\varphi \in {}_T\mathfrak{H}_{p',q'}^{1,2-s}(Q)$. Equation (3.28) follows because of the dense and continuous embedding ${}_T\mathfrak{H}_{p',q'}^{1,2-s}(Q) \overset{d}{\hookrightarrow} {}_T\mathfrak{H}_{p',q'}^{1-\theta,2-s}(Q)$. □

$$\begin{array}{ccc}
{}_0\mathfrak{H}_{p,q}^{1,s}(Q) & \xrightarrow{\mathcal{P}_{p,q}^{1,s}} & {}_T\mathfrak{H}_{p,q}^{0,s-2}(Q) \\
\big\downarrow d & & \big\downarrow d \\
{}_0\mathfrak{H}_{p,q}^{\theta,s}(Q) & \xrightarrow{\mathcal{P}_{p,q}^{\theta,s}} & {}_T\mathfrak{H}_{p,q}^{\theta-1,s-2}(Q) \\
\big\downarrow d & & \big\downarrow d \\
{}_0\mathfrak{H}_{p,q}^{0,s}(Q) & \xrightarrow{\mathcal{P}_{p,q}^{0,s}} & {}_T\mathfrak{H}_{p,q}^{-1,s-2}(Q)
\end{array}$$

FIGURE 2. Commutative diagram for the operators $\mathcal{P}_{p,q}^{0,s}$, $\mathcal{P}_{p,q}^{\theta,s}$, and $\mathcal{P}_{p,q}^{1,s}$.

We have a similar result for the nonhomogeneous case (3.11), which follows directly from the previous theorem. The conditions $0 \le s < 1/q$ and $0 \le \theta < 1/p$, which respectively imply that $2 - s > 1 + \frac{1}{q'}$ and $1 - \theta > \frac{1}{p'}$, guarantee that

$$\gamma_1 \in \mathcal{L}(\mathfrak{H}_{p',q'}^{-\theta,2-s}(Q), H_{p'}^{-\theta}(I; B_{q',q'}^{1/q-s}(\Gamma)))$$

and

$$\delta_0 \in \mathcal{L}(\mathfrak{H}_{p',q'}^{1-\theta,-s}(Q), H_{q',0,\sigma}^{-s}(\Omega))$$

are well-defined.

Corollary 3.8. *Let $p, q \in (1, \infty)$, $0 \le s < 1/q$, and $0 \le \theta < 1/p$. Given $f \in {}_T\mathfrak{H}_{p,q}^{\theta-1,s-2}(Q)$, $y_0 \in H_{q,0,\sigma}^{s}(\Omega)$, and $h \in H_p^{\theta}(I; B_{q,q}^{s-1/q}(\Gamma))$, there exists a unique very weak solution $y \in \mathfrak{H}_{p,q}^{\theta,s}(Q)$ of system (3.11) in the sense that the variational equation*

$$\begin{aligned}
&\langle y, -(\mathcal{I}_s + \kappa A_{q',1-s/2})\partial_t \varphi + \nu A_{q',1-s/2}\varphi \rangle_{\mathfrak{H}_{p,q}^{\theta,s}(Q)\times \mathfrak{H}_{p',q'}^{-\theta,-s}(Q)} \\
&= \langle f, \varphi \rangle_{{}_T\mathfrak{H}_{p,q}^{\theta-1,s-2}(Q)\times {}_T\mathfrak{H}_{p',q'}^{1-\theta,2-s}(Q)} \\
&\quad - \langle h, (-\kappa\partial_t + \nu)\gamma_1 \varphi \rangle_{H_p^{\theta}(I; B_{q,q}^{s-1/q}(\Gamma)) \times H_{p'}^{-\theta}(I; B_{q',q'}^{1/q-s}(\Gamma))} \\
&\quad + \langle y_0, \delta_0(\mathcal{I}_s + \kappa A_{q',1-s/2})\varphi \rangle_{H_{q,0,\sigma}^{s}(\Omega)\times H_{q',0,\sigma}^{-s}(\Omega)},
\end{aligned} \tag{3.30}$$

holds for every test function $\varphi \in {}_T\mathfrak{H}_{p',q'}^{1-\theta,2-s}(Q)$. Moreover, there exists a constant $C = C(\kappa, \nu, p, q, s, \theta, \Omega, \Gamma, T) > 0$ independent of y, y_0, f, and h such that

$$\|y\|_{\mathfrak{H}_{p,q}^{\theta,s}(Q)} \le C \left(\|y_0\|_{H_{q,0,\sigma}^{s}(\Omega)} + \|f\|_{{}_T\mathfrak{H}_{p,q}^{\theta-1,s-2}(Q)} + \|h\|_{H_p^{\theta}(I; B_{q,q}^{s-1/q}(\Gamma))} \right). \tag{3.31}$$

In particular, the existence and uniqueness of the very weak solution is guaranteed for $s \in [0,2]$, if we have a homogeneous boundary condition, that is, when $h \equiv 0$.

Proof. From the embedding theorem for the Bessel potential spaces (c.f., Theorem 3.4 in [27]) we have ${}_T\mathfrak{H}^{1-\theta,-s}_{p',q'}(Q) \hookrightarrow \mathcal{C}(\bar{I}; H^{-s}_{q',0,\sigma}(\Omega))$, for $0 \leq \theta < 1/p$ and any $s \in [0,2]$. Hence,

$$\delta_0 \in \mathcal{L}({}_T\mathfrak{H}^{1-\theta,-s}_{p',q'}(Q), H^{-s}_{q',0,\sigma}(\Omega))$$

is well-defined. This implies

$$(\mathcal{I}_s + \kappa A_{q,s/2})\delta_0^* \in \mathcal{L}(H^s_{q,0,\sigma}(\Omega), {}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)). \tag{3.32}$$

Thus, if we have a homogeneous boundary condition in (3.11), and f is replaced by $f + (\mathcal{I}_s + A_{q,s/2})\delta_0^* y_0 \in {}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)$ in Theorem 3.7, we obtain a unique very weak solution $y \in \mathfrak{H}^{\theta,s}_{p,q}(Q)$ satisfying (3.30). By our choice of θ, we have $\partial_t \in \mathcal{L}({}_T\mathfrak{H}^{1-\theta,2-s}_{p',q'}(Q), \mathfrak{H}^{-\theta,2-s}_{p',q'}(Q))$ by complex interpolation. We can characterize

$$\partial_t^* \in \mathcal{L}(\mathfrak{H}^{\theta,s-2}_{p,q}(Q), {}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q))$$

by duality as follows:

$$\langle \partial_t^* \psi, \varphi\rangle_{{}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q) \times {}_T\mathfrak{H}^{1-\theta,2-s}_{p',q'}(Q)} := \langle \psi, \partial_t \varphi\rangle_{\mathfrak{H}^{\theta,s-2}_{p,q}(Q) \times \mathfrak{H}^{-\theta,2-s}_{p',q'}(Q)}$$

for all $\psi \in \mathfrak{H}^{\theta,s-2}_{p,q}(Q)$ and $\varphi \in {}_T\mathfrak{H}^{1-\theta,2-s}_{p',q'}(Q)$. Moreover, from Section 4.7.1 in [31], we have

$$\gamma_1^* \in \mathcal{L}(H^\theta_p(I; B^{s-1/q}_{q,q}(\Gamma)), {}_T\mathfrak{H}^{\theta,s-2}_{p,q}(Q)) \tag{3.33}$$

for $1 < q < \infty$ and $0 \leq s < \frac{1}{q}$. Thus, due to the dense and continuous embedding ${}_T\mathfrak{H}^{\theta,2-s}_{p,q}(\Omega) \overset{d}{\hookrightarrow} {}_T\mathfrak{H}^{\theta-1,2-s}_{p,q}(\Omega)$, we obtain

$$(-\kappa\partial_t^* + \nu)\gamma_1^* \in \mathcal{L}(H^\theta_p(I; B^{s-1/q}_{q,q}(\Gamma)), {}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)). \tag{3.34}$$

Replacing f by $f - (-\kappa\partial_t^* + \nu)\gamma_1^* h + (\mathcal{I}_s + \kappa A_{q,s/2})\delta_0^* y_0 \in {}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)$ in Theorem 3.7 guarantees the existence and uniqueness of a very weak solution $y \in \mathfrak{H}^{\theta,s}_{p,q}(Q)$ satisfying (3.30). The estimate (3.31) follows from (3.29), (3.32), (3.33), and (3.34). □

For the case when $1/p < \theta \leq 1$, we use the results from Theorem 3.1 and Theorem 3.7.

Corollary 3.9. *Let $p, q \in (1,\infty)$, $s \in [0,2]$, and $1/p < \theta \leq 1$. Given $f \in {}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)$ and $y_0 \in H^s_{q,0,\sigma}(\Omega)$, there exists a unique very weak solution $y \in \mathfrak{H}^{\theta,s}_{p,q}(Q)$ of (3.1). Moreover, there is a constant $C = C(\kappa,\nu,p,q,s,\theta,\Omega,T) > 0$ independent of y, y_0, and f such that*

$$\|y\|_{\mathfrak{H}^{\theta,s}_{p,q}(Q)} \leq C\left(\|y_0\|_{H^s_{q,0,\sigma}(\Omega)} + \|f\|_{{}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)}\right). \tag{3.35}$$

Proof. Consider the following decomposition of (3.1):

$$\begin{cases} \partial_t y_1 - \kappa\partial_t\Delta y_1 - \nu\Delta y_1 + \nabla\mathfrak{p}_1 = f \quad \text{in } Q; \\ \text{div } y_1 = 0 \quad \text{in } Q, \qquad \gamma_0 y_1 = 0 \quad \text{on } \Sigma, \qquad \delta_0 y_1 = 0 \quad \text{in } \Omega, \end{cases} \tag{3.36}$$

$$\begin{cases} \partial_t y_2 - \kappa\partial_t\Delta y_2 - \nu\Delta y_2 + \nabla\mathfrak{p}_2 = 0 & \text{in } Q, \\ \operatorname{div} y_2 = 0 \quad \text{in } Q, \qquad \gamma_0 y_2 = 0 \quad \text{on } \Sigma, \qquad \delta_0 y_2 = y_0 & \text{in } \Omega. \end{cases} \tag{3.37}$$

Theorem 3.7 guarantees the existence and uniqueness of a very weak solution $y_1 \in {}_0\mathfrak{H}^{\theta,s}_{p,q}(Q)$ to (3.36) satisfying the estimate

$$\|y_1\|_{\mathfrak{H}^{\theta,s}_{p,q}(Q)} \le C\|f\|_{{}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)}. \tag{3.38}$$

Moreover, (3.37) has a unique solution $y_2 \in \mathfrak{H}^{1,s}_{p,q}(Q)$ according to Theorem 3.1 such that

$$\|y_2\|_{\mathfrak{H}^{\theta,s}_{p,q}(Q)} \le C\|y_2\|_{\mathfrak{H}^{1,s}_{p,q}(Q)} \le C\|y_0\|_{H^s_{q,0,\sigma}(\Omega)}. \tag{3.39}$$

Combining these, we see that $y = y_1 + y_2 \in {}_0\mathfrak{H}^{\theta,s}_{p,q}(Q) + \mathfrak{H}^{1,s}_{p,q}(Q) \subset \mathfrak{H}^{\theta,s}_{p,q}(Q)$ is a solution of (3.1). The uniqueness of solution to (3.1) follows from Theorem 3.7. Finally, (3.35) can be obtained from (3.38), (3.39), and the triangle inequality. □

4. Very weak solution of NSV with small data

We are now ready to present one of the goals of this paper. In particular, the very weak solutions of (NSV) will be described in this section. The contraction mapping principle will be utilized to show the existence and uniqueness of the solutions of (NSV) with a smallness condition on the initial data and the forcing function.

Let us now consider the Navier–Stokes–Voigt equations with a non-homogeneous initial condition

$$\begin{cases} \partial_t y - \kappa\partial_t\Delta y - \nu\Delta y + \operatorname{div}(y\otimes y) + \nabla\mathfrak{p} = f & \text{in } Q, \\ \operatorname{div} y = 0 \quad \text{in } Q, \qquad \gamma_0 y = 0 \text{ on } \Sigma, \qquad \delta_0 y = y_0 & \text{in } \Omega. \end{cases} \tag{4.1}$$

Similar formal computations from the previous section lead to the following variational formulation of (4.1):

$$(y, [-\partial_t\varphi + \kappa\Delta\partial_t\varphi - \nu\Delta\varphi])_Q = (f,\varphi)_Q + (y_0, \delta_0(\varphi+\kappa\Delta\varphi))_\Omega + (y\otimes y, \nabla\varphi)_Q. \tag{4.2}$$

Because of the divergence-free condition, we have $\operatorname{div}(y\otimes y) = (y\cdot\nabla)y$, hence,

$$(y\otimes y, \nabla\varphi)_Q = -((y\cdot\nabla)y, \varphi)_Q.$$

This leads to an equivalent variational formulation for (4.1) given by

$$(y, [-\partial_t\varphi + \kappa\Delta\partial_t\varphi - \nu\Delta\varphi])_Q = (f,\varphi)_Q + (y_0, \delta_0(\mathcal{I}+\kappa\Delta)\varphi)_\Omega - ((y\cdot\nabla)y, \varphi)_Q. \tag{4.3}$$

Given $f \in {}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)$ and $y_0 \in H^s_{q,0,\sigma}(\Omega)$, we say that $y \in \mathfrak{H}^{\theta,s}_{p,q}(Q)$ is a *very weak solution* of (4.1) if (4.2) or (4.3) is satisfied for all test functions $\varphi \in {}_T\mathfrak{H}^{1-\theta,2-s}_{p',q'}(Q)$. The main difference between the two variational equations will be apparent in relation to the parameters that we will consider below.

For the treatment of (4.2), we have the following assumptions. Consider the following conditions:

$$\bar{s} \in [0,1], \qquad \bar{q} \in \left[\frac{\bar{d}}{\bar{s}+1}, \infty\right) \cap (1,\infty). \tag{H1}$$

of a triple $(\bar{d}, \bar{s}, \bar{q})$. If $(1, \theta, p)$ and (d, s, q) both satisfy (H1), the Sobolev embedding implies

$$H_p^\theta(I; H_q^s(\Omega)) \hookrightarrow L_{\widehat{p}}(I; L_{\widehat{q}}(\Omega))$$

and

$$H_{p'}^{1-\theta}(I; H_{q'}^{1-s}(\Omega)) \hookrightarrow L_{(\widehat{p}/2)'}(I; L_{(\widehat{q}/2)'}(\Omega)),$$

where

$$\widehat{p} = \frac{2p}{p(1-\theta)+1} \quad \text{and} \quad \widehat{q} = \frac{2dq}{q(1-s)+d}.$$

It follows from Hölder's inequality that there exists a constant $C = C(p, q, s, \theta, \Omega) > 0$ such that

$$|(y_1 \otimes y_2, \nabla\varphi)_Q| \le C\|y_1\|_{L_{\widehat{p}}(I;L_{\widehat{q}}(\Omega))}\|y_2\|_{L_{\widehat{p}}(I;L_{\widehat{q}}(\Omega))}\|\nabla\varphi\|_{L_{(\widehat{p}/2)'}(I;L_{(\widehat{q}/2)'}(\Omega))},$$

for $y_1, y_2 \in \mathfrak{H}_{p,q}^{\theta,s}(Q)$ and $\varphi \in {}_T\mathfrak{H}_{p',q'}^{1-\theta,2-s}(Q)$. Because of the continuity of $\nabla : {}_T\mathfrak{H}_{p',q'}^{1-\theta,2-s}(Q) \to {}_T\mathfrak{H}_{p',q'}^{1-\theta,1-s}(Q)$, we then have

$$\|\nabla\varphi\|_{L_{(\widehat{p}/2)'}(I;L_{(\widehat{q}/2)'}(\Omega))} \le C\|\varphi\|_{{}_T\mathfrak{H}_{p',q'}^{1-\theta,2-s}(Q)}.$$

Thus, condition (H1) is sufficient for $y_1 \otimes y_2$ to be in ${}_T\mathfrak{H}_{p,q}^{\theta-1,s-1}(Q)$ whenever $y_1, y_2 \in \mathfrak{H}_{p,q}^{\theta,s}(Q)$.

For (4.3), assume $(1, \theta, p)$ satisfies (H1) and (d, s, q) satisfies one of the following:

$$d = 2, \qquad 1 < s < \frac{3}{2}, \qquad \frac{4}{2s+1} < q < \infty, \tag{H2.a}$$

$$d = 2, \qquad \frac{3}{2} \le s \le 2, \qquad 1 < q < \infty, \tag{H2.b}$$

$$d = 3, \qquad 1 < s \le 2, \qquad \frac{3}{s+1} < q < \infty, \tag{H2.c}$$

$$d \ge 4, \qquad 1 < s \le 2, \qquad \frac{d}{s+1} \le q < \infty, \tag{H2.d}$$

with q_0, q_1, and q_2 chosen appropriately, which shall be explained shortly below. Then,

$$H_p^\theta(I; H_q^s(\Omega)) \hookrightarrow L_{\widehat{p}}(I; L_{q_0}(\Omega)),$$
$$H_p^\theta(I; H_q^{s-1}(\Omega)) \hookrightarrow L_{\widehat{p}}(I; L_{q_1}(\Omega)),$$
$$H_{p'}^{1-\theta}(I; H_{q'}^{2-s}(\Omega)) \hookrightarrow L_{(\widehat{p}/2)'}(I; L_{q_2}(\Omega)).$$

Assuming that $\frac{1}{q_0} + \frac{1}{q_1} + \frac{1}{q_2} = 1$ holds, by Hölder's inequality we obtain

$$|((y_1 \cdot \nabla)y_2, \varphi)_Q| \le C\|y_1\|_{L_{\widehat{p}}(I;L_{q_0}(\Omega))}\|y_2\|_{L_{\widehat{p}}(I;L_{q_1}(\Omega))}\|\varphi\|_{L_{(\widehat{p}/2)'}(I;L_{q_2}(\Omega))},$$

with

$$\|\varphi\|_{L_{(\widehat{p}/2)'}(I;L_{q_2}(\Omega))} \le C\|\varphi\|_{{}_T\mathfrak{H}_{p',q'}^{1-\theta,2-s}(Q)}.$$

Hence, each condition in (H2) is sufficient for $(y_1 \cdot \nabla)y_2$ to be in ${}_T\mathfrak{H}_{p,q}^{\theta-1,s-2}(Q)$ provided that $y_1, y_2 \in \mathfrak{H}_{p,q}^{\theta,s}(Q)$.

In what follows, we show how to choose the parameters q_0, q_1, and q_2 used in the above arguments. For $1 < s \le 2$, consider

$$H_q^s(\Omega) \hookrightarrow L_{q_0}(\Omega), \quad H_q^{s-1}(\Omega) \hookrightarrow L_{q_1}(\Omega), \quad \text{and} \quad H_{q'}^{2-s}(\Omega) \hookrightarrow L_{q_2}(\Omega) \tag{4.5}$$

with $\frac{1}{q_0}+\frac{1}{q_1}+\frac{1}{q_2}=1$. By a Sobolev embedding, (4.5) is satisfied if $\frac{s}{d}\geq\frac{1}{q}-\frac{1}{q_0}$, $\frac{s-1}{d}\geq\frac{1}{q}-\frac{1}{q_1}$, and $\frac{2-s}{d}\geq\frac{1}{q'}-\frac{1}{q_2}$, or, equivalently, if

$$\frac{1}{q_0}\geq\frac{1}{q}-\frac{s}{d},\qquad \frac{1}{q_1}\geq\frac{1}{q}-\frac{s-1}{d},\qquad \text{and}\qquad \frac{1}{q_2}\geq 1-\frac{1}{q}-\frac{2-s}{d}. \tag{4.6}$$

Combining $\frac{1}{q_2}=1-\frac{1}{q_0}-\frac{1}{q_1}$ together with the first and second inequalities in (4.6), we obtain $1+\frac{2s-1}{d}-\frac{2}{q}\geq\frac{1}{q_2}$. Hence, $\frac{1}{q_2}$ will exist if $1>\frac{2}{q}-\frac{2s-1}{d}$ and $1+\frac{2s-1}{d}-\frac{2}{q}\geq 1-\frac{1}{q}-\frac{2-s}{d}$, that is, if

$$\frac{1}{2}\left(1-\frac{1}{d}\right)+\frac{s}{d}>\frac{1}{q}\quad\text{and}\quad\frac{s+1}{d}\geq\frac{1}{q}. \tag{4.7}$$

We consider different cases based on the bounds of $\frac{1}{q}$.

Case 1. $0<\frac{1}{q}\leq\frac{s-1}{d}$. In this case, $\frac{1}{q}-\frac{s}{d}<\frac{1}{q}-\frac{s-1}{d}\leq 0$. Choose

$$\frac{1}{q_0}=\frac{1}{2q},\qquad \frac{1}{q_1}=\frac{1}{2q},\qquad \frac{1}{q_2}=1-\frac{1}{q}. \tag{4.8}$$

Case 2. $\frac{s-1}{d}<\frac{1}{q}<\frac{s}{d}$. In this case, $\frac{1}{q}-\frac{s}{d}\leq 0<\frac{1}{q}-\frac{s-1}{d}$. Choose

$$\frac{1}{q_0}=\frac{1}{sq},\qquad \frac{1}{q_1}=\frac{1}{q}-\frac{s-1}{d},\qquad \frac{1}{q_2}=1-\frac{1}{q}-\frac{1}{sq}+\frac{s-1}{d}. \tag{4.9}$$

Notice that $0<\frac{1}{q_0}<\frac{1}{d}$ and $0<\frac{1}{q_1}\leq\frac{1}{d}$. Since $\frac{1}{sq}<\frac{1}{d}$ then, $0\leq 1-\frac{2}{d}<1-\frac{1}{q}+\frac{s-2}{d}\leq\frac{1}{q_2}<1$.

Case 3. $\frac{1}{q}=\frac{s}{d}$ and $1<s<2$ if $d=2$, or $1<s\leq 2$ if $d\geq 3$. Choose

$$\frac{1}{q_0}=\frac{1}{2d},\quad \frac{1}{q_1}=\frac{1}{d},\quad \frac{1}{q_2}=1-\frac{3}{2d}. \tag{4.10}$$

Case 4. For $d=2$ only and $\frac{s}{d}<\frac{1}{q}<\frac{s}{d}+\frac{1}{4}$ with $1<s<\frac{3}{2}$, or $\frac{s}{d}<\frac{1}{q}<1$ with $\frac{3}{2}\leq s<2$. Choose

$$\frac{1}{q_0}=\frac{1}{q}-\frac{s}{2},\quad \frac{1}{q_1}=\frac{1}{q}-\frac{s-1}{2},\quad \frac{1}{q_2}=s-\frac{2}{q}+\frac{1}{2}. \tag{4.11}$$

The conditions imply that $0<\frac{1}{q_0}<\frac{1}{2}$, $\frac{1}{2}<\frac{1}{q_0}<1$, and $0<\frac{1}{q_2}<\frac{1}{2}$.

Case 5. $\frac{s}{d}<\frac{1}{q}<\frac{s+1}{d}$ if $d=3$, or $\frac{s}{d}<\frac{1}{q}\leq\frac{s+1}{d}$ if $d\geq 4$. In this case, $0<\frac{1}{q}-\frac{s}{d}<\frac{1}{q}-\frac{s-1}{d}$. Choose

$$\frac{1}{q_0}=\frac{1}{q}-\frac{s}{d},\quad \frac{1}{q_1}=\frac{1}{q}-\frac{s-1}{d},\quad \frac{1}{q_2}=1-\frac{2}{q}+\frac{2s-1}{d}. \tag{4.12}$$

Note that $0<\frac{1}{q_0}\leq\frac{1}{d}$, $\frac{1}{d}\leq\frac{1}{q_1}\leq\frac{2}{d}$, and $0<\frac{1}{q_2}<1-\frac{1}{d}$. These observations are precisely those that have been summarized in (H2).

We are now in a position to define the following bilinear operators associated with the convection term. Let

$$\mathcal{B},\mathcal{B}_1,\mathcal{B}_2:\mathfrak{H}^{\theta,s}_{p,q}(Q)\times\mathfrak{H}^{\theta,s}_{p,q}(Q)\to{}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)$$

be given as follows:

$$\langle \mathcal{B}_1(y_1,y_2),\varphi\rangle_{{}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)\times {}_T\mathfrak{H}^{1-\theta,2-s}_{p',q'}(Q)} = -\langle y_1\otimes y_2, \nabla\varphi\rangle_{{}_T\mathfrak{H}^{\theta-1,s-1}_{p,q}(Q)\times {}_T\mathfrak{H}^{1-\theta,1-s}_{p',q'}(Q)},$$
$$\langle \mathcal{B}_2(y_1,y_2),\varphi\rangle_{{}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)\times {}_T\mathfrak{H}^{1-\theta,2-s}_{p',q'}(Q)} = \langle (y_1\cdot\nabla) y_2, \varphi\rangle_{{}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)\times {}_T\mathfrak{H}^{1-\theta,2-s}_{p',q'}(Q)},$$
$$\mathcal{B}(y_1,y_2) = \begin{cases} \mathcal{B}_1(y_1,y_2) & \text{if } s\in[0,1],\\ \mathcal{B}_2(y_1,y_2) & \text{if } s\in(1,2]. \end{cases}$$

Consider the closed ball

$$\overline{B}_r := \{y\in \mathfrak{H}^{\theta,s}_{p,q}(Q) : \|y\|_{\mathfrak{H}^{\theta,s}_{p,q}(Q)} \le r\}$$

and the map

$$S : \mathfrak{H}^{\theta,s}_{p,q}(Q)\to \mathfrak{H}^{\theta,s}_{p,q}(Q)$$

defined as follows. For a given $y\in \mathfrak{H}^{\theta,s}_{p,q}(Q)$, let $S(y)\in \mathfrak{H}^{\theta,s}_{p,q}(Q)$ be the solution to the variational equation

$$\begin{aligned}
&\langle S(y), -\partial_t\varphi + \kappa\Delta\partial_t\varphi - \nu\Delta\varphi\rangle_{\mathfrak{H}^{\theta,s}_{p,q}(Q)\times\mathfrak{H}^{-\theta,-s}_{p',q'}(Q)}\\
&= \langle f,\varphi\rangle_{{}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)\times {}_T\mathfrak{H}^{1-\theta,2-s}_{p',q'}(Q)} + \langle y_0, \delta_0(\mathcal{I}+\kappa A_{q',1-s/2})\varphi\rangle_{H^s_{q,0,\sigma}(\Omega)\times H^{-s}_{q',0,\sigma}(\Omega)}\\
&\quad + \langle \mathcal{B}(y,y),\varphi\rangle_{{}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)\times {}_T\mathfrak{H}^{1-\theta,2-s}_{p',q'}(Q)}
\end{aligned}$$

for all test functions $\varphi\in {}_T\mathfrak{H}^{1-\theta,2-s}_{p',q'}(Q)$. Using Theorem 3.7, Corollary 3.8, Corollary 3.9, and suitable assumptions (H1) or (H2), the existence and uniqueness of $S(y)$ is guaranteed. We can also find constants $C_i = C_i(\kappa,\nu,p,q,s,\theta,\Omega,T)>0$, for $i=1,2$, both independent of y, y_0, and f, such that

$$\|S(y)\|_{\mathfrak{H}^{\theta,s}_{p,q}(Q)} \le C_1\left(\|f\|_{{}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)} + \|y_0\|_{H^s_{q,0,\sigma}(\Omega)}\right) + C_2\|y\|^2_{\mathfrak{H}^{\theta,s}_{p,q}(Q)}.$$

Now, assume that

$$\|f\|_{{}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)} + \|y_0\|_{H^s_{q,0,\sigma}(\Omega)} \le \frac{3}{16C_1C_2}$$

and choose $r=\frac{1}{4C_2}$ so that whenever $y\in\overline{B}_r$, we have

$$\|S(y)\|_{\mathfrak{H}^{\theta,s}_{p,q}(Q)} \le \frac{1}{4C_2}.$$

Hence, $S(\overline{B}_r)\subset\overline{B}_r$. For each $y_1,y_2\in\overline{B}_r$, we have

$$\begin{aligned}
&\langle S(y_1)-S(y_2), -\partial_t\varphi + \kappa\Delta\partial_t\varphi - \nu\Delta\varphi\rangle_{\mathfrak{H}^{\theta,s}_{p,q}(Q)\times\mathfrak{H}^{-\theta,-s}_{p',q'}(Q)}\\
&= \langle \mathcal{B}(y_1,y_1)-\mathcal{B}(y_2,y_2),\varphi\rangle_{{}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)\times {}_T\mathfrak{H}^{1-\theta,2-s}_{p',q'}(Q)}\\
&= \langle \mathcal{B}(y_1,y_1-y_2)+\mathcal{B}(y_1-y_2,y_2),\varphi\rangle_{{}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)\times {}_T\mathfrak{H}^{1-\theta,2-s}_{p',q'}(Q)}.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\|S(y_1)-S(y_2)\|_{\mathfrak{H}^{\theta,s}_{p,q}(Q)} &\le C_2\left(\|y_1\|_{\mathfrak{H}^{\theta,s}_{p,q}(Q)} + \|y_2\|_{\mathfrak{H}^{\theta,s}_{p,q}(Q)}\right)\|y_1-y_2\|_{\mathfrak{H}^{\theta,s}_{p,q}(Q)}\\
&\le 2rC_2\|y_1-y_2\|_{\mathfrak{H}^{\theta,s}_{p,q}(Q)}\\
&= \frac{1}{2}\|y_1-y_2\|_{\mathfrak{H}^{\theta,s}_{p,q}(Q)}
\end{aligned}$$

where C_2 is the same constant as before. Therefore, S is a contraction on $\overline{B}_r$. The contraction principle implies the existence of a unique fixed point, hence a unique very weak solution, $y \in \overline{B}_r$ of (4.1). We summarize our observations in the following theorem.

Theorem 4.1. *Let $d \geq 2$ and assume $(1, \theta, p)$ satisfies (H1) with $\theta \neq \frac{1}{p}$. Suppose that (d, s, q) satisfies (H1) (resp. (d, s, q) satisfies (H2)). Let $f \in {}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)$ and $y_0 \in H^s_{q,0,\sigma}(\Omega)$. There exists a constant $\eta = \eta(\kappa, \nu, p, q, s, \theta, \Omega, T) > 0$ such that if*

$$\|f\|_{{}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)} + \|y_0\|_{H^s_{q,0,\sigma}(\Omega)} \leq \eta,$$

then (4.1) has a unique very weak solution $y \in \mathfrak{H}^{\theta,s}_{p,q}(Q)$ satisfying (4.2) (resp. (4.3)), and there exists a constant $C = C(\kappa, \nu, p, q, s, \theta, \Omega, T) > 0$ independent of y, y_0, and f such that

$$\|y\|_{\mathfrak{H}^{\theta,s}_{p,q}(Q)} \leq C \left(\|f\|_{{}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)} + \|y_0\|_{H^s_{q,0,\sigma}(\Omega)} \right).$$

Remark 4.2. Let us compare Theorem 4.1 with the results of [13]. According to Theorem 4.1, (4.1) has a unique solution for small forces and initial data if we are given $s = 1$, $1 < q < \infty$ for $d = 2$, or $\frac{d}{2} \leq q < \infty$ for $d \geq 3$ (c.f. Theorem 3.4 in [13]). Moreover, when $s = 2$, this is valid for $d \in \{2, 3\}$ with $1 < q < \infty$, or for $d \geq 4$ with $\frac{d}{3} \leq q < \infty$ (c.f. in Theorem 3.9 [13]).

5. Global-in-time solutions of NSV in 2D

The well-posedness of the two-dimensional NSVE for arbitrary initial data and source function will be established in this section. As mentioned in the introduction, we shall follow the framework utilized by Casas and Kunisch in [12]. We decompose (NSV) into its linear and nonlinear parts. The existence and uniqueness of solutions for the nonlinear part are shown using the Faedo–Galerkin approach. The results in Section 3 are combined with the nonlinear part to show the well-posedness of (NSV).

Throughout this section, we focus on the case $d = 2$. Letting $y := y_L + y_N$ and $\mathfrak{p} := \mathfrak{p}_L + \mathfrak{p}_N$, we can decompose (NSV) into its linear part

$$\begin{cases} \partial_t y_L - \kappa \partial_t \Delta y_L - \nu \Delta y_L + \nabla \mathfrak{p}_L = f_L \quad \text{in } Q, \\ \operatorname{div} y_L = 0 \quad \text{in } Q, \qquad \gamma_0 y_L = 0 \quad \text{on } \Sigma, \qquad \delta_0 y_L = y_{L0} \quad \text{in } \Omega, \end{cases} \tag{5.1}$$

and nonlinear part

$$\begin{cases} \partial_t y_N - \kappa \partial_t \Delta y_N - \nu \Delta y_N + \operatorname{div}(y_N \otimes y_N) + \operatorname{div}(y_L \otimes y_N) \\ + \operatorname{div}(y_N \otimes y_L) + \nabla \mathfrak{p}_N = f_N - \operatorname{div}(y_L \otimes y_L) \quad \text{in } Q, \\ \operatorname{div} y_N = 0 \quad \text{in } Q, \qquad \gamma_0 y_N = 0 \quad \text{on } \Sigma, \qquad \delta_0 y_N = y_{N0} \quad \text{in } \Omega. \end{cases} \tag{5.2}$$

For the nonlinear part, we consider the auxiliary partial differential equations given by

$$\begin{cases} \partial_t y_N - \kappa \Delta \partial_t y_N - \nu \Delta y_N + \operatorname{div}(y_N \otimes y_N) + \operatorname{div}(z_1 \otimes y_N) \\ + \operatorname{div}(y_N \otimes z_2) + \nabla p_N = g \quad \text{in } Q, \\ \operatorname{div} y_N = 0 \quad \text{in } Q, \qquad \gamma_0 y_N = 0 \quad \text{on } \Sigma, \qquad \delta_0 y_N = y_{N0} \quad \text{in } \Omega, \end{cases} \tag{5.3}$$

for some given functions $z_1, z_2 \in L_4(Q) = L_4(I; L_4(\Omega))$ and $g \in \mathfrak{H}^{0,-1}_{2,2}(Q)$. Note that (5.2) follows by taking $y_L = z_1 = z_2$, where y_L is the solution of (5.1). Similar to the previous formal computations, we call $y_N \in \mathfrak{H}^{1,1}_{2,2}(Q)$ a *weak solution* of (5.3) if

$$\begin{cases} \partial_t y_N + \kappa A_{2,1/2}\partial_t y_N + \nu A_{2,1/2} y_N + \operatorname{div}(y_N \otimes y_N) + \operatorname{div}(z_1 \otimes y_N) \\ + \operatorname{div}(y_N \otimes z_2) = g \quad \text{in } \mathfrak{H}^{0,-1}_{2,2}(Q), \\ \delta_0 y_N = y_{N0} \quad \text{in } H^1_{2,0,\sigma}(\Omega). \end{cases} \tag{5.4}$$

Theorem 5.1. *Let $d = 2$, $g \in \mathfrak{H}^{0,-1}_{2,2}(Q)$, $y_{N0} \in H^1_{2,0,\sigma}(\Omega)$, and $z_1, z_2 \in L_4(Q)$. There exists a unique weak solution $y_N \in \mathfrak{H}^{1,1}_{2,2}(Q)$ of (5.3). Moreover, there exist constants $c = c(\kappa, \nu, \Omega, T) > 0$ and $C = C(\kappa, \nu, \Omega, T) > 0$, both independent of y_N, y_{N0}, g, z_1, and z_2, such that*

$$\|y_N\|^2_{L_\infty(I;H^1_{2,0,\sigma}(\Omega))} \le C\left(\|y_{N0}\|^2_{H^1_{2,0,\sigma}(\Omega)} + \|g\|^2_{\mathfrak{H}^{0,-1}_{2,2}(Q)}\right) e^{c\|z_2\|^4_{L_4(Q)}}, \tag{5.5}$$

$$\begin{aligned} \|y_N\|^2_{\mathfrak{H}^{0,1}_{2,2}(Q)} \le{}& \frac{2}{\nu^2}\|g\|^2_{\mathfrak{H}^{0,-1}_{2,2}(Q)} + \|y_{N0}\|^2_{H^1_{2,0,\sigma}(\Omega)} \\ &+ C\|z_2\|^4_{L_4(Q)}\left(\|y_{N0}\|^2_{H^1_{2,0,\sigma}(\Omega)} + \|g\|^2_{\mathfrak{H}^{0,-1}_{2,2}(Q)}\right) e^{c\|z_2\|^4_{L_4(Q)}}, \end{aligned} \tag{5.6}$$

$$\begin{aligned} \|\partial_t y_N\|^2_{\mathfrak{H}^{0,1}_{2,2}(Q)} \le{}& \frac{4}{\kappa^2}\|g\|^2_{\mathfrak{H}^{0,-1}_{2,2}(Q)} + \frac{\nu}{\kappa}\|y_N\|^2_{\mathfrak{H}^{0,1}_{2,2}(Q)} \\ &+ \frac{c}{\kappa}\|y_N\|^2_{L_4(Q)}\left(\|y_N\|^2_{L_4(Q)} + \|z_1\|^2_{L_4(Q)} + \|z_2\|^2_{L_4(Q)}\right). \end{aligned} \tag{5.7}$$

Proof. We only show the formal computation of *a priori* estimates, leading to well-posedness via the standard Faedo–Galerkin method. For details on approximation and the limiting process, see for instance [6]. In the following inequalities, we use the generic constants $c = c(\kappa, \nu, \Omega, T) > 0$ and $C = C(\kappa, \nu, \Omega, T) > 0$, which may be different on each line.

Multiplying the first equation of (5.3) by y_N and integrating by parts over Ω, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|y_N(t)\|^2_{L_2(\Omega)} + \kappa\|\nabla y_N(t)\|^2_{L_2(\Omega)}\right) + \nu\|y_N(t)\|^2_{H^1_{2,0,\sigma}(\Omega)} \\ &= \langle g(t), y_N(t)\rangle_{H^{-1}_{2,0,\sigma}(\Omega)\times H^1_{2,0,\sigma}(\Omega)} + (y_N(t)\otimes y_N(t), \nabla z_2(t))_\Omega. \end{aligned} \tag{5.8}$$

Note that

$$(y_N(t)\otimes y_N(t), \nabla y_N(t))_\Omega = (z_1(t)\otimes y_N(t), \nabla y_N(t))_\Omega = 0$$

due to the anti-symmetry with respect to the second and third arguments. The integrals over Γ and the pressure are eliminated, thanks to y_N being divergence-free and since $\gamma_0 y_N = 0$ on Σ.

We estimate the duality pairing and the inner products on the right-hand side of (5.8). Applying the Hölder, Ladyzhenskaya, Poincaré, and Young inequalities, one has

$$\begin{aligned} &|(y_N(t)\otimes y_N(t), \nabla z_2(t))_\Omega| \\ &= |((y_N(t)\cdot\nabla)y_N(t), z_2(t))_\Omega| \\ &\le \|y_N(t)\|_{L_4(\Omega)}\|\nabla y_N(t)\|_{L_2(\Omega)}\|z_2(t)\|_{L_4(\Omega)} \end{aligned}$$

$$\le c\|y_N(t)\|_{L_2(\Omega)}^{1/2}\|\nabla y_N(t)\|_{L_2(\Omega)}^{3/2}\|z_2(t)\|_{L_4(\Omega)}$$
$$\le \frac{\nu}{4}\|\nabla y_N(t)\|_{L_2(\Omega)}^2 + c\|y_N(t)\|_{L_2(\Omega)}^2\|z_2(t)\|_{L_4(\Omega)}^4$$
$$\le \frac{\nu}{4}\|y_N(t)\|_{H^1_{2,0,\sigma}(\Omega)}^2 + c\|y_N(t)\|_{H^1_{2,0,\sigma}(\Omega)}^2\|z_2(t)\|_{L_4(\Omega)}^4. \tag{5.9}$$

Moreover, applying the Hölder inequality and then the Young inequality on the remaining term, we have

$$\langle g(t), y_N(t)\rangle_{H^{-1}_{2,0,\sigma}(\Omega)\times H^1_{2,0,\sigma}(\Omega)} \le \frac{\nu}{4}\|y_N(t)\|_{H^1_{2,0,\sigma}(\Omega)}^2 + \frac{1}{\nu}\|g(t)\|_{H^{-1}_{2,0,\sigma}(\Omega)}^2.$$

These inequalities, together with (5.8), imply the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|y_N(t)\|_{L_2(\Omega)}^2 + \kappa\|\nabla y_N(t)\|_{L_2(\Omega)}^2\right) + \nu\|y_N(t)\|_{H^1_{2,0}(\Omega)}^2$$
$$\le \frac{2}{\nu}\|g(t)\|_{H^{-1}_{2,0,\sigma}(\Omega)}^2 + c\|y_N(t)\|_{H^1_{2,0,\sigma}(\Omega)}^2\|z_2(t)\|_{L_4(\Omega)}^4.$$

Let $\tilde{\kappa} = \min\{1,\kappa\}$ so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|y_N(t)\|_{H^1_{2,0,\sigma}(\Omega)}^2 + \frac{\nu}{\tilde{\kappa}}\|y_N(t)\|_{H^1_{2,0,\sigma}(\Omega)}^2$$
$$\le \frac{2}{\nu\tilde{\kappa}}\|g(t)\|_{H^{-1}_{2,0,\sigma}(\Omega)}^2 + c\|y_N(t)\|_{H^1_{2,0,\sigma}(\Omega)}^2\|z_2(t)\|_{L_4(\Omega)}^4. \tag{5.10}$$

We can omit the non-negative term $\frac{\nu}{\tilde{\kappa}}\|y_N(t)\|_{H^1_{2,0}(\Omega)}^2$ and apply Grönwall's inequality to obtain (5.5). Integrate (5.10) over the interval I and use (5.5) to get

$$\frac{\nu}{\tilde{\kappa}}\int_0^T \|y_N(t)\|_{H^1_{2,0,\sigma}(\Omega)}^2\,\mathrm{d}t + \|y_N(T)\|_{H^1_{2,0,\sigma}(\Omega)}^2$$
$$\le \frac{2}{\nu\tilde{\kappa}}\|g\|_{\mathfrak{H}^{0,-1}_{2,2}(Q)}^2 + \|y_{N0}\|_{H^1_{2,0,\sigma}(\Omega)}^2$$
$$+ C\|z_2\|_{L_4(Q)}^4 e^{c\|z_2\|_{L_4(Q)}^4}\left(\|y_{N0}\|_{H^1_{2,0,\sigma}(\Omega)}^2 + \|g\|_{\mathfrak{H}^{0,-1}_{2,2}(Q)}^2\right).$$

Inequality (5.6) follows from this inequality.

We now estimate $\partial_t y_N$. Multiplying (5.3) by $\partial_t y_N$ and integrating by parts over Ω, we obtain

$$\|\partial_t y_N(t)\|_{L_2(\Omega)}^2 + \kappa\|\nabla\partial_t y_N(t)\|_{L_2(\Omega)}^2$$
$$= \langle g(t), y_N(t)\rangle_{H^{-1}_{2,0,\sigma}(\Omega)\times H^1_{2,0,\sigma}(\Omega)}$$
$$-\nu\int_0^T(\nabla y_N(t),\nabla\partial_t y_N(t))_\Omega\,\mathrm{d}t - \int_0^T(y_N(t)\otimes y_N(t),\nabla\partial_t y_N(t))_\Omega\,\mathrm{d}t$$
$$-\int_0^T(y_N(t)\otimes z_2(t),\nabla\partial_t y_N(t))_\Omega\,\mathrm{d}t - \int_0^T(z_1(t)\otimes y_N(t),\nabla\partial_t y_N(t))_\Omega\,\mathrm{d}t. \tag{5.11}$$

We infer from (5.5) and (5.6) that y_N is in $\mathfrak{H}^{0,1}_{2,2}(Q)\cap L_\infty(I;H^1_{2,0,\sigma}(\Omega))$. Moreover, the following inequalities are obtained using, once again, the Hölder, Ladyzhenskaya, and Young inequalities:

$$\int_0^T\langle g(t), y_N(t)\rangle_{H^{-1}_{2,0,\sigma}(\Omega)\times H^1_{2,0,\sigma}(\Omega)}\,\mathrm{d}t \le \|g\|_{\mathfrak{H}^{0,-1}_{2,2}(Q)}\|\partial_t y_N\|_{\mathfrak{H}^{0,1}_{2,2}(Q)}$$

$$\leq \frac{2}{\kappa}\|g\|^2_{\mathfrak{H}^{0,-1}_{2,2}(Q)} + \frac{\kappa}{8}\|\partial_t y_N\|^2_{\mathfrak{H}^{0,1}_{2,2}(Q)}$$

$$\nu\int_0^T (\nabla y_N(t), \nabla\partial_t y_N(t))_\Omega \, \mathrm{d}t = \frac{\nu}{2}\left(\|y_N(T)\|^2_{H^1_{2,0,\sigma}(\Omega)} - \|y_N(0)\|^2_{H^1_{2,0,\sigma}(\Omega)}\right)$$
$$\leq \frac{\nu}{2}\|y_N\|^2_{\mathfrak{H}^{0,1}_{2,2}(Q)}.$$

Furtheremore, since $H^1_{2,0,\sigma}(\Omega) \hookrightarrow L_4(\Omega)$, the antisymmetry with respect to the second and third arguments implies

$$\begin{aligned}
&-\int_0^T (y_N(t)\otimes y_N(t), \nabla\partial_t y_N(t))_\Omega \, \mathrm{d}t \\
&\quad\leq \int_0^T \|y_N(t)\|^2_{L_4(\Omega)}\|\nabla\partial_t y_N(t)\|_{L_2(\Omega)} \, \mathrm{d}t \\
&\quad\leq c\|y_N\|^2_{L_4(Q)}\|\partial_t y_N\|_{\mathfrak{H}^{0,1}_{2,2}(Q)} \\
&\quad\leq c\|y_N\|^4_{L_4(Q)} + \frac{\kappa}{8}\|\partial_t y_N\|^2_{\mathfrak{H}^{0,1}_{2,2}(Q)}, \\
&-\int_0^T (y_N(t)\otimes z_2(t), \nabla\partial_t y_N(t))_\Omega \, \mathrm{d}t \\
&\quad\leq \int_0^T \|y_N(t)\|_{L_4(\Omega)}\|\nabla\partial_t y_N(t)\|_{L_2(\Omega)}\|z_2(t)\|_{L_4(\Omega)} \, \mathrm{d}t \\
&\quad\leq \|y_N\|_{L_4(Q)}\|\nabla\partial_t y_N\|_{\mathfrak{H}^{0,1}_{2,2}(Q)}\|z_2(t)\|_{L_4(Q)} \\
&\quad\leq c\|y_N\|^2_{L_4(Q)}\|z_2(t)\|^2_{L_4(Q)} + \frac{\kappa}{8}\|\partial_t y_N\|^2_{\mathfrak{H}^{0,1}_{2,2}(Q)}, \\
&-\int_0^T (z_1(t)\otimes y_N(t), \nabla\partial_t y_N(t))_\Omega \, \mathrm{d}t \\
&\quad\leq \int_0^T \|y_N(t)\|_{L_4(\Omega)}\|\nabla\partial_t y_N(t)\|_{L_2(\Omega)}\|z_1(t)\|_{L_4(\Omega)} \, \mathrm{d}t \\
&\quad\leq c\|y_N\|^2_{L_4(Q)}\|z_1(t)\|^2_{L_4(Q)} + \frac{\kappa}{8}\|\partial_t y_N\|^2_{\mathfrak{H}^{0,1}_{2,2}(Q)}.
\end{aligned}$$

Integrating (5.11) over I and combining the computed upper bounds, we obtain

$$\begin{aligned}
&\|\partial_t y_N\|^2_{L_2(I;L_2(\Omega))} + \kappa\|\partial_t y_N\|^2_{\mathfrak{H}^{0,1}_{2,2}(Q)} \\
&\leq \frac{4}{\kappa}\|g\|^2_{\mathfrak{H}^{0,-1}_{2,2}(Q)} + \nu\|y_N\|^2_{\mathfrak{H}^{0,1}_{2,2}(Q)} \\
&\quad + c\|y_N\|^2_{L_4(Q)}\left(\|y_N\|^2_{L_4(Q)} + \|z_1\|^2_{L_4(Q)} + \|z_2\|^2_{L_4(Q)}\right).
\end{aligned}$$

This implies (5.7). □

Given two Banach spaces X and Y and a Hausdorff topological vector space Z, with continuous embeddings $X \hookrightarrow Z$ and $Y \hookrightarrow Z$, the sum $X + Y := \{x + y : x \in X, y \in Y\}$ is a Banach space when endowed with the norm

$$\|z\|_{X+Y} := \inf_{\substack{z=x+y \\ x\in X,\, y\in Y}} (\|x\|_X + \|y\|_Y), \quad \forall z \in X + Y.$$

Let us define the following Banach spaces:

$$
\begin{aligned}
{}_T\mathcal{F}^{\theta-1,s-2}_{p,q}(Q) &:= {}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q) + \mathfrak{H}^{0,-1}_{2,2}(Q),\\
\mathcal{H}^s_q(\Omega) &:= H^s_{q,0,\sigma}(\Omega) + H^1_{2,0,\sigma}(\Omega),\\
\mathcal{Y}^{\theta,s}_{p,q}(Q) &:= \mathfrak{H}^{\theta,s}_{p,q}(Q) + \mathfrak{H}^{1,1}_{2,2}(Q).
\end{aligned}
$$

We have $\mathcal{Y}^{\theta,s}_{p,q}(Q) \subset \mathcal{C}(\bar{I};\mathcal{H}^s_q(\Omega))$ for $p,q \in (1,\infty)$, $s \in [0,2]$, and $\theta \in (\frac{1}{p},1]$.

The embedding theorem for the space of Bessel potentials (cf. Theorem 3.4 in [27]) implies that

$$
H^\theta_p(I;H^s_q(\Omega)) \hookrightarrow L_4(I;L_4(\Omega)) = L_4(Q), \tag{5.12}
$$

under one of the following conditions:

$$
\begin{aligned}
&p \in [4,\infty), && q \in [4,\infty), && \theta \in [0,1], && s \in [0,2], && \text{(H3.a)}\\
&p \in (1,4), && q \in [4,\infty), && \theta \in \left[\frac{1}{p}-\frac{1}{4},1\right], && s \in [0,2], && \text{(H3.b)}\\
&p \in [4,\infty), && q \in (1,4), && \theta \in [0,1], && s \in \left[\frac{2}{q}-\frac{1}{2},2\right], && \text{(H3.c)}\\
&p \in (1,4), && q \in (1,4), && \theta \in \left[\frac{1}{p}-\frac{1}{4},1\right], && s \in \left[\frac{2}{q}-\frac{1}{2},2\right]. && \text{(H3.d)}
\end{aligned}
$$

Combining the results from (5.1) and (5.2), we obtain the following existence and uniqueness of very weak solutions of (4.1).

Theorem 5.2. *Assume one of (H3) and that $\theta \neq \frac{1}{p}$. Let $f \in {}_T\mathcal{F}^{\theta-1,s-2}_{p,q}(Q)$ and $y_0 \in \mathcal{H}^s_q(\Omega)$. There exists a unique weak solution $y \in \mathcal{Y}^{\theta,s}_{p,q}(Q)$ of (4.1) such that*

$$
\|y\|_{\mathcal{Y}^{\theta,s}_{p,q}(Q)} \le \eta\left(\|f\|_{{}_T\mathcal{F}^{\theta-1,s-2}_{p,q}(Q)} + \|y_0\|_{\mathcal{H}^s_q(\Omega)}\right), \tag{5.14}
$$

for some non-decreasing function $\eta : [0,\infty) \to [0,\infty)$ with $\eta(0) = 0$.

Proof. Let $f = f_L + f_N$ for some $f_L \in {}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q)$ and $f_N \in \mathfrak{H}^{0,-1}_{2,2}(Q)$, and let $y_0 = y_{L0} + y_{N0}$ for some $y_{L0} \in H^s_{q,0,\sigma}(\Omega)$ and $y_{N0} \in H^1_{2,0,\sigma}(\Omega)$. Corollary 3.8 and Corollary 3.9 guarantee that system (5.1) has a unique very weak solution $y_L \in \mathfrak{H}^{\theta,s}_{p,q}(Q)$. Let $g := f_N + \operatorname{div}(y_L \otimes y_L)$. Since one of (H3) is assumed, we are guaranteed that $\mathfrak{H}^{\theta,s}_{p,q}(Q) \hookrightarrow L_4(Q)$. It can be shown (by following the proof of Lemma 2.1 in Appendix A of [12]) that $\operatorname{div}(y_L \otimes y_L)$, and hence g, belong to $\mathfrak{H}^{0,-1}_{2,2}(Q)$. Taking $z_1 = z_2 = y_L$, it follows from Theorem 5.1 that system (5.2) has a unique very weak solution $y_N \in \mathfrak{H}^{1,1}_{2,2}(Q)$. Adding equations (5.1) and (5.2), we obtain that $y := y_L + y_N \in \mathcal{Y}^{\theta,s}_{p,q}(Q)$ is a very weak solution of (4.1). The estimate (5.14) will follow by taking the infimum of the sum of estimates (3.31), (5.6), and (5.7) over all representations $f = f_L + f_N$ and $y_0 = y_{L0} + y_{N0}$.

We now prove that the solution is unique. Suppose that $y_1 \in \mathcal{Y}^{\theta,s}_{p,q}(Q)$ and $y_2 \in \mathcal{Y}^{\theta,s}_{p,q}(Q)$ are very weak solutions of (4.1). Then, $\tilde{y} = y_1 - y_2 \in \mathcal{Y}^{\theta,s}_{p,q}(Q)$, and $\tilde{\mathfrak{p}} = \mathfrak{p}_1 + \mathfrak{p}_2$ is a very weak solution to

$$
\begin{cases}
\partial_t\tilde{y} - \kappa\Delta\partial_t\tilde{y} - \nu\Delta\tilde{y} + \nabla\tilde{\mathfrak{p}} = -\operatorname{div}(y_1 \otimes y_1) + \operatorname{div}(y_2 \otimes y_2) \quad \text{in } Q,\\
\operatorname{div}\,\tilde{y} = 0 \quad \text{in } Q, \qquad \gamma_0\tilde{y} = 0 \quad \text{on } \Sigma, \qquad \delta_0\tilde{y} = 0 \quad \text{in } \Omega.
\end{cases} \tag{5.15}
$$

The pressure term $\nabla\tilde{\mathfrak{p}}$ is eliminated due to the divergence-free solutions. Note that the right-hand side belongs to $\mathfrak{H}^{0,-1}_{2,2}(Q)$. Then, from Theorem 3.1, system (5.15) has a unique weak solution $y \in \mathfrak{H}^{1,1}_{2,2}(Q)$. Because of our choice of p, q, θ, and s, we have $\mathfrak{H}^{\theta,s}_{p,q}(Q) \hookrightarrow L_4(Q) \subset \mathfrak{H}^{0,-1}_{2,2}(Q)$. Hence, $\mathcal{Y}^{\theta,s}_{p,q}(Q) \subset \mathfrak{H}^{0,-1}_{2,2}(Q)$. This implies that $\tilde{y} = y$, that is, $\tilde{y} \in \mathfrak{H}^{1,1}_{2,2}(Q)$. Multiplying $\tilde{y}$ on both sides of (5.15), then using integration by parts, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\tilde{y}(t)\|^2_{L_2(\Omega)} + \kappa\|\nabla\tilde{y}(t)\|^2_{L_2(\Omega)}\right) + \nu\|\tilde{y}(t)\|^2_{H^1_{2,0,\sigma}(\Omega)}\\ &= (y_1(t)\otimes\tilde{y}(t), \nabla\tilde{y}(t))_\Omega + (\tilde{y}(t)\otimes y_2(t), \nabla\tilde{y}(t))_\Omega\\ &= (\tilde{y}(t)\otimes y_2(t), \nabla\tilde{y}(t))_\Omega.\end{aligned}$$

Following a similar approach as in (5.9) yields

$$\begin{aligned}&\frac{\tilde{\kappa}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{y}(t)\|^2_{H^1_{2,0,\sigma}(\Omega)} + \nu\|\tilde{y}(t)\|^2_{H^1_{2,0,\sigma}(\Omega)}\\ &\le \nu\|\tilde{y}(t)\|^2_{H^1_{2,0,\sigma}(\Omega)} + c\|\tilde{y}(t)\|^2_{H^1_{2,0,\sigma}(\Omega)}\|y_2(t)\|^4_{L_4(\Omega)}.\end{aligned}$$

Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{y}(t)\|^2_{H^1_{2,0,\sigma}(\Omega)} \le c\|\tilde{y}(t)\|^2_{H^1_{2,0,\sigma}(\Omega)}\|y_2(t)\|^4_{L_4(\Omega)}.$$

Applying Gronwall's inequality and the fact that $\tilde{y}(0) = 0$, we deduce that $\tilde{y} = 0$, proving the uniqueness of solutions. The last assertion follows from Theorem 3.7. □

Example 5.3. Additional conditions for θ and s in (H3) may be imposed for sources involving Radon measures. Denote the space of real and regular Borel measures in Ω by $\mathcal{M}(\Omega) := \mathcal{C}_0(\Omega)^*$, where $\mathcal{C}_0(\Omega)$ is the space of continuous functions on $\overline{\Omega}$ vanishing on the boundary Γ, endowed with the supremum norm. This duality identification follows from the classical Riesz–Radon theorem.

Consider the following spaces for the source function:

$$\begin{aligned}{}_TH^{\theta-1}_p(I;\mathcal{M}(\Omega)) &:= {}_TH^{1-\theta}_{p'}(I;\mathcal{C}_0(\Omega))^*,\\ \mathcal{M}(I;H^{s-2}_{q,0,\sigma}(\Omega)) &:= \mathcal{C}(\bar{I};H^{2-s}_{q',0,\sigma}(\Omega))^*,\\ \mathcal{M}(Q) &:= \mathcal{C}(\bar{I};\mathcal{C}_0(\Omega))^*.\end{aligned}$$

Note that we have the continuous embeddings ${}_TH^{1-\theta}_{p'}(I;\mathcal{C}_0(\Omega)) \hookrightarrow \mathcal{C}(\bar{I};\mathcal{C}_0(\Omega))$ and $H^{2-s}_{q',0,\sigma}(\Omega) \hookrightarrow \mathcal{C}_0(\Omega)$ when $0 \le \theta < \frac{1}{p}$ and $0 \le s < \frac{2}{q}$, respectively. Thus, by applying duality, $\mathcal{M}(Q) \hookrightarrow {}_TH^{\theta-1}_p(I;\mathcal{M}(\Omega))$ and $\mathcal{M}(\Omega) \hookrightarrow H^{s-2}_{q,0,\sigma}(\Omega)$ under the same restrictions on θ and s. Similarly, we have $\mathcal{M}(I;H^{s-2}_{q,0,\sigma}(\Omega)) \hookrightarrow {}_TH^{\theta-1}_p(I;H^{s-2}_{q,0,\sigma}(\Omega))$ when $0 \le \theta < \frac{1}{p}$. The implications of these observations are summarized in the following corollary, which follows directly from Theorem 5.2.

Corollary 5.4. *Assume one of* (H3) *and let* $y_0 \in \mathcal{H}^s_q(\Omega)$. *Consider one of the following cases:*

(i) $f \in {}_TH^{\theta-1}_p(I;\mathcal{M}(\Omega))$ *and* $0 \le s < \frac{2}{q}$,
(ii) $f \in \mathcal{M}(I;H^{s-2}_{q,0,\sigma}(\Omega))$ *and* $0 \le \theta < \frac{1}{p}$,
(iii) $f \in \mathcal{M}(Q)$, $0 \le s < \frac{2}{q}$, *and* $0 \le \theta < \frac{1}{p}$.

Then, there exists a unique very weak solution $y \in \mathcal{Y}^{\theta,s}_{p,q}(Q)$ *of* (4.1) *such that*

$$\|y\|_{\mathcal{Y}^{\theta,s}_{p,q}(Q)} \leq \eta\left(\|f\| + \|y_0\|_{\mathcal{H}^s_q(\Omega)}\right),$$

where the norm of f *is taken in the corresponding space, and* $\eta : [0,\infty) \to [0,\infty)$ *is a non-decreasing function with* $\eta(0) = 0$.

6. Sensitivity analysis of state equation in 2D

In this section, we show that the operator mapping the source function f to the solution y with a fixed initial condition y_0 is, in fact, of class $\mathcal{C}^\infty$. This is done using the implicit function theorem for Banach spaces.

Let us consider the framework of Theorem 5.2 throughout this section. Recall that the extended Stokes operator $A_{q,s/2} : H^s_{q,0,\sigma}(\Omega) \to H^{s-2}_{q,0,\sigma}(\Omega)$ is defined by

$$\langle A_{q,s/2}u, v\rangle_{H^{s-2}_{q,0,\sigma}(\Omega)\times H^{2-s}_{q',0,\sigma}(\Omega)} = \langle A^{s/2}_q u, A^{1-s/2}_{q'} v\rangle_{L_{q,\sigma}(\Omega)\times L_{q',\sigma}(\Omega)} \quad \forall v \in H^{2-s}_{q',0,\sigma}(\Omega)$$

and is linear and continuous for any $s \in [0,2]$. For $0 \leq \theta < \frac{1}{p}$, define the operator

$$\Lambda : \mathfrak{H}^{\theta,s}_{p,q}(Q) + {}_0\mathfrak{H}^{1,1}_{2,2}(Q) \to {}_T\mathfrak{H}^{\theta-1,s-2}_{p,q}(Q) + \mathfrak{H}^{0,-1}_{2,2}(Q)$$

according to

$$\Lambda y := \mathcal{P}^{\theta,s}_{p,q} y_1 + \mathcal{P}^{1,1}_{2,2} y_2,$$

for any decomposition $y = y_1 + y_2$, with $y_1 \in \mathfrak{H}^{\theta,s}_{p,q}(Q)$ and $y_2 \in {}_0\mathfrak{H}^{1,1}_{2,2}(Q)$. By density arguments, it can be shown that Λy is independent of the choice of representation of $y = y_1 + y_2$ and is also linear and continuous. Moreover, note that the mapping $\mathcal{Y}^{\theta,s}_{p,q}(Q) \times \mathcal{Y}^{\theta,s}_{p,q}(Q) \ni (u,u) \to \operatorname{div}(u \otimes u) \in {}_T\mathcal{F}^{\theta-1,s-2}_{p,q}(Q)$ is bilinear and continuous.

Let us define the mapping

$$\begin{aligned} &\mathcal{S} : [\mathfrak{H}^{\theta,s}_{p,q}(Q) + {}_0\mathfrak{H}^{1,1}_{2,2}(Q)] \times {}_T\mathcal{F}^{\theta-1,s-2}_{p,q}(Q) \to {}_T\mathcal{F}^{\theta-1,s-2}_{p,q}(Q), \\ &\mathcal{S}(y,f) := \Lambda y + \operatorname{div}(y\otimes y) - f - (\mathcal{I}_s + \kappa A_{q,s/2})\delta_0^* y_0 \end{aligned} \tag{6.1}$$

where $y_0 \in H^s_{q,0,\sigma}(\Omega)$ is given initial data. The observations provided in the previous paragraph imply that $\mathcal{S}$ is a $\mathcal{C}^\infty$-mapping.

Denote by $y_f \in \mathfrak{H}^{\theta,s}_{p,q}(Q) + {}_0\mathfrak{H}^{1,1}_{2,2}(Q)$ the solution of (4.1) for a given source function $f \in {}_T\mathcal{F}^{\theta-1,s-2}_{p,q}(Q)$ and initial data $y_0 \in H^s_{q,0,\sigma}(\Omega)$. Then, the derivative of $\mathcal{S}$ at y_f with respect to a direction $z \in \mathcal{Y}^{\theta,s}_{p,q}(Q)$, written as the mapping

$$\begin{aligned} &\partial_y\mathcal{S}(y_f,f) : {}_0\mathfrak{H}^{\theta,s}_{p,q}(Q) + {}_0\mathfrak{H}^{1,1}_{2,2}(Q) \to {}_T\mathcal{F}^{\theta-1,s-2}_{p,q}(Q) \\ &\partial_y\mathcal{S}(y_f,f)z := \Lambda z + \operatorname{div}(y_f\otimes z) + \operatorname{div}(z\otimes y_f) \end{aligned} \tag{6.2}$$

is linear and continuous. One can show that (6.2) is in fact an isomorphism by proving the existence and uniqueness of the solution z of

$$\begin{cases} \Lambda z + \operatorname{div}(y_f\otimes z) + \operatorname{div}(z\otimes y_f) = g, \\ \delta_0 z = 0, \end{cases} \tag{6.3}$$

as in the proof of Theorem 5.2. By the implicit function theorem, there exists a function $S : {}_T\mathcal{F}^{\theta-1,s-2}_{p,q}(Q) \to \mathfrak{H}^{\theta,s}_{p,q}(Q) + {}_0\mathfrak{H}^{1,1}_{2,2}(Q)$ of class $\mathcal{C}^\infty$ such that $\mathcal{S}(S(f),f) = 0$ for all $f \in {}_T\mathcal{F}^{\theta-1,s-2}_{p,q}(Q)$. By the uniqueness of solution of (4.1), we have $S(f) = y_f$.

Note that $z_g := S'(f)g$ satisfies (6.3) for $f, g \in {}_T\mathcal{F}_{p,q}^{\theta-1,s-2}(Q)$. Similar results hold for the case $\frac{1}{p} < \theta \leq 1$. We summarize our observations in the following theorem.

Theorem 6.1. *Assume that one of* (H3) *is satisfied and* $\theta \neq \frac{1}{p}$. *The nonlinear mapping*

$$S : {}_T\mathcal{F}_{p,q}^{\theta-1,s-2}(Q) \to \mathfrak{H}_{p,q}^{\theta,s}(Q) + {}_0\mathfrak{H}_{2,2}^{1,1}(Q)$$

that maps a force $f \in {}_T\mathcal{F}_{p,q}^{\theta-1,s-2}(Q)$ *to the corresponding very weak solution* $y \in \mathfrak{H}_{p,q}^{\theta,s}(Q) + {}_0\mathfrak{H}_{2,2}^{1,1}(Q)$ *of* (4.1) *with fixed initial data* $y_0 \in H_{q,0,\sigma}^s(\Omega)$ *is of class* $\mathcal{C}^\infty$. *Moreover, given* $f, g \in {}_T\mathcal{F}_{p,q}^{\theta-1,s-2}(Q)$, *the function* $z_g := S'(f)g \in {}_0\mathfrak{H}_{p,q}^{\theta,s}(Q) + {}_0\mathfrak{H}_{2,2}^{1,1}(Q)$ *is the unique solution of the linearized system*

$$\begin{cases} \partial_t z_g - \kappa \partial_t \Delta z_g - \nu \Delta z_g + \operatorname{div}(y_f \otimes z_g) + \operatorname{div}(z_g \otimes y_f) + \nabla \mathfrak{p}_g = g & \text{in } Q, \\ \operatorname{div} z_g = 0 \quad \text{in } Q, \qquad z_g = 0 \quad \text{on } \Sigma, \qquad z_g(0, \cdot) = 0 & \text{in } \Omega, \end{cases}$$

where $y_f = S(f)$.

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