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Optimal Control for Non-isothermal Binary Viscoelastic and Incompressible Flows with Stress Diffusion

OPTIMAL CONTROL FOR NON-ISOTHERMAL BINARY VISCOELASTIC AND INCOMPRESSIBLE FLOWS WITH STRESS DIFFUSION

GILBERT PERALTA

Abstract.
Distributed optimal control problems for a binary mixture of non-isothermal, incompressible, and non-Newtonian fluids under the framework of diffusive Johnson–Segalman models will be discussed. The flow is governed by a coupling of the two-dimensional Cahn–Hilliard equation for the order-parameter and chemical potential, the biharmonic heat equation with Voigt-type damping for the temperature, the incompressible Navier–Stokes equation for the mean velocity, and a Jeffreys-type differential constitutive equation for the viscoelastic stress tensor. The total Cauchy stress tensor for the model is given by the sum of the viscous stress, the contribution due to surface tension, and a quadratic function of the viscoelastic stress. The latter is based on a recent non-standard constitutive law for the Helmholtz free energy. The coefficients pertaining to diffusion processes depend on the concentration and temperature. We provide regularity results for the optimal control with various objective cost functionals. Such results rely on careful analysis of the corresponding linearized and adjoint problems. In particular, we study the strong, weak, and very weak solutions of the linearized and adjoint systems, and present the function spaces for the distributional time-derivatives.

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Cahn–Hilliard equation, Navier–Stokes equation, viscous Johnson–Segalman-type models, non-Newtonian flows, viscoelasticity, optimal control, weak and very weak solutions, regularity.

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Table of Contents

1. Introduction

This manuscript aims to provide an extensive analysis of optimal control problems for non-isothermal, incompressible, and viscoelastic multiphase flows. A typical example of such complex fluid flows is a binary fluid mixture consisting of a solvent and a diluted polymeric matter. The state variables of the model consist of the order-parameter for the volumetric fraction between the two concentrations, chemical potential, temperature, average velocity, pressure, and a viscoelastic stress tensor. We consider the case where the mobility, thermal conductivity, viscosity, and viscoelastic diffusion coefficients depend on the order-parameter and temperature. These assumptions are physically relevant since it has been observed that most viscoelastic rate-type fluids strongly depend on thermal effects (see, for instance, [53]).

The evolution of the concentration and velocity will be described using the Cahn–Hilliard equation with a regular potential and the Navier–Stokes equation. For the viscoelastic stress tensor, we will adapt a Jeffreys-type differential constitutive equation [54]. The complete model belongs to a class of non-isothermal Johnson–Segalman and PENE-Phan-Thien-Tanner/Peterlin-type models with an additional diffusion term in the dynamic equation for the viscoelastic stress tensor [44]. As the material derivative of the viscoelastic stress tensor is not objective, it must be rectified in terms of a commutator and an anti-commutator with the strain rate and vorticity tensors, respectively. The system allows for source terms, and in particular, mass conservation may not hold.

Regarding the temperature, we will use the convection-diffusion equation with biharmonic and Voigt-type regularizations. These provide a smoothing effect on the heat equation that is suitable for addressing the difficulty arising from the dependence of the diffusion coefficients on the temperature. Moreover, this approach will enable us to study the differentiability of the control-to-state operator. Further discussion on this matter will be provided below.

1.1. Model Formulation. We will now present the partial differential equations governing the binary fluid flow we are interested in. While we will not attempt to provide complete details of the derivation of the system, we will include relevant works from the current literature as references. The time interval will be denoted by $I := (0, T)$, where $T > 0$ is a fixed finite-time horizon. Let $\Omega \subset \mathbb{R}^2$ be a bounded and connected domain with a sufficiently smooth boundary Γ, and let $\boldsymbol{n}$ denote the unit vector that is outward normal to Γ. We set $\Omega_T := I \times \Omega$ for the time-space cylinder and $\Gamma_T := I \times \Gamma$ for its lateral boundary. We denote the order-parameter, chemical potential, relative temperature around some fixed value, velocity, pressure, and viscoelastic stress tensor by $\phi : \Omega_T \to \mathbb{R}$, $\mu : \Omega_T \to \mathbb{R}$, $\theta : \Omega_T \to \mathbb{R}$, $\boldsymbol{v} : \Omega_T \to \mathbb{R}^2$, $\widetilde{\mathrm{p}} : \Omega_T \to \mathbb{R}$, and $\mathbb{S} : \Omega_T \to \mathbb{R}^{2\times 2}_{\mathrm{s}}$ (the space of 2×2 symmetric matrices with real entries), respectively.

The evolution of the order-parameter ϕ can be obtained through the mass balance law:

$$\mathrm{D}_t\phi + \nabla \cdot \boldsymbol{j}_{\mathrm{m}} = f_{\mathrm{o}} \qquad \text{in } \Omega_T, \tag{1.1}$$

where $\boldsymbol{j}_{\mathrm{m}}$ is the mass flux and f_{o} is a concentration source or sink term. Here, $\mathrm{D}_t := \partial_t + \boldsymbol{v} \cdot \nabla$ denotes the material derivative. Using the Fickean law, the mass flux can be expressed as

$$\boldsymbol{j}_{\mathrm{m}} := -m(\phi, \theta)\nabla\mu, \tag{1.2}$$

where $m > 0$ is the diffusive mobility that depends on the order-parameter and temperature.

In the Cahn–Hilliard formulation [13, 14], the chemical potential is a differential with respect to the concentration of the following Ginzburg–Landau free energy functional:

$$G(\phi) = \int_\Omega \left(\frac{\alpha}{2}|\nabla\phi|^2 + F(\phi)\right) \mathrm{d}x.$$

The constant $\alpha > 0$ is related to the interfacial thickness, that is, the region where the two fluids mix. Also, the function F is commonly referred to as the Cahn–Hilliard potential. Here, we consider polynomial-like potentials (see Section 2.3). Thus, the chemical potential μ and the order-parameter ϕ are related via the semi-linear elliptic equation

$$\mu = \partial_\phi G(\phi) = -\alpha\Delta\phi + F'(\phi) \qquad \text{in } \Omega_T. \tag{1.3}$$

As we are in a temperature-dependent setting, a more realistic potential would be of the form $F(\phi, \theta)$, with singular and logarithmic terms dependent on both the order-parameter and temperature. However, singular potentials require more technical methods which are beyond the scope of the current manuscript. The techniques presented here can be extended to the case with bounded, temperature-dependent coefficients thanks to the Voigt and biharmonic regularizations that we impose for the heat equation. To simplify the presentation, we will focus on the case where the potential depends solely on ϕ (see also [11, 12] where the same simplification for the Ginzburg–Landau free energy was utilized). Nevertheless, the evolution of ϕ is influenced by θ through the diffusive mobility parameter m.

Concerning heat flux, we consider a higher-gradient extension of the classical Fourier law for heat conduction and a spatial relaxation or a Voigt-type damping as in [17]:

$$\mathrm{D}_t\theta - \tau\Delta\partial_t\theta + \nabla\cdot\boldsymbol{j}_\mathrm{h} = a_0\mathbf{g}\cdot\boldsymbol{v} + \mathbb{S} : \mathbb{D}\boldsymbol{v} + f_\mathrm{h} \qquad \text{in } \Omega_T, \tag{1.4}$$

where $\tau > 0$ is a constant, $\mathbf{g} \in \mathbb{R}^2$ is a (constant) gravitational force, and $a_0 \in \mathbb{R}$ is a constant expressing adiabatic heat effects obtained by linearization at some fixed temperature [46].

The terms on the right-hand side of (1.4) can be considered as either heat sources or sinks. The function f_h is an external heat source, and $\mathbb{S} : \mathbb{D}\boldsymbol{v}$ is the contribution due to elastic components, where $\mathbb{D}\boldsymbol{v} := \frac{1}{2}(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^\mathrm{t})$ is the symmetric part of the velocity gradient, with $^\mathrm{t}$ denoting transposition, called the strain rate or deformation tensor. Here, we assume that the traction on the boundary is only due to the normal component of the linear part of the viscoelastic stress tensor, see for instance [26] and [67]. This assumption neglects the viscous part and the capillary effects due to surface tension in the Cauchy stress tensor. This simplification enables us to apply a Hilbert space framework for the biharmonic-like heat equation and obtain the corresponding total energy identity, which holds globally in time (see Remark 3.2). If we were to take into account the complete stress tensor, then we would need to deal with local-in-time solutions.

For the heat flux, we take into account the effects of the gradient of the temperature curvature profile that diminishes the heat gradient evacuation. More precisely, we assume that the heat flux is given by

$$\boldsymbol{j}_\mathrm{h} = b\nabla\Delta\theta - \chi(\phi, \theta)\nabla\theta, \tag{1.5}$$

where $b > 0$ is the spatial retardation coefficient and χ is the thermal conductivity. In addition, we assume that the effects of the order-parameter on the first term are small so that we can take b to be a constant parameter. This formulation of the heat flux leads to an additional fourth-order term in the heat equation. For further details, in particular the justification via asymptotic analysis, we refer to [17]. Hence, the governing equation for the temperature can be viewed as a convection-diffusion-biharmonic equation.

As the mobility, thermal conductivity, viscosity, and viscoelastic stress diffusion parameters depend not only on the order-parameter but also on the temperature, the biharmonic term leads to better smoothness of the temperature, and this extra regularity plays a crucial role in the analysis of the optimal control problems. In this way, the temperature enjoys the same regularity as the order-parameter at the level of basic energy estimates.

If the parameters pertaining to diffusion are independent on θ, then the methods presented here are applicable even if $b = \tau = 0$; however, the regularity of θ would be of the same type as those with $\boldsymbol{v}$ and $\mathbb{S}$. The Voigt damping was also incorporated as a regularization for the optimal control of the three-dimensional Navier–Stokes equation in [4, 5]. Recently, Voigt-type regularizations for the pressure has been considered in [61]. For optimal control problems of viscoelastic semi-compressible flows but without the Voigt and bi-Laplace regularizations, we refer to [62].

For the momentum equation, we consider the following extension of the Navier–Stokes equation:

$$\mathrm{D}_t\boldsymbol{v} - \nabla\cdot\mathbb{T}_{\mathrm{s}} = \rho(\theta,\phi)\mathbf{g} + \boldsymbol{f}_{\mathrm{v}} + \boldsymbol{u}, \quad \nabla\cdot\boldsymbol{v} = 0 \qquad \text{in } \Omega_T. \tag{1.6}$$

Here, $\boldsymbol{f}_{\mathrm{v}}$ and $\boldsymbol{u}$ are the external body forces and control, respectively, and ρ as the equation of state. Avoiding technical difficulties, we will assume that the equation of state ρ is linear, see hypothesis $(\mathrm{A2})_s$ in Section 2.3. We followed the simplification discussed in [45, Section 54], where *density variations* are incorporated as body-forces for the momentum equation and are induced by temperature and concentration differences, but not by pressure.

The total Cauchy stress tensor consists of three parts

$$\mathbb{T}_{\mathrm{s}} := \mathbb{V}(\phi,\theta,\boldsymbol{v},\widetilde{\mathrm{p}}) + \mathbb{K}(\phi) + \mathbb{M}(\theta,\mathbb{S}). \tag{1.7}$$

These parts correspond to the viscous stress, the stress due to surface tension, and the contribution of viscoelastic stress, where

$$\mathbb{V}(\phi,\theta,\boldsymbol{v},\widetilde{\mathrm{p}}) := 2\nu(\phi,\theta)\mathbb{D}\boldsymbol{v} - \widetilde{\mathrm{p}}\mathbb{I} \tag{1.8}$$

$$\mathbb{K}(\phi) := \kappa\alpha\left(\frac{1}{2}|\nabla\phi|^2\mathbb{I} - \nabla\phi\otimes\nabla\phi\right) \tag{1.9}$$

$$\mathbb{M}(\theta,\mathbb{S}) := \sigma_0 a(\mathbb{S}^2 - \mathbb{S}) + \sigma\theta\mathbb{S} + a\mathrm{Tr}(\mathbb{S})\mathbb{S}, \tag{1.10}$$

with ν the kinematic viscosity, κ the capillary coefficient, $\mathbb{I}$ the identity tensor, a the rheological parameter, and Tr the trace operator. The stress tensor $\nabla\phi\otimes\nabla\phi$ represents the capillary forces due to surface tension on the interface between the fluid phases [49]. Also, σ_0 and σ are constants related to the nature of viscoelasticity, namely, the specific heat of phase transition and shear modulus related to relaxation mechanisms, respectively [60].

The quadratic term in $\mathbb{M}$ was obtained by applying a thermodynamic approach and resorting to a modified energy storage mechanism. With the *free energy* $\Psi(\theta,\mathbb{S}) := \sigma_0 a(\frac{1}{2}\mathbb{S}^2 - \mathbb{S}) + \sigma\theta\mathbb{S} + \frac{1}{2}a\mathrm{Tr}(\mathbb{S}^2)\mathbb{I}$, we can write $\mathbb{M}(\theta,\mathbb{S}) = \mathbb{S}\partial_{\mathbb{S}}\Psi(\theta,\mathbb{S})$. As source terms for the Navier–Stokes equation, $\nabla\cdot(\sigma_0 a(\mathbb{S}^2-\mathbb{S}))$ and $\nabla\cdot(\sigma\theta\mathbb{S})$ appeared in [7] and [8], respectively. Likewise, for $a = 1$, the term $\nabla\cdot(\mathrm{Tr}(\mathbb{S})\mathbb{S})$

appears in Peterlin-type models, see [11, 12] and the references therein. In the derivation of a priori estimates, it is crucial that the sign of the coefficient involving the last term is the same as the rheological constant. For a complete and more precise discussion in the case where $\sigma = 0$, we refer the reader to [8] in the isothermal case and to [7, 50] in the non-isothermal setting. The term $\sigma\theta\mathbb{S}$ was added to compensate the term $\mathbb{S} : \mathbb{D}\boldsymbol{v}$ in the heat equation. Such a bilinear term can be realized as the effect of the change in the temperature to the viscoelastic stress.

Finally, for the evolution of the viscoelastic stress, we consider a FENE-PTT-type (finite-extensible-nonlinear-elastic-Phan-Thien-Tanner) model as follows (we refer the reader to [18, 39, 28, 55, 59] and the bibliographies therein for other relevant references):

$$\mathrm{D}_{t,a}\mathbb{S} - \nabla\cdot\mathfrak{T}_{\mathrm{d}} = \lambda\mathbb{D}\boldsymbol{v} - \ell\mathbb{S} + \beta\mathrm{Tr}(\mathbb{S})(\mathbb{I} - \mathrm{Tr}(\mathbb{S})\mathbb{S}) + \mathbb{F}_{\mathrm{s}} \qquad \text{in } \Omega_T, \tag{1.11}$$

with $\lambda \sim 1/\mathrm{We}$, $\ell \sim 2(1-\mathrm{r})/(\mathrm{WeRe})$, and $\beta \sim 1/\mathrm{De}$, where $\sim$ means direct proportionality, De, Re, and We correspond to the Deborah, Reynolds, and Weissenberg numbers, and r is the ratio between the relaxation and retardation times. The operator $\mathrm{D}_{t,a} := \mathrm{D}_t + [\cdot, \mathbb{W}\boldsymbol{v}] - a\{\cdot, \mathbb{D}\boldsymbol{v}\}$ is the objective time derivative. The operators $[\cdot,\cdot]$ and $\{\cdot,\cdot\}$ act as the commutator and anti-commutator of two tensors $\mathbb{S}_1$ and $\mathbb{S}_2$, and are respectively defined as follows:

$$[\mathbb{S}_1, \mathbb{S}_2] := \mathbb{S}_1\mathbb{S}_2 - \mathbb{S}_2\mathbb{S}_1, \qquad \{\mathbb{S}_1, \mathbb{S}_2\} := \mathbb{S}_1\mathbb{S}_2 + \mathbb{S}_2\mathbb{S}_1.$$

Also, $\mathbb{W}\boldsymbol{v} := \frac{1}{2}(\nabla\boldsymbol{v} - (\nabla\boldsymbol{v})^{\mathrm{t}})$ denotes the anti-symmetric part of the velocity gradient, called the vorticity tensor. Thus, the full expression of the invariant-time derivative operator is

$$\mathrm{D}_{t,a}\mathbb{S} := \partial_t\mathbb{S} + (\boldsymbol{v}\cdot\nabla)\mathbb{S} + \mathbb{S}\mathbb{W}\boldsymbol{v} - \mathbb{W}\boldsymbol{v}\mathbb{S} - a(\mathbb{S}\mathbb{D}\boldsymbol{v} + \mathbb{D}\boldsymbol{v}\mathbb{S}).$$

For $a \in [-1, 1]$, $\mathrm{D}_{t,a}$ is called a Gordon–Schowalter derivative [38]. Particular cases are the corotational Jaumann–Zaremba derivative if $a = 0$ [43, 71] and the upper-convected Oldroyd derivative if $a = 1$ [54].

The tensor-valued function $\mathbb{F}_{\mathrm{s}}$ in (1.11) can be thought of as an external source term [34]. We shall take $\mathfrak{T}_{\mathrm{d}} := \varepsilon(\phi, \theta)\nabla\mathbb{S}$, giving us an additional diffusion term in the evolution governed by the viscoelastic stress tensor. The inclusion of a diffusion term was utilized for instance in [8, 19, 25, 28, 60, 70] for Oldroyd-B-type models and in [6, 11, 12] for Peterlin–Navier–Stokes systems. In the absence of the diffusion term, the evolution of the viscoelastic stress will be hyperbolic, see [48, 51] for non-isothermal single-phase viscoelastic fluids, and consequently, the regularizing effect of diffusion is not available and different tools are needed in the analysis of such systems. Note that if the initial viscoelastic stress tensor $\mathbb{S}(0)$ is symmetric and $\mathbb{F}_{\mathrm{s}}$ is symmetric-valued in I, then it follows from (1.11) that $\mathbb{S}(t)$ is symmetric for every $t \in I$. Thus, we obtain that $\mathbb{T}_{\mathrm{s}}^{\mathrm{t}} = \mathbb{T}_{\mathrm{s}}$, that is, the total stress tensor is symmetric [35].

Based on the following equation

$$\begin{aligned}\kappa\mu\nabla\phi &= \kappa\nabla\cdot\left(\frac{\alpha}{2}|\nabla\phi|^2\mathbb{I} + F(\phi)\mathbb{I}\right) - \kappa\alpha\nabla\cdot(\nabla\phi\otimes\nabla\phi)\\ &= \nabla\cdot\mathbb{K}(\phi) + \kappa\nabla F(\phi),\end{aligned} \tag{1.12}$$

we introduce a new *pressure* $\mathrm{p} := \widetilde{\mathrm{p}} + \kappa\nabla F(\phi)$. Furthermore, we introduce the following abbreviations:

$$\mathbb{J}(\boldsymbol{v}, \mathbb{S}) := [\mathbb{S}, \mathbb{W}\boldsymbol{v}] - a\{\mathbb{S}, \mathbb{D}\boldsymbol{v}\} \tag{1.13}$$

$$\mathbb{P}(\mathbb{S}) := -\ell\mathbb{S} + \beta\mathrm{Tr}(\mathbb{S})(\mathbb{I} - \mathrm{Tr}(\mathbb{S})\mathbb{S}). \tag{1.14}$$

The equations (1.1)–(1.14) lead us to the following coupled non-isothermal Cahn–Hilliard–Navier–Stokes and *generalized* diffusive Johnson–Segalman-type systems:

$$\left[\begin{array}{ll} \partial_t\phi + \boldsymbol{v}\cdot\nabla\phi - \nabla\cdot(m(\phi,\theta)\nabla\mu) = f_{\mathrm{o}} & \text{in } \Omega_T,\\ \mu = -\alpha\Delta\phi + F'(\phi) & \text{in } \Omega_T,\\ \partial_t(\theta - \tau\Delta\theta) + \boldsymbol{v}\cdot\nabla\theta - \nabla\cdot(\chi(\phi,\theta)\nabla\theta) + b\Delta^2\theta & \\ \qquad = a_0\mathbf{g}\cdot\boldsymbol{v} + \mathbb{S} : \mathbb{D}\boldsymbol{v} + f_{\mathrm{h}} & \text{in } \Omega_T,\\ \partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nabla\cdot(2\nu(\phi,\theta)\mathbb{D}\boldsymbol{v}) + \nabla\mathrm{p} & \\ \qquad = \nabla\cdot\mathbb{M}(\theta,\mathbb{S}) + \kappa\mu\nabla\phi + \rho(\phi,\theta)\mathbf{g} + \boldsymbol{f}_{\mathrm{v}} + \boldsymbol{u} & \text{in } \Omega_T,\\ \partial_t\mathbb{S} + (\boldsymbol{v}\cdot\nabla)\mathbb{S} + \mathbb{J}(\boldsymbol{v},\mathbb{S}) - \nabla\cdot(\varepsilon(\phi,\theta)\nabla\mathbb{S}) = \lambda\mathbb{D}\boldsymbol{v} + \mathbb{P}(\mathbb{S}) + \mathbb{F}_{\mathrm{s}} & \text{in } \Omega_T,\\ \nabla\cdot\boldsymbol{v} = 0 & \text{in } \Omega_T,\\ \partial_{\boldsymbol{n}}\phi = \partial_{\boldsymbol{n}}\mu = 0,\ \partial_{\boldsymbol{n}}\theta = \partial_{\boldsymbol{n}}\Delta\theta = 0,\ \boldsymbol{v} = \mathbf{0},\ \partial_{\boldsymbol{n}}\mathbb{S} = \mathbb{O} & \text{on } \Gamma_T,\\ \phi(0) = \phi_0,\ \theta(0) = \theta_0,\ \boldsymbol{v}(0) = \boldsymbol{v}_0,\ \mathbb{S}(0) = \mathbb{S}_0 & \text{in } \Omega. \end{array}\right. \tag{1.15}$$

Here, $\partial_{\boldsymbol{n}} = \boldsymbol{n}\cdot\nabla$ is the directional derivative normal to the boundary and $\mathbb{O}$ is the zero element in $\mathbb{R}^{2\times 2}_{\mathrm{s}}$. We mention that the lack of a Lyapunov structure and the presence of source terms in the energy identity provided in Remark 3.2 indicate some difficulties in the analysis of (1.15).

For the boundary conditions, we assume no-concentration flux, zero concentration-diffusion flux, zero heat-flux, zero diffusion-flux for the temperature, no-slip condition for the velocity, and a homogeneous Neumann condition for the viscoelastic stress. The latter is considered here for the sake of simplicity. Take note that for sufficiently smooth solutions, the boundary conditions $\partial_{\boldsymbol{n}}\phi = \partial_{\boldsymbol{n}}\mu = 0$ on Γ_T are equivalent to $\partial_{\boldsymbol{n}}\phi = \partial_{\boldsymbol{n}}\Delta\phi = 0$ on Γ_T. Indeed, this follows from the equation $\partial_{\boldsymbol{n}}\mu = -\alpha\partial_{\boldsymbol{n}}\Delta\phi + F''(\phi)\partial_{\boldsymbol{n}}\phi$ obtained from the second equation in (1.15). For the initial data, $\phi_0, \theta_0 : \Omega \to \mathbb{R}$, $\boldsymbol{v}_0 : \Omega \to \mathbb{R}^2$, and $\mathbb{S}_0 : \Omega \to \mathbb{R}^{2\times 2}_{\mathrm{s}}$ denote the initial order-parameter, temperature, velocity, and viscoelastic stress tensor, respectively.

If we set $\mathbb{S} = \mathbb{O}$ and the temperature to constant, then we obtain the classical Cahn–Hilliard–Navier–Stokes system [16, 40, 49], also known as Model H in dynamic critical phenomena [42]. If $-1 \leq a \leq 1$, then one has diffusive variants of the Johnson–Segalman models. In particular, for $a = 1$ and without the quadratic term $\mathbb{S}^2$, we obtain a diffusive version of the classical Oldroyd-B and Peterlin models. It was observed in [60] that the case $a = 1$ has good physical properties for the viscoelastic tensor under a thermodynamical framework. Nonetheless, we point out that the analysis presented in this work holds for any $a \in \mathbb{R}$ due to the quadratic dependence of $\mathbb{M}$ on $\mathbb{S}$. Also, the results of this paper can be easily adjusted to the aforementioned simpler models.

1.2. Statement of the Optimal Control Problem. We are interested in non-convex optimal control problems of the form

$$\min_{\boldsymbol{u}\in L^2(I;\boldsymbol{L}^2(\Omega))} J_{\mathrm{o}}(\phi) + J_{\mathrm{c}}(\mu) + J_{\mathrm{h}}(\theta) + J_{\mathrm{v}}(\boldsymbol{v}) + J_{\mathrm{s}}(\mathbb{S}) + \frac{\lambda_q}{2}\int_{\Omega_T} |\boldsymbol{u}|^2 \,\mathrm{d}x\,\mathrm{d}t \quad \text{subject to (1.15)}, \tag{1.16}$$

where $\lambda_q > 0$ is a Tikhonov regularization parameter. The precise definitions of each component of the tracking part and specific examples will be formulated in Section 6. The control $\boldsymbol{u}$ can be realized as the result of applying a mechanical stirring device on the binary viscous fluid.

Throughout this paper, the roman subscripts o, c, h, v, and s signify that the cost or source functions pertain to the order-parameter, chemical potential, heat, velocity, and viscoelastic stress, respectively. Though the control appears only in the equation governing the velocity, it is possible to include controls in the order-parameter and temperature through modification of the fluid composition or application of heat or cooling treatment as well (see Remark 6.2). Such control problems are also interesting, for instance, in the context of glass ceramic production (see the introduction of [58] on this matter).

There are several works dealing with optimal control problems for multiphase flows and phase-field type equations, see for example [20, 22, 23, 24, 30, 31, 32, 36]. Some of these include dynamic boundary conditions and singular potentials as well. For recent works on Cahn–Hilliard systems with source terms, we refer the reader to [21, 29, 47, 52] and the references therein. These are only selected lists and we refer to the references provided in these papers for additional related works. Order-parameter-dependent mobility for the Cahn–Hilliard–Navier–Stokes system was considered in [31], for which strong solutions were utilized in the analysis. For the Boussinesq system with temperature-dependent viscosity, we refer to [9]. The results of the latter paper were local-in-time and based on strong solutions as well. The biharmonic and Voigt regularizations in (1.15) of the heat equation allow us to pass from local to global results.

The theoretical frameworks used in [41, 69] for the Navier–Stokes equation and in [56] for the Cahn–Hilliard–Oberbeck–Boussinesq system will be employed here. In particular, we shall use the standard spectral Faedo–Galerkin method for the well-posedness of the nonlinear and linearized systems and the method of transposition for the analysis of the adjoint system. As we are dealing with the dynamics of complex binary fluids, where the governing partial differential equations are strongly coupled, it is expected that the computations here are more involved. In light of the state dependence of the coefficient functions related to diffusion, the associated evolution equations in the dual problem for the order-parameter and temperature involve gradient terms with coefficients that are gradients of the state variables. Such terms already appeared in the papers [9, 31] since these works considered state-dependent viscosities. The appropriate choice of various Sobolev inequalities plays a crucial step in the estimation of such bilinear terms.

To study the differentiability of the nonlinear operator that maps the controls to the solutions of the state system, we need strong solutions to (1.15) as in [1, 9, 31]. On one hand, this framework ensures that solutions to the state equation are unique, and on the other hand, the regularity of solutions allows a very weak formulation of the linearized system. Unlike [9, 31], we follow the ideas in [15] by using the results for the linearized system and the implicit function theorem in showing the differentiability of the control-to-state operator. The machinery provided here on the linearized and adjoint systems is more elaborate and, after appropriate adjustments, may be adapted to other optimal control problems of instationary semi-linear parabolic PDEs. We shall also exploit the results for the linearized system to prove existence of strong solutions for the nonlinear system (1.15) with improved time-regularity under minimal assumptions on the source terms and initial data (see Theorem A.4). We will utilize these results in Section 6 to improve the regularity of the optimal control through a bootstrapping argument.

Strong solutions for the state system may also yield strong solutions to the linearized system, and by a duality principle, the adjoint variables will have less regularity. Such situations arise when the objective functional involves high derivatives for the concentration (e.g. control of diffusion), a cost that involves the chemical potential at the terminal time, time-derivatives of the states, or limited regularity of the desired states. These may result in less regularity for the time-derivative of the component dual to the order-parameter and chemical potential. Although not considered here as it falls outside the scope of the current manuscript, it is also interesting to study objective functionals that involve the material derivatives.

Let us emphasize the limitations of this study. While the nonlinear system (1.15) may look somewhat complicated, it remains a "toy-model" for the optimal control of non-isothermal viscoelastic multi-phase flows. Logarithmic and singular potentials for the Cahn–Hilliard system were avoided in favor of polynomial-like potentials for more tractable analysis. However, it is important to note that that singular potentials were already been explored for Cahn–Hilliard–Navier–Stokes systems in [30, 31]. The heat equation was modified by including higher-order gradients. This modification aims to obtain global-in-time solutions; however, such solutions may not be physically realistic due to temperature discontinuities. We also deliberately chose a two-dimensional setting as three-dimensional (3D) problems are significantly more challenging. Due to the quadratic term for the viscoelastic stress in the Helmholtz free energy, the existence of weak solutions for the 3D case might be achievable using techniques from [8]. However, the complete verification is left for the interested reader. Lastly, questions regarding the existence of less regular solutions to (1.15), similar to those for the linearized system, are also compelling. In this direction, we refer the reader to [15, 57] where semigroup theory, interpolation methods, and maximal parabolic regularity were employed. Whether such approaches can be directly applied to (1.15) is an open problem to the best knowledge of the author.

Let us briefly lay down the plan for this paper. The notation for the function spaces and frequently used estimates are presented in Section 2. The well-posedness of the non-linear system (1.15) and the second-order differentiability of the control-to-state operator are the focus of Sections 3 and 4, respectively. The time-regularity of the solutions to the dual system obtained via transposition method will be discussed in Section 5. Applications to optimal control problems will be considered in Section 6. Finally, the complete details on the existence and uniqueness results for various solution concepts of the linearized system will be provided in Appendix A.

Further details on the goals and methods will be stated at the beginning of each sections.

2. Preliminaries and Notations

2.1. Inner Products and Differential Operators. Scalar, vector, and tensor quantities will be written in lowercase (a), boldface ($\boldsymbol{a}$), and blackboard bold ($\mathbb{A}$) fonts, respectively. We shall write vectors in a column format. Components of vectors and tensors will be in lowercase so that

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \qquad \mathbb{S} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}.$$

We denote by $\boldsymbol{v} \cdot \boldsymbol{w} := v_1 w_1 + v_2 w_2$, $\mathbb{S} : \mathbb{T} := \boldsymbol{s}_1 \cdot \boldsymbol{t}_1 + \boldsymbol{s}_2 \cdot \boldsymbol{t}_2$, and $\mathfrak{S} \therefore \mathfrak{T} := \mathbb{S}_1 : \mathbb{T}_1 + \mathbb{S}_2 : \mathbb{T}_2$ the (Frobenius) inner products of vectors $\boldsymbol{v} = [v_1\ v_2]^{\mathrm{t}}, \boldsymbol{w} = [w_1\ w_2]^{\mathrm{t}} \in \mathbb{R}^2$, tensors $\mathbb{S} = [\boldsymbol{s}_1\ \boldsymbol{s}_2], \mathbb{T} = [\boldsymbol{t}_1\ \boldsymbol{t}_2] \in \mathbb{R}^{2\times 2}$, and block-tensors $\mathfrak{S} = [\mathbb{S}_1\ \mathbb{S}_2]^{\mathrm{t}}, \mathfrak{T} = [\mathbb{T}_1\ \mathbb{T}_2]^{\mathrm{t}} \in \mathbb{R}^{4\times 2}$. Note that $\mathbb{S} : \mathbb{T} = \mathrm{Tr}(\mathbb{S}^{\mathrm{t}}\mathbb{T}) = \mathrm{Tr}(\mathbb{T}^{\mathrm{t}}\mathbb{S})$ and

$$\mathbb{S} : \mathbb{T} = \mathbb{T}^{\mathrm{t}} : \mathbb{S}^{\mathrm{t}} = \mathbb{S}^{\mathrm{t}} : \mathbb{T}^{\mathrm{t}} = \mathbb{T} : \mathbb{S}. \tag{2.1}$$

For the convenience of the reader, let us briefly recall the basic differential operators used in this paper. Consider smooth enough scalar-valued, vector-valued, and tensor-valued functions $\phi : \Omega \to \mathbb{R}$, $\boldsymbol{v} : \Omega \to \mathbb{R}^2$, and $\mathbb{S} : \Omega \to \mathbb{R}^{2\times 2}$, respectively. Then, the gradient fields $\nabla\phi : \Omega \to \mathbb{R}^2$, $\nabla\boldsymbol{v} : \Omega \to \mathbb{R}^{2\times 2}$, and $\nabla\mathbb{S} : \Omega \to \mathbb{R}^{4\times 2}$ are given by

$$\nabla\phi := \begin{bmatrix} \partial_1\phi \\ \partial_2\phi \end{bmatrix}, \quad \nabla\boldsymbol{v} := \begin{bmatrix} \partial_1 v_1 & \partial_1 v_2 \\ \partial_2 v_1 & \partial_2 v_2 \end{bmatrix},$$
$$\nabla\mathbb{S} := \begin{bmatrix} \partial_1\mathbb{S} \\ \partial_2\mathbb{S} \end{bmatrix}, \quad \partial_k\mathbb{S} := \begin{bmatrix} \partial_k s_{11} & \partial_k s_{12} \\ \partial_k s_{21} & \partial_k s_{22} \end{bmatrix}, \ k = 1, 2.$$

Given a vector field $\boldsymbol{w} : \Omega \to \mathbb{R}^2$, the actions of the convective derivative operator $\boldsymbol{w} \cdot \nabla := w_1\partial_1 + w_2\partial_2$ to a differentiable scalar, vector, and tensor valued functions are component-wise:

$$\boldsymbol{w} \cdot \nabla\phi := w_1\partial_1\phi + w_2\partial_2\phi, \quad (\boldsymbol{w} \cdot \nabla)\boldsymbol{v} := \begin{bmatrix} \boldsymbol{w} \cdot \nabla v_1 \\ \boldsymbol{w} \cdot \nabla v_2 \end{bmatrix},$$
$$(\boldsymbol{w} \cdot \nabla)\mathbb{S} := \begin{bmatrix} \boldsymbol{w} \cdot \nabla s_{11} & \boldsymbol{w} \cdot \nabla s_{12} \\ \boldsymbol{w} \cdot \nabla s_{21} & \boldsymbol{w} \cdot \nabla s_{22} \end{bmatrix}.$$

In particular, we have $(\boldsymbol{w} \cdot \nabla)\boldsymbol{v} = (\nabla\boldsymbol{v})^{\mathrm{t}}\boldsymbol{w}$. The divergence operators are defined by

$$\nabla \cdot \boldsymbol{v} = \partial_1 v_1 + \partial_2 v_2, \quad \nabla \cdot \mathbb{S} = \begin{bmatrix} \nabla \cdot \boldsymbol{s}_1 \\ \nabla \cdot \boldsymbol{s}_2 \end{bmatrix}, \quad \nabla \cdot \mathfrak{S} := \partial_1\mathbb{S}_1 + \partial_2\mathbb{S}_2,$$

where $\boldsymbol{v} = [v_1\ v_2]^{\mathrm{t}} : \Omega \to \mathbb{R}^2$ with $v_1, v_2 : \Omega \to \mathbb{R}$, $\mathbb{S} = [\boldsymbol{s}_1\ \boldsymbol{s}_2] : \Omega \to \mathbb{R}^{2\times 2}$ with $\boldsymbol{s}_1, \boldsymbol{s}_2 : \Omega \to \mathbb{R}^2$, and $\mathfrak{S} = [\mathbb{S}_1\ \mathbb{S}_2]^{\mathrm{t}} : \Omega \to \mathbb{R}^{4\times 2}$ with $\mathbb{S}_1, \mathbb{S}_2 : \Omega \to \mathbb{R}^{2\times 2}$. Finally, the Laplace operator given by $\Delta := \nabla \cdot \nabla = \partial_1^2 + \partial_2^2$ acts component-wise.

2.2. Function Spaces. The standard notation for the Lebesgue spaces and Sobolev spaces will be followed, namely, $L^p(\Omega)$ and $W^{s,p}(\Omega)$ for $s \geq 0$ and $1 \leq p \leq \infty$. The subspace of $W^{s,p}(\Omega)$ whose elements vanish on Γ in the sense of traces will be denoted by $W_0^{s,p}(\Omega)$, see [2] for details. Let $\langle \phi \rangle := \frac{1}{|\Omega|}\int_\Omega \phi \, \mathrm{d}x$ be the average of ϕ over Ω and set $L^p(\Omega)/\mathbb{R} := \{\phi \in L^p(\Omega) : \langle \phi \rangle = 0\}$. Likewise, we set $W^{s,p}(\Omega)/\mathbb{R} := W^{s,p}(\Omega) \cap (L^p(\Omega)/\mathbb{R})$ for $s > 0$. The dual spaces will be denoted by a negative superscript, for instance, $W^{-s,p}(\Omega) := W^{s,p/(p-1)}(\Omega)^*$ and $W_0^{-s,p}(\Omega) := W_0^{s,p/(p-1)}(\Omega)^*$ for $1 < p < \infty$. Such notation will be adapted to the function spaces discussed below.

With regard to function spaces with vanishing normal derivatives, we consider $W_{\boldsymbol{n}}^{2,2}(\Omega) := \{\phi \in W^{2,2}(\Omega) : \partial_{\boldsymbol{n}}\phi = 0 \text{ on } \Gamma\}$, $W_{\boldsymbol{n}}^{3,2}(\Omega) := \{\phi \in W^{3,2}(\Omega) : \partial_{\boldsymbol{n}}\phi = 0 \text{ on } \Gamma\}$, and $W_{\boldsymbol{n}}^{4,2}(\Omega) := \{\phi \in W^{4,2}(\Omega) : \partial_{\boldsymbol{n}}\phi = \partial_{\boldsymbol{n}}\Delta\phi = 0 \text{ on } \Gamma\}$. These are the function spaces pertaining to the order-parameter and temperature. Recall that the above are Hilbert spaces when equipped with the inner products induced by the following respective norms:

$$
\begin{aligned}
\|\phi\|_{W_{\boldsymbol{n}}^{2,2}} &:= (\|\phi\|_{L^2}^2 + \|\Delta\phi\|_{L^2}^2)^{\frac{1}{2}},\\
\|\phi\|_{W_{\boldsymbol{n}}^{3,2}} &:= (\|\phi\|_{L^2}^2 + \|\nabla\Delta\phi\|_{\boldsymbol{L}^2}^2)^{\frac{1}{2}},\\
\|\phi\|_{W_{\boldsymbol{n}}^{4,2}} &:= (\|\phi\|_{L^2}^2 + \|\Delta^2\phi\|_{L^2}^2)^{\frac{1}{2}}.
\end{aligned}
$$

Let $\boldsymbol{L}^p(\Omega) := L^p(\Omega)^2$, $\boldsymbol{W}^{s,p}(\Omega) := W^{s,p}(\Omega)^2$, and $\boldsymbol{W}_0^{s,p}(\Omega) := W_0^{s,p}(\Omega)^2$. Concerning velocity, we take the classical spaces of solenoidal or divergence-free vector fields $\boldsymbol{L}_\sigma^2(\Omega) := \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega) : \nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}$, $\boldsymbol{W}_{0,\sigma}^{1,2}(\Omega) := \boldsymbol{W}_0^{1,2}(\Omega) \cap \boldsymbol{L}_\sigma^2(\Omega)$, and $\boldsymbol{W}_{0,\sigma}^{2,2}(\Omega) := \boldsymbol{W}^{2,2}(\Omega) \cap \boldsymbol{W}_{0,\sigma}^{1,2}(\Omega)$. These are Hilbert spaces with the following norms:

$$
\|\boldsymbol{u}\|_{\boldsymbol{L}_\sigma^2} := \|\boldsymbol{u}\|_{\boldsymbol{L}^2}, \quad \|\boldsymbol{u}\|_{\boldsymbol{W}_{0,\sigma}^{1,2}} := \|\nabla\boldsymbol{u}\|_{(\boldsymbol{L}^2)^2}, \quad \|\boldsymbol{u}\|_{\boldsymbol{W}_{0,\sigma}^{2,2}} := \|\Delta\boldsymbol{u}\|_{\boldsymbol{L}^2}.
$$

For tensor-valued Lebesgue spaces, we let $\mathbb{L}_{\mathrm{s}}^p(\Omega)$, $\mathbb{W}_{\mathrm{s}}^{1,p}(\Omega)$, and $\mathbb{W}_{\boldsymbol{n},\mathrm{s}}^{2,2}(\Omega)$ be the subspaces consisting of all symmetric elements of $\mathbb{L}^p(\Omega) := \boldsymbol{L}^p(\Omega)^2$, $\mathbb{W}^{1,p}(\Omega) := \boldsymbol{W}^{1,p}(\Omega)^2$, and $\mathbb{W}_{\boldsymbol{n}}^{2,2}(\Omega) := \boldsymbol{W}_{\boldsymbol{n}}^{2,2}(\Omega)^2$, respectively. These will be the function spaces for the viscoelastic stress tensor.

In the derivation of a priori estimates, the following list of standard inequalities will be utilized throughout the manuscript:

$$
\begin{aligned}
\|\varphi - \langle\varphi\rangle\|_{L^2} &\leq c\|\nabla\varphi\|_{\boldsymbol{L}^2} && \forall \varphi \in W^{1,2}(\Omega), && (2.2)\\
\|\nabla\varphi\|_{\boldsymbol{W}^{1,2}} &\leq c\|\Delta\varphi\|_{L^2} && \forall \varphi \in W_{\boldsymbol{n}}^{2,2}(\Omega), && (2.3)\\
\|\varphi\|_{W^{2,2}} &\leq c(\|\varphi\|_{L^2} + \|\Delta\varphi\|_{L^2}) && \forall \varphi \in W_{\boldsymbol{n}}^{2,2}(\Omega), && (2.4)\\
\|\Delta\varphi\|_{W^{1,2}} &\leq c\|\nabla\Delta\varphi\|_{\boldsymbol{L}^2} && \forall \varphi \in W_{\boldsymbol{n}}^{3,2}(\Omega), && (2.5)\\
\|\Delta\varphi\|_{W^{2,2}} &\leq c\|\Delta^2\varphi\|_{L^2} && \forall \varphi \in W_{\boldsymbol{n}}^{4,2}(\Omega), && (2.6)\\
\|\boldsymbol{y}\|_{\boldsymbol{L}_\sigma^2} &\leq c\|\boldsymbol{y}\|_{\boldsymbol{W}_{0,\sigma}^{1,2}} && \forall \boldsymbol{y} \in \boldsymbol{W}_{0,\sigma}^{1,2}(\Omega), && (2.7)\\
\|\boldsymbol{y}\|_{\boldsymbol{W}_{0,\sigma}^{1,2}} &\leq c\|\boldsymbol{y}\|_{\boldsymbol{W}_{0,\sigma}^{2,2}} && \forall \boldsymbol{y} \in \boldsymbol{W}_{0,\sigma}^{2,2}(\Omega). && (2.8)
\end{aligned}
$$

Here, $c > 0$ is a constant depending only on the domain Ω.

The first inequality (2.2) is the well-known Poincaré–Wirtinger inequality. In particular, the norm of $W^{s,2}_{\boldsymbol{n}}(\Omega)$ for $s = 2, 3, 4$ is equivalent to that of $W^{s,2}(\Omega)$ by (2.3)–(2.6) and the norm of $\boldsymbol{W}^{2,2}_{0,\sigma}(\Omega)$ is equivalent to the one of $\boldsymbol{W}^{2,2}(\Omega)$ by (2.7) and (2.8). In addition to the above estimates, we shall frequently use the Gagliardo–Nirenberg and Agmon inequalities [3]

$$\|\varphi\|_{L^4} \le c\|\varphi\|_{L^2}^{1/2}\|\varphi\|_{W^{1,2}}^{1/2} \qquad \forall\varphi \in W^{1,2}(\Omega), \tag{2.9}$$

$$\|\varphi\|_{L^\infty} \le c\|\varphi\|_{L^2}^{1/2}\|\varphi\|_{W^{2,2}}^{1/2} \qquad \forall\varphi \in W^{2,2}(\Omega). \tag{2.10}$$

A general version of the Gagliardo–Nirenberg inequality is the following:

$$\|\varphi\|_{L^r} \le c(\|\varphi\|_{L^2} + \|\varphi\|_{L^2}^{2/r}\|\varphi\|_{W^{1,2}}^{(r-2)/r}) \qquad \forall\varphi \in W^{1,2}(\Omega),\ 2 < r < \infty. \tag{2.11}$$

Finally, Green's first identity and the Cauchy–Schwarz inequality give us

$$\|\nabla\varphi\|_{\boldsymbol{L}^2} \le \|\varphi\|_{L^2}^{1/2}\|\Delta\varphi\|_{L^2}^{1/2} \qquad \forall\varphi \in W^{2,2}_{\boldsymbol{n}}(\Omega), \tag{2.12}$$

$$\|\nabla\Delta\varphi\|_{\boldsymbol{L}^2} \le \|\Delta\varphi\|_{L^2}^{1/2}\|\Delta^2\varphi\|_{L^2}^{1/2} \qquad \forall\varphi \in W^{4,2}_{\boldsymbol{n}}(\Omega). \tag{2.13}$$

As we are dealing with non-stationary problems, we need to consider functions on an interval with values in a Banach space. Here, the Lebesgue–Bochner spaces $L^p(I;X)$ for $1 \le p \le \infty$ and the space of k-times continuously differentiable functions $C^k(\bar{I};X)$ taking values in a Banach space X will be utilized. The norms will be written as $\|\cdot\|_{L^p(X)}$ and $\|\cdot\|_{C^k(X)}$, respectively. Next, we need to consider Banach space-valued functions with time-derivatives that possibly lie in a larger function space. In this direction, let Y be another Banach space such that $X \hookrightarrow Y$, where the arrow represents a continuous embedding. This means that there is a constant $c > 0$ such that $\|x\|_Y \le c\|x\|_X$ for every $x \in X$. Let

$$W^{1,q,p}(I;X,Y) := \{w \in L^q(I;X) : \partial_t w \in L^p(I;Y)\},$$

endowed with the graph norm $\|w\|_{W^{1,q,p}(X,Y)} := \|w\|_{L^q(X)} + \|\partial_t w\|_{L^p(Y)}$. The time-derivative is taken in the sense of vector-valued distributions.

For simplicity, we denote $W^{1,q,p}(I;X) := W^{1,q,p}(I;X,X)$ and $W^{1,p}(I;X) := W^{1,p,p}(I;X)$. Also, we will set $W^{1,q,p}_0(I;X,Y) := \{u \in W^{1,q,p}(I;X,Y) : u(0) = 0\}$, considered as a subspace of $W^{1,q,p}(I;X,Y)$, $W^{1,q,p}_0(I;X) := W^{1,q,p}_0(I;X,X)$, $W^{1,p}_0(I;X) := W^{1,p,p}_0(I;X)$, and $W^{1,p}_{0,0}(I;X) := \{u \in W^{1,p}_0(I;X) : u(T) = 0\}$. Recall that time-evaluation of elements in $W^{1,q,p}(I;X,Y)$ is well-defined thanks to the continuous embeddings $W^{1,q,p}(I;X,Y) \hookrightarrow W^{1,\min\{q,p\}}(I;Y) \hookrightarrow C(\bar{I};Y)$ since $X \hookrightarrow Y$. We refer the reader to [73, Chapter 23] for the definitions and further details.

The space of linear and bounded operators from a Banach space X into another Banach space Y will be written by $\mathcal{L}(X,Y)$ and $\mathcal{L}_{\text{iso}}(X,Y)$ denote the open subset consisting of topological isomorphisms.

Now, we prove simple observations on commutators and anti-commutators. These will be utilized in the succeeding sections, for instance, in the well-posedness of the state system and the derivation of the adjoint problem to the linearized system.

Lemma 2.1. *Let $\boldsymbol{v} \in \boldsymbol{W}^{1,r}(\Omega)$, $\mathbb{S} \in \mathbb{L}^p_{\text{s}}(\Omega)$, and $\mathbb{T} \in \mathbb{L}^q_{\text{s}}(\Omega)$, where $p, q, r \in [1,\infty]$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \le 1$. Then, $[\mathbb{S}, \mathbb{W}\boldsymbol{v}] : \mathbb{T} = \nabla\boldsymbol{v} : [\mathbb{S},\mathbb{T}]$ and $\{\mathbb{S}, \mathbb{D}\boldsymbol{v}\} : \mathbb{T} = \nabla\boldsymbol{v} : \{\mathbb{S},\mathbb{T}\}$ in $L^1(\Omega)$.*

Proof. The terms on both sides of each equation lie in $L^1(\Omega)$ by the Hölder inequality. We have $\mathbb{S}\nabla \boldsymbol{v} : \mathbb{T} = \mathrm{Tr}((\nabla \boldsymbol{v})^{\mathrm{t}}\mathbb{S}\mathbb{T}) = \nabla \boldsymbol{v} : \mathbb{S}\mathbb{T}$ by symmetry of $\mathbb{S}$. Similarly, using (2.1), one has $\mathbb{S}(\nabla \boldsymbol{v})^{\mathrm{t}} : \mathbb{T} = (\nabla \boldsymbol{v})^{\mathrm{t}} : \mathbb{S}\mathbb{T} = \nabla \boldsymbol{v} : (\mathbb{S}\mathbb{T})^{\mathrm{t}} = \nabla \boldsymbol{v} : \mathbb{T}\mathbb{S}$ by symmetry of $\mathbb{S}$ and $\mathbb{T}$ once again. Thus, $\mathbb{S}\mathbb{W}\boldsymbol{v} : \mathbb{T} = \frac{1}{2}(\nabla \boldsymbol{v} : \mathbb{S}\mathbb{T} - \nabla \boldsymbol{v} : \mathbb{T}\mathbb{S}) = \frac{1}{2}\nabla \boldsymbol{v} : [\mathbb{S}, \mathbb{T}]$. Since $\mathbb{W}\boldsymbol{v}$ is anti-symmetric, we have $\mathbb{W}\boldsymbol{v}\mathbb{S} : \mathbb{T} = \mathbb{S}^{\mathrm{t}}(\mathbb{W}\boldsymbol{v})^{\mathrm{t}} : \mathbb{T}^{\mathrm{t}} = -\mathbb{S}\mathbb{W}\boldsymbol{v} : \mathbb{T}$. Therefore, $[\mathbb{S}, \mathbb{W}\boldsymbol{v}] : \mathbb{T} = \mathbb{S}\mathbb{W}\boldsymbol{v} : \mathbb{T} - \mathbb{W}\boldsymbol{v}\mathbb{S} : \mathbb{T} = 2\mathbb{S}\mathbb{W}\boldsymbol{v} : \mathbb{T} = \nabla \boldsymbol{v} : [\mathbb{S}, \mathbb{T}]$. The case of the anti-commutator is analogous and for this reason, we omit the details. □

2.3. REGULARITY ASSUMPTIONS FOR THE POTENTIAL AND COEFFICIENT FUNCTIONS. We shall consider the case where the Cahn–Hilliard potential F appearing in (1.15) is almost polynomial. More precisely, we assume for some $s \in \mathbb{N}$ that:

(A1)$_s$ *$F \in C^s(\mathbb{R})$, $F \geq 0$, and there exist constants $c_F > 0$ and $q \geq 1$ such that $|F'(\varphi)| \leq c_F(F(\varphi)+1)$, $F''(\varphi) \geq -c_F$, and $|F^{(s)}(\varphi)| \leq c_F(|\varphi|^q+1)$ for every $\varphi \in \mathbb{R}$.*

The conditions for the potential listed above have been utilized in [10, 18, 63] for instance. We emphasize here that the estimate for the derivative permits us to consider source functions in the Cahn–Hilliard equation that do not necessarily have zero mean. Hence, the mass of the fluids may not be conserved. The assumption that F is non-negative can be relaxed to boundedness from below. Take note that all other conditions hold if we add a constant on F, hence, we can assume without of loss of generality that $F \geq 0$. A typical example of a potential satisfying (A1) after adding a suitable constant is the Ginzburg–Landau–Wilson double-well potential $F(\varphi) = c_1\varphi^4 - c_2\varphi^2$ with $c_1, c_2 > 0$. This is typically used as an approximation of the more realistic logarithmic potential introduced in [27]. The last criterion in (A1)$_s$ also yields estimates on lower derivatives as stated in the following lemma.

Lemma 2.2. *Suppose that* (A1)$_s$ *holds. For each $k = 0, 1, \ldots, s$, there is $c_{F,k} > 0$ such that*

$$|F^{(k)}(\varphi)| \leq c_{F,k}(|\varphi|^{q-k+s} + 1) \quad \forall \varphi \in \mathbb{R}. \tag{2.14}$$

Proof. Note that (2.14) with $k = s$ is just (A1)$_s$ with $c_{F,s} = c_F$. If $k = s-1$, then from the mean-value theorem and the triangle inequality, for each $\varphi \in \mathbb{R}$ there is $0 \leq \theta_\varphi \leq 1$ such that

$$\begin{aligned} |F^{(s-1)}(\varphi)| &\leq |F^{(s)}(\theta_\varphi\varphi)\varphi| + |F^{(s-1)}(0)| \\ &\leq c_{F,s}(|\varphi|^{q+1} + |\varphi|) + |F^{(s-1)}(0)|. \end{aligned}$$

If $|\varphi| \leq 1$, then $|F^{(s-1)}(\varphi)| \leq c_{F,s}(|\varphi|^{q+1} + 1) + |F^{(s-1)}(0)|$. On the other hand, if $|\varphi| \geq 1$, then $|\varphi| \leq |\varphi|^{q+1}$ and so $|F^{(s-1)}(\varphi)| \leq 2c_{F,s}|\varphi|^{q+1} + |F^{(s-1)}(0)|$. Hence, one can take $c_{F,s-1} = 2c_{F,s} + |F^{(s-1)}(0)|$. The proofs for lower values of k are similar. □

The previous lemma paves the way to the following estimates on the derivatives of the potential in Lebesgue spaces.

Lemma 2.3. *Assume that* $(\mathrm{A1})_s$ *is satisfied. For each* $1 \le r < \infty$*, there exists* $c = c_{\Omega,r} > 0$ *such that for every* $k = 0, 1, \ldots, s$*,*

$$\|F^{(k)}(\phi)\|_{L^r} \le c(\|\phi\|_{W^{1,2}}^{q-k+s} + 1) \quad \forall \phi \in W^{1,2}(\Omega). \tag{2.15}$$

Furthermore, there is $c = c_\Omega > 0$ *such that*

$$\|F^{(k)}(\phi)\|_{L^\infty} \le c(\|\phi\|_{W^{2,2}}^{q-k+s} + 1) \quad \forall \phi \in W^{2,2}(\Omega). \tag{2.16}$$

Proof. In virtue of (2.14), the triangle inequality, and the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^\sigma(\Omega)$ for $1 \le \sigma < \infty$, we get

$$\begin{aligned}\|F^{(k)}(\phi)\|_{L^r} &\le c_{F,k}(\||\phi|^{q-k+s}\|_{L^r} + |\Omega|^{1/r}) \\ &\le \max_{0\le k\le 4}\{c_{F,k} + |\Omega|^{1/r}\}(\|\phi\|_{L^{r(q-k+s)}}^{q-k+s} + 1) \\ &\le c_{\Omega,r}(\|\phi\|_{W^{1,2}}^{q-k+s} + 1).\end{aligned}$$

The proof of (2.16) is the same, but now using the Sobolev embedding $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$. □

Let $C_b^s(\mathbb{R}^2)$ be the space of functions from $\mathbb{R}^2$ into $\mathbb{R}$ with bounded and continuous derivatives up to order s. For the order parameter-dependent coefficient functions and constants appearing in the model (1.15), we shall consider the following regularity and non-degeneracy hypotheses:

(A2)$_s$ *$\tau, b, \sigma_0, \sigma, \ell, \lambda, \beta > 0$, $a, a_0 \in \mathbb{R}$, $\rho(\phi,\theta) := b_1 + b_{\mathrm{o}}\phi + b_{\mathrm{h}}\theta$ with $b_1, b_{\mathrm{o}}, b_{\mathrm{h}} \in \mathbb{R}$, and $m, \chi, \nu, \varepsilon \in C_b^s(\mathbb{R}^2)$ are bounded from below by $m_0, \chi_0, \nu_0, \varepsilon_0 > 0$, respectively.*

We use the notation $|\cdot|_\infty$ for the supremum norm. For instance, $|m|_\infty := \sup_{(\phi,\theta)\in\mathbb{R}^2} |m(\phi,\theta)|$. The parameters λ, ℓ, and β are assumed to be constant, nonetheless, the methods presented here can be adapted to the case where they depend on (ϕ,θ) provided that we have similar conditions as for the parameters related to the diffusion processes. The analysis of such generalizations is straightforward since λ, ℓ, and β involve lower-order terms only, that is, $\mathbb{D}\boldsymbol{v}$, $\mathbb{S}$, $\mathrm{Tr}(\mathbb{S})$, and $\mathrm{Tr}(\mathbb{S})^2\mathbb{S}$.

Given a continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ and $(\phi,\theta) \in \mathbb{R}^2$, let $f'(\phi,\theta) : \mathbb{R}^2 \to \mathbb{R}$ be the bilinear operator

$$f'(\phi,\theta)(\psi,\eta) := f_\phi(\phi,\theta)\psi + f_\theta(\phi,\theta)\eta \quad \forall(\psi,\eta) \in \mathbb{R}^2 \tag{2.17}$$

with the subscripts denoting partial derivatives.

3. Well-Posedness of the State System

Let us denote the function spaces for a strong solution $(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S})$ and an initial data $(\phi_0, \theta_0, \boldsymbol{v}_0, \mathbb{S}_0)$ in the state system (1.15) as follows:

$$\begin{aligned}\mathcal{W}^2(\Omega_T) := {}& W^{1,2,2}(I; W_{\boldsymbol{n}}^{4,2}(\Omega), L^2(\Omega)) \times W^{1,2,2}(I; W_{\boldsymbol{n}}^{2,2}(\Omega), W_{\boldsymbol{n}}^{-2,2}(\Omega)) \\ &\times W^{1,2,2}(I; W_{\boldsymbol{n}}^{4,2}(\Omega), W_{\boldsymbol{n}}^{2,2}(\Omega)) \times W^{1,2,2}(I; \boldsymbol{W}_{0,\sigma}^{2,2}(\Omega), \boldsymbol{L}_\sigma^2(\Omega)) \\ &\times W^{1,2,2}(I; \mathbb{W}_{\boldsymbol{n},\mathrm{s}}^{2,2}(\Omega), \mathbb{L}_{\mathrm{s}}^2(\Omega)) \\ \mathcal{D}^2(\Omega) := {}& W_{\boldsymbol{n}}^{2,2}(\Omega) \times W_{\boldsymbol{n}}^{3,2}(\Omega) \times \boldsymbol{W}_{0,\sigma}^{1,2}(\Omega) \times \mathbb{W}_{\mathrm{s}}^{1,2}(\Omega).\end{aligned}$$

The corresponding product norms will be denoted by $\|\cdot\|_{\mathcal{W}^2}$ and $\|\cdot\|_{\mathcal{D}^2}$. The ambient space of controls will be written as $\boldsymbol{U} := L^2(I;\boldsymbol{L}^2(\Omega)) = L^2(\Omega_T)$. Given appropriate source functions, initial data, and a control, a quintuple $(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S}) \in \mathcal{W}^2(\Omega_T)$ with the associated pressure $\mathrm{p} \in L^2(I;W^{1,2}(\Omega)/\mathbb{R})$ is called a *strong solution* of (1.15) if the partial differential equations hold almost everywhere (a.e.) in Ω_T, the boundary conditions are satisfied a.e. on Γ_T, and the initial conditions are fulfilled a.e. in Ω.

The function spaces for the solution and initial data are compatible in the sense that

$$W^{1,2,2}(I;W^{4,2}_{\boldsymbol{n}}(\Omega),L^2(\Omega)) \hookrightarrow C(\bar{I};W^{2,2}_{\boldsymbol{n}}(\Omega)) \tag{3.1}$$

$$W^{1,2,2}(I;W^{4,2}_{\boldsymbol{n}}(\Omega),W^{2,2}_{\boldsymbol{n}}(\Omega)) \hookrightarrow C(\bar{I};W^{3,2}_{\boldsymbol{n}}(\Omega)) \tag{3.2}$$

$$W^{1,2,2}(I;\boldsymbol{W}^{2,2}_{0,\sigma}(\Omega),\boldsymbol{L}^2_{\sigma}(\Omega)) \hookrightarrow C(\bar{I};\boldsymbol{W}^{1,2}_{0,\sigma}(\Omega)) \tag{3.3}$$

$$W^{1,2,2}(I;\mathbb{W}^{2,2}_{\boldsymbol{n},\mathrm{s}}(\Omega),\mathbb{L}^2_{\mathrm{s}}(\Omega)) \hookrightarrow C(\bar{I};\mathbb{W}^{1,2}_{\mathrm{s}}(\Omega)) \tag{3.4}$$

due to classical interpolation theory. On the other hand, for the chemical potential, we have

$$W^{1,2,2}(I;W^{2,2}_{\boldsymbol{n}}(\Omega),W^{-2,2}_{\boldsymbol{n}}(\Omega)) \hookrightarrow C(\bar{I};L^2(\Omega)). \tag{3.5}$$

In virtue of the Agmon (2.10) and Gagliardo–Nirenberg (2.9) inequalities, it can be deduced from (3.1), (2.12), and (3.2) respectively that

$$W^{1,2,2}(I;W^{4,2}_{\boldsymbol{n}}(\Omega),L^2(\Omega)) \hookrightarrow L^4(I;W^{2,\infty}(\Omega)\cap W^{3,2}_{\boldsymbol{n}}(\Omega)) \tag{3.6}$$

$$W^{1,2,2}(I;W^{4,2}_{\boldsymbol{n}}(\Omega),W^{2,2}_{\boldsymbol{n}}(\Omega)) \hookrightarrow L^4(I;W^{3,4}(\Omega)). \tag{3.7}$$

Again, using the Gagliardo–Nirenberg inequality, (3.3), and (3.4), we have

$$W^{1,2,2}(I;\boldsymbol{W}^{2,2}_{0,\sigma}(\Omega),\boldsymbol{L}^2_{\sigma}(\Omega)) \hookrightarrow L^4(I;\boldsymbol{W}^{1,4}(\Omega)) \tag{3.8}$$

$$W^{1,2,2}(I;\mathbb{W}^{2,2}_{\boldsymbol{n},\mathrm{s}}(\Omega),\mathbb{L}^2_{\mathrm{s}}(\Omega)) \hookrightarrow L^4(I;\mathbb{W}^{1,4}(\Omega)). \tag{3.9}$$

Finally, thanks to (2.12) and (3.5), we obtain

$$W^{1,2,2}(I;W^{2,2}_{\boldsymbol{n}}(\Omega),W^{-2,2}_{\boldsymbol{n}}(\Omega)) \hookrightarrow L^4(I;W^{1,2}(\Omega)). \tag{3.10}$$

The above continuous embeddings will be utilized in later discussions.

We shall follow a standard spectral Faedo–Galerkin method for the existence and uniqueness of strong solutions to (1.15) as in [10, 18]. First, note that there are orthonormal bases $\{\varphi_j\}_{j=1}^{\infty}$, $\{\boldsymbol{y}_j\}_{j=1}^{\infty}$, and $\{\mathbb{Y}_j\}_{j=1}^{\infty}$ for $L^2(\Omega)$, $\boldsymbol{L}^2_{\sigma}(\Omega)$, and $\mathbb{L}^2_{\mathrm{s}}(\Omega)$, respectively, comprising of normalized eigenfunctions for the Neumann Laplacian $A_N := -\Delta : W^{2,2}_{\boldsymbol{n}}(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$, the Stokes operator $\boldsymbol{A}_S = -\boldsymbol{P}_{\sigma}\Delta : \boldsymbol{W}^{2,2}_{0,\sigma}(\Omega) \subset \boldsymbol{L}^2_{\sigma}(\Omega) \to \boldsymbol{L}^2_{\sigma}(\Omega)$, and the Neumann Laplacian on symmetric tensors $\mathbb{A}_N : \mathbb{W}^{2,2}_{\boldsymbol{n},\mathrm{s}}(\Omega) \subset \mathbb{L}^2_{\mathrm{s}}(\Omega) \to \mathbb{L}^2_{\mathrm{s}}(\Omega)$ (refer to [37] and [65] for details). Here, $\boldsymbol{P}_{\sigma} : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2_{\sigma}(\Omega)$ is the Leray–Helmholtz projector with respect to the orthogonal decomposition $\boldsymbol{L}^2(\Omega) = \boldsymbol{L}^2_{\sigma}(\Omega) \oplus \nabla(W^{1,2}(\Omega)/\mathbb{R})$. The existence of such bases follows from the fact that the Neumann Laplacian and the Stokes operator defined above are closed operators with compact resolvents.

Let Φ_k, $\boldsymbol{V}_k$, and $\mathcal{S}_k$ be the linear spans generated by $\{\varphi_j\}_{j=1}^{k}$, $\{\boldsymbol{y}_j\}_{j=1}^{k}$, and $\{\mathbb{Y}_j\}_{j=1}^{k}$. Denote the corresponding orthogonal projectors by $\widetilde{P}_{\Phi_k} : L^2(\Omega) \to \Phi_k$,

$\widetilde{\boldsymbol{P}}_{\boldsymbol{V}_k} : \boldsymbol{L}^2_\sigma(\Omega) \to \boldsymbol{V}_k$, and $\widetilde{\mathcal{P}}_{\mathcal{S}_k} : \mathbb{L}^2_{\mathrm{s}}(\Omega) \to \mathcal{S}_k$, where

$$\widetilde{P}_{\Phi_k}\phi := \sum_{j=1}^{k} (\phi, \varphi_j)_{L^2}\varphi_j, \quad \widetilde{\boldsymbol{P}}_{\boldsymbol{V}_k}\boldsymbol{v} := \sum_{j=1}^{k} (\boldsymbol{v}, \boldsymbol{y}_j)_{\boldsymbol{L}^2_\sigma}\boldsymbol{y}_j, \quad \widetilde{\mathcal{P}}_{\mathcal{S}_k}\mathbb{S} := \sum_{j=1}^{k} (\mathbb{S}, \mathbb{Y}_j)_{\mathbb{L}^2_{\mathrm{s}}}\mathbb{Y}_j,$$

for $\phi \in L^2(\Omega)$, $\boldsymbol{v} \in \boldsymbol{L}^2_\sigma(\Omega)$, and $\mathbb{S} \in \mathbb{L}^2_{\mathrm{s}}(\Omega)$. Let $E_{\Phi_k} : \Phi_k \to L^2(\Omega)$, $\boldsymbol{E}_{\boldsymbol{V}_k} : \boldsymbol{V}_k \to \boldsymbol{L}^2_\sigma(\Omega)$, and $\mathcal{E}_{\mathcal{S}_k} : \mathcal{S}_k \to \mathbb{L}^2_{\mathrm{s}}(\Omega)$ be the canonical injections. Then,

$$\begin{aligned} P_{\Phi_k} &:= E_{\Phi_k}\widetilde{P}_{\Phi_k}E_{\Phi_k} \in \mathcal{L}(\Phi_k, L^2(\Omega)), \\ \boldsymbol{P}_{\boldsymbol{V}_k} &:= \boldsymbol{E}_{\boldsymbol{V}_k}\widetilde{\boldsymbol{P}}_{\boldsymbol{V}_k}\boldsymbol{E}_{\boldsymbol{V}_k} \in \mathcal{L}(\boldsymbol{V}_k, \boldsymbol{L}^2_\sigma(\Omega)), \\ \mathcal{P}_{\mathcal{S}_k} &:= \mathcal{E}_{\mathcal{S}_k}\widetilde{\mathcal{P}}_{\mathcal{S}_k}\mathcal{E}_{\mathcal{S}_k} \in \mathcal{L}(\mathcal{S}_k, \mathbb{L}^2_{\mathrm{s}}(\Omega)). \end{aligned}$$

Let $P^*_{\Phi_k} \in \mathcal{L}(L^2(\Omega), \Phi_k)$, $\boldsymbol{P}^*_{\boldsymbol{V}_k} \in \mathcal{L}(\boldsymbol{L}^2_\sigma(\Omega), \boldsymbol{V}_k)$, and $\mathcal{P}^*_{\mathcal{S}_k} \in \mathcal{L}(\mathbb{L}^2_{\mathrm{s}}(\Omega), \mathcal{S}_k)$ be the associated Hilbert space adjoint operators.

Generic positive constants will be denoted by c or with a subscript to emphasize the dependence of such constants. In general, these quantities depend on at least one of the coefficient functions and the constants appearing in Lemma 2.2, the terminal time T, and the spatial domain Ω. These constants may also depend on the given source functions and initial data in (1.15), however, they do not depend on the unknown state variables.

Theorem 3.1. *Let* (A1)$_3$ *and* (A2)$_1$ *be satisfied. Suppose that we have initial data* $(\phi_0, \theta_0, \boldsymbol{v}_0, \mathbb{S}_0) \in \mathcal{D}^2(\Omega)$ *and source functions such that* $f_{\mathrm{o}} \in L^2(I; L^2(\Omega))$, $f_{\mathrm{h}} \in L^2(I; L^2(\Omega))$, $\boldsymbol{f}_{\mathrm{v}} \in L^2(I; \boldsymbol{L}^2(\Omega))$, *and* $\mathbb{F}_{\mathrm{s}} \in L^2(I; \mathbb{L}^2_{\mathrm{s}}(\Omega))$. *For each control* $\boldsymbol{u} \in \boldsymbol{U}$, *the nonlinear system* (1.15) *has a unique strong solution* $(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S}) \in \mathcal{W}^2(\Omega_T)$ *with an associated pressure* $\mathrm{p} \in L^2(I; W^{1,2}(\Omega)/\mathbb{R})$. *Moreover, there exists a monotone increasing and continuous function* $\mathfrak{C} : [0, \infty) \to [0, \infty)$ *that depend on the norms of the source functions in the given underlying spaces such that*

$$\|(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S})\|_{\mathcal{W}^2} + \|\mathrm{p}\|_{L^2(W^{1,2}/\mathbb{R})} \le \mathfrak{C}(\|(\phi_0, \theta_0, \boldsymbol{v}_0, \mathbb{S}_0)\|_{\mathcal{D}^2} + \|\boldsymbol{u}\|_{\boldsymbol{U}}). \tag{3.11}$$

Proof. Let us start with the local-in-time existence of solutions for the projected systems. For this, we consider unknown functions

$$\begin{aligned} \phi_k(t) &:= \sum_{j=1}^{k} \alpha_{jk}(t)\varphi_j, \quad & \theta_k(t) &:= \sum_{j=1}^{k} \beta_{jk}(t)\varphi_j, \\ \boldsymbol{v}_k(t) &:= \sum_{j=1}^{k} \gamma_{jk}(t)\boldsymbol{y}_j, \quad & \mathbb{S}_k(t) &:= \sum_{j=1}^{k} \eta_{jk}(t)\mathbb{Y}_j, \end{aligned}$$

where $\alpha_{jk}, \beta_{jk}, \gamma_{jk}, \eta_{jk} \in W^{1,2}(I)$, that satisfy the following finite-dimensional approximation of the state system:

$$\left[\begin{array}{ll} \partial_t \phi_k + P_{\Phi_k}^*\{\boldsymbol{v}_k \cdot \nabla \phi_k - \nabla \cdot (m_k \nabla \mu_k)\} = P_{\Phi_k}^* f_{\mathrm{o}} & \text{in } L^2(I;\Phi_k), \\ \mu_k = -\alpha \Delta \phi_k + P_{\Phi_k} F'(\phi_k) & \text{in } W^{1,2}(I;\Phi_k), \\ \partial_t(\theta_k - \tau \Delta \theta_k) + P_{\Phi_k}^*\{\boldsymbol{v}_k \cdot \nabla \theta_k - \nabla \cdot (\chi_k \nabla \theta_k)\} & \\ \qquad + b\Delta^2 \theta_k = P_{\Phi_k}^*\{a_0 \mathbf{g} \cdot \boldsymbol{v}_k + \mathbb{S}_k : \mathbb{D}\boldsymbol{v}_k + f_{\mathrm{h}}\} & \text{in } L^2(I;\Phi_k), \\ \partial_t \boldsymbol{v}_k + \boldsymbol{P}_{\boldsymbol{V}_k}^*\{(\boldsymbol{v}_k \cdot \nabla)\boldsymbol{v}_k - \nabla \cdot (2\nu_k \mathbb{D}\boldsymbol{v}_k)\} & \\ \qquad = \boldsymbol{P}_{\boldsymbol{V}_k}^*\{\nabla \cdot \mathbb{M}(\theta_k, \mathbb{S}_k) + \kappa \mu_k \nabla \phi_k + \rho(\phi_k, \theta_k)\mathbf{g}\} & \\ \qquad + \boldsymbol{P}_{\boldsymbol{V}_k}^*\{\boldsymbol{f}_{\mathrm{v}} + \boldsymbol{u}\} & \text{in } L^2(I;\boldsymbol{V}_k), \\ \partial_t \mathbb{S}_k + \mathcal{P}_{\mathcal{S}_k}^*\{(\boldsymbol{v}_k \cdot \nabla)\mathbb{S}_k + \mathbb{J}(\boldsymbol{v}_k, \mathbb{S}_k) - \nabla \cdot (\varepsilon_k \nabla \mathbb{S}_k)\} & \\ \qquad = \mathcal{P}_{\mathcal{S}_k}^*\{\lambda \mathbb{D}\boldsymbol{v}_k + \mathbb{P}(\mathbb{S}_k) + \mathbb{F}_{\mathrm{s}}\} & \text{in } L^2(I;\mathcal{S}_k), \\ \phi_k(0) = P_{\Phi_k}\phi_0, \ \theta_k(0) = P_{\Phi_k}\theta_0 & \text{in } \Omega, \\ \boldsymbol{v}_k(0) = \boldsymbol{P}_{\boldsymbol{V}_k}\boldsymbol{v}_0, \ \mathbb{S}_k(0) = \mathcal{P}_{\mathcal{S}_k}\mathbb{S}_0 & \text{in } \Omega, \end{array}\right. \tag{3.12}$$

where $m_k := m(\phi_k, \theta_k)$, $\chi_k := \chi(\phi_k, \theta_k)$, $\nu_k := \nu(\phi_k, \theta_k)$, and $\varepsilon_k := \varepsilon(\phi_k, \theta_k)$.

In virtue of the classical Cauchy–Lipschitz Theorem for ordinary differential equations, (3.12) possesses a unique maximal solution with components $\phi_k, \mu_k \in W^{1,2}(I_k;\Phi_k)$, $\boldsymbol{v}_k \in \boldsymbol{W}^{1,2}(I_k;\boldsymbol{V}_k)$, and $\mathbb{S}_k \in \mathbb{W}^{1,2}(I_k;\mathcal{S}_k)$ for some interval $I_k := [0, T_k) \subset I$. By a standard continuation argument, the succeeding uniform-in-time a priori estimates will establish that this solution exists on the whole time interval I. In what follows, $\delta > 0$ denotes a generic positive constant, typically chosen to be small.

Step 1. *Energy-type estimates.* Taking the test functions $\mu_k + \phi_k$ and $-\partial_t \phi_k$ in the first and second equations of (3.12), utilizing $(\boldsymbol{v}_k \cdot \nabla \phi_k, \phi_k)_{L^2} = 0$ in I_k due to $\nabla \cdot \boldsymbol{v}_k = 0$ in Ω_T and integration-by-parts, getting the sum of the resulting equations, and noting that $F \geq 0$ from $(\mathrm{A1})_3$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\int_\Omega |\phi_k|^2 \,\mathrm{d}x + \alpha \int_\Omega |\nabla \phi_k|^2 \,\mathrm{d}x + 2\int_\Omega |F(\phi_k)| \,\mathrm{d}x\right) \\ &\quad + \int_\Omega m_k |\nabla \mu_k|^2 \,\mathrm{d}x + \int_\Omega m_k \nabla \mu_k \cdot \nabla \phi_k \,\mathrm{d}x \\ &\quad = \int_\Omega f_{\mathrm{o}} \mu_k \,\mathrm{d}x + \int_\Omega f_{\mathrm{o}} \phi_k \,\mathrm{d}x - \int_\Omega (\boldsymbol{v}_k \cdot \nabla \phi_k)\mu_k \,\mathrm{d}x. \end{aligned} \tag{3.13}$$

Let us estimate the first two integrals on the right-hand side and the last integral on the left-hand side. We deduce from the Cauchy–Schwarz, Poincaré–Wirtinger (2.2), and Young inequalities that

$$\int_\Omega |f_{\mathrm{o}} \phi_k| \,\mathrm{d}x \leq \frac{1}{2}\|f_{\mathrm{o}}\|_{L^2}^2 + \frac{1}{2}\|\phi_k\|_{L^2}^2 \tag{3.14}$$

$$\begin{aligned} \int_\Omega |f_{\mathrm{o}} \mu_k| \,\mathrm{d}x &\leq \int_\Omega |f_{\mathrm{o}}(\mu_k - \langle \mu_k \rangle)| \,\mathrm{d}x + \int_\Omega |f_{\mathrm{o}} \langle \mu_k \rangle| \,\mathrm{d}x \\ &\leq \frac{m_0}{8}\|\nabla \mu_k\|_{\boldsymbol{L}^2}^2 + c\|f_{\mathrm{o}}\|_{L^2}^2 + |\Omega|^{1/2}\|f_{\mathrm{o}}\|_{L^2}|\langle \mu_k \rangle| \end{aligned} \tag{3.15}$$

$$\int_\Omega |m_k \nabla \mu_k \cdot \nabla \phi_k| \,\mathrm{d}x \le \frac{m_0}{8}\|\nabla \mu_k\|_{\boldsymbol{L}^2}^2 + c\|\nabla \phi_k\|_{\boldsymbol{L}^2}^2. \tag{3.16}$$

Let $K := \|f_\mathrm{o}\|_{L^2} + 1 \in L^2(I) \subset L^1(I)$. From (3.15), we need an estimate for the average of the approximation of the chemical potential. For this, we apply $|F'| \le c_F(|F|+1)$ from $(\mathrm{A1})_3$, $\langle \Delta\phi_k\rangle = 0$ in I_k, and Hölder inequality to the second equation in (3.12) so that

$$|\langle \mu_k\rangle| \le |\Omega|^{-1}\|F'(\phi_k)\|_{L^1} \le c(\|F(\phi_k)\|_{L^1} + 1). \tag{3.17}$$

Applying the test function $m_0\Delta\phi_k$ to the second equation of (3.12), taking note that $-F'' \le c_F$ from $(\mathrm{A1})_3$, and using $P_{\Phi_k}\Delta\phi_k = \Delta\phi_k$, we obtain

$$\begin{aligned} m_0\alpha\|\Delta\phi_k\|_{L^2}^2 &= m_0\int_\Omega (\nabla\mu_k - F''(\phi_k)\nabla\phi_k)\cdot\nabla\phi_k \,\mathrm{d}x \\ &\le \frac{m_0}{2}\|\nabla\mu_k\|_{\boldsymbol{L}^2}^2 + \frac{m_0}{2}(2c_F+1)\|\nabla\phi_k\|_{\boldsymbol{L}^2}^2. \end{aligned} \tag{3.18}$$

Using (3.14)–(3.17) in (3.13), taking the sum of the resulting inequality with (3.18), and applying $m_k \ge m_0$, we deduce that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\phi_k\|_{L^2}^2 + \alpha\|\nabla\phi_k\|_{\boldsymbol{L}^2}^2 + 2\|F(\phi_k)\|_{L^1}\right) + m_0\alpha\|\Delta\phi_k\|_{L^2}^2 + \frac{m_0}{4}\|\nabla\mu_k\|_{\boldsymbol{L}^2}^2 \\ &\quad\le -\int_\Omega (\boldsymbol{v}_k\cdot\nabla\phi_k)\mu_k \,\mathrm{d}x + cK(\|\phi_k\|_{L^2}^2 + \|\nabla\phi_k\|_{\boldsymbol{L}^2}^2 + \|F(\phi_k)\|_{L^1}) \\ &\qquad + c(\|f_\mathrm{o}\|_{L^2} + \|f_\mathrm{o}\|_{L^2}^2). \end{aligned} \tag{3.19}$$

We use the test function $\frac{\sigma}{\kappa}\theta_k$ to the third equation in (3.12) and apply $(\boldsymbol{v}\cdot\nabla\theta_k, \theta_k)_{L^2} = 0$ in I_k, to obtain

$$\begin{aligned} &\frac{\sigma}{2\kappa}\frac{d}{dt}\int_\Omega (|\theta_k|^2 + \tau|\nabla\theta_k|^2)\,\mathrm{d}x + \frac{\sigma}{\kappa}\int_\Omega (\chi_k|\nabla\theta_k|^2 + b|\Delta\theta_k|^2)\,\mathrm{d}x \\ &\quad = \frac{\sigma}{\kappa}\int_\Omega a_0\mathbf{g}\cdot\boldsymbol{v}_k\theta_k \,\mathrm{d}x + \frac{\sigma}{\kappa}\int_\Omega \theta_k\mathbb{S}_k : \mathbb{D}\boldsymbol{v}_k \,\mathrm{d}x + \frac{\sigma}{\kappa}\int_\Omega f_\mathrm{h}\theta_k \,\mathrm{d}x. \end{aligned} \tag{3.20}$$

The assumption $\chi_k \ge \chi_0$ and the Cauchy–Schwarz inequality applied to the first and last terms on the right-hand side in (3.20) lead to the estimate

$$\begin{aligned} &\frac{\sigma}{2\kappa}\frac{d}{dt}(\|\theta_k\|_{L^2}^2 + \tau\|\nabla\theta_k\|_{\boldsymbol{L}^2}^2) + \frac{\sigma\chi_0}{\kappa}\|\nabla\theta_k\|_{\boldsymbol{L}^2}^2 + \frac{\sigma b}{\kappa}\|\Delta\theta_k\|_{L^2}^2 \\ &\quad\le \frac{\sigma}{\kappa}\int_\Omega \theta_k\mathbb{S}_k : \mathbb{D}\boldsymbol{v}_k \,\mathrm{d}x + c(\|\theta_k\|_{L^2}^2 + \|\boldsymbol{v}_k\|_{\boldsymbol{L}^2_\sigma}^2 + \|f_\mathrm{h}\|_{L^2}^2). \end{aligned} \tag{3.21}$$

Taking the test function $\frac{1}{\kappa}\boldsymbol{v}_k$ to the fourth equation in (3.12), applying $((\boldsymbol{v}_k\cdot\nabla)\boldsymbol{v}_k, \boldsymbol{v}_k)_{\boldsymbol{L}^2} = 0$ in I_k, integrating by parts, and recalling the definition of $\mathbb{M}$ in (1.10), we get

$$\begin{aligned} &\frac{1}{2\kappa}\frac{d}{dt}\int_\Omega |\boldsymbol{v}_k|^2 \,\mathrm{d}x + \frac{1}{\kappa}\int_\Omega 2\nu_k|\mathbb{D}\boldsymbol{v}_k|^2 \,\mathrm{d}x = -\frac{\sigma_0 a}{\kappa}\int_\Omega \mathbb{S}_k^2 : \nabla\boldsymbol{v}_k \,\mathrm{d}x \\ &\quad + \frac{\sigma_0 a}{\kappa}\int_\Omega \mathbb{S}_k : \nabla\boldsymbol{v}_k \,\mathrm{d}x - \frac{\sigma}{\kappa}\int_\Omega \theta_k\mathbb{S}_k : \nabla\boldsymbol{v}_k \,\mathrm{d}x - \frac{a}{\kappa}\int_\Omega \mathrm{Tr}(\mathbb{S}_k)\mathbb{S}_k : \nabla\boldsymbol{v}_k \,\mathrm{d}x \\ &\quad + \int_\Omega \mu_k\nabla\phi_k\cdot\boldsymbol{v}_k \,\mathrm{d}x + \frac{1}{\kappa}\int_\Omega \rho(\phi_k,\theta_k)\mathbf{g}\cdot\boldsymbol{v}_k \,\mathrm{d}x + \frac{1}{\kappa}\int_\Omega (\boldsymbol{f}_\mathrm{v} + \boldsymbol{u})\cdot\boldsymbol{v}_k \,\mathrm{d}x. \end{aligned} \tag{3.22}$$

By Korn inequality and the condition $\nu_k \geq \nu_0$, one has

$$\frac{1}{\kappa}\int_\Omega 2\nu_k|\mathbb{D}\boldsymbol{v}_k|^2\,\mathrm{d}x \geq \frac{\nu_0}{\kappa}\|\nabla\boldsymbol{v}_k\|_{\mathbb{L}^2}^2. \tag{3.23}$$

With Young inequality, the second, sixth, and seventh integrals on the right-hand side of (3.22) can be estimated from above according to

$$\begin{aligned}
&\frac{\sigma_0 a}{\kappa}\int_\Omega |\mathbb{S}_k : \nabla\boldsymbol{v}_k|\,\mathrm{d}x + \frac{1}{\kappa}\int_\Omega |\rho(\phi_k,\theta_k)\mathbf{g}\cdot\boldsymbol{v}_k|\,\mathrm{d}x + \frac{1}{\kappa}\int_\Omega |(\boldsymbol{f}_{\mathrm{v}}+\boldsymbol{u})\cdot\boldsymbol{v}_k|\,\mathrm{d}x\\
&\quad\leq \frac{\nu_0}{2\kappa}\|\nabla\boldsymbol{v}_k\|_{\mathbb{L}^2}^2 + c(\|\mathbb{S}_k\|_{\mathbb{L}^2_{\mathrm{s}}}^2 + \|\boldsymbol{v}_k\|_{\boldsymbol{L}^2_\sigma}^2 + \|\phi_k\|_{L^2}^2 + \|\theta_k\|_{L^2}^2)\\
&\qquad + c(\|\boldsymbol{f}_{\mathrm{v}}\|_{\boldsymbol{L}^2}^2 + \|\boldsymbol{u}\|_{\boldsymbol{L}^2}^2 + |\mathbf{g}|^2).
\end{aligned} \tag{3.24}$$

Then, by plugging the inequalities (3.23) and (3.24) in (3.22), we have

$$\begin{aligned}
&\frac{1}{2\kappa}\frac{d}{dt}\|\boldsymbol{v}_k\|_{\boldsymbol{L}^2_\sigma}^2 + \frac{\nu_0}{2\kappa}\|\nabla\boldsymbol{v}_k\|_{\mathbb{L}^2}^2 \leq -\frac{\sigma_0 a}{\kappa}\int_\Omega \mathbb{S}_k^2 : \nabla\boldsymbol{v}_k\,\mathrm{d}x\\
&\quad - \frac{\sigma}{\kappa}\int_\Omega \theta_k\mathbb{S}_k : \mathbb{D}\boldsymbol{v}_k\,\mathrm{d}x - \frac{a}{\kappa}\int_\Omega \mathrm{Tr}(\mathbb{S}_k)\mathbb{S}_k : \nabla\boldsymbol{v}_k\,\mathrm{d}x + \int_\Omega \mu_k\nabla\phi_k\cdot\boldsymbol{v}_k\,\mathrm{d}x\\
&\quad + c(\|\phi_k\|_{L^2}^2 + \|\theta_k\|_{L^2}^2 + \|\boldsymbol{v}_k\|_{\boldsymbol{L}^2_\sigma}^2 + \|\mathbb{S}_k\|_{\mathbb{L}^2_{\mathrm{s}}}^2 + \|\boldsymbol{f}_{\mathrm{v}}\|_{\boldsymbol{L}^2}^2 + \|\boldsymbol{u}\|_{\boldsymbol{L}^2}^2 + |\mathbf{g}|^2).
\end{aligned} \tag{3.25}$$

Thanks to Lemma 2.1, we have $[\mathbb{S}_k, \mathbb{W}\boldsymbol{v}_k] : \mathbb{S}_k = \nabla\boldsymbol{v}_k : [\mathbb{S}_k,\mathbb{S}_k] = 0$ and $\{\mathbb{S}_k, \mathbb{D}\boldsymbol{v}_k\} : \mathbb{S}_k = \nabla\boldsymbol{v}_k : \{\mathbb{S}_k,\mathbb{S}_k\} = 2\nabla\boldsymbol{v}_k : \mathbb{S}_k^2$. Utilizing the test function $\frac{\sigma_0}{2\kappa}\mathbb{S}_k$ to the fifth equation in (3.12), using the previous equations for the commutator and anti-commutator, and $((\boldsymbol{v}_k\cdot\nabla)\mathbb{S}_k, \mathbb{S}_k)_{\mathbb{L}^2_{\mathrm{s}}} = 0$ in I_k, one has

$$\begin{aligned}
&\frac{\sigma_0}{4\kappa}\frac{d}{dt}\int_\Omega |\mathbb{S}_k|^2\,\mathrm{d}x + \frac{\sigma_0}{2\kappa}\int_\Omega \varepsilon_k|\nabla\mathbb{S}_k|^2\,\mathrm{d}x + \frac{\sigma_0\beta}{2\kappa}\int_\Omega |\mathrm{Tr}(\mathbb{S}_k)\mathbb{S}_k|^2\,\mathrm{d}x\\
&\quad = \frac{\sigma_0 a}{\kappa}\int_\Omega \nabla\boldsymbol{v}_k : \mathbb{S}_k^2\,\mathrm{d}x - \frac{\sigma_0\ell}{2\kappa}\int_\Omega |\mathbb{S}_k|^2\,\mathrm{d}x + \frac{\sigma_0\beta}{2\kappa}\int_\Omega |\mathrm{Tr}(\mathbb{S}_k)|^2\,\mathrm{d}x\\
&\qquad + \frac{\sigma_0}{2\kappa}\int_\Omega (\lambda\mathbb{D}\boldsymbol{v}_k + \mathbb{F}_{\mathrm{s}}) : \mathbb{S}_k\,\mathrm{d}x.
\end{aligned} \tag{3.26}$$

Here, we recall (1.13) and (1.14) for the definitions of $\mathbb{J}$ and $\mathbb{P}$. In virtue of the Young and Hölder inequalities, the last integral in (3.26) can be estimated by

$$\frac{\sigma_0}{2\kappa}\int_\Omega |(\lambda\mathbb{D}\boldsymbol{v}_k + \mathbb{F}_{\mathrm{s}}) : \mathbb{S}_k|\,\mathrm{d}x \leq \frac{\nu_0}{4\kappa}\|\nabla\boldsymbol{v}_k\|_{\mathbb{L}^2}^2 + c[(\lambda^2+1)\|\mathbb{S}_k\|_{\mathbb{L}^2_{\mathrm{s}}}^2 + \|\mathbb{F}_{\mathrm{s}}\|_{\mathbb{L}^2_{\mathrm{s}}}^2]. \tag{3.27}$$

Substituting (3.27) in (3.26) and noting that $\varepsilon_k \geq \varepsilon_0$, we obtain

$$\begin{aligned}
&\frac{\sigma_0}{4\kappa}\frac{d}{dt}\|\mathbb{S}_k\|_{\mathbb{L}^2_{\mathrm{s}}}^2 + \frac{\sigma_0\varepsilon_0}{2\kappa}\|\nabla\mathbb{S}_k\|_{(\mathbb{L}^2_{\mathrm{s}})^2}^2 - \frac{\nu_0}{4\kappa}\|\nabla\boldsymbol{v}_k\|_{\mathbb{L}^2}^2 + \frac{\sigma_0\beta}{2\kappa}\|\mathrm{Tr}(\mathbb{S}_k)\mathbb{S}_k\|_{\mathbb{L}^2_{\mathrm{s}}}^2\\
&\quad \leq \frac{\sigma_0 a}{\kappa}\int_\Omega \nabla\boldsymbol{v}_k : \mathbb{S}_k^2\,\mathrm{d}x + c(\|\mathbb{S}_k\|_{\mathbb{L}^2_{\mathrm{s}}}^2 + \|\mathrm{Tr}(\mathbb{S}_k)\|_{L^2}^2 + \|\mathbb{F}_{\mathrm{s}}\|_{\mathbb{L}^2_{\mathrm{s}}}^2).
\end{aligned} \tag{3.28}$$

Finally, in order to remove the integral term involving $\mathrm{Tr}(\mathbb{S}_k)$ in (3.25), we consider the test function $\frac{1}{2\kappa}\mathrm{Tr}(\mathbb{S}_k)\mathbb{I}$ in the fifth equation of (3.12). From Lemma 2.1, we have $[\mathbb{S}_k, \mathbb{W}\boldsymbol{v}_k] : \mathrm{Tr}(\mathbb{S}_k)\mathbb{I} = \mathrm{Tr}(\mathbb{S}_k)\nabla\boldsymbol{v}_k : [\mathbb{S}_k, \mathbb{I}] = 0$ and $\{\mathbb{S}_k, \mathbb{D}\boldsymbol{v}_k\} : \mathrm{Tr}(\mathbb{S}_k)\mathbb{I} = \mathrm{Tr}(\mathbb{S}_k)\nabla\boldsymbol{v}_k : \{\mathbb{S}_k, \mathbb{I}\} = 2\nabla\boldsymbol{v}_k : \mathrm{Tr}(\mathbb{S}_k)\mathbb{S}_k$. Moreover, it holds that

$$\int_\Omega (\boldsymbol{v}_{\boldsymbol{k}}\cdot\nabla)\mathbb{S}_k : \mathrm{Tr}(\mathbb{S}_k)\mathbb{I}\,\mathrm{d}x = \int_\Omega (\boldsymbol{v}_{\boldsymbol{k}}\cdot\nabla)\boldsymbol{d}(\mathbb{S}_k)\cdot\boldsymbol{d}(\mathbb{S}_k)\,\mathrm{d}x = 0,$$

where $\boldsymbol{d}(\mathbb{S}_k)$ is the vector with the diagonal of $\mathbb{S}_k$ as the entries, and $\nabla\mathbb{S}_k \therefore \nabla(\mathrm{Tr}(\mathbb{S}_k)\mathbb{I}) = |\nabla(\mathrm{Tr}(\mathbb{S}_k))|^2$. Thus, we have

$$\begin{aligned}
&\frac{1}{4\kappa}\frac{d}{dt}\int_\Omega |\mathrm{Tr}(\mathbb{S}_k)|^2\,\mathrm{d}x + \frac{1}{2\kappa}\int_\Omega \varepsilon|\nabla(\mathrm{Tr}(\mathbb{S}_k))|^2\,\mathrm{d}x + \frac{\beta}{2\kappa}\int_\Omega |\mathrm{Tr}(\mathbb{S}_k)|^4\,\mathrm{d}x\\
&\quad = \frac{1}{2\kappa}(2\beta-\ell)\int_\Omega |\mathrm{Tr}(\mathbb{S}_k)|^2\,\mathrm{d}x + \frac{a}{\kappa}\int_\Omega \mathrm{Tr}(\mathbb{S}_k)\mathbb{S}_k : \nabla\boldsymbol{v}_k\,\mathrm{d}x\\
&\quad + \frac{1}{2\kappa}\int_\Omega \mathbb{F}_\mathrm{s} : \mathrm{Tr}(\mathbb{S}_k)\mathbb{I}\,\mathrm{d}x. \qquad (3.29)
\end{aligned}$$

Here, we used the fact that $\lambda\mathbb{D}\boldsymbol{v}_k : \mathrm{Tr}(\mathbb{S}_k)\mathbb{I} = \lambda\mathrm{Tr}(\mathbb{S}_k)\nabla\cdot\boldsymbol{v}_k = 0$. Therefore, this leads us to

$$\begin{aligned}
&\frac{1}{4\kappa}\frac{d}{dt}\|\mathrm{Tr}(\mathbb{S}_k)\|_{L^2}^2 + \frac{\varepsilon_0}{2\kappa}\|\nabla(\mathrm{Tr}(\mathbb{S}_k))\|_{\boldsymbol{L}^2}^2 + \frac{\beta}{2\kappa}\|\mathrm{Tr}(\mathbb{S}_k)\|_{L^4}^4\\
&\quad \le \frac{a}{\kappa}\int_\Omega \mathrm{Tr}(\mathbb{S}_k)\mathbb{S}_k : \nabla\boldsymbol{v}_k\,\mathrm{d}x + c(\|\mathrm{Tr}(\mathbb{S}_k)\|_{L^2}^2 + \|\mathbb{F}_\mathrm{s}\|_{\mathbb{L}_\mathrm{s}^2}^2). \qquad (3.30)
\end{aligned}$$

Taking the sum of the estimates (3.19), (3.21), (3.25), (3.28), and (3.30), we see that the remaining integral terms cancel, and as a consequence, this leads to the differential inequality

$$\frac{1}{2}\frac{d}{dt}E_k + D_k \le c(S + KE_k) \quad \text{in } I_k, \qquad (3.31)$$

where $E_k, D_k : I_k \to [0,\infty)$ and $S : I \to [0,\infty)$ are given by

$$\begin{aligned}
E_k &:= \|\phi_k\|_{L^2}^2 + \alpha\|\nabla\phi_k\|_{\boldsymbol{L}^2}^2 + 2\|F(\phi_k)\|_{L^1} + \frac{\sigma}{\kappa}\|\theta_k\|_{L^2}^2\\
&\quad + \frac{\sigma\tau}{\kappa}\|\nabla\theta_k\|_{\boldsymbol{L}^2}^2 + \frac{1}{\kappa}\|\boldsymbol{v}_k\|_{\boldsymbol{L}_\sigma^2}^2 + \frac{\sigma_0}{2\kappa}\|\mathbb{S}_k\|_{\mathbb{L}_\mathrm{s}^2}^2 + \frac{1}{2\kappa}\|\mathrm{Tr}(\mathbb{S}_k)\|_{L^2}^2\\
D_k &:= m_0\alpha\|\Delta\phi_k\|_{L^2}^2 + \frac{m_0}{4}\|\nabla\mu_k\|_{\boldsymbol{L}^2}^2 + \frac{\sigma\chi_0}{\kappa}\|\nabla\theta_k\|_{\boldsymbol{L}^2}^2 + \frac{\sigma b}{\kappa}\|\Delta\theta_k\|_{L^2}^2 + \frac{\nu_0}{4\kappa}\|\nabla\boldsymbol{v}_k\|_{\mathbb{L}^2}^2\\
&\quad + \frac{\sigma_0\varepsilon_0}{2\kappa}\|\nabla\mathbb{S}_k\|_{(\mathbb{L}_\mathrm{s}^2)^2}^2 + \frac{\sigma_0\beta}{2\kappa}\|\mathrm{Tr}(\mathbb{S}_k)\mathbb{S}_k\|_{\mathbb{L}_\mathrm{s}^2}^2 + \frac{\varepsilon_0}{2\kappa}\|\nabla(\mathrm{Tr}(\mathbb{S}_k))\|_{\boldsymbol{L}^2}^2 + \frac{\beta}{2\kappa}\|\mathrm{Tr}(\mathbb{S}_k)\|_{L^4}^4\\
S &:= \|f_\mathrm{o}\|_{L^2} + \|f_\mathrm{o}\|_{L^2}^2 + \|f_\mathrm{h}\|_{L^2}^2 + \|\boldsymbol{f}_\mathrm{v}\|_{\boldsymbol{L}^2}^2 + \|\mathbb{F}_\mathrm{s}\|_{\mathbb{L}_\mathrm{s}^2}^2 + \|\boldsymbol{u}\|_{\boldsymbol{L}^2}^2 + |\mathbf{g}|^2.
\end{aligned}$$

Recall that $K \in L^1(I)$ and observe that $S \in L^1(I)$ from the assumptions on the source functions. Integrating (3.31) over $[0,t] \subset [0,t_k)$ yields

$$\begin{aligned}
E_k(t) &\le E_k(t) + 2\int_0^t D_k(s)\,\mathrm{d}s\\
&\le E_k(0) + 2c\int_0^t (S(s) + K(s)E_k(s))\,\mathrm{d}s \quad \text{for } t \in I_k. \qquad (3.32)
\end{aligned}$$

From the uniform boundedness of the orthogonal projectors P_{Φ_k}, $\boldsymbol{P}_{\boldsymbol{V}_k}$, and $\mathcal{P}_{\mathcal{S}_k}$, we have

$$E_k(0) \le c(\|\phi_0\|_{W^{1,2}}^2 + \|F(\phi_0)\|_{L^1} + \|\theta_0\|_{W^{1,2}}^2 + \|\boldsymbol{v}_0\|_{\boldsymbol{L}_\sigma^2}^2 + \|\mathbb{S}_0\|_{\mathbb{L}_\mathrm{s}^2}^2). \qquad (3.33)$$

Note that $\|F(\phi_0)\|_{L^1} \le c(\|\phi_0\|_{W^{1,2}}^{q+3} + 1)$ by (2.15) with $s = 3$ and $k = 0$. Applying the Grönwall Lemma to (3.32) and using (3.33), we conclude that there is a continuous function $\mathfrak{C} : [0,\infty) \to [0,\infty)$ as described by the theorem such that

$$\|\phi_k\|_{L^\infty(W^{1,2})} + \|F(\phi_k)\|_{L^\infty(L^1)} + \|\theta_k\|_{L^\infty(W^{1,2})} + \|\boldsymbol{v}_k\|_{L^\infty(\boldsymbol{L}^2_\sigma)} + \|\mathbb{S}_k\|_{L^\infty(\mathbb{L}^2_{\mathrm{s}})} \le c\|E_k\|_{L^\infty(I_k)} \le \mathfrak{C}_{0,\boldsymbol{u}} \tag{3.34}$$

where

$$\mathfrak{C}_{0,\boldsymbol{u}} := \mathfrak{C}(\|(\phi_0,\theta_0,\boldsymbol{v}_0,\mathbb{S}_0)\|_{W^{1,2}\times W^{1,2}\times \boldsymbol{L}^2_\sigma\times\mathbb{L}^2_{\mathrm{s}}} + \|\boldsymbol{u}\|_{\boldsymbol{U}}).$$

We point out that the function $\mathfrak{C}$ may differ in each appearance below. As a consequence of (3.32)–(3.34), we have the following estimate

$$\|\Delta\phi_k\|_{L^2(L^2)} + \|\nabla\mu_k\|_{L^2(\boldsymbol{L}^2)} + \|\Delta\theta_k\|_{L^2(L^2)} + \|\nabla\boldsymbol{v}_k\|_{L^2(\mathbb{L}^2)} + \|\nabla\mathbb{S}_k\|_{L^2((\mathbb{L}^2_{\mathrm{s}})^2)} \le c\|D_k\|_{L^1(I_k)} \le E_k(0) + c(\|S\|_{L^1(I)} + \|K\|_{L^1(I)}\|E_k\|_{L^\infty(I_k)}) \le \mathfrak{C}_{0,\boldsymbol{u}}. \tag{3.35}$$

From (3.17), (3.34), (3.35), and the Poincaré–Wirtinger inequality (2.2), we obtain

$$\|\mu_k\|_{L^2(W^{1,2})} \le c(\|\langle\mu\rangle\|_{L^2(I_k)} + \|\nabla\mu_k\|_{L^2(\boldsymbol{L}^2)}) \le \mathfrak{C}_{0,\boldsymbol{u}}. \tag{3.36}$$

Step 2. *Additional estimates on spatial derivatives.* This step focuses in deriving a priori estimates on higher-order spatial derivatives of the approximations.

$L^\infty(L^2)$, $L^2(L^2)$, and $L^2(L^2)$ estimates for $\Delta\phi_k$, $\Delta^2\phi_k$, and $\Delta\mu_k$. First, let us consider the Cahn–Hilliard equation. Take the test function $\Delta^2\phi_k$ to the first equation in (3.12) and distribute the divergence operator so that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta\phi_k|^2\,\mathrm{d}x + \int_\Omega \boldsymbol{v}_k\cdot\nabla\phi_k\Delta^2\phi_k\,\mathrm{d}x - \int_\Omega m_{k\phi}\nabla\phi_k\cdot\nabla\mu_k\Delta^2\phi_k\,\mathrm{d}x - \int_\Omega m_{k\theta}\nabla\theta_k\cdot\nabla\mu_k\Delta^2\phi_k\,\mathrm{d}x - \int_\Omega m_k\Delta\mu_k\Delta^2\phi_k\,\mathrm{d}x = \int_\Omega f_{\mathrm{o}}\Delta^2\phi_k\,\mathrm{d}x, \tag{3.37}$$

where $m_{k\phi} := m_\phi(\phi_k,\theta_k)$ and $m_{k\theta} := m_\theta(\phi_k,\theta_k)$. Without further notice, we also follow this kind of notation for the other state-dependent coefficients χ, ν, and ε.

By the Hölder and Young inequalities, the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$, and (2.3), we have the following estimates for the second and last integrals in equation (3.37):

$$\int_\Omega |f_{\mathrm{o}}\Delta^2\phi_k|\,\mathrm{d}x \le \delta\|\Delta^2\phi_k\|^2_{L^2} + c_\delta\|f_{\mathrm{o}}\|^2_{L^2} \tag{3.38}$$

$$\int_\Omega |(\boldsymbol{v}_k\cdot\nabla\phi_k)\Delta^2\phi_k|\,\mathrm{d}x \le \|\boldsymbol{v}_k\|_{\boldsymbol{L}^4}\|\nabla\phi_k\|_{\boldsymbol{L}^4}\|\Delta^2\phi_k\|_{L^2} \le \delta\|\Delta^2\phi_k\|^2_{L^2} + c_\delta\|\nabla\boldsymbol{v}_k\|^2_{\mathbb{L}^2}\|\Delta\phi_k\|^2_{L^2}. \tag{3.39}$$

Taking the Laplacian of μ_k in the second equation of (3.12) and using $\Delta P_{\Phi_k}F'(\phi_k) = P_{\Phi_k}\Delta F'(\phi_k)$, which can be easily shown by using the expansion of P_{Φ_k} in terms of the eigenfunctions of the Neumann Laplacian and by integrating by parts, we have

$$\Delta\mu_k = -\alpha\Delta^2\phi_k + P_{\Phi_k}\Delta F'(\phi_k). \tag{3.40}$$

Thus, the fifth integral in (3.37) can be bounded from below as

$$-\int_\Omega m_k\Delta\mu_k\Delta^2\phi_k\,\mathrm{d}x \ge \frac{\alpha m_0}{2}\|\Delta^2\phi_k\|^2_{L^2} - c|m|^2_\infty\|\Delta F'(\phi_k)\|^2_{L^2}. \tag{3.41}$$

Note that $\Delta F'(\phi_k) = \nabla \cdot (F''(\phi_k)\nabla\phi_k) = F''(\phi_k)\Delta\phi_k + F'''(\phi_k)|\nabla\phi_k|^2$. Thus, applying the Hölder inequality, the estimate (2.15), the Agmon inequality (2.10), and (2.6), we get

$$\begin{aligned}\|F''(\phi_k)\Delta\phi_k\|_{L^2}^2 &\le \|F''(\phi_k)\|_{L^2}^2\|\Delta\phi_k\|_{L^\infty}^2\\ &\le c(\|\phi_k\|_{W^{1,2}}^{2(q+1)}+1)\|\Delta\phi_k\|_{L^2}\|\Delta^2\phi_k\|_{L^2}\\ &\le \delta\|\Delta^2\phi_k\|_{L^2}^2 + c_\delta(\|\phi_k\|_{W^{1,2}}^{4(q+1)}+1)\|\Delta\phi_k\|_{L^2}^2. \end{aligned} \tag{3.42}$$

Similarly, by the Hölder inequality, (2.15), the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, and (2.3), we have

$$\begin{aligned}\|F'''(\phi_k)|\nabla\phi_k|^2\|_{L^2}^2 &\le c\|F'''(\phi_k)\|_{L^6}^2\|\nabla\phi_k\|_{\boldsymbol{L}^6}^4\\ &\le c(\|\phi_k\|_{W^{1,2}}^{2q}+1)\|\Delta\phi_k\|_{L^2}^4. \end{aligned} \tag{3.43}$$

Set $K_{1k} := \|\phi_k\|_{W^{1,2}}^{4(q+1)} + \|\phi_k\|_{W^{1,2}}^{2q} + 1$, so that $K_{1k} \in L^\infty(I_k)$ by (3.34). Thus, we deduce from (3.42) and (3.43) that

$$\begin{aligned}\|\Delta F'(\phi_k)\|_{L^2}^2 &\le 2\|F''(\phi_k)\Delta\phi_k\|_{L^2}^2 + 2\|F'''(\phi_k)|\nabla\phi_k|^2\|_{L^2}^2\\ &\le 2\delta\|\Delta^2\phi_k\|_{L^2}^2 + c_\delta K_{1k}(\|\Delta\phi_k\|_{L^2}^2 + \|\Delta\phi_k\|_{L^2}^4). \end{aligned} \tag{3.44}$$

Furthermore, according to (3.40) and (3.44) with $\delta = 1$, one has

$$\|\Delta\mu_k\|_{L^2}^2 - c\|\Delta^2\phi_k\|_{L^2}^2 \le cK_{1k}(\|\Delta\phi_k\|_{L^2}^2 + \|\Delta\phi_k\|_{L^2}^4). \tag{3.45}$$

Now, let us estimate the remaining integrals in (3.37) that involve the derivatives of m. Using the Gagliardo–Nirenberg inequality (2.9) and (2.3), we deduce that

$$\begin{aligned}&\int_\Omega |m_{k\phi}\nabla\phi_k\cdot\nabla\mu_k\Delta^2\phi_k|\,\mathrm{d}x \le c|m_\phi|_\infty\|\nabla\phi_k\|_{\boldsymbol{L}^4}\|\nabla\mu_k\|_{\boldsymbol{L}^4}\|\Delta^2\phi_k\|_{L^2}\\ &\quad\le c|m_\phi|_\infty\|\nabla\phi_k\|_{\boldsymbol{L}^2}^{1/2}\|\Delta\phi_k\|_{L^2}^{1/2}\|\nabla\mu_k\|_{\boldsymbol{L}^2}^{1/2}\|\Delta\mu_k\|_{L^2}^{1/2}\|\Delta^2\phi_k\|_{L^2}\\ &\quad\le \delta\|\Delta^2\phi_k\|_{L^2}^2 + \delta\|\Delta\mu_k\|_{L^2}^2 + c_\delta|m_\phi|_\infty^4\|\nabla\mu_k\|_{\boldsymbol{L}^2}^2\|\nabla\phi_k\|_{\boldsymbol{L}^2}^2\|\Delta\phi_k\|_{L^2}^2. \end{aligned} \tag{3.46}$$

Replacing $m_{k\phi}$ and $\nabla\phi_k$ by $m_{k\theta}$ and $\nabla\theta_k$ respectively in this estimate leads to

$$\begin{aligned}\int_\Omega |m_{k\theta}\nabla\theta_k\cdot\nabla\mu_k\Delta^2\phi_k|\,\mathrm{d}x &\le \delta\|\Delta^2\phi_k\|_{L^2}^2 + \delta\|\Delta\mu_k\|_{L^2}^2\\ &\quad + c_\delta|m_\theta|_\infty^4\|\nabla\mu_k\|_{\boldsymbol{L}^2}^2\|\nabla\theta_k\|_{\boldsymbol{L}^2}^2\|\Delta\theta_k\|_{L^2}^2. \end{aligned} \tag{3.47}$$

Using the estimates (3.38), (3.39), (3.44), (3.46), and (3.47) in (3.37), and then taking the sum of the resulting inequality with (3.45) multiplied by $2\delta_0 > 0$, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Delta\phi_k\|_{L^2}^2 + \left(\frac{\alpha m_0}{2} - 4\delta - 2c\delta|m|_\infty^2 - 2c\delta_0\right)\|\Delta^2\phi_k\|_{L^2}^2 + 2(\delta_0-\delta)\|\Delta\mu_k\|_{L^2}^2\\ &\quad\le c_{\delta,\delta_0}[K_{2k}(\|\Delta\phi_k\|_{L^2}^2 + \|\Delta\theta_k\|_{L^2}^2) + \|f_\circ\|_{L^2}^2] \end{aligned} \tag{3.48}$$

where

$$\begin{aligned}K_{2k} := K_{1k}\{&\|\Delta\phi_k\|_{L^2}^2 + |m_\phi|_\infty^4\|\nabla\mu_k\|_{\boldsymbol{L}^2}^2\|\nabla\phi_k\|_{\boldsymbol{L}^2}^2\\ &+ |m_\theta|_\infty^4\|\nabla\mu_k\|_{\boldsymbol{L}^2}^2\|\nabla\theta_k\|_{\boldsymbol{L}^2}^2 + \|\nabla\boldsymbol{v}_k\|_{\mathbb{L}^2}^2 + 1\}.\end{aligned}$$

Notice that $K_{2k} \in L^1(I_k)$ according to (3.34) and (3.35). If we choose $0 < \delta < \delta_0 < \alpha m_0/(8 + 4c|m|_\infty^2 + 4c)$ in (3.48), then it follows that there is $c > 0$ such that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\Delta\phi_k\|_{L^2}^2 + \frac{1}{c}\|\Delta^2\phi_k\|_{L^2}^2 + \frac{1}{c}\|\Delta\mu_k\|_{L^2}^2 \\ &\quad \le c[K_{2k}(\|\Delta\phi_k\|_{L^2}^2 + \|\Delta\theta_k\|_{L^2}^2) + \|f_{\mathrm{o}}\|_{L^2}^2]. \end{aligned} \tag{3.49}$$

$L^\infty(W^{1,2})$ and $L^2(L^2)$ estimates for $\Delta\theta_k$ and $\Delta^2\theta_k$. Now, we consider the regularized convection-diffusion equation. The test function $\Delta^2\theta_k$ applied to third equation in (3.12) along with the same procedure presented above for the Cahn–Hilliard system give us

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega (|\Delta\theta_k|^2 + \tau|\nabla\Delta\theta_k|^2)\,\mathrm{d}x + \int_\Omega \boldsymbol{v}_k\cdot\nabla\theta_k\Delta^2\theta_k\,\mathrm{d}x \\ &\quad - \int_\Omega \chi_{k\phi}\nabla\phi_k\cdot\nabla\theta_k\Delta^2\theta_k\,\mathrm{d}x - \int_\Omega \chi_{k\theta}|\nabla\theta_k|^2\Delta^2\theta_k\,\mathrm{d}x \\ &\quad - \int_\Omega \chi_k\Delta\theta_k\Delta^2\theta_k\,\mathrm{d}x + \int_\Omega b|\Delta^2\theta_k|^2\,\mathrm{d}x \\ &\quad = \int_\Omega a_0\mathbf{g}\cdot\boldsymbol{v}_k\Delta^2\theta_k\,\mathrm{d}x + \int_\Omega \mathbb{S}_k : \mathbb{D}\boldsymbol{v}_k\Delta^2\theta_k\,\mathrm{d}x + \int_\Omega f_{\mathrm{h}}\Delta^2\theta_k\,\mathrm{d}x. \end{aligned} \tag{3.50}$$

We can estimate the fifth integral by Young inequality and the term involving the heat source function and the convection term as in (3.38) and (3.39) as follows:

$$\int_\Omega |\chi_k\Delta\theta_k\Delta^2\theta_k|\,\mathrm{d}x \le \delta\|\Delta^2\theta_k\|_{L^2}^2 + c_\delta|\chi|_\infty^2\|\Delta\theta_k\|_{L^2}^2 \tag{3.51}$$

$$\int_\Omega |a_0\mathbf{g}\cdot\boldsymbol{v}_k\Delta^2\theta_k|\,\mathrm{d}x \le \delta\|\Delta^2\theta_k\|_{L^2}^2 + c_\delta|a_0|^2|\mathbf{g}|^2\|\nabla\boldsymbol{v}_k\|_{\mathbb{L}^2}^2 \tag{3.52}$$

$$\int_\Omega |f_{\mathrm{h}}\Delta^2\theta_k|\,\mathrm{d}x \le \delta\|\Delta^2\theta_k\|_{L^2}^2 + c_\delta\|f_{\mathrm{h}}\|_{L^2}^2 \tag{3.53}$$

$$\int_\Omega |(\boldsymbol{v}_k\cdot\nabla\theta_k)\Delta^2\theta_k|\,\mathrm{d}x \le \delta\|\Delta^2\theta_k\|_{L^2}^2 + c_\delta\|\nabla\boldsymbol{v}_k\|_{\mathbb{L}^2}^2\|\Delta\theta_k\|_{L^2}^2. \tag{3.54}$$

From the Hölder inequality, the embedding $W^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$, and (2.3), it holds that

$$\begin{aligned} \int_\Omega |\chi_{k\phi}\nabla\phi_k\cdot\nabla\theta_k\Delta^2\theta_k|\,\mathrm{d}x &\le c|\chi_\phi|_\infty\|\nabla\phi_k\|_{\boldsymbol{L}^4}\|\nabla\theta_k\|_{\boldsymbol{L}^4}\|\Delta^2\theta_k\|_{L^2} \\ &\le \delta\|\Delta^2\theta_k\|_{L^2}^2 + c_\delta|\chi_\phi|_\infty^2\|\Delta\phi_k\|_{L^2}^2\|\Delta\theta_k\|_{L^2}^2. \end{aligned} \tag{3.55}$$

By a similar argument, it holds that

$$\int_\Omega \chi_{k\theta}|\nabla\theta_k|^2\Delta^2\theta_k\,\mathrm{d}x \le \delta\|\Delta^2\theta_k\|_{L^2}^2 + c_\delta|\chi_\theta|_\infty^2\|\Delta\theta_k\|_{L^2}^4. \tag{3.56}$$

For the remaining term in (3.50) involving the velocity and viscoelastic tensor, we have

$$\begin{aligned} \int_\Omega |\mathbb{S}_k : \mathbb{D}\boldsymbol{v}_k\Delta^2\theta_k|\,\mathrm{d}x &\le c\|\mathbb{S}_k\|_{\mathbb{L}^4}\|\mathbb{D}\boldsymbol{v}_k\|_{\mathbb{L}^4}\|\Delta^2\theta_k\|_{L^2} \\ &\le c\|\mathbb{S}_k\|_{\mathbb{L}^4}\|\nabla\boldsymbol{v}_k\|_{\mathbb{L}^2}^{1/2}\|\Delta\boldsymbol{v}_k\|_{\boldsymbol{L}^2}^{1/2}\|\Delta^2\theta_k\|_{L^2} \end{aligned}$$

$$\leq \delta\|\Delta^2\theta_k\|_{L^2} + \frac{\nu_0}{8}\|\Delta \boldsymbol{v}_k\|_{\boldsymbol{L}^2}^2 + c_\delta\|\mathbb{S}_k\|_{\mathbb{L}_s^4}^4\|\nabla \boldsymbol{v}_k\|_{\mathbb{L}^2}^2. \tag{3.57}$$

Utilizing the estimates (3.51)–(3.57) in the equation (3.50), and then taking $0 < \delta < \frac{b}{7}$, one can see that there is a $c > 0$ such that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\Delta\theta_k\|_{L^2}^2 + \tau\|\nabla\Delta\theta_k\|_{\boldsymbol{L}^2}^2) + \frac{1}{c}\|\Delta^2\theta_k\|_{L^2} - \frac{\nu_0}{8}\|\Delta \boldsymbol{v}_k\|_{\boldsymbol{L}^2}^2 \\ \leq c[K_{3k}(\|\Delta\theta_k\|_{L^2}^2 + \|\nabla \boldsymbol{v}_k\|_{\mathbb{L}^2}^2) + \|f_{\mathrm{h}}\|_{L^2}^2]\end{aligned} \tag{3.58}$$

where

$$\begin{aligned}K_{3k} := |\chi|_\infty^2 + |a_0|^2|\mathbf{g}|^2 + \|\nabla \boldsymbol{v}_k\|_{\mathbb{L}^2}^2 \\ + |\chi_\phi|_\infty^2\|\Delta\phi_k\|_{L^2}^2 + |\chi_\theta|_\infty^2\|\Delta\theta_k\|_{L^2}^2 + \|\mathbb{S}_k\|_{\mathbb{L}_s^4}^4 + 1.\end{aligned}$$

Note that $K_{3k} \in L^1(I_k)$ thanks to (3.35) and $\|\mathbb{S}_k\|_{\mathbb{L}_s^4(\Omega)}^4 \in L^1(I_k)$ due to following embedding

$$L^\infty(I; L^2(\Omega)) \cap L^2(I; W^{1,2}(\Omega)) \hookrightarrow L^4(I; L^4(\Omega)). \tag{3.59}$$

$L^\infty(\mathbb{L}^2)$ and $L^2(\boldsymbol{L}_\sigma^2)$ estimates for $\nabla \boldsymbol{v}_k$ and $\Delta \boldsymbol{v}_k$. Next, we deal with the approximate Navier–Stokes equation. With the test function $\boldsymbol{A}_S\boldsymbol{v}_k = -\boldsymbol{P}_\sigma\Delta\boldsymbol{v}_k = -\Delta\boldsymbol{v}_k$ applied to the fourth equation in (3.12), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla \boldsymbol{v}_k|^2\,\mathrm{d}x - \int_\Omega (\boldsymbol{v}_k\cdot\nabla)\boldsymbol{v}_k\cdot\Delta\boldsymbol{v}_k\,\mathrm{d}x \\ &\quad + \int_\Omega 2\nu_{k\phi}(\mathbb{D}\boldsymbol{v}_k\nabla\phi_k)\cdot\Delta\boldsymbol{v}_k\,\mathrm{d}x + \int_\Omega 2\nu_{k\theta}(\mathbb{D}\boldsymbol{v}_k\nabla\theta_k)\cdot\Delta\boldsymbol{v}_k\,\mathrm{d}x + \int_\Omega \nu_k|\Delta\boldsymbol{v}_k|^2\,\mathrm{d}x \\ &\quad = -\int_\Omega (\sigma_0 a\nabla\cdot(\mathbb{S}_k^2))\cdot\Delta\boldsymbol{v}_k\,\mathrm{d}x + \int_\Omega (\sigma_0 a\nabla\cdot\mathbb{S}_k)\cdot\Delta\boldsymbol{v}_k\,\mathrm{d}x \\ &\quad - \int_\Omega (\nabla\cdot(\sigma\theta_k\mathbb{S}_k))\cdot\Delta\boldsymbol{v}_k\,\mathrm{d}x - \int_\Omega (a\nabla\cdot(\mathrm{Tr}(\mathbb{S}_k)\mathbb{S}_k))\cdot\Delta\boldsymbol{v}_k\,\mathrm{d}x \\ &\quad - \int_\Omega (\kappa\mu_k\nabla\phi_k + \rho(\phi_k,\theta_k)\mathbf{g} + \boldsymbol{f}_{\mathrm{v}} + \boldsymbol{u})\cdot\Delta\boldsymbol{v}_k\,\mathrm{d}x.\end{aligned} \tag{3.60}$$

In the fifth integral, we used the fact that $\nabla\cdot(2\mathbb{D}\boldsymbol{v}_k) = \Delta\boldsymbol{v}_k + \nabla\nabla\cdot\boldsymbol{v}_k = \Delta\boldsymbol{v}_k$ since $\nabla\cdot\boldsymbol{v}_k = 0$ in Ω_T. For the convection and surface tension terms, we apply the Hölder, Gagliardo–Nirenberg, and Young inequalities, and the embedding $W^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$, so that

$$\begin{aligned}\int_\Omega |(\boldsymbol{v}_k\cdot\nabla)\boldsymbol{v}_k\cdot\Delta\boldsymbol{v}_k|\,\mathrm{d}x &\leq \|\boldsymbol{v}_k\|_{\boldsymbol{L}^4}\|\nabla\boldsymbol{v}_k\|_{\mathbb{L}^4}\|\Delta\boldsymbol{v}_k\|_{\boldsymbol{L}^2} \\ &\leq c\|\boldsymbol{v}_k\|_{\boldsymbol{L}_\sigma^4}\|\nabla\boldsymbol{v}_k\|_{\mathbb{L}^2}^{1/2}\|\Delta\boldsymbol{v}_k\|_{\boldsymbol{L}^2}^{3/2} \leq \delta\|\Delta\boldsymbol{v}_k\|_{\boldsymbol{L}^2}^2 + c_\delta\|\boldsymbol{v}_k\|_{\boldsymbol{L}_\sigma^4}^4\|\nabla\boldsymbol{v}_k\|_{\mathbb{L}^2}^2\end{aligned} \tag{3.61}$$

$$\begin{aligned}\int_\Omega |\kappa\mu_k\nabla\phi_k\cdot\Delta\boldsymbol{v}_k|\,\mathrm{d}x &\leq c\|\mu_k\|_{L^4}\|\nabla\phi_k\|_{\boldsymbol{L}^4}\|\Delta\boldsymbol{v}_k\|_{\boldsymbol{L}^2} \\ &\leq \delta\|\Delta\boldsymbol{v}_k\|_{\boldsymbol{L}^2}^2 + c_\delta\|\mu_k\|_{W^{1,2}}^2\|\Delta\phi_k\|_{L^2}^2\end{aligned} \tag{3.62}$$

$$\int_\Omega |\rho(\phi_k,\theta_k)\mathbf{g}\cdot\Delta\boldsymbol{v}_k|\,\mathrm{d}x \leq \delta\|\Delta\boldsymbol{v}_k\|_{\boldsymbol{L}^2}^2 + c_\delta(\|\phi_k\|_{L^2}^2 + \|\theta_k\|_{L^2}^2 + |\mathbf{g}|^2). \tag{3.63}$$

Concerning the integrals involving the derivatives of the viscosity coefficient ν in (3.60), we have

$$\begin{aligned}\int_\Omega |2\nu_{k\phi}(\mathbb{D}\boldsymbol{v}_k\nabla\phi_k)\cdot\Delta\boldsymbol{v}_k|\,\mathrm{d}x &\le c|\nu_\phi|_\infty\|\mathbb{D}\boldsymbol{v}_k\|_{\mathbb{L}^4}\|\nabla\phi_k\|_{\boldsymbol{L}^4}\|\Delta\boldsymbol{v}_k\|_{\boldsymbol{L}^2}\\ &\le c|\nu_\phi|_\infty\|\nabla\phi_k\|_{\boldsymbol{L}^4}\|\nabla\boldsymbol{v}_k\|_{\mathbb{L}^2}^{1/2}\|\Delta\boldsymbol{v}_k\|_{\boldsymbol{L}^2}^{3/2}\\ &\le \delta\|\Delta\boldsymbol{v}_k\|_{\boldsymbol{L}^2}^2 + c_\delta|\nu_\phi|_\infty^4\|\nabla\phi_k\|_{\boldsymbol{L}^4}^4\|\nabla\boldsymbol{v}_k\|_{\mathbb{L}^2}^2 \end{aligned}\tag{3.64}$$

$$\int_\Omega |2\nu_{k\theta}(\mathbb{D}\boldsymbol{v}_k\nabla\theta_k)\cdot\Delta\boldsymbol{v}_k|\,\mathrm{d}x \le \delta\|\Delta\boldsymbol{v}_k\|_{\boldsymbol{L}^2}^2 + c_\delta|\nu_\theta|_\infty^4\|\nabla\theta_k\|_{\boldsymbol{L}^4}^4\|\nabla\boldsymbol{v}_k\|_{\mathbb{L}^2}^2. \tag{3.65}$$

Next, let us deal with the integrals in (3.60) that include the sources and viscoelastic tensor. Performing the divergence operator and applying Hölder inequality, we obtain

$$\begin{aligned}\int_\Omega |(\sigma_0 a\nabla\cdot(\mathbb{S}_k^2))\cdot\Delta\boldsymbol{v}_k|\,\mathrm{d}x &\le c\|\mathbb{S}_k\|_{\mathbb{L}^4_{\mathrm{s}}}\|\nabla\mathbb{S}_k\|_{(\mathbb{L}^4_{\mathrm{s}})^2}\|\Delta\boldsymbol{v}_k\|_{\boldsymbol{L}^2}\\ &\le c\|\mathbb{S}_k\|_{\mathbb{L}^4_{\mathrm{s}}}\|\nabla\mathbb{S}_k\|_{(\mathbb{L}^2_{\mathrm{s}})^2}^{1/2}\|\Delta\mathbb{S}_k\|_{\mathbb{L}^2_{\mathrm{s}}}^{1/2}\|\Delta\boldsymbol{v}_k\|_{\boldsymbol{L}^2}\\ &\le \delta\|\Delta\boldsymbol{v}_k\|_{\boldsymbol{L}^2}^2 + \frac{\varepsilon_0}{12}\|\Delta\mathbb{S}_k\|_{\mathbb{L}^2_{\mathrm{s}}}^2 + c_\delta\|\mathbb{S}_k\|_{\mathbb{L}^4_{\mathrm{s}}}^4\|\nabla\mathbb{S}_k\|_{(\mathbb{L}^2_{\mathrm{s}})^2}^2 \end{aligned}\tag{3.66}$$

$$\int_\Omega |\sigma_0 a(\nabla\cdot\mathbb{S}_k)\cdot\Delta\boldsymbol{v}_k|\,\mathrm{d}x \le \delta\|\Delta\boldsymbol{v}_k\|_{\boldsymbol{L}^2}^2 + c_\delta\|\nabla\mathbb{S}_k\|_{(\mathbb{L}^2_{\mathrm{s}})^2}^2 \tag{3.67}$$

$$\begin{aligned}\int_\Omega |\sigma(\nabla\cdot(\theta_k\mathbb{S}_k))\cdot\Delta\boldsymbol{v}_k|\,\mathrm{d}x &\le c(\|\theta_k\|_{L^4}\|\nabla\mathbb{S}_k\|_{(\mathbb{L}^4_{\mathrm{s}})^2} + \|\nabla\theta_k\|_{\boldsymbol{L}^4}\|\mathbb{S}_k\|_{\mathbb{L}^4_{\mathrm{s}}})\|\Delta\boldsymbol{v}_k\|_{\boldsymbol{L}^2}\\ &\le \delta\|\Delta\boldsymbol{v}_k\|_{\boldsymbol{L}^2}^2 + \frac{\varepsilon_0}{12}\|\Delta\mathbb{S}_k\|_{\mathbb{L}^2_{\mathrm{s}}}^2 + c_\delta\|\theta_k\|_{L^4}^4\|\nabla\mathbb{S}_k\|_{(\mathbb{L}^2_{\mathrm{s}})^2}^2\\ &\quad + c_\delta\|\mathbb{S}_k\|_{\mathbb{L}^4_{\mathrm{s}}}^2\|\Delta\theta_k\|_{L^2}^2 \end{aligned}\tag{3.68}$$

$$\begin{aligned}\int_\Omega |a(\nabla\cdot(\mathrm{Tr}(\mathbb{S}_k)\mathbb{S}_k))\cdot\Delta\boldsymbol{v}_k|\,\mathrm{d}x &\le \delta\|\Delta\boldsymbol{v}_k\|_{\boldsymbol{L}^2}^2 + \frac{\varepsilon_0}{12}\|\Delta\mathbb{S}_k\|_{\mathbb{L}^2_{\mathrm{s}}}^2\\ &\quad + c_\delta\|\mathbb{S}_k\|_{\mathbb{L}^4_{\mathrm{s}}}^4\|\nabla\mathbb{S}_k\|_{(\mathbb{L}^2_{\mathrm{s}})^2}^2 \end{aligned}\tag{3.69}$$

$$\int_\Omega |(\boldsymbol{f}_{\mathrm{v}}+\boldsymbol{u})\cdot\Delta\boldsymbol{v}_k|\,\mathrm{d}x \le \delta\|\Delta\boldsymbol{v}_k\|_{\boldsymbol{L}^2}^2 + c_\delta(\|\boldsymbol{f}_{\mathrm{v}}\|_{\boldsymbol{L}^2}^2 + \|\boldsymbol{u}\|_{\boldsymbol{L}^2}^2). \tag{3.70}$$

Thus, if we apply the estimates (3.61)–(3.70) in the equation (3.60) and choosing $0<\delta<\frac{\nu_0}{20}$, it follows that there exists a constant $c>0$ for which

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{v}_k\|_{\mathbb{L}^2}^2 + \frac{\nu_0}{2}\|\Delta\boldsymbol{v}_k\|_{\boldsymbol{L}^2}^2 - \frac{\varepsilon_0}{4}\|\Delta\mathbb{S}_k\|_{\mathbb{L}^2_{\mathrm{s}}}^2 \le cK_{4k}(\|\Delta\phi_k\|_{L^2}^2 + \|\Delta\theta_k\|_{L^2}^2)\\ &\quad + cK_{4k}(\|\phi_k\|_{L^2}^2 + \|\theta_k\|_{L^2}^2 + \|\nabla\boldsymbol{v}_k\|_{\mathbb{L}^2}^2 + \|\nabla\mathbb{S}_k\|_{(\mathbb{L}^2_{\mathrm{s}})^2}^2)\\ &\quad + c(\|\boldsymbol{f}_{\mathrm{v}}\|_{\boldsymbol{L}^2}^2 + \|\boldsymbol{u}\|_{\boldsymbol{L}^2}^2 + |\mathbf{g}|^2) \end{aligned}\tag{3.71}$$

where

$$\begin{aligned}K_{4k} := \|\boldsymbol{v}_k\|_{\boldsymbol{L}^4_\sigma}^4 + \|\mu_k\|_{W^{1,2}}^2 + |\nu_\phi|_\infty^4\|\nabla\phi_k\|_{\boldsymbol{L}^4}^4 + |\nu_\theta|_\infty^4\|\nabla\theta_k\|_{\boldsymbol{L}^4}^4\\ + \|\mathbb{S}_k\|_{\mathbb{L}^4_{\mathrm{s}}}^4 + \|\mathbb{S}_k\|_{\mathbb{L}^4_{\mathrm{s}}}^2 + \|\theta_k\|_{L^4}^4 + 1.\end{aligned}$$

From the embedding (3.59) it follows that $\|\theta_k\|_{L^4}^4, \|\boldsymbol{v}_k\|_{\boldsymbol{L}^4_\sigma}^4, \|\mathbb{S}_k\|_{\mathbb{L}^4_{\mathrm{s}}}^4 \in L^1(I_k)$, and in particular, $\|\mathbb{S}_k\|_{\mathbb{L}^4_{\mathrm{s}}}^2 \in L^2(I_k)\subset L^1(I_k)$. Analogously, one has $\|\nabla\phi_k\|_{\boldsymbol{L}^4}^4, \|\nabla\theta_k\|_{\boldsymbol{L}^4}^4 \in$

$L^1(I_k)$ due to

$$L^\infty(I; W^{1,2}(\Omega)) \cap L^2(I; W^{2,2}(\Omega)) \hookrightarrow L^4(I; W^{1,4}(\Omega)). \tag{3.72}$$

These observations together with (3.35) and (3.36) imply that $K_{4k} \in L^1(I_k)$.

$L^\infty((\mathbb{L}^2_s)^2)$ and $L^2(\mathbb{L}^2_s)$ estimates for $\nabla\mathbb{S}_k$ and $\Delta\mathbb{S}_k$. For the last part of this step, we apply the test function $-\Delta\mathbb{S}_k$ to the fifth equation of (3.12) in order to obtain the following:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\mathbb{S}_k|^2\,\mathrm{d}x &- \int_\Omega (\boldsymbol{v}_k\cdot\nabla)\mathbb{S}_k : \Delta\mathbb{S}_k\,\mathrm{d}x - \int_\Omega \mathbb{J}(\boldsymbol{v}_k,\mathbb{S}_k) : \Delta\mathbb{S}_k\,\mathrm{d}x\\ &+ \int_\Omega \varepsilon_{k\phi}\nabla\mathbb{S}_k\boxtimes\nabla\phi_k : \Delta\mathbb{S}_k\,\mathrm{d}x + \int_\Omega \varepsilon_{k\theta}\nabla\mathbb{S}_k\boxtimes\nabla\theta_k : \Delta\mathbb{S}_k\,\mathrm{d}x\\ &+ \int_\Omega \varepsilon_k|\Delta\mathbb{S}_k|^2\,\mathrm{d}x = -\int_\Omega \mathbb{P}(\mathbb{S}_k) : \Delta\mathbb{S}_k\,\mathrm{d}x - \int_\Omega (\lambda\mathbb{D}\boldsymbol{v} + \mathbb{F}_s) : \Delta\mathbb{S}_k\,\mathrm{d}x,\end{aligned} \tag{3.73}$$

where $\mathfrak{T}\boxtimes\boldsymbol{w} := w_1\mathbb{T}_1 + w_2\mathbb{T}_2$ for $\mathfrak{T} = [\mathbb{T}_1\ \mathbb{T}_2]^t \in \mathbb{R}^{4\times 2}$ and $\boldsymbol{w} = [w_1\ w_2]^t \in \mathbb{R}^2$. Similar to (3.61), (3.64), and (3.65), it can be deduced that

$$\int_\Omega |(\boldsymbol{v}_k\cdot\nabla)\mathbb{S}_k : \Delta\mathbb{S}_k|\,\mathrm{d}x \le \delta\|\Delta\mathbb{S}_k\|^2_{\mathbb{L}^2_s} + c\|\boldsymbol{v}_k\|^4_{\boldsymbol{L}^4_\sigma}\|\nabla\mathbb{S}_k\|^2_{(\mathbb{L}^2_s)^2} \tag{3.74}$$

$$\int_\Omega |\varepsilon_{k\phi}\nabla\mathbb{S}_k\boxtimes\nabla\phi_k : \Delta\mathbb{S}_k|\,\mathrm{d}x \le \delta\|\Delta\mathbb{S}_k\|^2_{\mathbb{L}^2_s} + c_\delta|\varepsilon_\phi|^4_\infty\|\nabla\phi_k\|^4_{\boldsymbol{L}^4}\|\nabla\mathbb{S}_k\|^2_{(\mathbb{L}^2_s)^2} \tag{3.75}$$

$$\int_\Omega |\varepsilon_{k\theta}\nabla\mathbb{S}_k\boxtimes\nabla\theta_k : \Delta\mathbb{S}_k|\,\mathrm{d}x \le \delta\|\Delta\mathbb{S}_k\|^2_{\mathbb{L}^2_s} + c_\delta|\varepsilon_\theta|^4_\infty\|\nabla\theta_k\|^4_{\boldsymbol{L}^4}\|\nabla\mathbb{S}_k\|^2_{(\mathbb{L}^2_s)^2}. \tag{3.76}$$

Finally, for the integrals involving the commutator, anti-commutator, trace operator, and those on the right-hand sides in (3.73), we apply the Gagliardo–Nirenberg and Young inequalities so that

$$\int_\Omega |\mathbb{J}(\boldsymbol{v}_k,\mathbb{S}_k) : \Delta\mathbb{S}_k|\,\mathrm{d}x \le \delta\|\Delta\mathbb{S}_k\|^2_{\mathbb{L}^2_s} + \frac{\nu_0}{8}\|\Delta\boldsymbol{v}_k\|^2_{\boldsymbol{L}^2} + c_\delta\|\mathbb{S}_k\|^4_{\mathbb{L}^4_s}\|\nabla\boldsymbol{v}_k\|^2_{\mathbb{L}^2} \tag{3.77}$$

$$\int_\Omega |(\lambda\mathbb{D}\boldsymbol{v}_k + \mathbb{F}_s) : \Delta\mathbb{S}_k|\,\mathrm{d}x \le \delta\|\Delta\mathbb{S}_k\|^2_{\mathbb{L}^2_s} + c_\delta(\lambda^2\|\nabla\boldsymbol{v}_k\|^2_{\mathbb{L}^2} + \|\mathbb{F}_s\|^2_{\mathbb{L}^2_s}) \tag{3.78}$$

$$\int_\Omega |\mathbb{P}(\mathbb{S}_k) : \Delta\mathbb{S}_k|\,\mathrm{d}x \le \delta\|\Delta\mathbb{S}_k\|^2_{\mathbb{L}^2_s} + c_\delta(\|\mathbb{S}_k\|^2_{\mathbb{L}^2_s} + \|\mathbb{S}_k\|^6_{\mathbb{L}^6_s}). \tag{3.79}$$

Using the general version of the Gagliardo–Nirenberg inequality (2.11), we obtain

$$\begin{aligned}\|\mathbb{S}_k\|^6_{\mathbb{L}^6_s} &\le c(\|\mathbb{S}_k\|_{\mathbb{L}^2_s} + \|\mathbb{S}_k\|^{1/3}_{\mathbb{L}^2_s}\|\nabla\mathbb{S}_k\|^{2/3}_{(\mathbb{L}^2_s)^2})^6\\ &\le c(\|\mathbb{S}_k\|^6_{\mathbb{L}^2_s} + \|\mathbb{S}_k\|^2_{\mathbb{L}^2_s}\|\nabla\mathbb{S}_k\|^4_{(\mathbb{L}^2_s)^2}).\end{aligned} \tag{3.80}$$

Hence, invoking the estimates (3.74)–(3.80) in the equation (3.73) and choosing $0 < \delta < \frac{\varepsilon_0}{12}$, we deduce the existence of $c > 0$ such that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\mathbb{S}_k\|^2_{(\mathbb{L}^2_s)^2} &+ \frac{\varepsilon_0}{2}\|\Delta\mathbb{S}_k\|^2_{\mathbb{L}^2_s} - \frac{\nu_0}{8}\|\Delta\boldsymbol{v}_k\|^2_{\boldsymbol{L}^2}\\ &\le c[K_{5k}(\|\nabla\boldsymbol{v}_k\|^2_{\mathbb{L}^2} + \|\mathbb{S}_k\|^2_{\mathbb{L}^2_s} + \|\nabla\mathbb{S}_k\|^2_{(\mathbb{L}^2_s)^2}) + \|\mathbb{F}_s\|^2_{\mathbb{L}^2_s}]\end{aligned} \tag{3.81}$$

where

$$K_{5k} := \|\boldsymbol{v}_k\|^4_{\boldsymbol{L}^4_\sigma} + \|\mathbb{S}_k\|^4_{\mathbb{L}^4_s} + |\varepsilon_\phi|^4_\infty\|\nabla\phi_k\|^4_{\boldsymbol{L}^4} + |\varepsilon_\theta|^4_\infty\|\nabla\theta_k\|^4_{\boldsymbol{L}^4}$$

$$+ \|\mathbb{S}_k\|_{\mathbb{L}^2_s}^4 + \|\mathbb{S}_k\|_{\mathbb{L}^2_s}^2 \|\nabla\mathbb{S}_k\|_{(\mathbb{L}^2_s)^2}^2 + 1.$$

By arguing as above, it can be shown that $K_{5k} \in L^1(I)$.

Getting the sum of the a priori estimates (3.49), (3.58), (3.71), and (3.81) leads to the inequality

$$\frac{1}{2}\frac{d}{dt}\widetilde{E}_k + \widetilde{D}_k \leq c[\widetilde{S}_k + \widetilde{K}_k(\|\phi_k\|_{L^2}^2 + \|\theta_k\|_{L^2}^2 + \|\mathbb{S}_k\|_{\mathbb{L}^2_s}^2 + \widetilde{E}_k)] \quad \text{in } I_k, \tag{3.82}$$

where $\widetilde{K}_k := K_{1k} + K_{2k} + \cdots + K_{5k} \in L^1(I_k)$ and the functions $\widetilde{E}_k, \widetilde{D}_k, \widetilde{S}_k : I_k \to [0,\infty)$ are given as follows:

$$\begin{aligned}
\widetilde{E}_k &:= \|\Delta\phi_k\|_{L^2}^2 + \|\Delta\theta_k\|_{L^2}^2 + \tau\|\nabla\Delta\theta_k\|_{\boldsymbol{L}^2}^2 + \|\nabla\boldsymbol{v}_k\|_{\mathbb{L}^2}^2 + \|\nabla\mathbb{S}_k\|_{(\mathbb{L}^2_s)^2}^2 \\
\widetilde{D}_k &:= \frac{1}{c}(\|\Delta^2\phi_k\|_{L^2}^2 + \|\Delta\mu_k\|_{L^2}^2 + \|\Delta^2\theta_k\|_{L^2}^2) + \frac{\nu_0}{4}\|\Delta\boldsymbol{v}_k\|_{\boldsymbol{L}^2}^2 + \frac{\varepsilon_0}{4}\|\Delta\mathbb{S}_k\|_{\mathbb{L}^2_s}^2 \\
\widetilde{S}_k &:= \|f_{\mathrm{o}}\|_{L^2}^2 + \|f_{\mathrm{h}}\|_{L^2}^2 + \|\boldsymbol{f}_{\mathrm{v}}\|_{\boldsymbol{L}^2}^2 + \|\mathbb{F}_{\mathrm{s}}\|_{\mathbb{L}^2_s}^2 + \|\boldsymbol{u}\|_{\boldsymbol{L}^2}^2 + |\mathbf{g}|^2.
\end{aligned}$$

Note that $\|\phi_k\|_{L^2}^2, \|\theta_k\|_{L^2}^2, \|\mathbb{S}_k\|_{\mathbb{L}^2_s}^2 \in L^\infty(I_k)$, see (3.34), and from our assumptions on the source functions, we have $\widetilde{S}_k \in L^1(I_k)$. Applying the Grönwall Lemma to (3.82), it can be deduced that

$$\begin{aligned}
&\|\Delta\phi_k\|_{L^\infty(L^2)} + \|\nabla\Delta\theta_k\|_{L^\infty(\boldsymbol{L}^2)} + \|\nabla\boldsymbol{v}_k\|_{L^\infty(\mathbb{L}^2)} + \|\nabla\mathbb{S}_k\|_{L^\infty((\mathbb{L}^2_s)^2)} + \|\Delta^2\phi_k\|_{L^2(L^2)} \\
&\quad + \|\Delta\mu_k\|_{L^2(L^2)} + \|\Delta^2\theta_k\|_{L^2(L^2)} + \|\Delta\boldsymbol{v}_k\|_{L^2(\boldsymbol{L}^2)} + \|\Delta\mathbb{S}_k\|_{L^2(\mathbb{L}^2_s)} \leq \widetilde{\mathfrak{C}}_{0,\boldsymbol{u}}
\end{aligned} \tag{3.83}$$

where $\widetilde{\mathfrak{C}}_{0,\boldsymbol{u}} = \mathfrak{C}(\|(\phi_0,\theta_0,\boldsymbol{v}_0,\mathbb{S}_0)\|_{\mathcal{D}^2} + \|\boldsymbol{u}\|_{\boldsymbol{U}})$ with $\mathfrak{C} : \mathbb{R} \to [0,\infty)$ a continuous and monotone increasing function depending continuously on the norms of the given source functions.

Step 3. *Estimates on temporal derivatives.* The previous step provides bounds for the time derivatives. For the order parameter, we have

$$\begin{aligned}
\|\partial_t\phi_k\|_{L^2(L^2)} &\leq c[\|\boldsymbol{v}_k\|_{L^\infty(\boldsymbol{W}^{1,2}_{0,\sigma})}\|\Delta\phi_k\|_{L^2(L^2)} + \|f_{\mathrm{o}}\|_{L^2(L^2)} \\
&\quad + (|m_\phi|_\infty\|\Delta\phi_k\|_{L^\infty(L^2)} + |m_\theta|_\infty\|\Delta\theta_k\|_{L^\infty(L^2)} + |m|_\infty)\|\Delta\mu_k\|_{L^2(L^2)}].
\end{aligned} \tag{3.84}$$

Applying the operator $(I - \tau\Delta)^{-1} : L^2(\Omega) \to W^{2,2}_{\boldsymbol{n}}(\Omega)$ to the third equation in (3.12), we have

$$\begin{aligned}
\|\partial_t\theta_k\|_{L^2(W^{2,2}_{\boldsymbol{n}})} &\leq c_\tau[\|\boldsymbol{v}_k\|_{L^\infty(\boldsymbol{W}^{1,2}_{0,\sigma})}\|\Delta\theta_k\|_{L^2(L^2)} + |\mathbf{g}|\|\boldsymbol{v}\|_{L^2(\boldsymbol{L}^2_\sigma)} \\
&\quad + (|\chi_\phi|_\infty\|\Delta\phi_k\|_{L^\infty(L^2)} + |\chi_\theta|_\infty\|\Delta\theta_k\|_{L^\infty(L^2)} + |\chi|_\infty)\|\Delta\theta_k\|_{L^2(L^2)} \\
&\quad + b\|\Delta^2\theta_k\|_{L^2(L^2)} + \|\mathbb{S}_k\|_{L^\infty(\mathbb{W}^{1,2}_s)}\|\Delta\boldsymbol{v}_k\|_{L^2(\boldsymbol{L}^2)} + \|f_{\mathrm{h}}\|_{L^2(L^2)}].
\end{aligned} \tag{3.85}$$

Concerning the velocity, we have the following estimate for the time-derivative

$$\begin{aligned}
\|\partial_t\boldsymbol{v}_k\|_{L^2(\boldsymbol{L}^2_\sigma)} &\leq c[\|\boldsymbol{v}_k\|_{L^\infty(\boldsymbol{W}^{1,2}_{0,\sigma})}\|\Delta\boldsymbol{v}_k\|_{L^2(\boldsymbol{L}^2)} \\
&\quad + (|\nu_\phi|_\infty\|\Delta\phi_k\|_{L^\infty(L^2)} + |\nu_\theta|_\infty\|\Delta\theta_k\|_{L^\infty(L^2)} + |\nu|_\infty)\|\Delta\boldsymbol{v}_k\|_{L^2(\boldsymbol{L}^2)} \\
&\quad + |a|(\sigma_0+1)\|\mathbb{S}_k\|_{L^\infty(\mathbb{W}^{1,2}_s)}\|\Delta\mathbb{S}_k\|_{L^2(\mathbb{L}^2_s)} + \sigma_0|a|\|\mathbb{S}_k\|_{L^2(\mathbb{W}^{1,2}_s)} \\
&\quad + \sigma\|\mathbb{S}_k\|_{L^\infty(\mathbb{W}^{1,2}_s)}\|\theta_k\|_{L^2(W^{2,2}_{\boldsymbol{n}})} + \kappa\|\mu_k\|_{L^2(W^{1,2})}\|\Delta\phi_k\|_{L^\infty(L^2)} \\
&\quad + (1 + \|\phi_k\|_{L^2(L^2)} + \|\theta_k\|_{L^2(L^2)})|\mathbf{g}| + \|\boldsymbol{f}_{\mathrm{v}}\|_{L^2(\boldsymbol{L}^2)} + \|\boldsymbol{u}\|_{L^2(\boldsymbol{L}^2)}].
\end{aligned} \tag{3.86}$$

Finally, the time-derivative of the viscoelastic stress tensor can be bounded from above according to

$$
\begin{aligned}
\|\partial_t \mathbb{S}_k\|_{L^2(\mathbb{L}^2_s)} &\le c[\|\boldsymbol{v}_k\|_{L^\infty(\boldsymbol{W}^{1,2}_{0,\sigma})}\|\Delta \mathbb{S}_k\|_{L^2(\mathbb{L}^2_s)} + \|\mathbb{F}_s\|_{\mathbb{L}^2_s} + \lambda\|\boldsymbol{v}_k\|_{L^2(\boldsymbol{W}^{1,2}_{0,\sigma})} \\
&\quad + (\ell+\beta)\|\mathbb{S}_k\|_{L^2(\mathbb{L}^2_s)} + \beta\|\mathbb{S}_k\|^3_{L^6(\mathbb{W}^{1,2}_s)} + (1+|a|)\|\mathbb{S}_k\|_{L^\infty(\mathbb{W}^{1,2}_s)}\|\Delta \boldsymbol{v}_k\|_{L^2(\boldsymbol{L}^2)} \\
&\quad + (|\varepsilon_\phi|_\infty\|\Delta\phi_k\|_{L^\infty(L^2)} + |\varepsilon_\theta|_\infty\|\Delta\theta_k\|_{L^\infty(L^2)} + |\varepsilon|_\infty)\|\Delta \mathbb{S}_k\|_{L^2(\mathbb{L}^2_s)}]. \qquad (3.87)
\end{aligned}
$$

Using the following relation for the chemical potential

$$
\partial_t\mu_k = -\alpha\Delta\partial_t\phi_k + P_{\Phi_k}F''(\phi_k)\partial_t\phi_k
$$

obtained by differentiating the second equation of (3.12) with respect to time, we have

$$
\|\partial_t\mu_k\|_{L^2(W^{-2,2}_{\boldsymbol{n}})} \le c(\alpha + \|F''(\phi_k)\|_{L^\infty(L^\infty)})\|\partial_t\phi_k\|_{L^2(L^2)}, \qquad (3.88)
$$

where $\|F''(\phi_k)\|_{L^\infty(L^\infty)} \le c(\|\phi_k\|^{q+1}_{L^\infty(W^{2,2}_{\boldsymbol{n}})} + 1)$ thanks to (2.16) with $s = 3$ and $k = 2$. The above uniform-in-time a priori estimates imply that the solutions to the Faedo–Galerkin approximations persist in the whole time interval I.

Step 4. *Passage to limit.* According to the Banach–Alaoglu–Bourbaki Theorem, there are subsequences (not relabelled) such that

$$
\begin{array}{llll}
\phi_k \overset{*}{\rightharpoonup} \phi & \text{in } L^\infty(I; W^{2,2}_{\boldsymbol{n}}(\Omega)), & \phi_k \rightharpoonup \phi & \text{in } L^2(I; W^{4,2}_{\boldsymbol{n}}(\Omega)),\\
\partial_t\phi_k \rightharpoonup \partial_t\phi & \text{in } L^2(I; L^2(\Omega)), & \theta_k \overset{*}{\rightharpoonup} \theta & \text{in } L^\infty(I; W^{3,2}_{\boldsymbol{n}}(\Omega)),\\
\theta_k \rightharpoonup \theta & \text{in } L^2(I; W^{4,2}_{\boldsymbol{n}}(\Omega)), & \partial_t\theta_k \rightharpoonup \partial_t\theta & \text{in } L^2(I; W^{2,2}_{\boldsymbol{n}}(\Omega)),\\
\mu_k \overset{*}{\rightharpoonup} \mu & \text{in } L^\infty(I; L^2(\Omega)), & \mu_k \rightharpoonup \mu & \text{in } L^2(I; W^{2,2}_{\boldsymbol{n}}(\Omega)),\\
\partial_t\mu_k \rightharpoonup \partial_t\mu & \text{in } L^2(I; W^{-2,2}_{\boldsymbol{n}}(\Omega)), & \boldsymbol{v}_k \overset{*}{\rightharpoonup} \boldsymbol{v} & \text{in } L^\infty(I; \boldsymbol{W}^{1,2}_{0,\sigma}(\Omega)),\\
\boldsymbol{v}_k \rightharpoonup \boldsymbol{v} & \text{in } L^2(I; \boldsymbol{W}^{2,2}_{0,\sigma}(\Omega)), & \partial_t\boldsymbol{v}_k \rightharpoonup \partial_t\boldsymbol{v} & \text{in } L^2(I; \boldsymbol{L}^2_\sigma(\Omega)),\\
\mathbb{S}_k \overset{*}{\rightharpoonup} \mathbb{S} & \text{in } L^\infty(I; \mathbb{W}^{1,2}_s(\Omega)), & \mathbb{S}_k \rightharpoonup \mathbb{S} & \text{in } L^2(I; \mathbb{W}^{2,2}_{\boldsymbol{n},s}(\Omega)),\\
\partial_t\mathbb{S}_k \rightharpoonup \partial_t\mathbb{S} & \text{in } L^2(I; \mathbb{L}^2_s(\Omega)). & &
\end{array}
$$

By further extraction of a subsequence, we have $\phi_k \to \phi$ and $\theta_k \to \theta$ almost everywhere in Ω_T and one can obtain from the Aubin–Lions–Simon Lemma [64] the strong convergence $\phi_k \to \phi$ in $L^2(I; W^{3,2}_{\boldsymbol{n}}(\Omega))$, $\mu_k \to \mu$ in $L^2(I; W^{1,2}(\Omega))$, $\theta_k \to \theta$ in $L^2(I; W^{3,2}_{\boldsymbol{n}}(\Omega))$, $\boldsymbol{v}_k \to \boldsymbol{v}$ in $L^2(I; \boldsymbol{W}^{1,2}_{0,\sigma}(\Omega))$, and $\mathbb{S}_k \to \mathbb{S}$ in $L^2(I; \mathbb{W}^{1,2}_s(\Omega))$.

With the above convergence, we now pass to the limit to the first five equations in the approximate system (3.12). First, let us consider the convection terms. This is standard, however, we provide briefly the details. Indeed, note that for each $\boldsymbol{w} \in L^2(I; \boldsymbol{W}^{1,2}_{0,\sigma}(\Omega))$, we have by the triangle inequality and the Hölder inequality

$$
\begin{aligned}
&\int_{\Omega_T} |(\boldsymbol{v}_k\cdot\nabla)\boldsymbol{v}_k\cdot\boldsymbol{w} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}|\,\mathrm{d}x \\
&\quad \le (\|\boldsymbol{v}_k - \boldsymbol{v}\|_{L^2(\boldsymbol{L}^4)}\|\nabla\boldsymbol{v}_k\|_{L^\infty(\mathbb{L}^2)} + \|\boldsymbol{v}_k\|_{L^\infty(\boldsymbol{L}^4)}\|\nabla\boldsymbol{v}_k - \nabla\boldsymbol{v}\|_{L^2(\mathbb{L}^2)})\|\boldsymbol{w}\|_{L^2(\boldsymbol{L}^4)} \\
&\quad \le c\|\boldsymbol{v}_k\|_{L^\infty(\boldsymbol{W}^{1,2}_{0,\sigma})}\|\boldsymbol{v}_k - \boldsymbol{v}\|_{L^2(\boldsymbol{W}^{1,2}_{0,\sigma})}\|\boldsymbol{w}\|_{L^2(\boldsymbol{W}^{1,2}_{0,\sigma})} \to 0.
\end{aligned}
$$

Hence, $(\boldsymbol{v}_k\cdot\nabla)\boldsymbol{v}_k \rightharpoonup (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ in $L^2(I;\boldsymbol{W}^{-1,2}_{0,\sigma}(\Omega))$. The other convection terms $\boldsymbol{v}_k\cdot\nabla\phi_k$, $\boldsymbol{v}_k\cdot\nabla\theta_k$, and $(\boldsymbol{v}_k\cdot\nabla)\mathbb{S}_k$ can be treated similarly. Also, the bilinear terms $\mu_k\nabla\phi_k$, $[\mathbb{S}_k,\mathbb{W}\boldsymbol{v}]$, $\{\mathbb{S}_k,\mathbb{D}\boldsymbol{v}\}$, $\nabla\cdot(\mathbb{S}_k^2)$, $\nabla\cdot(\theta_k\mathbb{S}_k)$, $\nabla\cdot(\mathrm{Tr}(\mathbb{S}_k)\mathbb{S}_k)$, and $\mathbb{S}_k:\mathbb{D}\boldsymbol{v}_k$, as well as the trilinear term $\mathrm{Tr}(\mathbb{S}_k)(\mathbb{I}-\mathrm{Tr}(\mathbb{S}_k)\mathbb{S}_k)$ can be considered analogously.

Let $\varphi\in L^2(I;W^{1,2}(\Omega))$. By continuity of m, we have $m_k = m(\phi_k,\theta_k)\to m(\phi,\theta)$ almost everywhere in Ω_T. Thus, by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}\int_{\Omega_T} &|m(\phi_k,\theta_k)\nabla\varphi - m(\phi,\theta)\nabla\varphi|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\le \int_{\Omega_T} |m(\phi_k,\theta_k)-m(\phi,\theta)|^2|\nabla\varphi|^2\,\mathrm{d}x\,\mathrm{d}t\to 0,\end{aligned}$$

that is, $m(\phi_k,\theta_k)\nabla\varphi\to m(\phi,\theta)\nabla\varphi$ in $L^2(I;\boldsymbol{L}^2(\Omega))$. Due to the convergence $\nabla\mu_k\to\nabla\mu$ in $L^2(I;\boldsymbol{L}^2(\Omega))$, one has

$$\int_{\Omega_T} m(\phi_k,\theta_k)\nabla\mu_k\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\to\int_{\Omega_T} m(\phi,\theta)\nabla\mu\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t.$$

As a result, $\nabla\cdot(m(\phi_k,\theta_k)\nabla\mu_k)\rightharpoonup\nabla\cdot(m(\phi,\theta)\nabla\mu)$ in $L^2(I;W^{-1,2}(\Omega))$. The treatment for the terms $\nabla\cdot(\chi(\phi_k,\theta_k)\nabla\theta_k)$, $\nabla\cdot(\nu(\phi_k,\theta_k)\mathbb{D}\boldsymbol{v}_k)$, and $\nabla\cdot(\varepsilon(\phi_k,\theta_k)\nabla\mathbb{S}_k)$ are completely the same.

Observe that there exists a constant $M=M(\widetilde{\mathfrak{C}}_{0,\boldsymbol{u}})>0$ such that $\|\phi_k\|_{C(\bar\Omega_T)}+\|\phi\|_{C(\bar\Omega_T)}\le M$ for each positive integer k, thanks to the continuous embeddings $W^{1,2,2}(I;W^{4,2}_{\boldsymbol{n}}(\Omega),L^2(\Omega))\hookrightarrow C(\bar I;W^{2,2}_{\boldsymbol{n}}(\Omega))\hookrightarrow C(\bar I;C(\bar\Omega))=C(\bar\Omega_T)$. Hence, by the mean-value theorem

$$\int_{\Omega_T}|F'(\phi_k)-F'(\phi)|^2\,\mathrm{d}x\,\mathrm{d}t\le\max_{|\psi|\le M}|F''(\psi)|^2\int_{\Omega_T}|\phi_k-\phi|^2\,\mathrm{d}x\,\mathrm{d}t\to 0,$$

so that $F'(\phi_k)\to F'(\phi)$ in $L^2(I;L^2(\Omega))$.

For the remaining linear terms, we have $\rho(\theta_k,\phi_k)\to\rho(\theta,\phi)$ in $L^2(I;L^2(\Omega))$, $\Delta\phi_k\rightharpoonup\Delta\phi$ in $L^2(I;W^{2,2}_{\boldsymbol{n}}(\Omega))$, $\Delta^2\theta_k\rightharpoonup\Delta^2\theta$ in $L^2(I;L^2(\Omega))$, and $\nabla\cdot\mathbb{S}_k\to\nabla\cdot\mathbb{S}$ in $L^2(I;\mathbb{L}^2_{\mathrm{s}}(\Omega))$. These convergence, along with those presented above for the time-derivatives, imply that the first five equations in (3.12) converge to the corresponding equations in the system (1.15) with respect to the weak topologies of $L^2(I;W^{-1,2}(\Omega))$, $L^2(I;L^2(\Omega))$, $L^2(I;W^{-1,2}(\Omega))$, $L^2(I;\boldsymbol{W}^{-1,2}_{0,\sigma}(\Omega))$, and $L^2(I;\mathbb{W}^{-1,2}_{\mathrm{s}}(\Omega))$.

Now, let us pass to the limit of the initial conditions in (3.12). For the order parameter, note that the map $\psi\mapsto\psi(0):W^{1,2,2}(I;W^{4,2}_{\boldsymbol{n}}(\Omega),L^2(\Omega))\to W^{2,2}_{\boldsymbol{n}}(\Omega)$ is continuous, hence, weakly continuous. Thus, $\phi_k(0)\rightharpoonup\phi(0)$ in $W^{2,2}_{\boldsymbol{n}}(\Omega)$ since $\phi_k\rightharpoonup\phi$ in $W^{1,2,2}(I;W^{4,2}_{\boldsymbol{n}}(\Omega),L^2(\Omega))$. From $\phi_{k0}\to\phi_0$ in $W^{2,2}_{\boldsymbol{n}}(\Omega)$, it follows that $\phi(0)=\phi_0$ by uniqueness of weak limits. In a similar manner, it can be shown that $\theta(0)=\theta_0$, $\boldsymbol{v}(0)=\boldsymbol{v}_0$, and $\mathbb{S}(0)=\mathbb{S}_0$. Therefore, the existence of a strong solution has been established.

Step 5. *Uniqueness of solution and a priori estimate.* The proof that the solution constructed above is unique is very similar to the well-posedness of the linearized system, and thus, we remove the details and refer the reader to the succeeding section and the Appendix A. Alternatively, one may adapt the proof of uniqueness for strong solutions in the two-dimensional isothermal case provided in [18, Section 5]. Although the proof given there deals only with diffusion coefficients depending on ϕ, the arguments can be extended for coefficients depending also on θ as θ has better regularity than ϕ. Also, the dependence of ε on (ϕ, θ) appearing in the evolution equation for $\mathbb{S}$ will not pose an issue as one can derive the corresponding a priori estimates similar to those of the Navier–Stokes part. Finally, for the evolution equation for θ, one may proceed as in the Cahn–Hilliard part.

The a priori estimate (3.11) without the pressure, that is,

$$\|(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S})\|_{\mathcal{W}^2} \leq \mathfrak{C}(\|(\phi_0, \theta_0, \boldsymbol{v}_0, \mathbb{S}_0)\|_{\mathcal{D}^2} + \|\boldsymbol{u}\|_{\boldsymbol{U}}), \tag{3.89}$$

follows by combining the estimates (3.34)–(3.36) and (3.83)–(3.88), passing to the limit inferior to the resulting inequality, and invoking the lower semicontinuity of the norm with respect to weak and weak* topologies. Finally, the existence and uniqueness of a pressure $\mathrm{p} \in L^2(I; W^{1,2}(\Omega)/\mathbb{R})$ is a consequence of the classical de Rham Theorem and the a priori estimate for the pressure can be obtained from the Navier–Stokes equation in (1.15) and (3.89). □

Conditions for initial data and source functions leading to strong solutions with additional regularity will be presented in the Appendix A (see Theorem A.4). For the meantime, we end this section by stating the following energy identity satisfied by strong solutions.

Remark 3.2. Taking the sum of the starting integral identities in the above proof, namely, (3.13), (3.20), (3.22), (3.26), and (3.29), but only using the test function μ_k and not $\phi_k + \mu_k$ in the case of (3.13), and passing to the limit $k \to \infty$, it follows that the strong solution satisfies the following basic energy identity:

$$\frac{1}{2}\int_\Omega E(t)\,\mathrm{d}x + \int_\Omega D(t)\,\mathrm{d}x = \frac{1}{2}\int_\Omega E(0)\,\mathrm{d}x + \int_\Omega S(t)\,\mathrm{d}x \qquad t \in I,$$

where the energy, dissipation, and source terms as functions of time are given by

$$\begin{aligned}
E &:= \alpha|\nabla\phi|^2 + 2|F(\phi)| + \frac{\sigma}{\kappa}|\theta|^2 + \frac{\sigma\tau}{\kappa}|\Delta\theta|^2 + \frac{1}{\kappa}|\boldsymbol{v}|^2 + \frac{\sigma_0}{2\kappa}|\mathbb{S}|^2 + \frac{1}{2\kappa}|\mathrm{Tr}(\mathbb{S})|^2 \\
D &:= m(\phi,\theta)|\nabla\mu|^2 + \frac{\sigma\chi(\phi,\theta)}{\kappa}|\nabla\theta|^2 + \frac{\sigma b}{\kappa}|\Delta\theta|^2 + \frac{2\nu(\phi,\theta)}{\kappa}|\mathbb{D}\boldsymbol{v}|^2 \\
&\quad + \frac{\sigma_0\varepsilon(\phi,\theta)}{2\kappa}|\nabla\mathbb{S}|^2 + \frac{\sigma_0\beta}{2\kappa}|\mathrm{Tr}(\mathbb{S})\mathbb{S}|^2 + \frac{\sigma_0\ell}{2\kappa}|\mathbb{S}|^2 + \frac{\varepsilon(\phi,\theta)}{2\kappa}|\nabla(\mathrm{Tr}(\mathbb{S}))|^2 \\
&\quad + \frac{\beta}{2\kappa}|\mathrm{Tr}(\mathbb{S})|^4 + \frac{1}{2\kappa}(\ell - \sigma_0\beta - 2\beta)|\mathrm{Tr}(\mathbb{S})|^2 - \frac{\sigma_0}{\kappa}\left(a + \frac{\lambda}{2}\right)\mathbb{D}\boldsymbol{v} : \mathbb{S} \\
S &:= f_{\mathrm{o}}\mu + \frac{\sigma}{\kappa}f_{\mathrm{h}}\theta + \frac{1}{\kappa}(\sigma a_0\theta\mathbf{g} + \rho(\phi,\theta)\mathbf{g} + \boldsymbol{f}_{\mathrm{v}} + \boldsymbol{u})\cdot\boldsymbol{v} + \frac{1}{2\kappa}\mathbb{F}_{\mathrm{s}} : (\sigma_0\mathbb{S} + \mathrm{Tr}(\mathbb{S})\mathbb{I}).
\end{aligned}$$

Here, the term energy is in a purely mathematical sense. For a more physically relevant elastic energy incorporating a logarithmic term that ensures the positive-definiteness of the conformation tensor, we refer to [8].

Observe that $\nu_0 \geq \frac{\sigma_0}{4}(|a| + \frac{\lambda}{2})$ and $\ell \geq \max\{|a| + \frac{\lambda}{2}, (\sigma_0 + 2)\beta\}$ imply the non-negativity of D. This can be easily verified from the Cauchy–Schwarz inequality. In this case, D can be thought as the total energy dissipation and S as the contribution of the sources to the energy. These can be utilized in the study of attractors and asymptotic behavior of solutions to the system (1.15), see for instance [33] in the case of isothermal two-phase flows without stress diffusion.

4. The Linearized System and Control-to-State Operator

The directional derivatives of the operator that maps a control to a strong solution of (1.15) can be represented in terms of the linearization around a fixed solution of the PDE system. Moreover, by duality principle, the regularity of the adjoint variables, and hence the optimal control, depends on the type of solutions for the linearized system. Note that the nature of the cost functional dictates what formulation for the linearized system is needed.

We will consider the strong, weak, and very weak formulations to the linearized problem. For the sake of brevity of this section, definitions and proofs of existence and uniqueness concerning such solutions are presented in the Appendix A. Here, we focus on the continuity with respect to the weak topologies and the second-order differentiability under the strong topologies of the control-to-state operator. These are established by applying a sequential compactness argument and the implicit function theorem, respectively.

4.1. Well-posedness of the Linearized System. Let us set $W^{0,2}_{\boldsymbol{n}}(\Omega) := L^2(\Omega)$, $W^{1,2}_{\boldsymbol{n}}(\Omega) := W^{1,2}(\Omega)$, $\mathbb{W}^{0,2}_{\boldsymbol{n},\mathrm{s}}(\Omega) := \mathbb{L}^2_{\mathrm{s}}(\Omega)$, and $\mathbb{W}^{1,2}_{\boldsymbol{n},\mathrm{s}}(\Omega) := \mathbb{W}^{1,2}_{\mathrm{s}}(\Omega)$. We introduce the following function spaces concerned with the strong, weak, and very weak solutions to the linearized system:

$$\begin{aligned}\mathcal{V}^k(\Omega_T) := {} & W^{1,2,2}(I; W^{k+2,2}_{\boldsymbol{n}}(\Omega), W^{k-2,2}_{\boldsymbol{n}}(\Omega)) \times L^2(I; W^{k,2}_{\boldsymbol{n}}(\Omega)) \\ & \times W^{1,2,2}(I; W^{k+2,2}_{\boldsymbol{n}}(\Omega), W^{k,2}_{\boldsymbol{n}}(\Omega)) \times W^{1,2,2}(I; \boldsymbol{W}^{k,2}_{0,\sigma}(\Omega), \boldsymbol{W}^{k-2,2}_{0,\sigma}(\Omega)) \\ & \times W^{1,2,2}(I; \mathbb{W}^{k,2}_{\boldsymbol{n},\mathrm{s}}(\Omega), \mathbb{W}^{k-2,2}_{\boldsymbol{n},\mathrm{s}}(\Omega)), \quad k = 2, 1, 0.\end{aligned}$$

Roughly speaking, the spatial regularity reduces by one from strong to weak and from weak to very weak. Also, take note that the second factor (pertaining to the linearized chemical potential) of the function spaces $\mathcal{V}^2(\Omega_T)$ and $\mathcal{W}^2(\Omega_T)$ differ. However, it is also possible to take $\mathcal{W}^2(\Omega_T)$ as the space of strong solutions for the linearized system. In line with this, we also set

$$\begin{aligned}\mathcal{W}^k(\Omega_T) := {} & W^{1,2,2}(I; W^{k+2,2}_{\boldsymbol{n}}(\Omega), W^{k-2,2}_{\boldsymbol{n}}(\Omega)) \times W^{1,2,2}(I; W^{k,2}_{\boldsymbol{n}}(\Omega), W^{k-4,2}_{\boldsymbol{n}}(\Omega)) \\ & \times W^{1,2,2}(I; W^{k+2,2}_{\boldsymbol{n}}(\Omega), W^{k,2}_{\boldsymbol{n}}(\Omega)) \times W^{1,2,2}(I; \boldsymbol{W}^{k,2}_{0,\sigma}(\Omega), \boldsymbol{W}^{k-2,2}_{0,\sigma}(\Omega)) \\ & \times W^{1,2,2}(I; \mathbb{W}^{k,2}_{\boldsymbol{n},\mathrm{s}}(\Omega), \mathbb{W}^{k-2,2}_{\boldsymbol{n},\mathrm{s}}(\Omega)), \quad k = 1, 0.\end{aligned}$$

Thus, $\mathcal{W}^k(\Omega_T) \hookrightarrow \mathcal{V}^k(\Omega_T)$ for $k = 0, 1, 2$. Also, observe that $\mathcal{V}^2(\Omega_T) \hookrightarrow \mathcal{V}^1(\Omega_T) \hookrightarrow \mathcal{V}^0(\Omega_T)$ and $\mathcal{W}^2(\Omega_T) \hookrightarrow \mathcal{W}^1(\Omega_T) \hookrightarrow \mathcal{W}^0(\Omega_T)$.

Associated to the above function spaces for the solutions are the spaces for the initial data $\mathcal{D}^2(\Omega)$ as defined in Section 3 in the case of strong solutions, and

$$\mathcal{D}^k(\Omega) := W^{k,2}_{\boldsymbol{n}}(\Omega) \times W^{k+1,2}_{\boldsymbol{n}}(\Omega) \times \boldsymbol{W}^{k-1,2}_{0,\sigma}(\Omega) \times \mathbb{W}^{k-1,2}_{\boldsymbol{n},\mathrm{s}}(\Omega), \quad k = 1, 0,$$

in the case of weak and very weak solutions. Likewise, the sources in the linearized system will be elements of the dual of the following:

$$\mathcal{U}^k(\Omega_T) := L^2(I; W^{k,2}_{\boldsymbol{n}}(\Omega)) \times L^2(I; W^{k-2,2}_{\boldsymbol{n}}(\Omega)) \times L^2(I; W^{k,2}_{\boldsymbol{n}}(\Omega))$$

$$\times L^2(I;\boldsymbol{W}^{k,2}_{0,\sigma}(\Omega)) \times L^2(I;\mathbb{W}^{k,2}_{\boldsymbol{n},\mathrm{s}}(\Omega)), \quad k = 0,1,2.$$

We define the function spaces $\mathcal{Y}^k(\Omega_T)$ to be $\mathcal{U}^k(\Omega_T)$ but with the second factor replaced by the dual of the second factor in $\mathcal{W}^{2-k}(\Omega_T)$, that is,

$$\mathcal{Y}^k(\Omega_T) := L^2(I;W^{k,2}_{\boldsymbol{n}}(\Omega)) \times W^{1,2,2}(I;W^{2-k,2}_{\boldsymbol{n}}(\Omega), W^{-2-k,2}_{\boldsymbol{n}}(\Omega))^* \\ \times L^2(I;W^{k,2}_{\boldsymbol{n}}(\Omega)) \times L^2(I;\boldsymbol{W}^{k,2}_{0,\sigma}(\Omega)) \times L^2(I;\mathbb{W}^{k,2}_{\boldsymbol{n},\mathrm{s}}(\Omega)), \quad k = 0,1,2.$$

Hence, $\mathcal{U}^k(\Omega_T) \hookrightarrow \mathcal{Y}^k(\Omega_T)$. Also, note that $\mathcal{U}^2(\Omega_T) \hookrightarrow \mathcal{U}^1(\Omega_T) \hookrightarrow \mathcal{U}^0(\Omega_T)$, $\mathcal{Y}^2(\Omega_T) \hookrightarrow \mathcal{Y}^1(\Omega_T) \hookrightarrow \mathcal{Y}^0(\Omega_T)$, and $\mathcal{D}^2(\Omega) \hookrightarrow \mathcal{D}^1(\Omega) \hookrightarrow \mathcal{D}^0(\Omega)$. For the function spaces we have discussed, the associated product norms will be denoted by $\|\cdot\|_{\mathcal{U}^k}$, $\|\cdot\|_{\mathcal{Y}^k}$, $\|\cdot\|_{\mathcal{W}^k}$, $\|\cdot\|_{\mathcal{V}^k}$, and $\|\cdot\|_{\mathcal{D}^k}$.

Given fixed source functions $(f_\mathrm{o}, f_\mathrm{h}, \boldsymbol{f}_\mathrm{v}, \mathbb{F}_\mathrm{s})$ and initial data $(\phi_0, \theta_0, \boldsymbol{v}_0, \mathbb{S}_0)$ as in Theorem 3.1, we introduce the nonlinear operator

$$\mathcal{N} = (\mathcal{N}_1, \mathcal{N}_0) : \mathcal{W}^2(\Omega_T) \times \boldsymbol{U} \to \mathcal{U}^0(\Omega_T)^* \times \mathcal{D}^2(\Omega)$$

as follows: (i) the first coordinate function $\mathcal{N}_1$ is obtained by subtracting both sides of the first five equations in (1.15) by the corresponding terms on the right, with the Leray–Helmholtz orthogonal projector $\boldsymbol{P}_\sigma$ applied to the fourth equation, and (ii) the second coordinate function $\mathcal{N}_0$ is obtained by adapting the same process as in (i) to the initial data. In short, we make the right-hand sides of the differential equations and initial data to be zero. Note that the application of $\boldsymbol{P}_\sigma$ to the Navier–Stokes equation eliminates the pressure p.

In virtue of Theorem 3.1, given a control $\boldsymbol{u} \in \boldsymbol{U}$, there is a unique $(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S}) \in \mathcal{W}^2(\Omega_T)$ such that

$$\mathcal{N}(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S}, \boldsymbol{u}) = 0. \tag{4.1}$$

Define the control-to-state operator $\mathcal{T} : \boldsymbol{U} \to \mathcal{W}^2(\Omega_T)$ by

$$\mathcal{T}(\boldsymbol{u}) = (\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S})$$

if and only if (4.1) holds. From the definition of $\mathcal{N}$, notice that $\mathcal{T}(\boldsymbol{P}_\sigma \boldsymbol{u}) = \mathcal{T}(\boldsymbol{u})$ for every $\boldsymbol{u} \in \boldsymbol{U}$. This means that the divergence-free part of the control is the one that matters in the operator $\mathcal{T}$. In particular, $\mathcal{T}$ is not injective.

Using a sequential compactness argument, we establish the following weak continuity of the control-to-state operator.

Theorem 4.1. *The nonlinear operator $\mathcal{T} : \boldsymbol{U} \to \mathcal{W}^2(\Omega_T)$ is continuous with respect to weak topologies, that is, if $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}$ in $\boldsymbol{U}$, then $\mathcal{T}(\boldsymbol{u}_k) \rightharpoonup \mathcal{T}(\boldsymbol{u})$ in $\mathcal{W}^2(\Omega_T)$.*

Proof. We follow the proof in [56, Lemma 2]. First, note that continuity and sequential continuity with respect to the weak topologies in $\boldsymbol{U}$ and in $\mathcal{W}^2(\Omega_T)$ are equivalent since both are reflexive separable spaces. Suppose that $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}$ in $\boldsymbol{U}$ and let $(\phi_k, \mu_k, \theta_k, \boldsymbol{v}_k, \mathbb{S}_k) := \mathcal{T}(\boldsymbol{u}_k)$ for each $k \in \mathbb{N}$. Then, $\{\boldsymbol{u}_k\}_{k=1}^\infty$ is bounded in $\boldsymbol{U}$ and so $\{\mathcal{T}(\boldsymbol{u}_k)\}_{k=1}^\infty$ is bounded in $\mathcal{W}^2(\Omega_T)$ according to Theorem 3.1. Thus, there is a subsequence such that $\mathcal{T}(\boldsymbol{u}_{k_j}) \rightharpoonup (\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S})$ in $\mathcal{W}^2(\Omega_T)$. Applying the Aubin–Lions–Lemma, we deduce that $\mathcal{T}(\boldsymbol{u}_{k_j}) \to (\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S})$ in $L^2(I; W^{3,2}_{\boldsymbol{n}}(\Omega) \times W^{1,2}(\Omega) \times W^{3,2}_{\boldsymbol{n}}(\Omega) \times \boldsymbol{W}^{1,2}_{0,\sigma}(\Omega) \times \mathbb{W}^{1,2}_{\mathrm{s}}(\Omega))$. Following Step 4 in the proof of Theorem 3.1, passing $k_j \to \infty$ to the equation $\mathcal{N}_1(\phi_{k_j}, \mu_{k_j}, \theta_{k_j}, \boldsymbol{v}_{k_j}, \mathbb{S}_{k_j}, \boldsymbol{u}_{k_j}) = 0$, we obtain $\mathcal{N}_1(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S}, \boldsymbol{u}) = 0$.

Next, we claim that $\mathcal{N}_0(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S},\boldsymbol{u}) = 0$. Observe that the linear operator $(\phi,\theta,\boldsymbol{v},\mathbb{S}) \mapsto (\phi(0),\theta(0),\boldsymbol{v}(0),\mathbb{S}(0))$ is continuous from

$$\begin{aligned} &W^{1,2,2}(I;W^{4,2}_{\boldsymbol{n}}(\Omega),L^2(\Omega)) \times W^{1,2,2}(I;W^{4,2}_{\boldsymbol{n}}(\Omega),W^{2,2}_{\boldsymbol{n}}(\Omega)) \\ &\quad \times W^{1,2,2}(I;\boldsymbol{W}^{2,2}_{0,\sigma}(\Omega),\boldsymbol{L}^2_\sigma(\Omega)) \times W^{1,2,2}(I;\mathbb{W}^{2,2}_{\boldsymbol{n},\mathrm{s}}(\Omega),\mathbb{L}^2_{\mathrm{s}}(\Omega)) \end{aligned}$$

into $\mathcal{D}^2(\Omega)$ thanks to the continuous embeddings (3.1)–(3.4). As a result, we have $(\phi_0,\theta_0,\boldsymbol{v}_0,\mathbb{S}_0) = (\phi_{k_j}(0),\theta_{k_j}(0),\boldsymbol{v}_{k_j}(0),\mathbb{S}_{k_j}(0)) \rightharpoonup (\phi(0),\theta(0),\boldsymbol{v}(0),\mathbb{S}(0))$ in $\mathcal{D}^2(\Omega)$, and this proves the claim. Hence, $\mathcal{N}(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S},\boldsymbol{u}) = 0$, that is, $\mathcal{T}(\boldsymbol{u}) = (\phi,\mu,\theta,\boldsymbol{v},\mathbb{S})$. Since $(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S})$ is uniquely determined by $\boldsymbol{u}$, we obtain that the whole sequence $\{\mathcal{T}(\boldsymbol{u}_k)\}_{k=1}^\infty$ converges weakly to $\mathcal{T}(\boldsymbol{u})$ in $\boldsymbol{U}$. □

To study the differentiability properties of $\mathcal{T}$, we will consider the linearized system

$$\left[\begin{array}{ll} \partial_t\psi + \boldsymbol{w}\cdot\nabla\phi + \boldsymbol{v}\cdot\nabla\psi & \\ \qquad -\nabla\cdot(m'(\phi,\theta)(\psi,\eta)\nabla\mu + m(\phi,\theta)\nabla\xi) = h_{\mathrm{o}} & \text{in } \Omega_T, \\ \xi + \alpha\Delta\psi - F''(\phi)\psi = h_{\mathrm{c}} & \text{in } \Omega_T, \\ \partial_t(\eta - \tau\Delta\eta) + \boldsymbol{w}\cdot\nabla\theta + \boldsymbol{v}\cdot\nabla\eta & \\ \qquad -\nabla\cdot(\chi'(\phi,\theta)(\psi,\eta)\nabla\theta + \chi(\phi,\theta)\nabla\eta) & \\ \qquad + b\Delta^2\eta - \mathbb{T}:\mathbb{D}\boldsymbol{v} - \mathbb{S}:\mathbb{D}\boldsymbol{w} - a_0\mathbf{g}\cdot\boldsymbol{w} = h_{\mathrm{h}} & \text{in } \Omega_T, \\ \partial_t\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{w} & \\ \qquad -\nabla\cdot(2\nu'(\phi,\theta)(\psi,\eta)\mathbb{D}\boldsymbol{v} + 2\nu(\phi,\theta)\mathbb{D}\boldsymbol{w}) + \nabla\mathrm{q} & \\ \qquad -\nabla\cdot(\sigma\eta\mathbb{S} + \mathbb{M}_{\mathbb{S}}(\theta,\mathbb{S})\mathbb{T}) - \kappa(\xi\nabla\phi + \mu\nabla\psi) & \\ \qquad -(b_{\mathrm{o}}\psi + b_{\mathrm{h}}\eta)\mathbf{g} = \boldsymbol{h}_{\mathrm{v}} & \text{in } \Omega_T, \\ \partial_t\mathbb{T} + (\boldsymbol{w}\cdot\nabla)\mathbb{S} + (\boldsymbol{v}\cdot\nabla)\mathbb{T} + \mathbb{J}(\boldsymbol{w},\mathbb{S}) + \mathbb{J}(\boldsymbol{v},\mathbb{T}) & \\ \qquad -\nabla\cdot(\varepsilon'(\phi,\theta)(\psi,\eta)\nabla\mathbb{S} + \varepsilon(\phi,\theta)\nabla\mathbb{T}) - \lambda\mathbb{D}\boldsymbol{w} - \mathbb{P}'(\mathbb{S})\mathbb{T} = \mathbb{H}_{\mathrm{s}} & \text{in } \Omega_T, \\ \nabla\cdot\boldsymbol{w} = 0 & \text{in } \Omega_T, \\ \partial_{\boldsymbol{n}}\psi = \partial_{\boldsymbol{n}}\xi = 0,\ \partial_{\boldsymbol{n}}\eta = \partial_{\boldsymbol{n}}\Delta\eta = 0,\ \boldsymbol{w} = \mathbf{0},\ \partial_{\boldsymbol{n}}\mathbb{T} = \mathbb{O} & \text{on } \Gamma_T, \\ \psi(0) = \psi_0,\ \eta(0) = \eta_0,\ \boldsymbol{w}(0) = \boldsymbol{w}_0,\ \mathbb{T}(0) = \mathbb{T}_0 & \text{in } \Omega, \end{array}\right. \tag{4.2}$$

where m', χ', ν', and ε' are defined as in (2.17) and

$$\mathbb{M}_{\mathbb{S}}(\theta,\mathbb{S})\mathbb{T} := 2\sigma_0 a\mathbb{S}\mathbb{T} - \sigma_0 a\mathbb{T} + \sigma\theta\mathbb{T} + a\mathrm{Tr}(\mathbb{T})\mathbb{S} + a\mathrm{Tr}(\mathbb{S})\mathbb{T} \tag{4.3}$$

$$\mathbb{P}'(\mathbb{S})\mathbb{T} := -\ell\mathbb{T} + \beta(\mathrm{Tr}(\mathbb{T})\mathbb{I} - 2\mathrm{Tr}(\mathbb{S})\mathrm{Tr}(\mathbb{T})\mathbb{S} - \mathrm{Tr}(\mathbb{S})^2\mathbb{T}). \tag{4.4}$$

By introducing the linear operator-valued mapping

$$\begin{aligned} \mathcal{A} = (\mathcal{A}_1,\mathcal{A}_0) : \mathcal{W}^2(\Omega_T) &\to \mathcal{L}(\mathcal{V}^1(\Omega_T),\mathcal{U}^1(\Omega_T)^* \times \mathcal{D}^1(\Omega)) \\ &\simeq \mathcal{L}(\mathcal{V}^1(\Omega_T),\mathcal{U}^1(\Omega_T)^*) \times \mathcal{L}(\mathcal{V}^1(\Omega_T),\mathcal{D}^1(\Omega)) \end{aligned}$$

having the components in such a way that for given tuples $X := (\phi,\mu,\theta,\boldsymbol{v},\mathbb{S}) \in \mathcal{W}^2(\Omega_T)$ and $Y := (\psi,\xi,\eta,\boldsymbol{w},\mathbb{T}) \in \mathcal{V}^1(\Omega_T)$, the actions $\mathcal{A}_1(X)Y$ and $\mathcal{A}_0(X)Y$ correspond to the first five equations on the left-hand side and the initial data in (4.2), respectively. Once again, the projector $\boldsymbol{P}_\sigma$ is applied to the linearized Navier–Stokes equation. With this, we can write the linear system (4.2) in a concise form as follows:

$$\mathcal{A}(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S})(\psi,\xi,\eta,\boldsymbol{w},\mathbb{T}) = ((h_{\mathrm{o}},h_{\mathrm{c}},h_{\mathrm{h}},\boldsymbol{h}_{\mathrm{v}},\mathbb{H}_{\mathrm{s}}),(\psi_0,\eta_0,\boldsymbol{w}_0,\mathbb{T}_0)).$$

The nonlinear map $\mathcal{A}$ is well-defined. Indeed, this follows from Theorem 4.2 below.

We are now in position to state one of the main results of this section.

Theorem 4.2. *Let $k = 0,1,2$. Suppose that* (A1)$_3$ *and* (A2)$_1$ *hold if $k = 0,1$ or* (A1)$_4$ *and* (A2)$_2$ *hold if $k = 2$. Then, we have*

$$\mathcal{A} : \mathcal{W}^2(\Omega_T) \to \mathcal{L}_{\mathrm{iso}}(\mathcal{V}^k(\Omega_T),\mathcal{U}^{2-k}(\Omega_T)^* \times \mathcal{D}^k(\Omega)) \cap \mathcal{L}_{\mathrm{iso}}(\mathcal{W}^k(\Omega_T),\mathcal{Y}^{2-k}(\Omega_T)^* \times \mathcal{D}^k(\Omega)).$$

Proof. This is a consequence of Theorems A.1, A.2, and A.3 in Appendix A. □

In other words, this theorem states that under suitable conditions on the sources and initial data, the system (4.2) admits either a strong, weak, or very weak solution.

Remark 4.3. In the linear system (4.2), if $\boldsymbol{h}_{\mathrm{v}}$ lies in $L^2(I;\boldsymbol{L}^2(\Omega))$, $L^2(I;\boldsymbol{W}_0^{-1,2}(\Omega))$ or $L^2(I;\boldsymbol{W}_0^{-2,2}(\Omega))$, then it follows that there exists a unique associated pressure q that belongs to either of the function spaces $L^2(I;W^{1,2}(\Omega)/\mathbb{R})$, $W_{0,0}^{-1,2}(I;L^2(\Omega)/\mathbb{R})$ or $W_{0,0}^{-1,2}(I;W_0^{-1,2}(\Omega)/\mathbb{R})$, respectively, in virtue of de Rham's theorem. The first is classical, while the second and third cases follow from the following generalization, whose proof can be obtained from the closed range theorem.

Theorem 4.4. (De Rham) *Let k be a positive integer. Given*

$$\boldsymbol{L} \in W_{0,0}^{-1,2}(I;\boldsymbol{W}_0^{-k,2}(\Omega)),$$

we have

$$\langle \boldsymbol{L},\boldsymbol{w}\rangle_{W_{0,0}^{-1,2}(\boldsymbol{W}_0^{-k,2}),W_{0,0}^{1,2}(\boldsymbol{W}_0^{k,2})} = 0 \quad \forall \boldsymbol{w} \in W_{0,0}^{1,2}(I;\boldsymbol{W}_0^{k,2}(\Omega)\cap \boldsymbol{L}_\sigma^2(\Omega))$$

if and only if there is a unique $\mathrm{p} \in W_{0,0}^{-1,2}(I;W_0^{1-k,2}(\Omega)/\mathbb{R})$ *such that* $\nabla \mathrm{p} = \boldsymbol{L}$ *in the sense of distributions:*

$$\begin{aligned}&\langle \boldsymbol{L},\boldsymbol{v}\rangle_{W_{0,0}^{-1,2}(\boldsymbol{W}_0^{-k,2}),W_{0,0}^{1,2}(\boldsymbol{W}_0^{k,2})}\\ &\quad = -\langle \mathrm{p},\nabla\cdot\boldsymbol{v}\rangle_{W_{0,0}^{-1,2}(W_0^{1-k,2}/\mathbb{R}),W_{0,0}^{1,2}(W_0^{k-1,2}/\mathbb{R})} \quad \forall \boldsymbol{v} \in W_{0,0}^{1,2}(I;\boldsymbol{W}_0^{k,2}(\Omega)).\end{aligned}$$

In addition, there is a constant $c > 0$ such that

$$\|\mathrm{p}\|_{W_{0,0}^{-1,2}(W_0^{1-k,2}/\mathbb{R})} \le c\|\boldsymbol{L}\|_{W_{0,0}^{-1,2}(\boldsymbol{W}_0^{-k,2})}.$$

4.2. DIFFERENTIABILITY OF THE CONTROL-TO-STATE OPERATORS. Before establishing that the control-to-state operator $\mathcal{T}$ defined in the previous subsection is twice differentiable, we prepare with two lemmas.

Lemma 4.5. *If* (A1)$_6$ *holds, then the map $p_F : \phi \mapsto F'(\phi)$ satisfies*

$$p_F \in C^2(W^{1,2,2}(I;W_{\boldsymbol{n}}^{4,2}(\Omega),L^2(\Omega)),W^{1,2,2}(I;W_{\boldsymbol{n}}^{2,2}(\Omega),L^2(\Omega))).$$

Moreover, for every $\phi,\psi_1,\psi_2 \in W^{1,2,2}(I;W_{\boldsymbol{n}}^{4,2}(\Omega),L^2(\Omega))$, we have

$$p_F'(\phi)\psi_1 = F''(\phi)\psi_1, \qquad p_F'(\phi)(\psi_1,\psi_2) = F'''(\phi)\psi_1\psi_2.$$

Proof. Let us first derive point-wise identities in the time-space domain Ω_T. For an integer j, the variable $s_j \in [0,1]$ appearing below depends on $(t,x) \in \Omega_T$. Consider $\phi, \psi_1, \psi_2 \in W^{1,2,2}(I; W^{4,2}_{\boldsymbol{n}}(\Omega), L^2(\Omega))$, and by scaling, we can assume without loss of generality that we have $\|\psi_k\|_{W^{1,2,2}(W^{4,2}_{\boldsymbol{n}},L^2)} \le 1$ for $k = 1,2$.

Let $r_1 := F'(\phi+\psi_1) - F'(\phi) - F''(\phi)\psi_1$. From the classical differentiation rules and Taylor's Theorem, it can be deduced that

$$
\begin{aligned}
r_1 &= F^{(3)}(\phi + s_1\psi_1)\psi_1^2 \\
\partial_t r_1 &= [F''(\phi+\psi_1) - F''(\phi) - F'''(\phi)\psi_1]\partial_t\phi + [F''(\phi+\psi_1) - F''(\phi)]\partial_t\psi_1 \\
&= F^{(4)}(\phi + s_2\psi_1)\psi_1^2\partial_t\phi + F'''(\phi + s_3\psi_1)\psi_1\partial_t\psi_1.
\end{aligned}
$$

Similarly, the gradient of r_1 can be expressed as

$$
\begin{aligned}
\nabla r_1 &= [F''(\phi+\psi_1) - F''(\phi) - F'''(\phi)\psi_1]\nabla\phi + [F''(\phi+\psi_1) - F''(\phi)]\nabla\psi_1 \\
&= F^{(4)}(\phi + s_2\psi_1)\psi_1^2\nabla\phi + F'''(\phi + s_3\psi_1)\psi_1\nabla\psi_1.
\end{aligned}
$$

Thus, $\partial_{\boldsymbol{n}} r_1 = 0$ on Γ_T since $\partial_{\boldsymbol{n}}\phi = \partial_{\boldsymbol{n}}\psi_1 = 0$ on Γ_T. Finally, the Laplacian of r_1 can be written as

$$
\begin{aligned}
\Delta r_1 = \nabla\cdot\nabla r_1 &= [F'''(\phi+\psi_1) - F'''(\phi) - F^{(4)}(\phi)\psi_1]|\nabla\phi|^2 \\
&\quad + [F''(\phi+\psi_1) - F''(\phi) - F'''(\phi)\psi_1]\Delta\phi + 2[F'''(\phi+\psi_1) - F'''(\phi)]\nabla\phi\cdot\nabla\psi_1 \\
&\quad + [F''(\phi+\psi_1) - F''(\phi)]\Delta\psi_1 + F'''(\phi+\psi_1)|\nabla\psi_1|^2 \\
&= F^{(5)}(\phi + s_4\psi_1)\psi_1^2|\nabla\phi|^2 + F^{(4)}(\phi + s_5\psi_1)\psi_1^2\Delta\phi + 2F^{(4)}(\phi + s_6\psi_1)\psi_1\nabla\phi\cdot\nabla\psi_1 \\
&\quad + F'''(\phi + s_3\psi_1)\psi_1\Delta\psi_1 + F'''(\phi+\psi_1)|\nabla\psi_1|^2.
\end{aligned}
$$

From the Hölder inequality, there is $c_\phi = c(\|\phi\|_{W^{1,2,2}(W^{4,2}_{\boldsymbol{n}},L^2)}) > 0$ independent of ψ_1 such that

$$
\begin{aligned}
\|r_1\|_{W^{1,2}(L^2)} &\le c_\phi(\|\psi_1\|^2_{L^4(L^4)} + \|\psi_1\|^2_{L^\infty(L^\infty)}\|\partial_t\phi\|_{L^2(L^2)}) \\
&\quad + c_\phi(\|\psi_1\|_{L^\infty(L^\infty)}\|\partial_t\psi\|_{L^2(L^2)}) \\
\|\nabla r_1\|_{L^2(\boldsymbol{L}^2)} &\le c_\phi(\|\psi_1\|^2_{L^\infty(L^\infty)}\|\nabla\phi\|_{L^2(\boldsymbol{L}^2)} + \|\psi_1\|_{L^\infty(L^\infty)}\|\nabla\psi\|_{L^2(\boldsymbol{L}^2)}) \\
\|\Delta r_1\|_{L^2(L^2)} &\le c_\phi\|\psi_1\|^2_{L^\infty(L^\infty)}(\|\nabla\phi\|^2_{L^4(\boldsymbol{L}^4)} + \|\Delta\phi\|_{L^2(L^2)}) \\
&\quad + c_\phi(\|\psi_1\|_{L^\infty(L^\infty)}\|\nabla\phi\|_{L^4(\boldsymbol{L}^4)}\|\nabla\psi_1\|_{L^4(\boldsymbol{L}^4)} + \|\nabla\psi_1\|^2_{L^4(\boldsymbol{L}^4)}).
\end{aligned}
$$

Hence, we deduce from these estimates and the continuous embeddings $W^{1,2,2}(I; W^{4,2}_{\boldsymbol{n}}(\Omega), L^2(\Omega)) \hookrightarrow L^\infty(I; W^{1,4}(\Omega)) \hookrightarrow L^\infty(I; L^\infty(\Omega))$ that

$$
\|r_1\|_{L^\infty(L^\infty)} \le c\|r_1\|_{L^\infty(W^{1,4})} \le c\|r_1\|_{W^{1,2,2}(W^{2,2}_{\boldsymbol{n}},L^2)} \le c_\phi\|\psi_1\|^2_{W^{1,2,2}(W^{4,2}_{\boldsymbol{n}},L^2)}. \tag{4.5}
$$

The last inequality along with the local Lipschitz continuity of F''' imply that p_F is continuously differentiable and $p_F'(\phi)\psi_1 = F''(\phi)\psi_1$.

Let $r_2 := \widetilde{r}_2\psi_1$, where $\widetilde{r}_2 := F''(\phi+\psi_2) - F''(\phi) - F'''(\phi)\psi_2$. Then, $\partial_t r_2 = \psi_1\partial_t\widetilde{r}_2 + \widetilde{r}_2\partial_t\psi_1$, $\nabla r_2 = \psi_1\nabla\widetilde{r}_2 + \widetilde{r}_2\nabla\psi_1$, and $\Delta r_2 = \psi_1\Delta\widetilde{r}_2 + 2\nabla\widetilde{r}_2\cdot\nabla\psi_1 + \widetilde{r}_2\Delta\psi_1$. The expansions of $\partial_t\widetilde{r}_2$, $\nabla\widetilde{r}_2$, and $\Delta\widetilde{r}_2$ can be handled in the same manner as those presented above for r_1, however, the order of derivatives of F appearing on the right-hand sides are increased by 1. In particular, $\partial_{\boldsymbol{n}} r_2 = 0$ on Γ_T. Hence, (4.5) holds with r_1 and ψ_1 replaced by $\widetilde{r}_2$ and ψ_2, respectively. Moreover,

$$\begin{aligned}
\|r_2\|_{W^{1,2}(L^2)} &\le c_\phi(\|\psi_1\|_{L^\infty(L^\infty)}\|\widetilde{r}_2\|_{W^{1,2}(L^2)} + \|\widetilde{r}_2\|_{L^\infty(L^\infty)}\|\partial_t\psi_1\|_{L^2(L^2)})\\
\|\nabla r_2\|_{L^2(\boldsymbol{L}^2)} &\le c_\phi(\|\psi_1\|_{L^\infty(L^\infty)}\|\nabla\widetilde{r}_2\|_{L^2(\boldsymbol{L}^2)} + \|\widetilde{r}_2\|_{L^\infty(L^\infty)}\|\nabla\psi_1\|_{L^2(\boldsymbol{L}^2)})\\
\|\Delta r_2\|_{L^2(L^2)} &\le c_\phi(\|\psi_1\|_{L^\infty(L^\infty)}\|\Delta\widetilde{r}_2\|_{L^2(L^2)} + \|\nabla\widetilde{r}_2\|_{L^2(\boldsymbol{L}^4)}\|\nabla\psi_1\|_{L^\infty(\boldsymbol{L}^4)})\\
&\quad + c_\phi(\|\widetilde{r}_2\|_{L^\infty(L^\infty)}\|\Delta\psi_1\|_{L^2(L^2)}).
\end{aligned}$$

As a consequence of the previous estimates for r_2 and (4.5) for $\widetilde{r}_2$, one can obtain

$$\|r_2\|_{W^{1,2,2}(W^{2,2}_{\boldsymbol{n}},L^2)} \le c_\phi\|\psi_2\|^2_{W^{1,2,2}(W^{4,2}_{\boldsymbol{n}},L^2)},$$

for some $c_\phi > 0$ independent of ψ_1 and ψ_2. Therefore, p_F is continuously differentiable and $p''_F(\phi)(\psi_1,\psi_2) = F''(\phi)\psi_1\psi_2$, thanks to the local Lipschitz continuity of $F^{(4)}$. □

Remark 4.6. It follows immediately from Lemma 4.5 that

$$p_F \in C^2(W^{1,2,2}(I;W^{4,2}_{\boldsymbol{n}}(\Omega),L^2(\Omega)),X(\Omega_T)),$$

whenever $W^{1,2,2}(I;W^{2,2}_{\boldsymbol{n}}(\Omega),L^2(\Omega)) \hookrightarrow X(\Omega_T)$.

Lemma 4.7. *If $f \in C^5(\mathbb{R}^2)$ and $d_f : (\phi,\theta,\mu) \mapsto \nabla\cdot(f(\phi,\theta)\nabla\mu)$, then*

$$d_f \in C^2([W^{1,2,2}(I;W^{4,2}_{\boldsymbol{n}}(\Omega),L^2(\Omega))]^2 \times L^2(I;W^{2,2}_{\boldsymbol{n}}(\Omega)),L^2(I;L^2(\Omega))).$$

Furthermore, the action of the first and second derivatives are given by

$$d'_f(\phi,\theta,\mu)(\psi_1,\eta_1,\xi_1) = \nabla\cdot(f'(\phi,\theta)(\psi_1,\eta_1)\nabla\mu + f(\phi,\theta)\nabla\xi_1) \tag{4.6}$$

$$\begin{aligned}
d''_f(\phi,\theta,\mu)(\psi_1,\eta_1,\xi_1)(\psi_2,\eta_2,\xi_2) &= \nabla\cdot(f'(\phi,\theta)(\psi_1,\eta_1)\nabla\xi_2 + f'(\phi,\theta)(\psi_2,\eta_2)\nabla\xi_1)\\
&\quad + \nabla\cdot(f''(\phi,\theta)(\psi_1,\eta_1)(\psi_2,\eta_2)\nabla\mu)
\end{aligned} \tag{4.7}$$

for $(\phi,\theta,\mu),(\psi_1,\eta_1,\xi_1),(\psi_2,\eta_2,\xi_2) \in [W^{1,2,2}(I;W^{4,2}_{\boldsymbol{n}}(\Omega),L^2(\Omega))]^2 \times L^2(I;W^{2,2}_{\boldsymbol{n}}(\Omega))$, where

$$f''(\phi,\theta)(\psi_1,\eta_1)(\psi_2,\eta_2) := \psi_2 f'_\phi(\phi,\theta)(\psi_1,\eta_1) + \eta_2 f'_\theta(\phi,\theta)(\psi_1,\eta_1).$$

Proof. Let us denote the right-hand sides of (4.6) and (4.7) by δ_1 and δ_2. Again, by scaling, we may assume without loss of generality that

$$\|(\psi_k,\eta_k,\xi_k)\|_{[W^{1,2,2}(W^{4,2}_{\boldsymbol{n}},L^2)]^2\times L^2(W^{2,2}_{\boldsymbol{n}})} \le 1$$

for $k = 1,2$. Setting $r_j := f(\phi+\psi_j,\theta+\eta_j) - f(\phi,\eta) - f'(\phi,\theta)(\psi_j,\eta_j)$ for $j = 1,2$ and $s_1 := f(\phi+\psi_1,\theta+\eta_1) - f(\phi,\theta)$, we can write

$$\begin{aligned}
d_{f,1} &:= d_f(\phi+\psi_1,\theta+\eta_1,\mu+\xi_1) - d_f(\phi,\theta,\mu) - \delta_1\\
&= \nabla\cdot(r_1\nabla\mu + s_1\nabla\xi_1) = \nabla r_1\cdot\nabla\mu + r_1\Delta\mu + \nabla s_1\cdot\nabla\xi_1 + s_1\Delta\xi_1.
\end{aligned}$$

As in the proof of Lemma 4.5, it can be shown that there exists a constant $c_{f,\phi,\theta,\mu} > 0$ independent of ψ_1 and ψ_2 such that

$$\|r_1\|_{W^{1,2,2}(W^{4,2}_{\boldsymbol{n}},L^2)} + \|s_1\|^2_{W^{1,2,2}(W^{4,2}_{\boldsymbol{n}},L^2)} \le c_{f,\phi,\theta,\mu}\|(\psi_1,\eta_1)\|^2_{[W^{1,2,2}(W^{4,2}_{\boldsymbol{n}},L^2)]^2}.$$

This estimate implies that

$$\begin{aligned}
\|d_{f,1}\|_{L^2(L^2)} &\le \|\nabla r_1\|_{L^\infty(\boldsymbol{L}^4)}\|\nabla\mu\|_{L^2(\boldsymbol{L}^4)} + \|r_1\|_{L^\infty(L^\infty)}\|\Delta\mu\|_{L^2(L^2)}\\
&\quad + \|\nabla s_1\|_{L^\infty(\boldsymbol{L}^4)}\|\nabla\xi_1\|_{L^2(\boldsymbol{L}^4)} + \|s_1\|_{L^\infty(L^\infty)}\|\Delta\xi_1\|_{L^2(L^2)}
\end{aligned}$$

$$\leq c_{f,\phi,\theta,\mu}\|(\psi_1,\eta_1,\xi_1)\|^2_{[W^{1,2,2}(W^{4,2}_{\boldsymbol{n}},L^2)]^2\times L^2(W^{2,2}_{\boldsymbol{n}})}.$$

For the action of the second-order derivative, by setting $q_2 := f'(\phi+\psi_2,\theta+\eta_2)(\psi_1,\eta_1) - f'(\phi,\theta)(\psi_1,\psi_2) - f''(\phi,\theta)(\psi_1,\eta_1)(\psi_2,\eta_2)$ and $s_2 := f'(\phi+\psi_2,\theta+\eta_2)(\psi_1,\eta_1) - f'(\phi,\theta)(\psi_1,\eta_1)$, we deduce that

$$\begin{aligned} d_{f,2} &:= d_f'(\phi+\psi_2,\theta+\eta_2,\mu+\xi_2)(\psi_1,\eta_1,\xi_1) - d_f'(\phi,\theta,\mu)(\psi_1,\eta_1,\xi_1) - \delta_2 \\ &= \nabla\cdot(q_2\nabla\mu + r_2\nabla\xi_1 + s_2\nabla\xi_2) \\ &= \nabla q_2\cdot\nabla\mu + q_2\Delta\mu + \nabla r_2\cdot\nabla\xi_1 + r_2\Delta\xi_1 + \nabla s_2\cdot\nabla\xi_2 + s_2\Delta\xi_2. \end{aligned}$$

A similar argument as above leads to

$$\begin{aligned} &\|r_2\|_{W^{1,2,2}(W^{4,2}_{\boldsymbol{n}},L^2)} + \|q_2\|_{W^{1,2,2}(W^{4,2}_{\boldsymbol{n}},L^2)} + \|s_2\|^2_{W^{1,2,2}(W^{4,2}_{\boldsymbol{n}},L^2)} \\ &\qquad \leq c_{f,\phi,\theta,\mu}\|(\psi_2,\eta_2)\|^2_{[W^{1,2,2}(W^{4,2}_{\boldsymbol{n}},L^2)]^2}, \end{aligned}$$

and as a result,

$$\|d_{f,2}\|_{L^2(L^2)} \leq c_{f,\phi,\theta,\mu}\|(\psi_2,\eta_2,\xi_2)\|^2_{[W^{1,2,2}(W^{4,2}_{\boldsymbol{n}},L^2)]^2\times L^2(W^{2,2}_{\boldsymbol{n}})}.$$

From these, we obtain (4.6) and (4.7), and due to the local Lipschitz continuity of f''', we deduce that d_f is twice continuously differentiable. $\square$

Remark 4.8. Theorem 4.7 holds as well when the function space $L^2(I;W^{2,2}_{\boldsymbol{n}}(\Omega))$ pertaining to the variable μ is replaced by $W^{1,2,2}(I;W^{2,2}_{\boldsymbol{n}}(\Omega),W^{-2,2}_{\boldsymbol{n}}(\Omega))$. This is due to the fact that the latter space is embedded in the former space.

We are now in position to prove the main result of this subsection.

Theorem 4.9. *Let* (A1)$_6$ *and* (A2)$_5$ *be satisfied. Then,* $\mathcal{T}\in C^2(\boldsymbol{U},\mathcal{W}^2(\Omega_T))\cap C^2(\boldsymbol{U},\mathcal{V}^2(\Omega_T))$. *Moreover, if* $\mathcal{P}:\boldsymbol{U}\to\mathcal{U}^0(\Omega_T)^*$ *is given by* $\mathcal{P}\boldsymbol{u} := (0,0,\boldsymbol{P}_\sigma\boldsymbol{u},\mathbb{O})^{\mathrm{t}}$, *then the action of the first-order and second-order derivatives of* $\mathcal{T}$ *are given by*

$$\begin{aligned} \mathcal{T}'(\boldsymbol{u})\boldsymbol{h} &= [\mathcal{A}_1(\mathcal{T}(\boldsymbol{u}))]^{-1}\mathcal{P}\boldsymbol{h} \qquad \forall\,\boldsymbol{h}\in\boldsymbol{U}, \\ \mathcal{T}''(\boldsymbol{u})(\boldsymbol{h}_1,\boldsymbol{h}_2), &= -[\mathcal{A}_1(\mathcal{T}(\boldsymbol{u}))]^{-1}[\mathcal{A}_1(\mathcal{T}(\boldsymbol{u}))]'(\mathcal{T}'(\boldsymbol{u})\boldsymbol{h}_1,\mathcal{T}'(\boldsymbol{u})\boldsymbol{h}_2) \qquad \forall\,\boldsymbol{h}_1,\boldsymbol{h}_2\in\boldsymbol{U}. \end{aligned}$$

Proof. It suffices to show that

$$\begin{aligned} \mathcal{N} \in\, & C^2(\mathcal{W}^2(\Omega_T)\times\boldsymbol{U},\mathcal{U}^0(\Omega_T)^*\times\mathcal{D}^2(\Omega)) \\ & \cap C^2(\mathcal{W}^2(\Omega_T)\times\boldsymbol{U},\mathcal{Y}^0(\Omega_T)^*\times\mathcal{D}^2(\Omega)). \end{aligned} \tag{4.8}$$

Indeed, if this is the case, then given $\boldsymbol{u}^*\in U$, there exists a unique $(\phi^*,\mu^*,\theta^*,\boldsymbol{v}^*,\mathbb{S}^*)\in\mathcal{W}^2(\Omega_T)$ such that $\mathcal{N}(\phi^*,\mu^*,\theta^*,\boldsymbol{v}^*,\mathbb{S}^*,\boldsymbol{u}^*)=0$, and from Theorem A.3, we have

$$\begin{aligned} &\frac{\partial\mathcal{N}(\phi^*,\mu^*,\theta^*,\boldsymbol{v}^*,\mathbb{S}^*,\boldsymbol{u}^*)}{\partial(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S})} = \mathcal{A}(\phi^*,\mu^*,\theta^*,\boldsymbol{v}^*,\mathbb{S}^*) \\ &\quad \in \mathcal{L}_{\mathrm{iso}}(\mathcal{W}^2(\Omega_T),\mathcal{Y}^0(\Omega_T)^*\times\mathcal{D}^2(\Omega))\cap\mathcal{L}_{\mathrm{iso}}(\mathcal{V}^2(\Omega_T),\mathcal{U}^0(\Omega_T)^*\times\mathcal{D}^2(\Omega)). \end{aligned}$$

From the implicit function theorem for Banach spaces, see [72, Section 4.7] for instance, it will follow that $\mathcal{T}\in C^2(\boldsymbol{U},\mathcal{W}^2(\Omega_T))\cap C^2(\boldsymbol{U},\mathcal{V}^2(\Omega_T))$. To show that (4.8) is satisfied, we only need to establish the twice continuous differentiability of the diffusion terms and the derivative F' of the Cahn–Hilliard potential since the other expressions in (1.15) are either bounded linear, bilinear, or trilinear forms. However, these were already done in Lemma 4.7 and Remark 4.6 with $X(\Omega_T) = W^{1,2,2}(I; W^{2,2}_{\boldsymbol{n}}(\Omega), W^{-2,2}_{\boldsymbol{n}}(\Omega))$ in the case of the pair $(\mathcal{W}^2(\Omega_T), \mathcal{Y}^0(\Omega_T)^*)$ or $X(\Omega_T) = L^2(I; W^{2,2}_{\boldsymbol{n}}(\Omega))$ for the pair $(\mathcal{V}^2(\Omega_T), \mathcal{U}^0(\Omega_T)^*)$. The representations of the actions of the derivatives of $\mathcal{T}$ presented above can be deduced by implicit differentiation. □

The action of the first-order derivative $\mathcal{T}'(\boldsymbol{u})$ stated in Theorem 4.9 is nothing but the solution of the system linearized about $\mathcal{T}(\boldsymbol{u})$ in the direction of a control $\boldsymbol{h}$. Likewise, the action of the second-order derivative $\mathcal{T}''(\boldsymbol{u})$ corresponds to the solution of the linearized system, but with right-hand sides that correspond to the actions of the second-order derivatives of the nonlinear terms in the state system (1.15) at $(\boldsymbol{h}_1, \boldsymbol{h}_2)$. Take note that the second-order actions of the bilinear and trilinear terms can be easily calculated. Thus, in principle, the linear system can be written explicitly with the help of Lemma 4.5 and Lemma 4.7. However, this PDE system is a bit tedious and messy to write, and for this reason we leave the task to the interested reader. Nonetheless, the above representation of the second-order derivative allows the study of second-order necessary and sufficient and conditions for local optimality. We do not also pursue this issue here and refer to [56] for the case with control constraints.

5. The Adjoint System

For this section, we study the dual problem to the linearized system. Similar to Theorem 4.2 for the linearized system, we aim to analyze solutions of the adjoint system with varying order of regularity. However, unlike the linearized system, the Faedo–Galerkin method will not be used, with the exception of Theorems 5.9 and 5.12.

We begin with the *dual result* to Theorem 4.2 (see Theorem 5.1 below for the precise formulation). This theorem establishes solutions to the adjoint problem in the spaces $\mathcal{U}^{2-k}(\Omega_T)$ or $\mathcal{Y}^{2-k}(\Omega_T)$ for $k = 0, 1, 2$. Since these function spaces do not involve time-derivatives, we need to determine the regularity of the time-derivatives of the adjoint states and deduce the corresponding stability estimates. This will be done by separately examining the evolution equations for each adjoint variable.

The dependence of the diffusion coefficients on (ϕ, θ) introduces gradient terms with coefficients involving the gradients of the state variables. The meticulous estimation of these terms is important in determining the appropriate function spaces for the time-derivatives of the adjoint states. Additionally, to account for the inclusion of time-derivatives of the states in the cost functional, we will employ function spaces for the source terms in the adjoint problem that can handle such functionals.

Let $\mathcal{W}^k_0(\Omega_T) := \{(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S}) \in \mathcal{W}^k(\Omega_T) : \phi(0) = \theta(0) = 0, \boldsymbol{w}(0) = \mathbf{0}, \mathbb{S}(0) = \mathbb{O}\}$, treated as a closed subspace of $\mathcal{W}^k(\Omega_T)$. We define the subspace $\mathcal{V}^k_0(\Omega_T)$ of $\mathcal{V}^k(\Omega_T)$ in a similar fashion. Then, notice that

$$\mathcal{A}_1(\mathcal{T}(\boldsymbol{u})) \in \mathcal{L}_{\text{iso}}(\mathcal{V}^k_0(\Omega_T), \mathcal{U}^{2-k}(\Omega_T)^*) \cap \mathcal{L}_{\text{iso}}(\mathcal{W}^k_0(\Omega_T), \mathcal{Y}^{2-k}(\Omega_T)^*)$$

for $k = 0, 1, 2$ in virtue of Theorem 4.2. Hence, for the inverse of the adjoint operator, it holds that

$$[\mathcal{A}_1(\mathcal{T}(\boldsymbol{u}))]^{-*} := ([\mathcal{A}_1(\mathcal{T}(\boldsymbol{u}))]^*)^{-1}$$

$$\in \mathcal{L}_{\rm iso}(\mathcal{V}_0^k(\Omega_T)^*, \mathcal{U}^{2-k}(\Omega_T)) \cap \mathcal{L}_{\rm iso}(\mathcal{W}_0^k(\Omega_T)^*, \mathcal{Y}^{2-k}(\Omega_T)). \quad (5.1)$$

This observation will be the foundation of the analysis for the adjoint system.

The variational formulation for the dual problem to the linearized system is expressed as follows: Given $(g_{\rm o}, g_{\rm c}, g_{\rm h}, \boldsymbol{g}_{\rm v}, \mathbb{G}_{\rm s}) \in \mathcal{V}_0^k(\Omega_T)^*$ ($\mathcal{W}_0^k(\Omega_T)^*$, respectively), determine $(\varphi, \zeta, \vartheta, \boldsymbol{y}, \mathbb{Y}) \in \mathcal{U}^{2-k}(\Omega_T)$ ($\mathcal{Y}^{2-k}(\Omega_T)$, respectively) such that the variational equation

$$\begin{aligned}&\langle \mathcal{A}_1(\mathcal{T}(\boldsymbol{u}))^*(\varphi, \zeta, \vartheta, \boldsymbol{y}, \mathbb{Y}), (\psi, \xi, \eta, \boldsymbol{w}, \mathbb{T})\rangle_{(\mathcal{V}_0^k)^*, \mathcal{V}_0^k} \\ &\quad = \langle (g_{\rm o}, g_{\rm c}, g_{\rm h}, \boldsymbol{g}_{\rm v}, \mathbb{G}_{\rm s}), (\psi, \xi, \eta, \boldsymbol{w}, \mathbb{T})\rangle_{(\mathcal{V}_0^k)^*, \mathcal{V}_0^k} \quad \forall (\psi, \xi, \eta, \boldsymbol{w}, \mathbb{T}) \in \mathcal{V}_0^k(\Omega_T)\end{aligned} \quad (5.2)$$

holds, with appropriate modifications in the case of $\mathcal{W}_0^k(\Omega_T)^*$ and $\mathcal{Y}^{2-k}(\Omega_T)$. The analysis on the linearized system, in particular (5.1), immediately leads us to the following well-posedness theorem for this problem.

Theorem 5.1. *Let $k = 0, 1, 2$. Suppose that* (A1)$_3$ *and* (A2)$_1$ *are satisfied when $k = 0, 1$ or* (A1)$_4$ *and* (A2)$_2$ *are satisfied when $k = 2$. For each $(g_{\rm o}, g_{\rm c}, g_{\rm h}, \boldsymbol{g}_{\rm v}, \mathbb{G}_{\rm s}) \in \mathcal{V}_0^k(\Omega_T)^*$ ($\mathcal{W}_0^k(\Omega_T)^*$, respectively), there exists a unique variational solution $(\varphi, \zeta, \vartheta, \boldsymbol{y}, \mathbb{Y}) \in \mathcal{U}^{2-k}(\Omega_T)$ ($\mathcal{Y}^{2-k}(\Omega_T)$, respectively) to the adjoint problem* (5.2)*. Furthermore, for $c_k := \|[\mathcal{A}_1(\mathcal{T}(\boldsymbol{u}))]^{-*}\|_{\mathcal{L}((\mathcal{V}_0^k)^*, \mathcal{U}^{2-k})}$ we have*

$$\|(\varphi, \zeta, \vartheta, \boldsymbol{y}, \mathbb{Y})\|_{\mathcal{U}^{2-k}} \le c_k \|(g_{\rm o}, g_{\rm c}, g_{\rm h}, \boldsymbol{g}_{\rm v}, \mathbb{G}_{\rm s})\|_{(\mathcal{V}_0^k)^*} \quad (5.3)$$

and a similar estimate holds in the case of the function spaces $\mathcal{Y}^{2-k}(\Omega_T)$ and $\mathcal{W}_0^k(\Omega_T)^*$.

For $(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S}) \in \mathcal{W}^2(\Omega_T)$, $\gamma = \phi, \theta$, and suitable $(\varphi, \vartheta, \boldsymbol{y}, \mathbb{Y})$, we introduce the notation

$$\begin{aligned}\mathfrak{d}_\gamma(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S})(\varphi, \vartheta, \boldsymbol{y}, \mathbb{Y}) &:= m_\gamma(\phi, \theta)\nabla\mu \cdot \nabla\varphi \\ &+ \chi_\gamma(\phi, \theta)\nabla\theta \cdot \nabla\vartheta + 2\nu_\gamma(\phi, \theta)\mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{y} + \varepsilon_\gamma(\phi, \theta)\nabla\mathbb{S} \mathbin{\therefore} \nabla\mathbb{Y}.\end{aligned} \quad (5.4)$$

Note that such terms arise due to the dependence of the diffusion coefficients to the order parameter and temperature, which are obviously not present in the constant-coefficient case. The regularity of $\mathfrak{d}_\gamma(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S})(\varphi, \vartheta, \boldsymbol{y}, \mathbb{Y})$ depends on the nature of the solution to the adjoint system. We shall look at this in the succeeding discussions.

Let us calculate formally the actions for the dual operators of $\mathbb{M}_\mathbb{S}(\theta, \mathbb{S})$ and $\mathbb{P}'(\mathbb{S})$ as defined in (4.3) and (4.4). For smooth enough $\mathbb{X}$ and $\mathbb{Y}$, we have

$$\begin{aligned}\int_\Omega \mathbb{M}_\mathbb{S}(\theta, \mathbb{S})\mathbb{T} : \mathbb{X}\, {\rm d}x &= \int_\Omega [2\sigma_0 a {\rm Tr}((\mathbb{S}\mathbb{T})^{\rm t}\mathbb{X}) - \mathbb{T} : (\sigma_0 a \mathbb{X})]\, {\rm d}x \\ &+ \int_\Omega [\mathbb{T} : (\sigma\theta\mathbb{X}) + a(\mathbb{T} : \mathbb{I})(\mathbb{S} : \mathbb{X}) + \mathbb{T} : (a{\rm Tr}(\mathbb{S})\mathbb{X})]\, {\rm d}x \\ &= \int_\Omega \mathbb{T} : \{\sigma_0 a(2\mathbb{S} - \mathbb{I})\mathbb{X} + \sigma\theta\mathbb{X} + a[(\mathbb{S} : \mathbb{X})\mathbb{I} + {\rm Tr}(\mathbb{S})\mathbb{X}]\}\, {\rm d}x \\ \int_\Omega \mathbb{P}'(\mathbb{S})\mathbb{T} : \mathbb{Y}\, {\rm d}x &= \int_\Omega \mathbb{T} : (-\ell\mathbb{Y}) + \beta(\mathbb{T} : \mathbb{I})(\mathbb{I} : \mathbb{Y})\, {\rm d}x \\ &- \int_\Omega \beta[2(\mathbb{T} : \mathbb{I})({\rm Tr}(\mathbb{S})\mathbb{S} : \mathbb{Y}) - \mathbb{T} : ({\rm Tr}(\mathbb{S})^2\mathbb{Y})]\, {\rm d}x\end{aligned}$$

$$= \int_\Omega \mathbb{T} : \{-\ell\mathbb{Y} + \beta\mathrm{Tr}(\mathbb{Y})\mathbb{I} - \beta[2(\mathrm{Tr}(\mathbb{S})\mathbb{S} : \mathbb{Y})\mathbb{I} - \mathrm{Tr}(\mathbb{S})^2\mathbb{Y}]\}\,\mathrm{d}x.$$

These imply that we have

$$\mathbb{M}_{\mathbb{S}}(\theta, \mathbb{S})^*\mathbb{X} = \sigma_0 a(2\mathbb{S} - \mathbb{I})\mathbb{X} + \sigma\theta\mathbb{X} + a[(\mathbb{S} : \mathbb{X})\mathbb{I} + \mathrm{Tr}(\mathbb{S})\mathbb{X}] \tag{5.5}$$

$$\mathbb{P}'(\mathbb{S})^*\mathbb{Y} = -\ell\mathbb{Y} + \beta[\mathrm{Tr}(\mathbb{Y})\mathbb{I} - 2(\mathrm{Tr}(\mathbb{S})\mathbb{S} : \mathbb{Y})\mathbb{I} - \mathrm{Tr}(\mathbb{S})^2\mathbb{Y}]. \tag{5.6}$$

With these, we will see in the proof of the succeeding theorem that the strong formulation of the variational equation (5.2) with $(g_\mathrm{o}, g_\mathrm{c}, g_\mathrm{h}, \boldsymbol{g}_\mathrm{v}, \mathbb{G}_\mathrm{s})$ given by (5.8) below, is the linear system

$$\left[\begin{array}{ll}
-\partial_t\varphi - \boldsymbol{v}\cdot\nabla\varphi + \mathfrak{d}_\phi(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S})(\varphi,\vartheta,\boldsymbol{y},\mathbb{Y}) + \alpha\Delta\zeta & \\
\quad - F''(\phi)\zeta + \kappa\boldsymbol{y}\cdot\nabla\mu - b_\mathrm{o}\mathbf{g}\cdot\boldsymbol{y} = g_\mathrm{o} & \text{in } \Omega_T,\\
\zeta - \nabla\cdot(m(\phi,\theta)\nabla\varphi) - \kappa\boldsymbol{y}\cdot\nabla\phi = g_\mathrm{c} & \text{in } \Omega_T,\\
-\partial_t(\vartheta - \tau\Delta\vartheta) - \boldsymbol{v}\cdot\nabla\vartheta - \nabla\cdot(\chi(\phi,\theta)\nabla\vartheta) + b\Delta^2\vartheta & \\
\quad + \mathfrak{d}_\theta(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S})(\varphi,\vartheta,\boldsymbol{y},\mathbb{Y}) + \sigma\mathbb{S} : \mathbb{D}\boldsymbol{y} - b_\mathrm{h}\mathbf{g}\cdot\boldsymbol{y} = g_\mathrm{h} & \text{in } \Omega_T,\\
-\partial_t\boldsymbol{y} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{y} + (\nabla\boldsymbol{v})\boldsymbol{y} - \nabla\cdot(2\nu(\phi,\theta)\mathbb{D}\boldsymbol{y}) & \\
\quad - \nabla\cdot([\mathbb{S},\mathbb{Y}] - a\{\mathbb{S},\mathbb{Y}\} - \lambda\mathbb{Y} - \vartheta\mathbb{S}) & \\
\quad + \varphi\nabla\phi + \vartheta\nabla\theta + \nabla\mathbb{S}\odot\mathbb{Y} - a_0\vartheta\mathbf{g} + \nabla\mathrm{s} = \boldsymbol{g}_\mathrm{v} & \text{in } \Omega_T,\\
-\partial_t\mathbb{Y} - (\boldsymbol{v}\cdot\nabla)\mathbb{Y} + 2\mathbb{W}\boldsymbol{v}\mathbb{Y} - 2a\mathbb{D}\boldsymbol{v}\mathbb{Y} - \nabla\cdot(\varepsilon(\phi,\theta)\nabla\mathbb{Y}) & \\
\quad - \mathbb{P}'(\mathbb{S})^*\mathbb{Y} + \mathbb{M}_\mathbb{S}(\theta,\mathbb{S})^*\mathbb{D}\boldsymbol{y} - \vartheta\mathbb{D}\boldsymbol{v} = \mathbb{G}_\mathrm{s} & \text{in } \Omega_T,\\
\nabla\cdot\boldsymbol{y} = 0 & \text{in } \Omega_T,\\
\partial_{\boldsymbol{n}}\varphi = \partial_{\boldsymbol{n}}\Delta\varphi = 0,\ \partial_{\boldsymbol{n}}\vartheta = \partial_{\boldsymbol{n}}\Delta\vartheta = 0,\ \boldsymbol{y} = \mathbf{0},\ \partial_{\boldsymbol{n}}\mathbb{Y} = \mathbb{O} & \text{on } \Gamma_T,\\
\varphi(T) = \varphi_T,\ \vartheta(T) - \tau\Delta\vartheta(T) = \vartheta_T,\ \boldsymbol{y}(T) = \boldsymbol{y}_T,\ \mathbb{Y}(T) = \mathbb{Y}_T & \text{in } \Omega,
\end{array}\right. \tag{5.7}$$

where $\nabla\mathbb{S}\odot\mathbb{Y} = [\partial_j\mathbb{S} : \mathbb{Y}]_{j=1}^2$. Here, the given functions are the coefficients and strong solution $(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S})$ of (1.15), the source term $(g_\mathrm{o}, g_\mathrm{c}, g_\mathrm{h}, \boldsymbol{g}_\mathrm{v}, \mathbb{G}_\mathrm{s})$, and the terminal data $(\varphi_T, \vartheta_T, \boldsymbol{y}_T, \mathbb{Y}_T)$, while the unknown adjoint state is $(\varphi, \zeta, \vartheta, \boldsymbol{y}, \mathbb{Y})$. As usual, for low regular adjoint variables, the spatial and temporal derivatives appearing above are to be understood at least in the sense of distributions (refer to the proof of Theorem 5.2 below for the precise definitions). Owing to the regularity of the state variables, these terms are not general distributions per se, but are elements of suitable Lebesgue–Bochner spaces or negative-index Sobolev spaces with respect to time.

Assuming additional regularity on the data in the adjoint system, one can establish additional smoothness of the dual variables. In this direction, we shall consider the following decomposition for the data appearing in the adjoint system:

$$(g_\mathrm{o}, g_\mathrm{c}, g_\mathrm{h}, \boldsymbol{g}_\mathrm{v}, \mathbb{G}_\mathrm{s}) = (g_{d\mathrm{o}}, g_\mathrm{c}, g_{d\mathrm{h}}, \boldsymbol{g}_{d\mathrm{v}}, \mathbb{G}_{d\mathrm{s}}) + e_T(\varphi_T, \vartheta_T, \boldsymbol{y}_T, \mathbb{Y}_T), \tag{5.8}$$

where $e_T : \mathcal{D}^k(\Omega)^* \to \mathcal{V}_0^k(\Omega_T)^*$ is defined by

$$\begin{aligned}
&\langle e_T(\varphi_T, \vartheta_T, \boldsymbol{y}_T, \mathbb{Y}_T), (\psi, \xi, \eta, \boldsymbol{w}, \mathbb{T})\rangle_{(\mathcal{V}_0^k)^*, \mathcal{V}_0^k}\\
&\quad := \langle(\varphi_T, \vartheta_T, \boldsymbol{y}_T, \mathbb{Y}_T), (\psi(T), \eta(T), \boldsymbol{w}(T), \mathbb{T}(T))\rangle_{(\mathcal{D}^k)^*, \mathcal{D}^k}.
\end{aligned}$$

Note that if $(\psi, \xi, \eta, \boldsymbol{w}, \mathbb{T}) \in \mathcal{V}_0^k(\Omega_T)$, then $(\psi, \eta, \boldsymbol{w}, \mathbb{T}) \in C(\bar{I}; \mathcal{D}^k(\Omega))$. Thus, we can easily see that $e_T \in \mathcal{L}(\mathcal{D}^k(\Omega)^*, \mathcal{V}_0^k(\Omega_T)^*)$. Observe that e_T is independent with respect to the variable ξ. In concrete terms, the first tuple on the right-hand side of the decomposition (5.8) corresponds to terms arising from the cost functional integrated over the time-space domain Ω_T, while the second tuple is designated for terms arising from integration over Ω at the terminal time T.

The precise regularity conditions for the first tuple on the right-hand side of (5.8) will be presented below. In general, this tuple will be taken in the dual of a Hilbert space $\mathcal{X}^k(\Omega_T)$, for which

$$\begin{aligned}\mathcal{V}_0^k(\Omega_T) \hookrightarrow \mathcal{X}^k(\Omega_T) := X_{\rm o}^k(\Omega_T) \times L^2(I; W_{\boldsymbol{n}}^{k,2}(\Omega)) \\ \times X_{\rm h}^k(\Omega_T) \times \boldsymbol{X}_{\rm v}^k(\Omega_T) \times \mathbb{X}_{\rm s}^k(\Omega_T),\end{aligned} \tag{5.9}$$

for $k = 0, 1, 2$. All throughout, we assume that $C^\infty(\bar{I}; C^\infty(\bar{\Omega}) \cap \boldsymbol{W}_{\boldsymbol{n}}^{4,2}(\Omega))$ is dense in $X_{\rm o}^k(\Omega_T)$ and $X_{\rm h}^k(\Omega_T)$, $C^\infty(\bar{I}; C_0^\infty(\Omega)^2)$ is dense in $\boldsymbol{X}_{\rm v}^k(\Omega_T)$, and $C^\infty(\bar{I}; C^\infty(\bar{\Omega})^{2\times 2} \cap \mathbb{L}_{\rm s}^2(\Omega))$ is dense in $\mathbb{X}_{\rm s}^k(\Omega_T)$. Furthermore, it is assumed that

$$B_N = I + \tau A_N \in \mathcal{L}_{\rm iso}(X_{\rm h}^k(\Omega_T), B_N X_{\rm h}^k(\Omega_T)), \quad k = 0, 1, 2. \tag{5.10}$$

We note that the function space $\mathcal{X}^k(\Omega_T)$ appears, for instance, when the cost functional in the optimal control problem involves time-derivatives (see Section 6).

Recall that given a finite collection $\{(X_k, \|\cdot\|_{X_k})\}_{k=1}^n$ of Banach spaces that are continuously embedded in some Hausdorff topological vector space, the sum $X_1 + \cdots + X_n := \{x_1 + \cdots + x_k : x_1 \in X_1, \ldots, x_n \in X_n\}$ is again a Banach space when equipped with the norm

$$\|x\|_{X_1+\cdots+X_n} := \inf_{\substack{x = x_1 + \cdots + x_n \\ x_1 \in X_1, \ldots, x_n \in X_n}} \sum_{k=1}^n \|x_k\|_{X_k}. \tag{5.11}$$

The following theorem is concerned with the time-regularity of the very weak solutions to the adjoint system.

Theorem 5.2. *Let* $(\mathrm{A1})_4$ *and* $(\mathrm{A2})_2$ *hold. Suppose that we have source functions* $(g_{\rm do}, g_{\rm c}, g_{\rm dh}, \boldsymbol{g}_{\rm dv}, \mathbb{G}_{\rm ds}) \in \mathcal{X}^2(\Omega_T)^*$ *and initial data* $(\varphi_T, \vartheta_T, \boldsymbol{y}_T, \mathbb{Y}_T) \in \mathcal{D}^2(\Omega)^*$. *Then, the adjoint system* (5.7) *admits a unique solution* $(\varphi, \zeta, \vartheta, \boldsymbol{y}, \mathbb{Y}) \in \mathcal{U}^0(\Omega_T)$ *such that*

$$\begin{aligned}&\|(\varphi, \zeta, \vartheta, \boldsymbol{y}, \mathbb{Y})\|_{\mathcal{U}^0} + \|\partial_t \varphi\|_{L^2(W_{\boldsymbol{n}}^{-4,2}) + L^1(W_{\boldsymbol{n}}^{-2,2}) + (X_{\rm o}^2)^*} \\ &\quad + \|\partial_t \vartheta\|_{L^2(W_{\boldsymbol{n}}^{-2,2}) + L^1(L^2) + (B_N X_{\rm h}^2)^*} + \|\partial_t \boldsymbol{y}\|_{L^2(\boldsymbol{W}_{0,\sigma}^{-2,2}) + (\boldsymbol{X}_{\rm v}^2)^*} \\ &\quad + \|\partial_t \mathbb{Y}\|_{L^2(\mathbb{W}_{\boldsymbol{n},{\rm s}}^{-2,2}) + (\mathbb{X}_{\rm s}^2)^*} \le c\|(g_{\rm do}, g_{\rm c}, g_{\rm dh}, \boldsymbol{g}_{\rm dv}, \mathbb{G}_{\rm ds})\|_{(\mathcal{X}^2)^*} \\ &\quad + c\|(\varphi_T, \vartheta_T, \boldsymbol{y}_T, \mathbb{Y}_T)\|_{(\mathcal{D}^2)^*}\end{aligned}$$

for some $c = c(\|(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S})\|_{\mathcal{W}^2}) > 0$.

Proof. We shall denote the space-time variable by $\omega = (t, x) \in \Omega_T$. By assumption, it follows that the tuple defined in (5.8) lies in $\mathcal{V}_0^2(\Omega_T)^*$. Hence, Theorem 5.1 tells us that (5.2) has a unique solution $(\varphi, \zeta, \vartheta, \boldsymbol{y}, \mathbb{Y}) \in \mathcal{U}^0(\Omega_T)$. Moreover, we obtain from (5.3) that

$$\begin{aligned}&\|(\varphi, \zeta, \vartheta, \boldsymbol{y}, \mathbb{Y})\|_{\mathcal{U}^0} \\ &\quad \le \widetilde{c}_0 := c[\|(g_{\rm do}, g_{\rm c}, g_{\rm dh}, \boldsymbol{g}_{\rm dv}, \mathbb{G}_{\rm ds})\|_{(\mathcal{X}^2)^*} + c_T\|(\varphi_T, \vartheta_T, \boldsymbol{y}_T, \mathbb{Y}_T)\|_{(\mathcal{D}^2)^*}],\end{aligned} \tag{5.12}$$

where $c_T := \|e_T\|_{\mathcal{L}((\mathcal{D}^2)^*,(\mathcal{V}_0^2)^*)}$, thanks to $\mathcal{X}^2(\Omega_T)^* \hookrightarrow \mathcal{V}_0^2(\Omega_T)^*$ from the assumption (5.9). The time-regularity of the state variables will be established by duality arguments. The main idea is to take a test function in $\mathcal{V}_0^2(\Omega_T)$ where all components vanish except one.

Let us start by determining the regularity of $\partial_t\varphi$. Taking test functions $\psi \in W_0^{1,2,2}(I; W_{\boldsymbol{n}}^{4,2}(\Omega), L^2(\Omega)) \hookrightarrow X_\mathrm{o}^2(\Omega_T)$, $\xi = \eta = 0$, $\boldsymbol{w} = \mathbf{0}$, and $\mathbb{T} = \mathbb{O}$ in (5.2) leads to the equation

$$\begin{aligned}
&\int_{\Omega_T} (\partial_t\psi)\varphi \,\mathrm{d}\omega + \int_{\Omega_T} (\boldsymbol{v}\cdot\nabla\psi)\varphi \,\mathrm{d}\omega - \int_{\Omega_T} \nabla\cdot(m_\phi(\phi,\theta)\psi\nabla\mu)\varphi \,\mathrm{d}\omega \\
&+ \langle \zeta, \alpha\Delta\psi - F''(\phi)\psi\rangle_{L^2(W_{\boldsymbol{n}}^{-2,2}),L^2(W_{\boldsymbol{n}}^{2,2})} - \int_{\Omega_T} \nabla\cdot(\chi_\phi(\phi,\theta)\psi\nabla\theta)\vartheta \,\mathrm{d}\omega \\
&- \int_{\Omega_T} \nabla\cdot(2\nu_\phi(\phi,\theta)\psi\mathbb{D}\boldsymbol{v})\cdot\boldsymbol{y}\,\mathrm{d}\omega - \int_{\Omega_T} \kappa\mu\nabla\psi\cdot\boldsymbol{y}\,\mathrm{d}\omega - \int_{\Omega_T} b_\mathrm{o}\psi\mathbf{g}\cdot\boldsymbol{y}\,\mathrm{d}\omega \\
&- \int_{\Omega_T} \nabla\cdot(\varepsilon_\phi(\phi,\theta)\psi\nabla\mathbb{S}) : \mathbb{Y}\,\mathrm{d}\omega = \langle g_{d\mathrm{o}}, \psi\rangle_{(X_\mathrm{o}^2)^*,X_\mathrm{o}^2} + \langle \varphi_T, \psi(T)\rangle_{W_{\boldsymbol{n}}^{-2,2}\times W_{\boldsymbol{n}}^{2,2}}.
\end{aligned} \tag{5.13}$$

Alternatively, one can take the sum of the very weak formulation (A.2)–(A.5) of the linearized system and the one obtained by testing the equation for ξ by ζ, and then apply a tuple of test functions where all components vanish except for ψ.

Suppose that $\psi \in C_0^\infty(I; W_{\boldsymbol{n}}^{4,2}(\Omega)) \subset W_0^{1,2,2}(I; W_{\boldsymbol{n}}^{4,2}(\Omega), L^2(\Omega))$. Using the anti-symmetry of the trilinear term with respect to the second and third arguments induced by the convective derivative, we have $\boldsymbol{v}\cdot\nabla\varphi \in L^2(I; W_{\boldsymbol{n}}^{-2,2}(\Omega))$ since

$$\begin{aligned}
\langle \boldsymbol{v}\cdot\nabla\varphi, \psi\rangle_{L^2(W_{\boldsymbol{n}}^{-2,2}),L^2(W_{\boldsymbol{n}}^{2,2})} &:= -\int_{\Omega_T} (\boldsymbol{v}\cdot\nabla\psi)\varphi\,\mathrm{d}\omega \\
&\le c\|\boldsymbol{v}\|_{L^\infty(\boldsymbol{W}_{0,\sigma}^{1,2})}\|\varphi\|_{L^2(L^2)}\|\psi\|_{L^2(W_{\boldsymbol{n}}^{2,2})}
\end{aligned}$$

and $C_0^\infty(I; W_{\boldsymbol{n}}^{4,2}(\Omega))$ is dense in $L^2(I; W_{\boldsymbol{n}}^{2,2}(\Omega))$. Performing the divergence operator, we obtain

$$\begin{aligned}
\nabla\cdot(m_\phi(\phi,\theta)\psi\nabla\mu) = {}& (m_{\phi\phi}(\phi,\theta)\psi\nabla\phi + m_{\phi\theta}(\phi,\theta)\psi\nabla\theta + m_\phi(\phi,\theta)\nabla\psi)\cdot\nabla\mu \\
&+ m_\phi(\phi,\theta)\psi\Delta\mu.
\end{aligned}$$

In virtue of the embeddings $W_{\boldsymbol{n}}^{2,2}(\Omega) \hookrightarrow W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$ and the previous equation, one has

$$\begin{aligned}
&\langle m_\phi(\phi,\theta)\nabla\mu\cdot\nabla\varphi, \psi\rangle_{L^1(W_{\boldsymbol{n}}^{-2,2}),L^\infty(W_{\boldsymbol{n}}^{2,2})} := -\int_{\Omega_T} \nabla\cdot(m_\phi(\phi,\theta)\psi\nabla\mu)\varphi\,\mathrm{d}\omega \\
&\quad \le c(\|\phi\|_{L^\infty(W^{1,4})} + \|\theta\|_{L^\infty(W^{1,4})} + 1)\|\mu\|_{L^2(W_{\boldsymbol{n}}^{2,2})}\|\varphi\|_{L^2(L^2)}\|\psi\|_{L^\infty(W_{\boldsymbol{n}}^{2,2})}.
\end{aligned} \tag{5.14}$$

Thus, we deduce that $m_\phi(\phi,\theta)\nabla\mu\cdot\nabla\varphi \in L^1(I; W_{\boldsymbol{n}}^{-2,2}(\Omega))$. By following a similar process, it can be shown that $\chi_\phi(\phi,\theta)\nabla\theta\cdot\nabla\vartheta$, $2\nu_\phi(\phi,\theta)\mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{y}$, $\varepsilon_\phi(\phi,\theta)\nabla\mathbb{S} \mathbin{\therefore} \nabla\mathbb{Y} \in L^1(I; W_{\boldsymbol{n}}^{-2,2}(\Omega))$ with bounds analogous to the one given by (5.14).

Performing the estimation as in (A.24), we obtain

$$\begin{aligned}
\langle F''(\phi)\zeta, \psi\rangle_{L^2(W_{\boldsymbol{n}}^{-2,2}),L^2(W_{\boldsymbol{n}}^{2,2})} &:= \langle \zeta, F''(\phi)\psi\rangle_{L^2(W_{\boldsymbol{n}}^{-2,2}),L^2(W_{\boldsymbol{n}}^{2,2})} \\
&\le c_\phi\|\zeta\|_{L^2(W_{\boldsymbol{n}}^{-2,2})}\|\psi\|_{L^2(W_{\boldsymbol{n}}^{2,2})}
\end{aligned}$$

so that $F''(\phi)\zeta \in L^2(I;W_{\boldsymbol{n}}^{-2,2}(\Omega))$. Likewise, we have $\alpha\Delta\zeta \in L^2(I;W_{\boldsymbol{n}}^{-4,2}(\Omega))$ according to

$$\begin{aligned}\langle \alpha\Delta\zeta,\psi\rangle_{L^2(W_{\boldsymbol{n}}^{-4,2}),L^2(W_{\boldsymbol{n}}^{4,2})} &:= \langle \zeta,\alpha\Delta\psi\rangle_{L^2(W_{\boldsymbol{n}}^{-2,2}),L^2(W_{\boldsymbol{n}}^{2,2})}\\ &\le c\|\zeta\|_{L^2(W_{\boldsymbol{n}}^{-2,2})}\|\psi\|_{L^2(I;W_{\boldsymbol{n}}^{4,2})}.\end{aligned}$$

We have $\kappa\boldsymbol{y}\cdot\nabla\mu \in L^{4/3}(I;W_{\boldsymbol{n}}^{-2,2}(\Omega)) \hookrightarrow L^1(I;W_{\boldsymbol{n}}^{-2,2}(\Omega))$ thanks to the following inequality

$$\begin{aligned}\langle \kappa\boldsymbol{y}\cdot\nabla\mu,\psi\rangle_{L^{4/3}(W_{\boldsymbol{n}}^{-2,2}),L^4(W_{\boldsymbol{n}}^{2,2})} &:= -\int_{\Omega_T}\kappa\mu\nabla\psi\cdot\boldsymbol{y}\,\mathrm{d}\omega\\ &\le c\|\mu\|_{L^4(W^{1,2})}\|\boldsymbol{y}\|_{L^2(\boldsymbol{L}^2_\sigma)}\|\psi\|_{L^4(W_{\boldsymbol{n}}^{2,2})}.\end{aligned}$$

Finally, it can be easily seen that $b_{\mathrm{o}}\mathbf{g}\cdot\boldsymbol{y} \in L^2(I;L^2(\Omega))$ and $\|b_{\mathrm{o}}\mathbf{g}\cdot\boldsymbol{y}\|_{L^2(L^2)} \le c\|\boldsymbol{y}\|_{L^2(\boldsymbol{L}^2_\sigma)}$.

Therefore, from the above inequalities and $g_{do} \in X_{\mathrm{o}}^2(\Omega_T)^*$, we obtain that $\partial_t\varphi \in L^2(I;W_{\boldsymbol{n}}^{-4,2}(\Omega)) + L^1(I;W_{\boldsymbol{n}}^{-2,2}(\Omega)) + X_{\mathrm{o}}^2(\Omega_T)^*$, and thanks to (5.11) and (5.12), we have

$$\|\partial_t\varphi\|_{L^2(W_{\boldsymbol{n}}^{-4,2})+L^1(W_{\boldsymbol{n}}^{-2,2})+(X_{\mathrm{o}}^2)^*} \le \widetilde{c}_0, \tag{5.15}$$

In addition, the first equation in (5.7) holds in this sum of function spaces. For brevity, we shall adopt the above argument to the other state variables without further comments.

Next, we estimate the norm of ζ. By taking the test functions $\psi = \eta = 0$, $\xi \in L^2(I;W_{\boldsymbol{n}}^{2,2}(\Omega))$, $\boldsymbol{v} = 0$, and $\mathbb{S} = \mathbb{O}$ in (5.2), we get

$$\begin{aligned}&\langle \zeta,\xi\rangle_{L^2(W_{\boldsymbol{n}}^{-2,2}),L^2(W_{\boldsymbol{n}}^{2,2})} - \int_{\Omega_T}\nabla\cdot(m(\phi,\theta)\nabla\xi)\varphi\,\mathrm{d}\omega\\ &\qquad - \int_{\Omega_T}\kappa\xi\nabla\phi\cdot\boldsymbol{y}\,\mathrm{d}\omega = \langle g_{\mathrm{c}},\xi\rangle_{L^2(W_{\boldsymbol{n}}^{-2,2}),L^2(W_{\boldsymbol{n}}^{2,2})}.\end{aligned}$$

Note that $\kappa\boldsymbol{y}\cdot\nabla\phi \in L^2(I;W^{-1,2}(\Omega))$ and $\nabla\cdot(m(\phi,\theta)\nabla\varphi) \in L^2(I;W_{\boldsymbol{n}}^{-2,2}(\Omega))$ since

$$\begin{aligned}&\langle \kappa\boldsymbol{y}\cdot\nabla\phi,\xi\rangle_{L^2(W^{-1,2}),L^2(W^{1,2})} := -\int_{\Omega_T}\kappa\xi\nabla\phi\cdot\boldsymbol{y}\,\mathrm{d}\omega\\ &\quad\le c\|\boldsymbol{y}\|_{L^2(\boldsymbol{L}^2_\sigma)}\|\phi\|_{L^\infty(W^{1,4})}\|\xi\|_{L^2(W^{1,2})}\\ &\langle \nabla\cdot(m(\phi,\theta)\nabla\varphi),\xi\rangle_{L^2(W_{\boldsymbol{n}}^{-2,2}),L^2(W_{\boldsymbol{n}}^{2,2})} := \int_{\Omega_T}\nabla\cdot(m(\phi,\theta)\nabla\xi)\varphi\,\mathrm{d}\omega\\ &\quad\le c(\|\phi\|_{L^\infty(W^{1,4})} + \|\theta\|_{L^\infty(W^{1,4})} + 1)\|\varphi\|_{L^2(L^2)}\|\xi\|_{L^2(W_{\boldsymbol{n}}^{2,2})}.\end{aligned}$$

As a consequence of these inequalities and (5.12), ζ enjoys the estimate

$$\|\zeta\|_{L^2(W_{\boldsymbol{n}}^{-2,2})} \le c(\|\boldsymbol{y}\|_{L^2(\boldsymbol{L}^2_\sigma)} + \|\varphi\|_{L^2(L^2)} + \|g_{\mathrm{c}}\|_{L^2(W_{\boldsymbol{n}}^{-2,2})}) \le \widetilde{c}_0. \tag{5.16}$$

Now, we consider the regularity of $\partial_t\vartheta$. Let $\gamma \in W_0^{1,2,2}(I;W_{\boldsymbol{n}}^{2,2}(\Omega),L^2(\Omega))$. Using $\psi = \xi = 0$, $\eta = B_N^{-1}\gamma \in W_0^{1,2,2}(I;W_{\boldsymbol{n}}^{4,2}(\Omega),W_{\boldsymbol{n}}^{2,2}(\Omega)) \hookrightarrow X_{\mathrm{h}}^2(\Omega_T)$, $\boldsymbol{v} = \mathbf{0}$, and $\mathbb{T} = \mathbb{O}$ in (5.2) yield the equation

$$\int_{\Omega_T}(\partial_t\gamma)\vartheta\,\mathrm{d}\omega + \int_{\Omega_T}(\boldsymbol{v}\cdot\nabla B_N^{-1}\gamma)\vartheta\,\mathrm{d}\omega$$

$$
\begin{aligned}
&- \int_{\Omega_T} \nabla \cdot (\chi_\theta(\phi,\theta) B_N^{-1}\gamma \nabla\theta + \chi(\phi,\theta)\nabla B_N^{-1}\gamma)\vartheta \,\mathrm{d}\omega \\
&+ \int_{\Omega_T} (b\Delta^2 B_N^{-1}\gamma)\vartheta \,\mathrm{d}\omega - \int_{\Omega_T} \nabla \cdot (m_\theta(\phi,\theta) B_N^{-1}\gamma\nabla\mu)\varphi \,\mathrm{d}\omega \\
&- \int_{\Omega_T} \nabla \cdot (2\nu_\theta(\phi,\theta) B_N^{-1}\gamma \mathbb{D}\boldsymbol{v}) \cdot \boldsymbol{y} \,\mathrm{d}\omega - \int_{\Omega_T} \nabla \cdot (\sigma B_N^{-1}\gamma \mathbb{S}) \cdot \boldsymbol{y} \,\mathrm{d}\omega \\
&- \int_{\Omega_T} b_{\mathrm{h}} B_N^{-1}\gamma \mathbf{g} \cdot \boldsymbol{y} \,\mathrm{d}\omega - \int_{\Omega_T} \nabla \cdot (\varepsilon_\theta(\phi,\theta) B_N^{-1}\gamma \nabla\mathbb{S}) : \mathbb{Y} \,\mathrm{d}\omega \\
&= \langle g_{\mathrm{dh}}, B_N^{-1}\gamma\rangle_{(X_{\mathrm{h}}^2)^*, X_{\mathrm{h}}^2} + \langle \vartheta_T, B_N^{-1}\gamma(T)\rangle_{W_{\boldsymbol{n}}^{-3,2}, W_{\boldsymbol{n}}^{3,2}}.
\end{aligned} \tag{5.17}
$$

Suppose that $\gamma \in C_0^\infty(I; W_{\boldsymbol{n}}^{2,2}(\Omega)) \subset W_0^{1,2,2}(I; W_{\boldsymbol{n}}^{2,2}(\Omega), L^2(\Omega))$. Since $B_N^{-*} := (B_N^{-1})^* : W_{\boldsymbol{n}}^{-2,2}(\Omega) \to L^2(\Omega)$ is continuous, we have $B_N^{-*}(\boldsymbol{v}\cdot\nabla\vartheta) \in L^2(I; L^2(\Omega))$ by duality due to

$$
\begin{aligned}
(B_N^{-*}(\boldsymbol{v}\cdot\nabla\vartheta), \gamma)_{L^2(L^2)} &:= -\int_{\Omega_T} (\boldsymbol{v}\cdot\nabla B_N^{-1}\gamma)\vartheta \,\mathrm{d}\omega \\
&\le c\|\boldsymbol{v}\|_{L^\infty(\boldsymbol{W}_{0,\sigma}^{1,2})} \|\vartheta\|_{L^2(L^2)} \|\gamma\|_{L^2(L^2)}.
\end{aligned}
$$

As in the previous discussion, we have $B_N^{-*}(\chi_\theta(\phi,\theta)\nabla\theta\cdot\nabla\vartheta)$, $B_N^{-*}(m_\theta(\phi,\theta)\nabla\mu\cdot\nabla\varphi)$, $B_N^{-*}(2\nu_\theta(\phi,\theta)\mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{y})$, $B_N^{-*}(\varepsilon_\theta(\phi,\theta)\nabla\mathbb{S} \therefore \nabla\mathbb{Y}) \in L^1(I; L^2(\Omega))$ and $B_N^{-*}\nabla\cdot(\chi(\phi,\theta)\nabla\vartheta) \in L^2(I; L^2(\Omega))$. Moreover, the norms can be estimated following (5.14).

Observe that $\nabla\cdot(B_N^{-1}\gamma\mathbb{S}) = \mathbb{S}\nabla B_N^{-1}\gamma + B_N^{-1}\gamma\nabla\cdot\mathbb{S}$. Hence, $B_N^{-*}(\sigma\mathbb{S} : \mathbb{D}\boldsymbol{y}) \in L^2(I; L^2(\Omega))$ and $B_N^{-*}(b\Delta^2\vartheta) \in L^2(I; W_{\boldsymbol{n}}^{-2,2}(\Omega))$ since

$$
\begin{aligned}
(B_N^{-*}(\sigma\mathbb{S} : \mathbb{D}\boldsymbol{y}), \gamma)_{L^2(L^2)} &:= -\int_{\Omega_T} \sigma(\nabla\cdot(B_N^{-1}\gamma\mathbb{S}))\cdot\boldsymbol{y} \,\mathrm{d}\omega \\
&\le c\|\mathbb{S}\|_{L^\infty(\mathbb{W}_{\mathrm{s}}^{1,2})} \|\boldsymbol{y}\|_{L^2(\boldsymbol{L}_\sigma^2)} \|\gamma\|_{L^2(L^2)} \\
\langle B_N^{-*}(b\Delta^2\vartheta), \gamma\rangle_{L^2(W_{\boldsymbol{n}}^{-2,2}), L^2(W_{\boldsymbol{n}}^{2,2})} &:= \int_{\Omega_T} b(\Delta^2 B_N^{-1}\gamma)\vartheta \,\mathrm{d}\omega \\
&\le c\|\vartheta\|_{L^2(L^2)} \|\gamma\|_{L^2(W_{\boldsymbol{n}}^{2,2})}.
\end{aligned}
$$

Next, $B_N^{-*}(b_{\mathrm{h}}\mathbf{g}\cdot\boldsymbol{y}) \in L^2(I; W_{\boldsymbol{n}}^{2,2}(\Omega))$ by the boundedness of $B_N^{-*} : L^2(\Omega) \to W_{\boldsymbol{n}}^{2,2}(\Omega)$, and we have

$$
\|B_N^{-*}(b_{\mathrm{h}}\mathbf{g}\cdot\boldsymbol{y})\|_{L^2(W_{\boldsymbol{n}}^{2,2})} \le c\|\boldsymbol{y}\|_{L^2(\boldsymbol{L}_\sigma^2)}.
$$

Finally, $B_N^{-*}g_{\mathrm{dh}} \in (B_N X_{\mathrm{h}}^2(\Omega_T))^*$ according to (5.10) and

$$
\begin{aligned}
\langle B_N^{-*}g_{\mathrm{dh}}, \gamma\rangle_{(B_N X_{\mathrm{h}}^2)^*, B_N X_{\mathrm{h}}^2} &:= \langle g_{\mathrm{dh}}, B_N^{-1}\gamma\rangle_{(X_{\mathrm{h}}^2)^*, X_{\mathrm{h}}^2} \\
&\le \|g_{\mathrm{dh}}\|_{(X_{\mathrm{h}}^2)^*} \|B_N^{-1}\|_{\mathcal{L}(B_N X_{\mathrm{h}}^2, X_{\mathrm{h}}^2)} \|\gamma\|_{B_N X_{\mathrm{h}}^2}.
\end{aligned} \tag{5.18}
$$

Thus, it follows that $\partial_t\vartheta \in L^2(I; W_{\boldsymbol{n}}^{-2,2}(\Omega)) + L^1(I; L^2(\Omega)) + (B_N X_{\mathrm{h}}^2(\Omega_T))^*$ and the third equation in (5.7) is satisfied in this space. Moreover, upon combining the above estimates along with that of (5.12) and using (5.11), we get

$$
\|\partial_t\vartheta\|_{L^2(W_{\boldsymbol{n}}^{-2,2}) + L^1(L^2) + (B_N X_{\mathrm{h}}^2)^*} \le \widetilde{c}_0. \tag{5.19}
$$

Now, we consider the case of $\partial_t \boldsymbol{y}$. For this, we take the test functions $\boldsymbol{w} \in W_0^{1,2,2}(I; \boldsymbol{W}^{2,2}_{0,\sigma}(\Omega), \boldsymbol{L}^2_\sigma(\Omega)) \hookrightarrow \boldsymbol{X}^2_{\mathrm{v}}(\Omega_T)$, $\psi = \xi = \eta = 0$, and $\mathbb{T} = \mathbb{O}$ in (5.2) so that

$$\begin{aligned}
&\int_{\Omega_T} (\partial_t \boldsymbol{w}) \cdot \boldsymbol{y} \,\mathrm{d}\omega + \int_{\Omega_T} [(\boldsymbol{v} \cdot \nabla) \cdot \boldsymbol{w}] \cdot \boldsymbol{y} \,\mathrm{d}\omega + \int_{\Omega_T} \boldsymbol{w} \cdot (\nabla \boldsymbol{v}) \boldsymbol{y} \,\mathrm{d}\omega \\
&- \int_{\Omega_T} \nabla \cdot (2\nu(\phi, \theta) \mathbb{D}\boldsymbol{w}) \cdot \boldsymbol{y} \,\mathrm{d}\omega + \int_{\Omega_T} (\boldsymbol{w} \cdot \nabla \phi) \varphi \,\mathrm{d}\omega + \int_{\Omega_T} (\boldsymbol{w} \cdot \nabla \theta) \vartheta \,\mathrm{d}\omega \\
&- \int_{\Omega_T} \mathbb{S} : \mathbb{D}\boldsymbol{w} \vartheta \,\mathrm{d}\omega - \int_{\Omega_T} a_0 \vartheta \mathbf{g} \cdot \boldsymbol{w} \,\mathrm{d}\omega + \int_{\Omega_T} (\boldsymbol{w} \cdot \nabla) \mathbb{S} : \mathbb{Y} \,\mathrm{d}\omega \\
&+ \int_{\Omega_T} \mathbb{J}(\boldsymbol{w}, \mathbb{S}) : \mathbb{Y} \,\mathrm{d}\omega - \int_{\Omega_T} \lambda \mathbb{D}\boldsymbol{w} : \mathbb{Y} \,\mathrm{d}\omega \\
&= \langle \boldsymbol{g}_{\mathrm{dv}}, \boldsymbol{w} \rangle_{(\boldsymbol{X}^2_{\mathrm{v}})^*, \boldsymbol{X}^2_{\mathrm{v}}} + \langle \boldsymbol{y}_T, \boldsymbol{w}(T) \rangle_{\boldsymbol{W}^{-1,2}_{0,\sigma}, \boldsymbol{W}^{1,2}_{0,\sigma}}.
\end{aligned} \tag{5.20}$$

Let $\boldsymbol{w} \in C_0^\infty(I; \boldsymbol{W}^{2,2}_{0,\sigma}(\Omega)) \subset W_0^{1,2,2}(I; \boldsymbol{W}^{2,2}_{0,\sigma}(\Omega), \boldsymbol{L}^2_\sigma(\Omega))$. As before, it can be shown that $-(\boldsymbol{v} \cdot \nabla)\boldsymbol{y}$, $(\nabla \boldsymbol{v})\boldsymbol{y}$, $-\nabla \cdot (2\nu(\phi, \theta)\mathbb{D}\boldsymbol{y})$, and $\nabla \cdot (\vartheta \mathbb{S})$, with definitions as in the second, third, fourth, and seventh integrals in (5.20), are elements of $L^2(I; \boldsymbol{W}^{-2,2}_{0,\sigma}(\Omega))$. Likewise, $\varphi \nabla \phi$, $\lambda \nabla \cdot \mathbb{Y} \in L^2(I; \boldsymbol{W}^{-1,2}_{0,\sigma}(\Omega))$, and these correspond to the fifth and eleventh term in (5.20). For the sixth and eighth integrals, it holds that $\vartheta \nabla \theta$, $a_0 \vartheta \mathbf{g} \in L^2(I; \boldsymbol{L}^2(\Omega))$.

It remains to consider the ninth and tenth integrals in (5.20). First, observe that for sufficiently smooth $\boldsymbol{w}$, $\mathbb{S}$, and $\mathbb{Y}$, one has

$$\begin{aligned}
&\int_{\Omega_T} (\boldsymbol{w} \cdot \nabla) \mathbb{S} : \mathbb{Y} \,\mathrm{d}\omega = \int_{\Omega_T} \boldsymbol{w} \cdot (\nabla \mathbb{S} \odot \mathbb{Y}) \,\mathrm{d}\omega \\
&\int_{\Omega_T} \mathbb{J}(\boldsymbol{w}, \mathbb{S}) : \mathbb{Y} \,\mathrm{d}\omega = \int_{\Omega_T} ([\mathbb{S}, \mathbb{W}\boldsymbol{w}] : \mathbb{Y} - a\{\mathbb{S}, \mathbb{D}\boldsymbol{w}\} : \mathbb{Y}) \,\mathrm{d}\omega \\
&\quad = \int_{\Omega_T} \nabla \boldsymbol{w} : ([\mathbb{S}, \mathbb{Y}] - a\{\mathbb{S}, \mathbb{Y}\}) \,\mathrm{d}\omega = - \int_{\Omega_T} \boldsymbol{w} \cdot (\nabla \cdot ([\mathbb{S}, \mathbb{Y}] - a\{\mathbb{S}, \mathbb{Y}\})) \,\mathrm{d}\omega.
\end{aligned}$$

Here, we utilized Lemma 2.1 for the terms involving the commutator and anti-commutator. With these, we define $\nabla \mathbb{S} \odot \mathbb{Y}$ and $-\nabla \cdot ([\mathbb{S}, \mathbb{Y}] - a\{\mathbb{S}, \mathbb{Y}\})$ by the left-hand sides of these equations, and take note that both of these lie in $L^2(I; \boldsymbol{W}^{-2,2}_{0,\sigma}(\Omega))$. Hence, we obtain $\partial_t \boldsymbol{y} \in L^2(I; \boldsymbol{W}^{-2,2}_{0,\sigma}(\Omega)) + \boldsymbol{X}^2_{\mathrm{v}}(\Omega_T)^*$ and the fourth equation of (5.7) holds with respect to this function space. From the estimates that can be derived from the norms of the previous terms along with (5.12), we deduce from (5.11) that

$$\|\partial_t \boldsymbol{y}\|_{L^2(\boldsymbol{W}^{-2,2}_{0,\sigma}) + (\boldsymbol{X}^2_{\mathrm{v}})^*} \le \widetilde{c}_0. \tag{5.21}$$

To finish the proof of the theorem, we now consider $\partial_t \mathbb{Y}$. We take the test functions $\psi = \xi = \eta = 0$, $\boldsymbol{w} = \boldsymbol{0}$, and $\mathbb{T} \in W_0^{1,2,2}(I; \mathbb{W}^{2,2}_{\boldsymbol{n},\mathrm{s}}(\Omega), \mathbb{L}^2_{\mathrm{s}}(\Omega)) \hookrightarrow \mathbb{X}^2_{\mathrm{s}}(\Omega)$ in (5.2) to obtain

$$\begin{aligned}
&\int_{\Omega_T} \partial_t \mathbb{T} : \mathbb{Y} \,\mathrm{d}\omega + \int_{\Omega_T} (\boldsymbol{v} \cdot \nabla) \mathbb{T} : \mathbb{Y} \,\mathrm{d}\omega + \int_{\Omega_T} \mathbb{J}(\boldsymbol{v}, \mathbb{T}) : \mathbb{Y} \,\mathrm{d}\omega \\
&- \int_{\Omega_T} \nabla \cdot (\varepsilon(\phi, \theta) \nabla \mathbb{T}) : \mathbb{Y} \,\mathrm{d}\omega - \int_{\Omega_T} \mathbb{P}'(\mathbb{S}) \mathbb{T} : \mathbb{Y} \,\mathrm{d}\omega
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\Omega_T} \nabla \cdot (\mathbb{M}_{\mathbb{S}}(\theta, \mathbb{S})\mathbb{T}) \cdot \boldsymbol{y} \,\mathrm{d}\omega - \int_{\Omega_T} \mathbb{T} : \mathbb{D}\boldsymbol{v}\vartheta \,\mathrm{d}\omega \\
&= \langle \mathbb{G}_{ds}, \mathbb{T} \rangle_{(\mathbb{X}^2_{\mathrm{s}})^*, \mathbb{X}^2_{\mathrm{s}}} + \langle \mathbb{Y}_T, \mathbb{T}(T) \rangle_{\mathbb{W}_{\mathrm{s}}^{-1,2}, \mathbb{W}_{\mathrm{s}}^{1,2}}. \qquad (5.22)
\end{aligned}
$$

Let $\mathbb{T} \in C_0^\infty(I; \mathbb{W}^{2,2}_{\boldsymbol{n},\mathrm{s}}(\Omega)) \subset W_0^{1,2,2}(I; \mathbb{W}^{2,2}_{\boldsymbol{n},\mathrm{s}}(\Omega), \mathbb{L}^2_{\mathrm{s}}(\Omega))$. In the above integrals, except for the third, we can deduce that $(\boldsymbol{v} \cdot \nabla)\mathbb{Y}$, $\nabla \cdot (\varepsilon(\phi, \theta)\nabla \mathbb{Y})$, $\mathbb{M}_{\mathbb{S}}(\theta, \mathbb{S})^*\mathbb{D}\boldsymbol{y}$, $\vartheta\mathbb{D}\boldsymbol{v} \in L^2(I; \mathbb{W}^{-2,2}_{\boldsymbol{n},\mathrm{s}}(\Omega))$ and $\mathbb{P}'(\mathbb{S})^*\mathbb{Y} \in L^2(I; \mathbb{W}_{\mathrm{s}}^{-1,2}(\Omega))$, see (5.5) and (5.6).

Consider the third integral in (5.22). Notice that $\mathbb{T}\mathbb{W}\boldsymbol{v} : \mathbb{Y} = (\mathbb{T}\mathbb{W}\boldsymbol{v})^{\mathrm{t}} : \mathbb{Y}^{\mathrm{t}} = -\mathbb{W}\boldsymbol{v}\mathbb{T} : \mathbb{Y}$ by the anti-symmetry of $\mathbb{W}\boldsymbol{v}$ and the symmetry of $\mathbb{T}$ and $\mathbb{Y}$. Thus,

$$
[\mathbb{T}, \mathbb{W}\boldsymbol{v}] : \mathbb{Y} = -2\mathbb{Y} : \mathbb{W}\boldsymbol{v}\mathbb{T} = -2\mathrm{Tr}(\mathbb{Y}\mathbb{W}\boldsymbol{v}\mathbb{T}) = 2\mathrm{Tr}((\mathbb{W}\boldsymbol{v}\mathbb{Y})^{\mathrm{t}}\mathbb{T}) = 2\mathbb{W}\boldsymbol{v}\mathbb{Y} : \mathbb{T}.
$$

In a similar way, $\{\mathbb{T}, \mathbb{W}\boldsymbol{v}\} : \mathbb{Y} = 2\mathbb{D}\boldsymbol{v}\mathbb{Y} : \mathbb{T}$. Hence,

$$
\int_{\Omega_T} \mathbb{J}(\boldsymbol{v}, \mathbb{T}) : \mathbb{Y} \,\mathrm{d}\omega = \int_{\Omega_T} (2\mathbb{W}\boldsymbol{v}\mathbb{Y} - 2a\mathbb{D}\boldsymbol{v}\mathbb{Y}) : \mathbb{T} \,\mathrm{d}\omega
$$

and $2\mathbb{W}\boldsymbol{v}\mathbb{Y} - 2a\mathbb{D}\boldsymbol{v}\mathbb{Y} \in L^2(I; \mathbb{W}^{-2,2}_{\boldsymbol{n},\mathrm{s}}(\Omega))$.

The above observations imply that $\partial_t \mathbb{Y} \in L^2(I; \mathbb{W}^{-2,2}_{\boldsymbol{n},\mathrm{s}}(\Omega)) + \mathbb{X}^2_{\mathrm{s}}(\Omega_T)^*$, so that the fifth equation of (5.7) holds, and it can be established from (5.11) that

$$
\|\partial_t \mathbb{Y}\|_{L^2(\mathbb{W}^{-2,2}_{\boldsymbol{n},\mathrm{s}}) + (\mathbb{X}^2_{\mathrm{s}})^*} \le \widetilde{c}_0. \qquad (5.23)
$$

Taking the sum of the estimates (5.15), (5.16), (5.19), (5.21), and (5.23) leads to the desired estimate for the very weak solution as stated by the theorem. □

Remark 5.3. If we have the continuous embeddings $X^2_{\mathrm{o}}(\Omega_T)^* \hookrightarrow L^1(I; W_{\boldsymbol{n}}^{-4,2}(\Omega))$, $(B_N X^2_{\mathrm{h}}(\Omega_T))^* \hookrightarrow L^1(I; W_{\boldsymbol{n}}^{-2,2}(\Omega))$, $\boldsymbol{X}^2_{\mathrm{v}}(\Omega_T)^* \hookrightarrow L^2(I; \boldsymbol{W}^{-2,2}_{0,\sigma}(\Omega))$, and $\mathbb{X}^2_{\mathrm{s}}(\Omega_T)^* \hookrightarrow L^2(I; \mathbb{W}^{-2,2}_{\boldsymbol{n},\mathrm{s}}(\Omega))$ in Theorem 5.2, then the variational solution to (5.7) satisfies

$$
\begin{aligned}
&\varphi \in W^{1,2,1}(I; L^2(\Omega), W_{\boldsymbol{n}}^{-4,2}(\Omega)), && \vartheta \in W^{1,2,1}(I; L^2(\Omega), W_{\boldsymbol{n}}^{-2,2}(\Omega)), \\
&\boldsymbol{y} \in W^{1,2,2}(I; \boldsymbol{L}^2_\sigma(\Omega), \boldsymbol{W}^{-2,2}_{0,\sigma}(\Omega)), && \mathbb{Y} \in W^{1,2,2}(I; \mathbb{L}^2_{\mathrm{s}}(\Omega), \mathbb{W}^{-2,2}_{\boldsymbol{n},\mathrm{s}}(\Omega)).
\end{aligned}
$$

The integrability of the time-derivatives of φ and ϑ will be improved to $\frac{4}{3}$ in the case of weak solutions, see Remark 5.5 below.

Let us discuss the existence and regularity of the associated pressure s in the fourth equation of the adjoint system (5.7). In addition to the conditions stated in Remark 5.3, suppose that we have $\boldsymbol{g}_{d\mathrm{v}} \in L^2(I; \boldsymbol{W}_0^{-2,2}(\Omega))$ and $\boldsymbol{X}^2_{\mathrm{v}}(\Omega_T)^* \hookrightarrow L^2(I; \boldsymbol{W}_0^{-2,2}(\Omega))$. Note that $\partial_t \boldsymbol{y} \in W^{-1,2}_{0,0}(I; \boldsymbol{L}^2(\Omega)) \hookrightarrow W^{-1,2}_{0,0}(I; \boldsymbol{W}_0^{-2,2}(\Omega))$ in the distributional sense due to $\boldsymbol{y} \in L^2(I; \boldsymbol{L}^2(\Omega))$. Revisiting the proof of Theorem 5.2, we can see that all terms on the left-hand side of the differential equation for $\boldsymbol{y}$, except for the time-derivate, are elements of $L^2(I; \boldsymbol{W}_0^{-2,2}(\Omega)) \hookrightarrow W^{-1,2}_{0,0}(I; \boldsymbol{W}_0^{-2,2}(\Omega))$. Therefore, according to Theorem 4.4, there is an associated pressure $\mathrm{s} \in W^{-1,2}_{0,0}(I; W_0^{-1,2}(\Omega)/\mathbb{R})$. Furthermore, from the a priori estimate in Theorem 5.2, we have

$$
\begin{aligned}
\|\mathrm{s}\|_{W^{-1,2}_{0,0}(W_0^{-1,2}/\mathbb{R})} \le c(&\|(g_{d\mathrm{o}}, g_{\mathrm{c}}, g_{d\mathrm{h}}, \mathbb{G}_{ds})\|_{(X^2_{\mathrm{o}})^* \times L^2(W_{\boldsymbol{n}}^{-2,2}) \times (X^2_{\mathrm{h}})^* \times (\mathbb{X}^2_{\mathrm{s}})^*} \\
&+ \|\boldsymbol{g}_{d\mathrm{v}}\|_{L^2(\boldsymbol{W}_0^{-2,2})} + \|(\varphi_T, \vartheta_T, \boldsymbol{y}_T, \mathbb{Y}_T)\|_{(\mathcal{D}^2)^*}).
\end{aligned}
$$

Next, we consider the time-regularity of the weak solutions to the adjoint system.

Theorem 5.4. *Let* (A1)$_3$ *and* (A2)$_1$ *be satisfied. Consider source functions and initial data for which* $(g_{\mathrm{do}}, g_{\mathrm{c}}, g_{\mathrm{dh}}, \boldsymbol{g}_{\mathrm{dv}}, \mathbb{G}_{\mathrm{ds}}) \in \mathcal{X}^1(\Omega_T)^*$ *and* $(\varphi_T, \vartheta_T, \boldsymbol{y}_T, \mathbb{Y}_T) \in \mathcal{D}^1(\Omega)^*$. *Then, the adjoint system* (5.7) *possesses a unique solution* $(\varphi, \zeta, \vartheta, \boldsymbol{y}, \mathbb{Y}) \in \mathcal{U}^1(\Omega_T)$ *such that*

$$
\begin{aligned}
&\|(\varphi, \zeta, \vartheta, \boldsymbol{y}, \mathbb{Y})\|_{\mathcal{U}^1} + \|\partial_t \varphi\|_{L^2(W_{\boldsymbol{n}}^{-3,2}) + L^{4/3}(W_{\boldsymbol{n}}^{-2,2}) + (X_{\mathrm{o}}^1)^*} \\
&\quad + \|\partial_t \vartheta\|_{L^2(W^{-1,2}) + L^{4/3}(L^2) + (B_N X_{\mathrm{h}}^1)^*} + \|\partial_t \boldsymbol{y}\|_{L^2(\boldsymbol{W}_{0,\sigma}^{-1,2}) + (\boldsymbol{X}_{\mathrm{v}}^1)^*} \\
&\quad + \|\partial_t \mathbb{Y}\|_{L^2(\mathbb{W}_{\mathrm{s}}^{-1,2}) + (\mathbb{X}_{\mathrm{s}}^1)^*} \le c\|(g_{\mathrm{do}}, g_{\mathrm{c}}, g_{\mathrm{dh}}, \boldsymbol{g}_{\mathrm{dv}}, \mathbb{G}_{\mathrm{ds}})\|_{(\mathcal{X}^1)^*} \\
&\quad + c\|(\varphi_T, \vartheta_T, \boldsymbol{y}_T, \mathbb{Y}_T)\|_{(\mathcal{D}^1)^*}
\end{aligned}
$$

for a constant $c = c(\|(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S})\|_{\mathcal{W}^2}) > 0$.

Proof. From Theorem 5.1, (5.2) admits a unique solution such that

$$
\begin{aligned}
&\|(\varphi, \zeta, \vartheta, \boldsymbol{y}, \mathbb{Y})\|_{\mathcal{U}^1} \\
&\quad \le \widetilde{c}_1 := c[\|(g_{\mathrm{do}}, g_{\mathrm{c}}, g_{\mathrm{dh}}, \boldsymbol{g}_{\mathrm{dv}}, \mathbb{G}_{\mathrm{ds}})\|_{(\mathcal{X}^1)^*} + c_T\|(\varphi_T, \vartheta_T, \boldsymbol{y}_T, \mathbb{Y}_T)\|_{(\mathcal{D}^1)^*}],
\end{aligned} \tag{5.24}
$$

where $c_T := \|e_T\|_{\mathcal{L}((\mathcal{D}^1)^*, (\mathcal{V}_0^1)^*)}$. In what follows, we establish the regularity of each term appearing in the adjoint system (5.7) and apply (5.11) in deriving the a priori estimates for the time-derivatives.

First, we deal with $\partial_t \varphi$. Note that $\boldsymbol{v} \cdot \nabla \varphi \in L^2(I; W^{-1,2}(\Omega))$ since

$$
\int_{\Omega_T} (\boldsymbol{v} \cdot \nabla \varphi)\psi \,\mathrm{d}\omega \le \|\boldsymbol{v}\|_{L^\infty(\boldsymbol{W}_{0,\sigma}^{1,2})} \|\varphi\|_{L^2(W^{1,2})} \|\psi\|_{L^2(W^{1,2})}.
$$

Likewise, we have $m_\phi(\phi, \theta)\nabla\mu \cdot \nabla\varphi \in L^{4/3}(I; W_{\boldsymbol{n}}^{-2,2}(\Omega))$ and $2\nu_\phi(\phi, \theta)\mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{y} \in L^2(I; W_{\boldsymbol{n}}^{-2,2}(\Omega))$ due to

$$
\begin{aligned}
&\int_{\Omega_T} m_\phi(\phi, \theta)\psi \nabla\mu \cdot \nabla\varphi \,\mathrm{d}\omega \le c\|\mu\|_{L^4(W^{1,2})} \|\varphi\|_{L^2(W^{1,2})} \|\psi\|_{L^4(W_{\boldsymbol{n}}^{2,2})} \\
&\int_{\Omega_T} 2\nu_\phi(\phi, \theta)\psi \mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{y} \,\mathrm{d}\omega \le c\|\boldsymbol{v}\|_{L^\infty(\boldsymbol{W}_{0,\sigma}^{1,2})} \|\boldsymbol{y}\|_{L^2(\boldsymbol{W}_{0,\sigma}^{1,2})} \|\psi\|_{L^2(W_{\boldsymbol{n}}^{2,2})}.
\end{aligned}
$$

Similarly, $\chi_\phi(\phi, \theta)\nabla\theta \cdot \nabla\vartheta, \varepsilon_\phi(\phi, \theta)\nabla\mathbb{S} \therefore \nabla\mathbb{Y} \in L^2(I; W_{\boldsymbol{n}}^{-2,2}(\Omega))$. From the following estimates for the duality pairings

$$
\begin{aligned}
\langle \zeta, \alpha\Delta\psi \rangle_{L^2(W^{-1,2}), L^2(W^{1,2})} &\le c\|\zeta\|_{L^2(W^{-1,2})} \|\psi\|_{L^2(W_{\boldsymbol{n}}^{3,2})} \\
\langle \zeta, F''(\phi)\psi \rangle_{L^2(W^{-1,2}), L^2(W^{1,2})} &\le c_\phi\|\zeta\|_{L^2(W^{-1,2})} \|\psi\|_{L^2(W^{1,2})}
\end{aligned}
$$

we obtain $\alpha\Delta\zeta \in L^2(I; W_{\boldsymbol{n}}^{-3,2}(\Omega))$ and $F''(\phi)\zeta \in L^2(I; W^{-1,2}(\Omega))$. Thanks to the estimate

$$
\int_{\Omega_T} \kappa\psi \nabla\mu \cdot \boldsymbol{y} \,\mathrm{d}\omega \le c\|\mu\|_{L^4(W^{1,2})} \|\boldsymbol{y}\|_{L^2(\boldsymbol{W}_{0,\sigma}^{1,2})} \|\psi\|_{L^4(W^{1,2})}
$$

we have $\kappa\boldsymbol{y} \cdot \nabla\mu \in L^{4/3}(I; W^{-1,2}(\Omega))$. Finally, $b_{\mathrm{o}}\mathbf{g} \cdot \boldsymbol{y} \in L^2(I; W^{1,2}(\Omega))$. These information show that $\partial_t\varphi \in L^2(I; W_{\boldsymbol{n}}^{-3,2}(\Omega)) + L^{4/3}(I; W_{\boldsymbol{n}}^{-2,2}(\Omega)) + X_{\mathrm{o}}^1(\Omega_T)^*$, and from the above estimates along with that of (5.24), one has

$$
\|\partial_t \varphi\|_{L^2(W_{\boldsymbol{n}}^{-3,2}) + L^{4/3}(W_{\boldsymbol{n}}^{-2,2}) + (X_{\mathrm{o}}^1)^*} \le \widetilde{c}_1. \tag{5.25}
$$

Since $\kappa\boldsymbol{y} \cdot \nabla\phi \in L^2(I; L^2(\Omega))$ and $\nabla \cdot (m\nabla\varphi) \in L^2(I; W^{-1,2}(\Omega))$, we obtain

$$
\|\zeta\|_{L^2(W^{-1,2})} \le c(\|\boldsymbol{y}\|_{L^2(\boldsymbol{W}_{0,\sigma}^{1,2})} + \|\varphi\|_{L^2(W^{1,2})} + \|g_{\mathrm{c}}\|_{L^2(W^{-1,2})}) \le \widetilde{c}_1. \tag{5.26}
$$

With regard to $\partial_t\vartheta$, we shall utilize the continuity of $B_N^{-*} : W^{s,2}_{\boldsymbol{n}}(\Omega) \to W^{s+2,2}_{\boldsymbol{n}}(\Omega)$ for $s = -2, -1, 0, 1$. Hence, $B_N^{-*}(\boldsymbol{v}\cdot\nabla\vartheta) \in L^2(I; W^{1,2}(\Omega))$. On the other hand, using the analysis presented in the previous paragraph, it can be shown that $B_N^{-*}(m_\theta(\phi,\theta)\nabla\mu\cdot\nabla\varphi) \in L^{4/3}(I; L^2(\Omega))$, $B_N^{-*}(\chi_\theta(\phi,\theta)\nabla\theta\cdot\nabla\vartheta)$, $B_N^{-*}(2\nu_\theta(\phi,\theta)\mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{y})$, $B_N^{-*}(\varepsilon_\theta(\phi,\theta)\nabla\mathbb{S} \therefore \nabla\mathbb{Y}) \in L^2(I; L^2(\Omega))$, and also, $B_N^{-*}\nabla\cdot(\chi(\phi,\theta)\nabla\vartheta) \in L^2(I; W^{1,2}(\Omega))$. Moreover, we have $B_N^{-*}(b\Delta^2\vartheta) \in L^2(I; W^{-1,2}(\Omega))$, $B_N^{-*}(\sigma\mathbb{S} : \mathbb{D}\boldsymbol{y}) \in L^2(I; W^{1,2}(\Omega))$, and $B_N^{-*}(b_{\mathrm{h}}\mathbf{g}\cdot\boldsymbol{y}) \in L^2(I; W^{3,2}_{\boldsymbol{n}}(\Omega))$. As a consequence, $\partial_t\vartheta \in L^2(I; W^{-1,2}(\Omega)) + L^{4/3}(I; L^2(\Omega)) + (B_N X^1_{\mathrm{h}}(\Omega_T))^*$ and

$$\|\partial_t\vartheta\|_{L^2(W^{-1,2})+L^{4/3}(L^2)+(B_N X^1_{\mathrm{h}})^*} \le \widetilde{c}_1 \tag{5.27}$$

by following a similar argument as in (5.18) and (5.19).

Thanks to the additional spatial regularity of the adjoint variables, the spatial regularity for each term appearing on the equation involving $\partial_t\boldsymbol{y}$ in (5.7) is increased by 1 compared to the one given by Theorem 5.2. More precisely, one can show that $(\boldsymbol{v}\cdot\nabla)\boldsymbol{y}$, $(\nabla\boldsymbol{v})\boldsymbol{y}$, $\nabla\cdot(2\nu(\phi,\theta)\mathbb{D}\boldsymbol{y})$, $\nabla\cdot(\vartheta\mathbb{S})$, $\nabla\mathbb{S}\odot\mathbb{Y}$, $\nabla\cdot([\mathbb{S},\mathbb{Y}] - a\{\mathbb{S},\mathbb{Y}\}) \in L^2(I; \boldsymbol{W}^{-1,2}_{0,\sigma}(\Omega))$, $\varphi\nabla\phi$, $\lambda\nabla\cdot\mathbb{Y} \in L^2(I; \boldsymbol{L}^2(\Omega))$, and $\vartheta\nabla\theta$, $a_0\vartheta\mathbf{g} \in L^2(I; \boldsymbol{W}^{1,2}(\Omega))$. Therefore, we obtain $\partial_t\boldsymbol{y} \in L^2(I; \boldsymbol{W}^{-1,2}_{0,\sigma}(\Omega)) + \boldsymbol{X}^1_{\mathrm{v}}(\Omega_T)^*$ and

$$\|\partial_t\boldsymbol{y}\|_{L^2(\boldsymbol{W}^{-1,2}_{0,\sigma})+(\boldsymbol{X}^1_{\mathrm{v}})^*} \le \widetilde{c}_1. \tag{5.28}$$

With the same reasoning, it can be deduced that $(\boldsymbol{v}\cdot\nabla)\mathbb{Y}$, $\nabla\cdot(\varepsilon(\phi,\theta)\nabla\mathbb{Y})$, $2\mathbb{W}\boldsymbol{v}\mathbb{Y} - 2a\mathbb{D}\boldsymbol{v}\mathbb{Y}$, $\mathbb{M}_{\mathbb{S}}(\theta,\mathbb{S})^*\mathbb{D}\boldsymbol{y}$, $\vartheta\mathbb{D}\boldsymbol{v} \in L^2(I; \mathbb{W}^{-1,2}_{\mathrm{s}}(\Omega))$, and $\mathbb{P}'(\mathbb{S})^*\mathbb{Y} \in L^2(I; \mathbb{L}^2_{\mathrm{s}}(\Omega))$. These imply that $\partial_t\mathbb{Y} \in L^2(I; \mathbb{W}^{-1,2}_{\mathrm{s}}(\Omega)) + \mathbb{X}^1_{\mathrm{s}}(\Omega_T)^*$, and moreover,

$$\|\partial_t\mathbb{Y}\|_{L^2(\mathbb{W}^{-1,2}_{\mathrm{s}})+(\mathbb{X}^1_{\mathrm{s}})^*} \le \widetilde{c}_1. \tag{5.29}$$

Therefore, the a priori estimate as stated by the theorem for the time-derivatives of the adjoint variables follows from (5.25)–(5.29). □

Remark 5.5. If we have the continuous embeddings $X^1_{\mathrm{o}}(\Omega_T)^* \hookrightarrow L^{4/3}(I; W^{-3,2}_{\boldsymbol{n}}(\Omega))$, $(B_N X^1_{\mathrm{h}}(\Omega_T))^* \hookrightarrow L^{4/3}(I; W^{-1,2}(\Omega))$, $\boldsymbol{X}^1_{\mathrm{v}}(\Omega_T)^* \hookrightarrow L^2(I; \boldsymbol{W}^{-1,2}_{0,\sigma}(\Omega))$, and $\mathbb{X}^1_{\mathrm{s}}(\Omega_T)^* \hookrightarrow L^2(I; \mathbb{W}^{-1,2}_{\mathrm{s}}(\Omega))$, then the variational solution given in Theorem 5.4 satisfies

$$\varphi \in W^{1,2,4/3}(I; W^{1,2}(\Omega), W^{-3,2}_{\boldsymbol{n}}(\Omega)), \qquad \vartheta \in W^{1,2,4/3}(I; W^{1,2}(\Omega), W^{-1,2}(\Omega)),$$
$$\boldsymbol{y} \in W^{1,2,2}(I; \boldsymbol{W}^{1,2}_{0,\sigma}(\Omega), \boldsymbol{W}^{-1,2}_{0,\sigma}(\Omega)), \qquad \mathbb{Y} \in W^{1,2,2}(I; \mathbb{W}^{1,2}_{\mathrm{s}}(\Omega), \mathbb{W}^{-1,2}_{\mathrm{s}}(\Omega)).$$

By a density argument, it can be shown that the terminal conditions in (5.7) are satisfied. More precisely, $\varphi(T) = \varphi_T$ in $W^{-1,2}(\Omega)$, $\vartheta(T) - \tau\Delta\vartheta(T) = \vartheta_T$ in $W^{-2,2}_{\boldsymbol{n}}(\Omega)$, $\boldsymbol{y}(T) = \boldsymbol{y}_T$ in $\boldsymbol{L}^2_\sigma(\Omega)$, and $\mathbb{Y}(T) = \mathbb{Y}_T$ in $\mathbb{L}^2_{\mathrm{s}}(\Omega)$. We refer the reader to [56] or [69] for the details. We point out that the integrability of the time-derivatives of φ and ϑ will be improved to 2 in the case of strong solutions, refer to Remark 5.7 below.

Arguing as in the discussion succeeding Remark 5.3, if the conditions of Remark 5.5 hold and we have $\boldsymbol{g}_{\mathrm{dv}} \in L^2(I; \boldsymbol{W}^{-1,2}_0(\Omega))$ and $\boldsymbol{X}^1_{\mathrm{v}}(\Omega_T)^* \hookrightarrow L^2(I; \boldsymbol{W}^{-1,2}_0(\Omega))$, then Theorem 4.4 implies that the associated pressure satisfies the regularity $\mathrm{s} \in W^{-1,2}_{0,0}(I; L^2(\Omega)/\mathbb{R})$ and

$$\begin{aligned}\|\mathrm{s}\|_{W^{-1,2}_{0,0}(L^2/\mathbb{R})} \le c(&\|(g_{\mathrm{do}}, g_{\mathrm{c}}, g_{\mathrm{dh}}, \mathbb{G}_{ds})\|_{(X^1_{\mathrm{o}})^*\times L^2(W^{-1,2})\times(X^1_{\mathrm{h}})^*\times(\mathbb{X}^1_{\mathrm{s}})^*} \\ &+ \|\boldsymbol{g}_{\mathrm{dv}}\|_{L^2(\boldsymbol{W}^{-1,2}_0)} + \|(\varphi_T, \vartheta_T, \boldsymbol{y}_T, \mathbb{Y}_T)\|_{(\mathcal{D}^1)^*}).\end{aligned}$$

For the next theorem, we shall improve the time-regularity presented in Theorem 5.4 under additional conditions on the sources and the data.

Theorem 5.6. *Suppose that* (A1)$_3$ *and* (A2)$_2$ *hold. Assume that* $(g_{\rm do}, g_{\rm c}, g_{\rm dh}, \boldsymbol{g}_{\rm dv}, \mathbb{G}_{\rm ds}) \in \mathcal{X}^0(\Omega_T)^*$ *and* $(\varphi_T, \vartheta_T, \boldsymbol{y}_T, \mathbb{Y}_T) \in \mathcal{D}^0(\Omega)^*$. *The adjoint system* (5.7) *has a unique solution* $(\varphi, \zeta, \vartheta, \boldsymbol{y}, \mathbb{Y}) \in \mathcal{U}^2(\Omega_T)$ *satisfying*

$$\begin{aligned}&\|(\varphi, \zeta, \vartheta, \boldsymbol{y}, \mathbb{Y})\|_{\mathcal{U}^2} + \|\partial_t\varphi\|_{L^2(W^{-2,2}_{\boldsymbol{n}})+(X^0_{\rm o})^*}\\ &\quad + \|\partial_t\vartheta\|_{L^2(L^2)+(B_N X^0_{\rm h})^*} + \|\partial_t\boldsymbol{y}\|_{L^2(\boldsymbol{L}^2_\sigma)+(\boldsymbol{X}^0_{\rm v})^*} + \|\partial_t\mathbb{Y}\|_{L^2(\mathbb{L}^2_{\rm s})+(\mathbb{X}^0_{\rm s})^*}\\ &\quad \le c(\|(g_{\rm do}, g_{\rm c}, g_{\rm dh}, \boldsymbol{g}_{\rm dv}, \mathbb{G}_{\rm ds})\|_{(\mathcal{X}^0)^*} + \|(\varphi_T, \vartheta_T, \boldsymbol{y}_T, \mathbb{Y}_T)\|_{(\mathcal{D}^0)^*})\end{aligned}$$

for some $c = c(\|(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S})\|_{\mathcal{W}^2}) > 0$.

Proof. As usual, we start with Theorem 5.1 to obtain a solution to (5.2) satisfying

$$\begin{aligned}&\|(\varphi, \zeta, \vartheta, \boldsymbol{y}, \mathbb{Y})\|_{\mathcal{U}^0}\\ &\quad \le \widetilde{c}_2 := c[\|(g_{\rm do}, g_{\rm c}, g_{\rm dh}, \boldsymbol{g}_{\rm dv}, \mathbb{G}_{\rm ds})\|_{(\mathcal{X}^0)^*} + c_T\|(\varphi_T, \vartheta_T, \boldsymbol{y}_T, \mathbb{Y}_T)\|_{(\mathcal{D}^0)^*}],\end{aligned} \tag{5.30}$$

with $c_T := \|e_T\|_{\mathcal{L}((\mathcal{D}^0)^*, (\mathcal{V}^0_0)^*)}$. As in the previous theorems, we shall apply (5.11) to obtain the estimates for the time-derivatives.

Let us estimate $\partial_t\varphi$. First, we have $\boldsymbol{v}\cdot\nabla\varphi$, $F''(\phi)\zeta \in L^2(I; L^2(\Omega))$, and $\alpha\Delta\zeta \in L^2(I; W^{-2,2}_{\boldsymbol{n}}(\Omega))$. On the other hand, $m_\phi(\phi, \theta)\nabla\mu\cdot\nabla\varphi \in L^2(I; W^{-2,2}_{\boldsymbol{n}}(\Omega))$ since

$$\begin{aligned}\int_{\Omega_T} m_\phi(\phi, \theta)\psi\nabla\mu\cdot\nabla\varphi\psi \,{\rm d}\omega &= -\int_{\Omega_T} \mu m_\phi(\phi, \theta)\psi\Delta\varphi \,{\rm d}\omega\\ &- \int_{\Omega_T} \mu[(m_{\phi\phi}(\phi, \theta)\psi\nabla\phi + m_{\phi\theta}(\phi, \theta)\psi\nabla\theta + m_\phi(\phi, \theta)\nabla\psi)\cdot\nabla\varphi] \,{\rm d}\omega\\ &\le c\|\mu\|_{L^\infty(L^2)}(\|\phi\|_{L^\infty(W^{1,4})} + \|\theta\|_{L^\infty(W^{1,4})} + 1)\|\varphi\|_{L^2(W^{2,2}_{\boldsymbol{n}})}\|\psi\|_{L^2(W^{2,2}_{\boldsymbol{n}})}.\end{aligned}$$

Likewise, one has $2\nu_\phi(\phi, \theta)\mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{y}$, $\chi_\phi(\phi, \theta)\nabla\theta\cdot\nabla\vartheta$, $\varepsilon_\phi(\phi, \theta)\nabla\mathbb{S} \therefore \nabla\mathbb{Y} \in L^2(I; W^{-1,2}(\Omega))$ since

$$\int_{\Omega_T} 2\nu_\phi(\phi, \theta)\psi\mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{y} \,{\rm d}\omega \le c\|\boldsymbol{v}\|_{L^\infty(\boldsymbol{W}^{1,2}_{0,\sigma})}\|\boldsymbol{y}\|_{L^2(\boldsymbol{W}^{2,2}_{0,\sigma})}\|\psi\|_{L^2(W^{1,2})}$$

and by analogous estimates in the cases of the temperature and the tensors. Also, we have $b_{\rm o}\mathbf{g}\cdot\boldsymbol{y} \in L^2(I; W^{2,2}(\Omega))$ and $\kappa\boldsymbol{y}\cdot\nabla\mu \in L^2(I; W^{-1,2}(\Omega))$ due to

$$\int_{\Omega_T} \kappa\mu\nabla\psi\cdot\boldsymbol{y} \,{\rm d}\omega \le c\|\mu\|_{L^\infty(L^2)}\|\boldsymbol{y}\|_{L^2(\boldsymbol{W}^{2,2}_{0,\sigma})}\|\psi\|_{L^2(W^{1,2})}.$$

These imply that $\partial_t\varphi \in L^2(I; W^{-2,2}_{\boldsymbol{n}}(\Omega)) + X^0_{\rm o}(\Omega_T)^*$ and

$$\|\partial_t\varphi\|_{L^2(W^{-2,2}_{\boldsymbol{n}})+(X^0_{\rm o})^*} \le \widetilde{c}_2. \tag{5.31}$$

For the estimate of ζ, note that $\kappa\boldsymbol{y}\cdot\nabla\phi \in L^2(I; W^{1,2}(\Omega))$. Indeed, this follows from $\kappa\boldsymbol{y}\cdot\nabla\phi \in L^2(I; L^2(\Omega))$ and $\nabla(\boldsymbol{y}\cdot\nabla\phi) = \nabla\boldsymbol{y}\nabla\phi + \nabla^2\phi\boldsymbol{y}$ so that

$$\|\nabla(\boldsymbol{y}\cdot\nabla\phi)\|_{L^2(\boldsymbol{L}^2)} \le c\|\boldsymbol{y}\|_{L^2(\boldsymbol{W}^{2,2}_{0,\sigma})}\|\phi\|_{L^\infty(W^{2,2}_{\boldsymbol{n}})}.$$

Also, $\nabla\cdot(m(\phi, \theta)\nabla\varphi) \in L^2(I; L^2(\Omega))$ since

$$\|\nabla\cdot(m(\phi, \theta)\nabla\varphi)\|_{L^2(L^2)} \le c(\|\phi\|_{L^\infty(W^{1,4})} + \|\theta\|_{L^\infty(W^{1,4})} + 1)\|\varphi\|_{L^2(W^{2,2}_{\boldsymbol{n}})}.$$

Hence, we obtain

$$\|\zeta\|_{L^2(L^2)} \le c(\|\boldsymbol{y}\|_{L^2(\boldsymbol{W}^{2,2}_{0,\sigma})} + \|\varphi\|_{L^2(W^{2,2}_{\boldsymbol{n}})} + \|g_{\mathrm{c}}\|_{L^2(L^2)}) \le \widetilde{c}_2. \tag{5.32}$$

Concerning $\partial_t\vartheta$, let us observe the following: $B_N^{-*}(\boldsymbol{v}\cdot\nabla\vartheta)$, $B_N^{-*}\nabla\cdot(\chi(\phi,\theta)\nabla\vartheta)$, $B_N^{-*}(\sigma\mathbb{S}:\mathbb{D}\boldsymbol{y})$, $B_N^{-*}(b_{\mathrm{h}}\mathbf{g}\cdot\boldsymbol{y}) \in L^2(I;W^{2,2}_{\boldsymbol{n}}(\Omega))$, $B_N^{-*}(m_\theta(\phi,\theta)\nabla\mu\cdot\nabla\varphi)$, $B_N^{-*}(b\Delta^2\vartheta) \in L^2(I;L^2(\Omega))$, and $B_N^{-*}(\chi_\theta(\phi,\theta)\nabla\theta\cdot\nabla\vartheta)$, $B_N^{-*}(2\nu_\theta(\phi,\theta)\mathbb{D}\boldsymbol{v}:\mathbb{D}\boldsymbol{y})$, $B_N^{-*}(\varepsilon_\theta(\phi,\theta)\nabla\mathbb{S}\,\therefore\,\nabla\mathbb{Y}) \in L^2(I;W^{1,2}(\Omega))$. Thus, with the same reasoning as above, it follows that $\partial_t\vartheta \in L^2(I;L^2(\Omega)) + (B_N X^0_{\mathrm{h}}(\Omega_T))^*$ and

$$\|\partial_t\vartheta\|_{L^2(L^2)+(B_N X^0_{\mathrm{h}})^*} \le \widetilde{c}_2. \tag{5.33}$$

Revisiting the proof of Theorem 5.4 in the case of $\partial_t\boldsymbol{y}$, it is enough to recognize that $(\boldsymbol{v}\cdot\nabla)\boldsymbol{y}$, $(\nabla\boldsymbol{v})\boldsymbol{y}$, $\nabla\cdot(2\nu(\phi,\theta)\mathbb{D}\boldsymbol{y})$, $\nabla\cdot(\vartheta\mathbb{S})$, $\nabla\mathbb{S}\odot\mathbb{Y}$, $\nabla\cdot([\mathbb{S},\mathbb{Y}]-a\{\mathbb{S},\mathbb{Y}\}) \in L^2(I;\boldsymbol{L}^2(\Omega))$, in order to conclude that $\partial_t\boldsymbol{y} \in L^2(I;\boldsymbol{L}^2_\sigma(\Omega)) + \boldsymbol{X}^0_{\mathrm{v}}(\Omega_T)^*$. In addition,

$$\|\partial_t\boldsymbol{y}\|_{L^2(\boldsymbol{L}^2_\sigma)+(\boldsymbol{X}^0_{\mathrm{v}})^*} \le \widetilde{c}_2. \tag{5.34}$$

Similarly, since $(\boldsymbol{v}\cdot\nabla)\mathbb{Y}$, $\nabla\cdot(\varepsilon(\phi,\theta)\nabla\mathbb{Y})$, $2\mathbb{W}\boldsymbol{v}\mathbb{Y} - 2a\mathbb{D}\boldsymbol{v}\mathbb{Y}$, $\mathbb{M}_{\mathbb{S}}(\theta,\mathbb{S})^*\mathbb{D}\boldsymbol{y}$, $\vartheta\mathbb{D}\boldsymbol{v} \in L^2(I;\mathbb{L}^2_{\mathrm{s}}(\Omega))$, we have $\partial_t\mathbb{Y} \in L^2(I;\mathbb{L}^2_{\mathrm{s}}(\Omega)) + \mathbb{X}^0_{\mathrm{s}}(\Omega_T)^*$ and

$$\|\partial_t\mathbb{Y}\|_{L^2(\mathbb{L}^2_{\mathrm{s}})+(\mathbb{X}^0_{\mathrm{s}})^*} \le \widetilde{c}_2. \tag{5.35}$$

Taking the sum of (5.31)–(5.35) completes the proof of the theorem. □

Remark 5.7. If $X^0_{\mathrm{o}}(\Omega_T)^* \hookrightarrow L^2(I;W^{-2,2}_{\boldsymbol{n}}(\Omega))$, $(B_N X^0_{\mathrm{h}}(\Omega_T))^* \hookrightarrow L^2(I;L^2(\Omega))$, $\boldsymbol{X}^0_{\mathrm{v}}(\Omega_T)^* \hookrightarrow L^2(I;\boldsymbol{L}^2(\Omega))$, and $\mathbb{X}^0_{\mathrm{s}}(\Omega_T)^* \hookrightarrow L^2(I;\mathbb{L}^2_{\mathrm{s}}(\Omega))$ in the previous theorem, then the variational solution to (5.7) enjoys the following regularity:

$$\begin{aligned}
&\varphi \in W^{1,2,2}(I;W^{2,2}_{\boldsymbol{n}}(\Omega),W^{-2,2}_{\boldsymbol{n}}(\Omega)), &&\vartheta \in W^{1,2,2}(I;W^{2,2}_{\boldsymbol{n}}(\Omega),L^2(\Omega)),\\
&\boldsymbol{y} \in W^{1,2,2}(I;\boldsymbol{W}^{2,2}_{0,\sigma}(\Omega),\boldsymbol{L}^2_\sigma(\Omega)), &&\mathbb{Y} \in W^{1,2,2}(I;\mathbb{W}^{2,2}_{\boldsymbol{n},\mathrm{s}}(\Omega),\mathbb{L}^2_{\mathrm{s}}(\Omega)).
\end{aligned}$$

If $\boldsymbol{g}_{\mathrm{dv}} \in L^2(I;\boldsymbol{L}^2(\Omega))$, then the classical de Rham's Theorem gives us an associated pressure such that $\mathrm{s} \in L^2(I;W^{1,2}(\Omega)/\mathbb{R})$ with

$$\begin{aligned}
\|\mathrm{s}\|_{L^2(W^{1,2}/\mathbb{R})} \le c(&\|(g_{\mathrm{do}},g_{\mathrm{c}},g_{\mathrm{dh}},\mathbb{G}_{\mathrm{ds}})\|_{(X^0_{\mathrm{o}})^*\times L^2(L^2)\times(X^0_{\mathrm{h}})^*\times(\mathbb{X}^0_{\mathrm{s}})^*}\\
&+\|\boldsymbol{g}_{\mathrm{dv}}\|_{L^2(\boldsymbol{L}^2)} + \|(\varphi_T,\vartheta_T,\boldsymbol{y}_T,\mathbb{Y}_T)\|_{(\mathcal{D}^0)^*}).
\end{aligned}$$

Our next set of results deals on the additional regularity of the components with respect to the dual of the non-isothermal Cahn–Hilliard part of the adjoint system. We start with the following auxiliary lemma.

Lemma 5.8. *Suppose that we have ϕ, $\theta \in W^{1,2,2}(I;W^{4,2}_{\boldsymbol{n}}(\Omega),L^2(\Omega))$ and $\mu \in W^{1,2,2}(I;W^{2,2}_{\boldsymbol{n}}(\Omega),W^{-2,2}_{\boldsymbol{n}}(\Omega))$. Consider the backward-in-time linear system*

$$\left[\begin{aligned}
&-\partial_t\widetilde{\varphi} + m_\phi(\phi,\theta)\nabla\mu\cdot\nabla\widetilde{\varphi} + \alpha\Delta\widetilde{\zeta} = \widetilde{g}_{\mathrm{o}} \text{ in } \Omega_T,\\
&\widetilde{\zeta} - \nabla\cdot(m(\phi,\theta)\nabla\widetilde{\varphi}) = \widetilde{g}_{\mathrm{c}} \text{ in } \Omega_T,\\
&\partial_{\boldsymbol{n}}\widetilde{\varphi} = \partial_{\boldsymbol{n}}\Delta\widetilde{\varphi} = 0 \text{ on } \Gamma_T, \quad \widetilde{\varphi}(T) = \widetilde{\varphi}_T \text{ in } \Omega.
\end{aligned}\right. \tag{5.36}$$

Let $k=0,1$. If $\widetilde{g}_{\mathrm{o}} \in L^2(I;W^{k-1,2}(\Omega))$, $\widetilde{g}_{\mathrm{c}} \in L^2(I;W^{k+1,2}(\Omega))$, and $\widetilde{\varphi}_T \in W^{k+1,2}_{\boldsymbol{n}}(\Omega)$, then

$$\widetilde{\varphi} \in W^{1,2,2}(I;W^{k+3,2}_{\boldsymbol{n}}(\Omega),W^{k-1,2}(\Omega)), \quad \widetilde{\zeta} \in L^2(I;W^{k+1,2}(\Omega)) \tag{5.37}$$

and there exists $c > 0$ depending on the norms of (ϕ, θ, μ), but not on the solution, the sources, and the terminal data, such that

$$\begin{aligned}&\|\widetilde{\varphi}\|_{W^{1,2,2}(W_{\boldsymbol{n}}^{k+3,2},W^{k-1,2})} + \|\widetilde{\zeta}\|_{L^2(W^{k+1,2})}\\ &\quad \le c(\|\widetilde{g}_{\mathrm{c}}\|_{L^2(W^{k-1,2})} + \|\widetilde{g}_{\mathrm{h}}\|_{L^2(W^{k+1,2})} + \|\widetilde{\varphi}_T\|_{W_{\boldsymbol{n}}^{k+1,2}}).\end{aligned}$$

Proof. We only provide the derivation of the a priori estimates needed for the Faedo–Galerkin method. By testing the first equation of (5.36) with $\widetilde{\varphi} - \Delta\widetilde{\varphi}$ and integrating by parts, we have

$$\begin{aligned}&-\frac{1}{2}\frac{d}{dt}\int_\Omega |\widetilde{\varphi}|^2 + |\nabla\widetilde{\varphi}|^2 \,\mathrm{d}x + \int_\Omega m_\phi(\phi,\theta)\nabla\mu\cdot\nabla\widetilde{\varphi}(\widetilde{\varphi} - \Delta\widetilde{\varphi})\,\mathrm{d}x\\ &\quad + \int_\Omega \alpha\nabla\widetilde{\zeta}\cdot\nabla(\Delta\widetilde{\varphi} - \widetilde{\varphi})\,\mathrm{d}x = \langle \widetilde{g}_{\mathrm{o}}, \widetilde{\varphi} - \Delta\widetilde{\varphi}\rangle_{W^{-1,2},W^{1,2}}. \end{aligned}\tag{5.38}$$

For the right-hand side and the second integral, we get

$$\begin{aligned}&|\langle \widetilde{g}_{\mathrm{o}}, \widetilde{\varphi} - \Delta\widetilde{\varphi}\rangle_{W^{-1,2},W^{1,2}}|\\ &\quad \le \delta\|\nabla\Delta\widetilde{\varphi}\|_{\boldsymbol{L}^2}^2 + c_\delta(\|\widetilde{\varphi}\|_{L^2}^2 + \|\nabla\widetilde{\varphi}\|_{\boldsymbol{L}^2}^2 + \|\widetilde{g}_{\mathrm{o}}\|_{W^{-1,2}}^2)\end{aligned}\tag{5.39}$$

$$\begin{aligned}&\int_\Omega |m_\phi(\phi,\theta)\nabla\mu\cdot\nabla\widetilde{\varphi}(\widetilde{\varphi} - \Delta\widetilde{\varphi})|\,\mathrm{d}x\\ &\quad \le \delta\|\nabla\Delta\widetilde{\varphi}\|_{\boldsymbol{L}^2}^2 + c\|\Delta\mu\|_{L^2}^2(\|\widetilde{\varphi}\|_{L^2}^2 + \|\nabla\widetilde{\varphi}\|_{\boldsymbol{L}^2}^2).\end{aligned}\tag{5.40}$$

Let us write $\nabla\widetilde{\zeta} = \pi_\phi + \pi_\theta + \pi$, where $\pi_\gamma := \nabla(m_\gamma(\phi,\theta)\nabla\gamma\cdot\nabla\widetilde{\varphi})$ for $\gamma = \phi, \theta$ and $\pi = \Delta\widetilde{\varphi}(m_\phi(\phi,\theta)\nabla\phi + m_\theta(\phi,\theta)\nabla\theta) + m(\phi,\theta)\nabla\Delta\widetilde{\varphi} + \nabla\widetilde{g}_{\mathrm{c}}$. Then,

$$\begin{aligned}&\int_\Omega \alpha\pi\cdot\nabla(\Delta\widetilde{\varphi} - \widetilde{\varphi})\,\mathrm{d}x \ge \frac{\alpha m_0}{2}\|\nabla\Delta\widetilde{\varphi}\|_{\boldsymbol{L}^2}^2\\ &\quad - c[(\|\nabla\phi\|_{\boldsymbol{L}^\infty}^2 + \|\nabla\theta\|_{\boldsymbol{L}^\infty}^2)\|\Delta\widetilde{\varphi}\|_{L^2}^2 + \|\nabla\widetilde{\varphi}\|_{\boldsymbol{L}^2}^2 + \|\nabla\widetilde{g}_{\mathrm{c}}\|_{\boldsymbol{L}^2}^2].\end{aligned}\tag{5.41}$$

Performing the gradient in π_γ, we obtain the expression

$$\pi_\gamma = m_{\gamma\phi}(\phi,\theta)(\nabla\gamma\cdot\nabla\phi)\nabla\widetilde{\varphi} + m_{\gamma\theta}(\phi,\theta)(\nabla\gamma\cdot\nabla\theta)\nabla\widetilde{\varphi} + m_\gamma(\phi,\theta)(\nabla^2\gamma\nabla\widetilde{\varphi} + \nabla^2\widetilde{\varphi}\nabla\gamma).$$

So, for each $\gamma = \phi, \theta$, one has

$$\begin{aligned}&\int_\Omega |\alpha\pi_\gamma\cdot\nabla(\Delta\widetilde{\varphi} - \widetilde{\varphi})|\,\mathrm{d}x\\ &\quad \le \delta\|\nabla\Delta\widetilde{\varphi}\|_{\boldsymbol{L}^2}^2 + c_\delta\|\nabla\gamma\|_{\boldsymbol{L}^\infty}^2(\|\nabla\phi\|_{\boldsymbol{L}^\infty}^2 + \|\nabla\theta\|_{\boldsymbol{L}^\infty}^2)\|\nabla\widetilde{\varphi}\|_{\boldsymbol{L}^2}^2\\ &\quad + c_\delta(\|\nabla^2\gamma\|_{\mathbb{L}^4}^2 + \|\nabla\gamma\|_{\boldsymbol{L}^\infty}^2 + 1)(\|\widetilde{\varphi}\|_{L^2}^2 + \|\nabla\widetilde{\varphi}\|_{\boldsymbol{L}^2}^2 + \|\Delta\widetilde{\varphi}\|_{L^2}^2).\end{aligned}\tag{5.42}$$

Note that $\|\Delta\widetilde{\varphi}\|_{L^2}^2 \le \|\nabla\Delta\widetilde{\varphi}\|_{\boldsymbol{L}^2}\|\nabla\widetilde{\varphi}\|_{\boldsymbol{L}^2}$ by Green identity and the Cauchy–Schwarz inequality. Thus,

$$\begin{aligned}&c(\|\nabla\phi\|_{\boldsymbol{L}^\infty}^2 + \|\nabla\theta\|_{\boldsymbol{L}^\infty}^2)\|\Delta\widetilde{\varphi}\|_{L^2}^2\\ &\quad \le \delta\|\nabla\Delta\widetilde{\varphi}\|_{\boldsymbol{L}^2}^2 + c_\delta(\|\nabla\phi\|_{\boldsymbol{L}^\infty}^4 + \|\nabla\theta\|_{\boldsymbol{L}^\infty}^4)\|\nabla\widetilde{\varphi}\|_{\boldsymbol{L}^2}^2\end{aligned}\tag{5.43}$$

$$\begin{aligned}&c_\delta(\|\nabla^2\gamma\|_{\mathbb{L}^4}^2 + \|\nabla\gamma\|_{\boldsymbol{L}^\infty}^2 + 1)\|\Delta\widetilde{\varphi}\|_{L^2}^2\\ &\quad \le \delta\|\nabla\Delta\widetilde{\varphi}\|_{\boldsymbol{L}^2}^2 + c_\delta(\|\nabla^2\gamma\|_{\mathbb{L}^4}^4 + \|\nabla\gamma\|_{\boldsymbol{L}^\infty}^4 + 1)\|\nabla\widetilde{\varphi}\|_{\boldsymbol{L}^2}^2.\end{aligned}\tag{5.44}$$

Utilizing (5.39)–(5.44) in the equation (5.38), as well as (see (3.1) and (3.6))

$$\phi, \theta \in L^2(I; W_{\boldsymbol{n}}^{4,2}(\Omega)) \cap L^4(I; W^{2,\infty}(\Omega) \cap W_{\boldsymbol{n}}^{3,2}(\Omega)) \cap L^\infty(I; W_{\boldsymbol{n}}^{2,2}(\Omega)),$$

$$\mu \in L^2(I; W^{2,2}_{\boldsymbol{n}}(\Omega)), \tag{5.45}$$

we obtain a $K \in L^1(I)$ satisfying (A.6) and

$$\begin{aligned} -\frac{1}{2}\frac{d}{dt}\|\widetilde{\varphi}\|^2_{W^{1,2}} + \left(\frac{\alpha m_0}{2} - 5\delta\right)\|\nabla\Delta\widetilde{\varphi}\|^2_{\boldsymbol{L}^2} \\ \leq c_\delta(K\|\widetilde{\varphi}\|^2_{W^{1,2}} + \|\widetilde{g}_{\mathrm{o}}\|^2_{W^{-1,2}} + \|\widetilde{g}_{\mathrm{c}}\|^2_{W^{1,2}}) \end{aligned} \tag{5.46}$$

$$\|\widetilde{\zeta}\|^2_{W^{1,2}} \leq c\|\nabla\Delta\widetilde{\varphi}\|^2_{\boldsymbol{L}^2} + c(K\|\widetilde{\varphi}\|^2_{W^{1,2}} + \|\widetilde{g}_{\mathrm{c}}\|^2_{W^{1,2}}). \tag{5.47}$$

Choosing $0 < \delta < \frac{\alpha m_0}{10}$, invoking Grönwall Lemma to (5.46), and applying the result to (5.47), the obtained estimates and the one that can be derived for $\|\partial_t\widetilde{\varphi}\|_{L^2(W^{-1,2})}$ will lead to (5.37) for $k = 0$ in the Faedo–Galerkin method.

Next, we show (5.37) for $k = 1$. Using the test function $\Delta^2\widetilde{\varphi}$ in (5.36), we have

$$\begin{aligned} -\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta\widetilde{\varphi}|^2\,\mathrm{d}x + \int_\Omega m_\phi(\phi,\theta)\nabla\mu\cdot\nabla\widetilde{\varphi}\Delta^2\widetilde{\varphi}\,\mathrm{d}x \\ + \int_\Omega \alpha\Delta\widetilde{\zeta}\Delta^2\widetilde{\varphi}\,\mathrm{d}x = \int_\Omega \widetilde{g}_{\mathrm{o}}\Delta^2\widetilde{\varphi}\,\mathrm{d}x. \end{aligned} \tag{5.48}$$

The second and fourth integrals can be estimated as follows:

$$\int_\Omega |m_\phi(\phi,\theta)\nabla\mu\cdot\nabla\widetilde{\varphi}\Delta^2\widetilde{\varphi}|\,\mathrm{d}x \leq \delta\|\Delta^2\widetilde{\varphi}\|^2_{L^2} + c_\delta\|\Delta\mu\|^2_{L^2}\|\Delta\widetilde{\varphi}\|^2_{L^2} \tag{5.49}$$

$$\int_\Omega |\widetilde{g}_{\mathrm{o}}\Delta^2\widetilde{\varphi}|\,\mathrm{d}x \leq \delta\|\Delta^2\widetilde{\varphi}\|^2_{L^2} + c_\delta\|\widetilde{g}_{\mathrm{o}}\|^2_{L^2}. \tag{5.50}$$

For the third integral, let us note that $\Delta\widetilde{\zeta} = \Delta\nabla\cdot(m(\phi,\theta)\nabla\widetilde{\varphi}) + \Delta\widetilde{g}_{\mathrm{c}}$. Utilizing the classical differentiation rules, the expansion of $\Delta\nabla\cdot(m(\phi,\theta)\nabla\widetilde{\varphi})$ is given by

$$\Delta\nabla\cdot(m(\phi,\theta)\nabla\widetilde{\varphi}) = \sum_{\gamma_1=\phi,\theta}\Delta(m_{\gamma_1}(\phi,\theta)\nabla\widetilde{\varphi}\cdot\nabla\gamma_1) + \Delta(m(\phi,\theta)\Delta\widetilde{\varphi}) \tag{5.51}$$

$$\begin{aligned} &= \sum_{\gamma_1,\gamma_2=\phi,\theta}\nabla\cdot[m_{\gamma_1\gamma_2}(\phi,\theta)(\nabla\gamma_1\cdot\nabla\widetilde{\varphi})\nabla\gamma_2] \\ &+ \sum_{\gamma_1=\phi,\theta}\nabla\cdot[m_{\gamma_1}(\phi,\theta)(\nabla^2\widetilde{\varphi}\nabla\gamma_1 + \nabla^2\gamma_1\nabla\widetilde{\varphi} + \Delta\widetilde{\varphi}\nabla\gamma_1)] + \nabla\cdot(m(\phi,\theta)\nabla\Delta\widetilde{\varphi}) \\ &= \sum_{\gamma_1,\gamma_2,\gamma_3=\phi,\theta} m_{\gamma_1\gamma_2\gamma_3}(\phi,\theta)(\nabla\gamma_1\cdot\nabla\widetilde{\varphi})(\nabla\gamma_2\cdot\nabla\gamma_3) \\ &+ \sum_{\gamma_1,\gamma_2=\phi,\theta} m_{\gamma_1\gamma_2}(\phi,\theta)[2(\nabla^2\gamma_1\nabla\widetilde{\varphi})\cdot\nabla\gamma_2 + 2(\nabla^2\widetilde{\varphi}\nabla\gamma_1)\cdot\nabla\gamma_2 \\ &\qquad + (\nabla\gamma_1\cdot\nabla\widetilde{\varphi})\Delta\gamma_2 + \Delta\widetilde{\varphi}\nabla\gamma_1\cdot\nabla\gamma_2] \\ &+ \sum_{\gamma_1=\phi,\theta} m_{\gamma_1}(\phi,\theta)(\nabla\Delta\gamma_1\cdot\nabla\widetilde{\varphi} + 2\nabla^2\widetilde{\varphi}:\nabla^2\gamma_1 + 3\nabla\Delta\widetilde{\varphi}\cdot\nabla\gamma_1 + \Delta\widetilde{\varphi}\Delta\gamma_1) \\ &+ m(\phi,\theta)\Delta^2\widetilde{\varphi}. \end{aligned}$$

Here, we used $\nabla\cdot\nabla^2\gamma = \Delta\nabla\gamma = \nabla\Delta\gamma$ in the weak sense. Let us write $\Delta\nabla\cdot(m(\phi,\theta)\nabla\widetilde{\varphi}) =: \widetilde{\pi} + m(\phi,\theta)\Delta^2\widetilde{\varphi}$, where $\widetilde{\pi}$ are the terms involving the sums in (5.51).

Thus,

$$\int_\Omega \alpha\Delta\widetilde{\zeta}\Delta^2\widetilde{\varphi}\,\mathrm{d}x \geq \frac{m_0\alpha}{2}\|\Delta^2\widetilde{\varphi}\|_{L^2}^2 - c(\|\widetilde{\pi}\|_{L^2}^2 + \|\Delta\widetilde{g}_{\mathrm{c}}\|_{L^2}^2). \tag{5.52}$$

Our next task is to estimate each term appearing in the sums of $\widetilde{\pi}$. For these, we apply Hölder inequality so that for $\gamma_1, \gamma_2, \gamma_3 = \phi, \theta$, we have the following:

$$\int_\Omega |m_{\gamma_1}(\phi,\theta)\nabla\Delta\gamma_1\cdot\nabla\widetilde{\varphi}|^2\,\mathrm{d}x \leq c\|\Delta^2\gamma_1\|_{L^2}^2\|\Delta\widetilde{\varphi}\|_{L^2}^2 \tag{5.53}$$

$$\begin{aligned}\int_\Omega |m_{\gamma_1}(\phi,\theta)\nabla\Delta\widetilde{\varphi}\cdot\nabla\gamma_1|^2\,\mathrm{d}x &\leq c\|\nabla\gamma_1\|_{\boldsymbol{L}^\infty}^2\|\nabla\Delta\widetilde{\varphi}\|_{\boldsymbol{L}^2}^2\\ &\leq \delta\|\Delta^2\widetilde{\varphi}\|_{L^2}^2 + c_\delta\|\nabla\gamma_1\|_{\boldsymbol{L}^\infty}^4\|\Delta\widetilde{\varphi}\|_{L^2}^2\end{aligned} \tag{5.54}$$

$$\int_\Omega |m_{\gamma_1\gamma_2}(\phi,\theta)(\nabla^2\gamma_1\nabla\widetilde{\varphi})\cdot\nabla\gamma_2|^2\,\mathrm{d}x \leq c\|\nabla^2\gamma_1\|_{\mathbb{L}^\infty_{\mathrm{s}}}^2\|\Delta\gamma_2\|_{L^2}^2\|\Delta\widetilde{\varphi}\|_{L^2}^2 \tag{5.55}$$

$$\int_\Omega |m_{\gamma_1\gamma_2}(\phi,\theta)(\nabla\gamma_1\cdot\nabla\widetilde{\varphi})\Delta\gamma_2|^2\,\mathrm{d}x \leq c\|\Delta\gamma_1\|_{L^2}^2\|\nabla\Delta\gamma_2\|_{\boldsymbol{L}^2}^2\|\Delta\widetilde{\varphi}\|_{L^2}^2 \tag{5.56}$$

$$\begin{aligned}&\int_\Omega |m_{\gamma_1\gamma_2\gamma_3}(\phi,\theta)(\nabla\gamma_1\cdot\nabla\widetilde{\varphi})(\nabla\gamma_2\cdot\nabla\gamma_3)|^2\,\mathrm{d}x\\ &\quad\leq c\|\Delta\gamma_1\|_{L^2}^2\|\Delta\gamma_2\|_{L^2}^2\|\Delta\gamma_3\|_{L^2}^2\|\Delta\widetilde{\varphi}\|_{L^2}^2\end{aligned} \tag{5.57}$$

$$\begin{aligned}&\int_\Omega |m_{\gamma_1}(\phi,\theta)\nabla^2\widetilde{\varphi}:\nabla^2\gamma_1|^2 + |m_{\gamma_1}(\phi,\theta)\Delta\widetilde{\varphi}\Delta\gamma_1|^2\,\mathrm{d}x\\ &\quad\leq c\|\nabla^2\gamma_1\|_{\mathbb{L}^\infty_{\mathrm{s}}}^2(\|\widetilde{\varphi}\|_{L^2}^2 + \|\Delta\widetilde{\varphi}\|_{L^2}^2)\end{aligned} \tag{5.58}$$

$$\begin{aligned}&\int_\Omega |m_{\gamma_1\gamma_2}(\phi,\theta)(\nabla^2\widetilde{\varphi}\nabla\gamma_1)\cdot\nabla\gamma_2|^2 + |m_{\gamma_1\gamma_2}(\phi,\theta)\Delta\widetilde{\varphi}\nabla\gamma_1\cdot\nabla\gamma_2|^2\,\mathrm{d}x\\ &\quad\leq c(\|\nabla\gamma_1\|_{\boldsymbol{L}^\infty}^4 + \|\nabla\gamma_2\|_{\boldsymbol{L}^\infty}^4)(\|\widetilde{\varphi}\|_{L^2}^2 + \|\Delta\widetilde{\varphi}\|_{L^2}^2).\end{aligned} \tag{5.59}$$

Here, we utilized (2.13) in (5.54). Using (5.53)–(5.59) in (5.51), $\widetilde{\pi}$ can be estimated as follows

$$\|\widetilde{\pi}\|_{L^2}^2 \leq \delta\|\Delta^2\widetilde{\varphi}\|_{L^2}^2 + c_\delta K\|\widetilde{\varphi}\|_{W^{2,2}_{\boldsymbol{n}}}^2 \tag{5.60}$$

for some $K \in L^1(I)$ satisfying (A.6) due to (5.45).

Plugging (5.60) in (5.52), and using the resulting inequality together with (5.49) and (5.50) in (5.48), we deduce for some $c > 0$ that

$$\begin{aligned}-\frac{1}{2}\frac{d}{dt}\|\Delta\widetilde{\varphi}\|_{L^2}^2 &+ \left(\frac{\alpha m_0}{2} - c\delta\right)\|\Delta^2\widetilde{\varphi}\|_{L^2}^2\\ &\leq c_\delta(K\|\widetilde{\varphi}\|_{W^{2,2}_{\boldsymbol{n}}}^2 + \|\widetilde{g}_{\mathrm{o}}\|_{L^2}^2 + \|\Delta\widetilde{g}_{\mathrm{c}}\|_{L^2}^2)\end{aligned} \tag{5.61}$$

$$\|\Delta\widetilde{\zeta}\|_{L^2}^2 \leq c\|\Delta^2\widetilde{\varphi}\|_{L^2}^2 + c(K\|\widetilde{\varphi}\|_{W^{2,2}_{\boldsymbol{n}}}^2 + \|\Delta\widetilde{g}_{\mathrm{c}}\|_{L^2}^2), \tag{5.62}$$

after suitably modifying $K \in L^1(I)$ that still satisfies (A.6). As above, the estimates (5.61) and (5.62) imply that $\widetilde{\varphi}$ satisfies (5.37) for $k = 1$ and $\Delta\widetilde{\zeta} \in L^2(I; L^2(\Omega))$. With this regularity of $\widetilde{\varphi}$ and expanding as in (5.51) but with ∇^2 instead of Δ, it can be shown that (5.37) with $k = 1$ holds for $\widetilde{\zeta}$. This completes the proof of the lemma. □

Theorem 5.9. *Consider the frameworks of Theorem 5.6 and Remark 5.7, and let $k = 0, 1$. If $g_{\mathrm{do}} \in L^2(I; W^{k-1,2}(\Omega))$, $g_{\mathrm{c}} \in L^2(I; W^{k+1,2}(\Omega))$, and $\varphi_T \in W^{k+1,2}_{\boldsymbol{n}}(\Omega)$, then φ and ζ satisfies (5.37), provided that $\phi \in L^\infty(I; W^{1,\infty}(\Omega))$ if $k = 1$.*

Proof. We apply Lemma 5.8 with $\widetilde{g}_{\mathrm{c}} := g_{\mathrm{c}} + \kappa \boldsymbol{y} \cdot \nabla\phi$ and

$$
\begin{aligned}
\widetilde{g}_{\mathrm{o}} &= g_{\mathrm{do}} + \boldsymbol{v} \cdot \nabla\varphi - \chi_\phi(\phi,\theta)\nabla\theta \cdot \nabla\vartheta - 2\nu_\phi(\phi,\theta)\mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{y} \\
&\quad - \varepsilon_\phi(\phi,\theta)\nabla\mathbb{S} \therefore \nabla\mathbb{Y} + F''(\phi)\zeta - \kappa\boldsymbol{y} \cdot \nabla\mu + b_{\mathrm{o}}\mathbf{g} \cdot \boldsymbol{y}.
\end{aligned}
$$

Suppose that $k = 0$. The proof of Theorem 5.6 shows that $\widetilde{g}_{\mathrm{o}} \in L^2(I; W^{-1,2}(\Omega))$ and $\widetilde{g}_{\mathrm{c}} \in L^2(I; W^{1,2}(\Omega))$. Hence, (5.37) with $k = 0$ hold for φ and ζ. In particular, $\varphi \in L^\infty(I; W^{1,2}(\Omega)) \cap L^2(I; W^{1,\infty}(\Omega))$.

Consider the case $k = 1$. Note that $\boldsymbol{y} \in L^2(I; \boldsymbol{W}^{2,2}_{0,\sigma}(\Omega)) \cap L^\infty(I; \boldsymbol{W}^{1,2}_{0,\sigma}(\Omega))$ from Remark 5.7. Thus, we have $\boldsymbol{y} \cdot \nabla\mu \in L^2(I; L^2(\Omega))$ due to

$$
\|\boldsymbol{y} \cdot \nabla\mu\|_{L^2(L^2)} \le c\|\boldsymbol{y}\|_{L^\infty(\boldsymbol{L}^4)}\|\nabla\mu\|_{L^2(\boldsymbol{L}^4)} \le c\|\boldsymbol{y}\|_{L^\infty(\boldsymbol{W}^{1,2}_{0,\sigma})}\|\mu\|_{L^2(W^{2,2}_{\boldsymbol{n}})}.
$$

In the same manner, one has $\boldsymbol{v} \cdot \nabla\varphi \in L^2(I; L^2(\Omega))$. Also, $\chi_\phi(\phi,\theta)\nabla\theta \cdot \nabla\vartheta \in L^2(I; L^2(\Omega))$ since

$$
\begin{aligned}
\|\chi_\phi(\phi,\theta)\nabla\theta \cdot \nabla\vartheta\|_{L^2(L^2)} &\le c\|\nabla\theta\|_{L^4(\boldsymbol{L}^4)}\|\nabla\vartheta\|_{L^4(\boldsymbol{L}^4)} \\
&\le c\|\theta\|_{W^{1,2,2}(W^{2,2}_{\boldsymbol{n}}, L^2)}\|\vartheta\|_{W^{1,2,2}(W^{2,2}_{\boldsymbol{n}}, L^2)}.
\end{aligned}
$$

Similarly, we have $2\nu_\phi(\phi,\theta)\mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{y}$, $\varepsilon_\phi(\phi,\theta)\nabla\mathbb{S} \therefore \nabla\mathbb{Y} \in L^2(I; L^2(\Omega))$. Finally, the remaining terms $F''(\phi)\zeta$ and $b_{\mathrm{o}}\mathbf{g} \cdot \boldsymbol{y}$ also lie in $L^2(I; L^2(\Omega))$ based on the proof of Theorem 5.6. These result into $\widetilde{g}_{\mathrm{o}} \in L^2(I; L^2(\Omega))$.

We claim that $\boldsymbol{y} \cdot \nabla\phi \in L^2(I; W^{2,2}(\Omega))$. Indeed, using the following equation

$$
\nabla^2(\boldsymbol{y} \cdot \nabla\phi) = \sum_{j=1,2} [y_j \nabla^2\partial_j\phi + (\nabla y_j)(\nabla\partial_j\phi)^{\mathrm{t}} + (\nabla\partial_j\phi)^{\mathrm{t}}\nabla y_j + \partial_j\phi\nabla^2 y_j]
$$

we obtain that

$$
\begin{aligned}
\|\nabla^2(\boldsymbol{y} \cdot \nabla\phi)\|_{L^2(\mathbb{L}^2_{\mathrm{s}})} &\le c(\|\boldsymbol{y}\|_{L^\infty(\boldsymbol{L}^4)}\|\nabla^3\phi\|_{L^2((\mathbb{L}^4_{\mathrm{s}})^2)} \\
&\quad + \|\nabla\boldsymbol{y}\|_{L^\infty(\mathbb{L}^2)}\|\nabla^2\phi\|_{L^2(\mathbb{L}^\infty_{\mathrm{s}})} + \|\nabla\phi\|_{L^\infty(\boldsymbol{L}^\infty)}\|\nabla^2\boldsymbol{y}\|_{L^2((\mathbb{L}^2)^2)}) \\
&\le c\|\boldsymbol{y}\|_{L^2(\boldsymbol{W}^{2,2}_{0,\sigma}) \cap L^\infty(\boldsymbol{W}^{1,2}_{0,\sigma})}\|\phi\|_{L^2(W^{4,2}_{\boldsymbol{n}}) \cap L^\infty(W^{1,\infty})}.
\end{aligned}
$$

Consequently, $\widetilde{g}_{\mathrm{c}} \in L^2(I; W^{2,2}(\Omega))$. Therefore, φ and ζ satisfy (5.37) with $k = 1$ due to Lemma 5.8. □

Corollary 5.10. *If the conditions of Theorem 5.9 hold for $k = 1$ and $\partial_t g_{\mathrm{c}} \in W^{1,2,2}(I; W^{2,2}(\Omega), W^{-2,2}_{\boldsymbol{n}}(\Omega))$, then $\zeta \in W^{1,2,2}(I; W^{2,2}(\Omega), W^{-2,2}_{\boldsymbol{n}}(\Omega))$.*

Proof. We already know from Theorem 5.9 that $\zeta \in L^2(I; W^{2,2}(\Omega))$. Observe that we have

$$
\partial_t\zeta = \nabla \cdot (m'(\phi,\theta)(\partial_t\phi, \partial_t\theta)\nabla\varphi) + \nabla \cdot (m(\phi,\theta)\nabla\partial_t\varphi) + \kappa\partial_t\boldsymbol{y} \cdot \nabla\phi + \kappa\boldsymbol{y} \cdot \nabla\partial_t\phi + \partial_t g_{\mathrm{c}}
$$

in the sense of distributions. Let us recall from the previous theorem that $\partial_t\varphi \in L^2(I; L^2(\Omega))$. Thus, we have $\nabla \cdot (m(\phi,\theta)\nabla\partial_t\varphi), \kappa\boldsymbol{y} \cdot \nabla\partial_t\phi \in L^2(I; W^{-2,2}_{\boldsymbol{n}}(\Omega))$ due to

$$
\langle \nabla \cdot (m(\phi,\theta)\nabla\partial_t\varphi), \psi \rangle_{L^2(W^{-2,2}_{\boldsymbol{n}}), L^2(W^{2,2}_{\boldsymbol{n}})} = \int_{\Omega_T} \partial_t\varphi \nabla \cdot (m(\phi,\theta)\nabla\psi) \,\mathrm{d}\omega
$$

$$\leq c_{\phi,\theta}\|\partial_t\varphi\|_{L^2(L^2)}\|\psi\|_{L^2(W^{2,2}_{\boldsymbol{n}})}$$

$$\langle\kappa\boldsymbol{y}\cdot\nabla\partial_t\phi,\psi\rangle_{L^2(W^{-2,2}_{\boldsymbol{n}}),L^2(W^{2,2}_{\boldsymbol{n}})} = -\int_{\Omega_T}\kappa\partial_t\phi\boldsymbol{y}\cdot\nabla\psi\,\mathrm{d}\omega$$

$$\leq c\|\partial_t\phi\|_{L^2(L^2)}\|\boldsymbol{y}\|_{L^\infty(\boldsymbol{W}^{1,2}_{0,\sigma})}\|\psi\|_{L^2(W^{2,2}_{\boldsymbol{n}})}$$

for each $\psi \in L^2(I;W^{2,2}_{\boldsymbol{n}}(\Omega))$. Furthermore, we have $\nabla\cdot(m'(\phi,\theta)(\partial_t\phi,\partial_t\theta)\nabla\varphi) \in L^2(I;W^{-2,2}_{\boldsymbol{n}}(\Omega))$ and $\kappa\partial_t\boldsymbol{y}\cdot\nabla\phi \in L^2(I;W^{-1,2}(\Omega))$ since

$$\int_{\Omega_T}|m'(\phi,\theta)(\partial_t\phi,\partial_t\theta)\nabla\varphi\cdot\nabla\psi|\,\mathrm{d}\omega$$

$$\leq c(\|\partial_t\phi\|_{L^2(L^2)}+\|\partial_t\theta\|_{L^2(L^2)})\|\varphi\|_{L^\infty(W^{2,2}_{\boldsymbol{n}})}\|\psi\|_{L^2(W^{2,2}_{\boldsymbol{n}})}$$

$$\int_{\Omega_T}|(\kappa\partial_t\boldsymbol{y}\cdot\nabla\phi)\psi|\,\mathrm{d}\omega \leq c\|\partial_t\boldsymbol{y}\|_{L^2(\boldsymbol{L}^2_\sigma)}\|\phi\|_{L^\infty(W^{2,2})}\|\psi\|_{L^2(W^{1,2})}.$$

Since $\partial_t g_{\mathrm{c}} \in L^2(I;W^{-2,2}_{\boldsymbol{n}}(\Omega))$, we conclude that $\partial_t\zeta \in L^2(I;W^{-2,2}_{\boldsymbol{n}}(\Omega))$. □

The next auxiliary lemma is concerned with the regularity of solutions to a linear backward-in-time biharmonic problem with Voigt-type damping.

Lemma 5.11. *Consider the initial-boundary value problem*

$$\left[\begin{array}{l} -\partial_t(\widetilde{\vartheta}-\tau\Delta\widetilde{\vartheta})+b\Delta^2\widetilde{\vartheta}=\widetilde{g}_{\mathrm{h}} \text{ in } \Omega_T,\\ \partial_{\boldsymbol{n}}\widetilde{\vartheta}=\partial_{\boldsymbol{n}}\Delta\widetilde{\vartheta}=0 \text{ on } \Gamma_T, \quad \widetilde{\vartheta}(T)-\tau\Delta\widetilde{\vartheta}(T)=\widetilde{\vartheta}_T \text{ in } \Omega. \end{array}\right.$$

Let $k=0,1$. If $\widetilde{g}_{\mathrm{h}} \in L^2(I;W^{k-1,2}(\Omega))$ and $\widetilde{\vartheta}_T \in W^{k,2}(\Omega)$, then

$$\widetilde{\vartheta} \in W^{1,2,2}(I;W^{k+3,2}_{\boldsymbol{n}}(\Omega),W^{k+1,2}_{\boldsymbol{n}}(\Omega))$$

and we have

$$\|\widetilde{\vartheta}\|_{W^{1,2,2}(W^{k+3,2}_{\boldsymbol{n}},W^{k+1,2}_{\boldsymbol{n}})} \leq c(\|\widetilde{g}_{\mathrm{h}}\|_{L^2(W^{k-1,2})}+\|\widetilde{\vartheta}_T\|_{W^{k,2}}).$$

Proof. Taking the test function $\widetilde{\vartheta}-\tau\Delta\widetilde{\vartheta}$ and applying Young's inequality, one has the a priori estimate

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega|\widetilde{\vartheta}|^2+2\tau|\nabla\widetilde{\vartheta}|^2+\tau^2|\Delta\widetilde{\vartheta}|^2\,\mathrm{d}x$$

$$+\frac{b}{2}(\|\Delta\widetilde{\vartheta}\|^2_{L^2}+\tau\|\nabla\Delta\widetilde{\vartheta}\|^2_{\boldsymbol{L}^2}) \leq c(\|\widetilde{\vartheta}\|^2_{W^{1,2}}+\|\widetilde{g}_{\mathrm{h}}\|^2_{W^{-1,2}})$$

for some constant $c>0$. On the other hand, if we choose the test function $\Delta^2\widetilde{\vartheta}$, then we have

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega|\Delta\widetilde{\vartheta}|^2+\tau|\nabla\Delta\widetilde{\vartheta}|^2\,\mathrm{d}x+\frac{b}{2}\|\Delta^2\widetilde{\vartheta}\|^2_{L^2} \leq c\|\widetilde{g}_{\mathrm{h}}\|^2_{L^2}.$$

We can write the terminal condition as $\widetilde{\vartheta}(T)=B_N^{-1}\widetilde{\vartheta}_T$, and hence, we have the estimate $\|\widetilde{\vartheta}(T)\|_{W^{k+2,2}_{\boldsymbol{n}}} \leq \|B_N^{-1}\|_{\mathcal{L}(W^{k,2},W^{k+2,2}_{\boldsymbol{n}})}\|\widetilde{\vartheta}_T\|_{W^{k,2}}$ for $k=0,1$. Utilizing a standard Faedo–Galerkin approach, the above a priori estimates will lead to the conclusion of the lemma. □

Theorem 5.12. *Let $k = 0, 1$ and suppose that the conditions of Theorem 5.6, Remark 5.7, and Theorem 5.9 hold. If $g_{\rm dh} \in L^2(I; W^{k-1,2}(\Omega))$ and $\vartheta_T \in W^{k,2}(\Omega)$, then $\vartheta \in W^{1,2,2}(I; W^{k+3,2}_{\boldsymbol{n}}(\Omega), W^{k+1,2}_{\boldsymbol{n}}(\Omega))$.*

Proof. The theorem follows from Lemma 5.11 with $\widetilde{\vartheta}_T = \vartheta_T$ and

$$\widetilde{g}_{\rm h} = g_{\rm h} + \boldsymbol{v}\cdot\nabla\vartheta + \nabla\cdot(\chi(\phi,\theta)\nabla\vartheta) - \mathfrak{d}_\theta(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S})(\varphi,\vartheta,\boldsymbol{y},\mathbb{Y}) - \sigma\mathbb{S} : \mathbb{D}\boldsymbol{y} + b_{\rm h}\mathbf{g}\cdot\boldsymbol{y}.$$

To see this, it is enough to show that $\widetilde{g}_{\rm h} \in L^2(I; W^{k-1,2}(\Omega))$. Indeed, it is obvious that $b_{\rm h}\mathbf{g}\cdot\boldsymbol{y} \in L^2(I; W^{2,2}(\Omega))$. Also, $\boldsymbol{v}\cdot\nabla\vartheta$, $\sigma\mathbb{S} : \mathbb{D}\boldsymbol{y}$, $\nabla\cdot(\chi(\phi,\theta)\nabla\vartheta) \in L^2(I; L^2(\Omega))$ thanks to Remark 5.7 and the estimates

$$\begin{aligned}
&\|\boldsymbol{v}\cdot\nabla\vartheta\|_{L^2(L^2)} \le c\|\boldsymbol{v}\|_{L^\infty(\boldsymbol{W}^{1,2}_{0,\sigma})}\|\Delta\vartheta\|_{L^2(L^2)}\\
&\|\sigma\mathbb{S} : \mathbb{D}\boldsymbol{y}\|_{L^2(L^2)} \le c\|\mathbb{S}\|_{L^\infty(\mathbb{W}^{1,2}_{\rm s})}\|\boldsymbol{y}\|_{L^2(\boldsymbol{W}^{2,2}_{0,\sigma})}\\
&\|\nabla\cdot(\chi(\phi,\theta)\nabla\vartheta)\|_{L^2(L^2)} \le c(\|\phi\|_{L^\infty(W^{1,4})} + \|\theta\|_{L^\infty(W^{1,4})} + 1)\|\Delta\vartheta\|_{L^2(L^2)}.
\end{aligned}$$

Here, the last inequality was obtained by distributing the divergence and invoking the Hölder inequality.

To determine the regularity of $\mathfrak{d}_\theta(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S})(\varphi,\vartheta,\boldsymbol{y},\mathbb{Y})$, we first consider the case where $k = 0$. Then, $m_\theta(\phi,\theta)\nabla\mu\cdot\nabla\varphi \in L^2(I; L^{4/3}(\Omega)) \hookrightarrow L^2(I; W^{-1,2}(\Omega))$ since $\varphi \in L^\infty(I; W^{1,2}(\Omega))$ by Theorem 5.9 and

$$\|m_\theta(\phi,\theta)\nabla\mu\cdot\nabla\varphi\|_{L^2(L^{4/3})} \le c\|\nabla\mu\|_{L^2(\boldsymbol{L}^4)}\|\nabla\varphi\|_{L^\infty(\boldsymbol{L}^2)} \le c\|\mu\|_{L^2(W^{2,2}_{\boldsymbol{n}})}\|\varphi\|_{L^\infty(W^{1,2})}.$$

Recall from the proof of Theorem 5.6 that $\chi_\theta(\phi,\theta)\nabla\theta\cdot\nabla\vartheta$, $2\nu_\theta(\phi,\theta)\mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{y}$, $\varepsilon_\theta(\phi,\theta)\nabla\mathbb{S} \therefore \nabla\mathbb{Y} \in L^2(I; W^{-1,2}(\Omega))$. Consequently, we established that $\widetilde{g}_{\rm h} \in L^2(I; W^{-1,2}(\Omega))$ when $k = 0$.

Now, assume $k = 1$. Since $\varphi \in W^{1,2,2}(I; W^{4,2}_{\boldsymbol{n}}(\Omega), L^2(\Omega))$, we apply (3.1) along with the estimate

$$\|m_\theta(\phi,\theta)\nabla\mu\cdot\nabla\varphi\|_{L^2(L^2)} \le c\|\nabla\mu\|_{L^2(\boldsymbol{L}^4)}\|\nabla\varphi\|_{L^\infty(\boldsymbol{L}^4)} \le c\|\mu\|_{L^2(W^{2,2}_{\boldsymbol{n}})}\|\varphi\|_{L^\infty(W^{2,2}_{\boldsymbol{n}})}$$

to get $m_\theta(\phi,\theta)\nabla\mu\cdot\nabla\varphi \in L^2(I; L^2(\Omega))$. The rest of the terms appearing in $\mathfrak{d}_\theta(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S})(\varphi,\vartheta,\boldsymbol{y},\mathbb{Y})$ lie in $L^2(I; L^2(\Omega))$ by utilizing the same argument as in Theorem 5.9. Therefore, we have verified that $\widetilde{g}_{\rm h} \in L^2(I; L^2(\Omega))$ when $k = 1$. □

6. Applications to Optimal Control Problems

For the cost functional in the control problem (1.16), we shall consider quadratic tracking-type costs with the allowable regularity for the state components. Thanks to the square-integrability of the time-derivatives of the strong solution, we can include them in the cost functionals.

6.1. Cost Functionals with Time Derivatives and High Spatial Derivatives. First, we consider cost functionals involving the temporal derivatives, the biharmonic operator for the order parameter and the temperature, and the Laplacian of the chemical potential, velocity, and viscoelastic stress on the space-time domain. In particular, we define

$$J_{{\rm o},\Omega_T}(\phi) := \frac{1}{2}\int_{\Omega_T} \lambda_{{\rm do},4}|\Delta^2\phi - \phi_4|^2 + \lambda_{{\rm do},5}|\partial_t\phi - \phi_5|^2\,{\rm d}\omega,$$

$$J_{\mathrm{c},\Omega_T}(\mu) := \frac{1}{2}\int_{\Omega_T} \lambda_{d\mathrm{c},2}|\Delta\mu - \mu_2|^2 \,\mathrm{d}\omega$$
$$J_{\mathrm{h},\Omega_T}(\theta) := \frac{1}{2}\int_{\Omega_T} \lambda_{d\mathrm{h},4}|\Delta^2\theta - \theta_4|^2 + \lambda_{d\mathrm{h},5}|\partial_t\theta - \theta_5|^2 \,\mathrm{d}\omega$$
$$+ \frac{1}{2}\int_{\Omega_T} \lambda_{d\mathrm{h},6}|\nabla\partial_t\theta - \boldsymbol{\theta}_6|^2 + \lambda_{d\mathrm{h},7}|\Delta\partial_t\theta - \theta_7|^2 \,\mathrm{d}\omega$$
$$J_{\mathrm{v},\Omega_T}(\boldsymbol{v}) := \frac{1}{2}\int_{\Omega_T} \lambda_{d\mathrm{v},2}|\Delta\boldsymbol{v} - \boldsymbol{v}_2|^2 + \lambda_{d\mathrm{v},3}|\partial_t\boldsymbol{v} - \boldsymbol{v}_3|^2 \,\mathrm{d}\omega$$
$$J_{\mathrm{s},\Omega_T}(\mathbb{S}) := \frac{1}{2}\int_{\Omega_T} \lambda_{d\mathrm{s},2}|\Delta\mathbb{S} - \mathbb{S}_2|^2 + \frac{1}{2}\int_{\Omega_T} \lambda_{d\mathrm{s},3}|\partial_t\mathbb{S} - \mathbb{S}_3|^2 \,\mathrm{d}\omega.$$

With regard to the tracking-type functionals at the terminal time, we consider the following:

$$J_{\mathrm{o},\Omega}(\phi) := \frac{1}{2}\int_{\Omega} \lambda_{T\mathrm{o},2}|\Delta\phi(T) - \phi_{T,2}|^2 \,\mathrm{d}x$$
$$J_{\mathrm{c},\Omega}(\mu) := \frac{1}{2}\int_{\Omega} \lambda_{T\mathrm{c},0}|\mu(T) - \mu_T|^2 \,\mathrm{d}x$$
$$J_{\mathrm{h},\Omega}(\theta) := \frac{1}{2}\int_{\Omega} \lambda_{T\mathrm{h},3}|\nabla\Delta\theta(T) - \boldsymbol{\theta}_{T,3}|^2 \,\mathrm{d}x$$
$$J_{\mathrm{v},\Omega}(\boldsymbol{v}) := \frac{1}{2}\int_{\Omega} \lambda_{T\mathrm{v},1}|\nabla\boldsymbol{v}(T) - \mathbb{V}_{T,1}|^2 \,\mathrm{d}x$$
$$J_{\mathrm{s},\Omega}(\mathbb{S}) := \frac{1}{2}\int_{\Omega} \lambda_{T\mathrm{s},1}|\nabla\mathbb{S}(T) - \mathfrak{S}_{T,1}|^2 \,\mathrm{d}x.$$

The above λ parameters are assumed to be positive, unless stated otherwise. In practice, some of the parameters are set to zero depending on the goal of a particular problem. We suppose that the desired states are at least square-integrable either in Ω_T or Ω. For instance, with regard to the term involving the parameters $\lambda_{d\mathrm{o},4}$ and $\lambda_{T\mathrm{o},2}$, we have $\phi_4 \in L^2(I; L^2(\Omega))$ and $\phi_{T,2} \in L^2(\Omega)$. With these, the cost functionals are well-defined thanks to $(\varphi, \mu, \theta, \boldsymbol{v}, \mathbb{S}) \in \mathcal{W}^2(\Omega_T)$.

Let us write the reduced cost functional by

$$\begin{aligned}\mathcal{J}(\boldsymbol{u}) &:= \mathcal{G}(\varphi(\boldsymbol{u}), \mu(\boldsymbol{u}), \theta(\boldsymbol{u}), \boldsymbol{v}(\boldsymbol{u}), \mathbb{S}(\boldsymbol{u})) + \frac{\lambda_q}{2}\int_{\Omega_T} |\boldsymbol{u}|^2 \,\mathrm{d}\omega \\ &:= J_{\mathrm{o}}(\phi(\boldsymbol{u})) + J_{\mathrm{c}}(\mu(\boldsymbol{u})) + J_{\mathrm{h}}(\theta(\boldsymbol{u})) + J_{\mathrm{v}}(\boldsymbol{v}(\boldsymbol{u})) + J_{\mathrm{s}}(\mathbb{S}(\boldsymbol{u})) + \frac{\lambda_q}{2}\int_{\Omega_T} |\boldsymbol{u}|^2 \,\mathrm{d}\omega\end{aligned}$$

where $(\varphi(\boldsymbol{u}), \mu(\boldsymbol{u}), \theta(\boldsymbol{u}), \boldsymbol{v}(\boldsymbol{u}), \mathbb{S}(\boldsymbol{u})) = \mathcal{T}(\boldsymbol{u})$ and $J_{\mathrm{k}} := J_{\mathrm{k},\Omega_T} + J_{\mathrm{k},\Omega}$ for $\mathrm{k} = \mathrm{o}, \mathrm{c}, \mathrm{h}, \mathrm{v}, \mathrm{s}$. An element $\boldsymbol{u} \in \boldsymbol{U}$ is called a *global minimizer* of $\mathcal{J}$ if $\mathcal{J}(\boldsymbol{u}) \le \mathcal{J}(\boldsymbol{z})$ for all $\boldsymbol{z} \in \boldsymbol{U}$.

Using classical sequential compactness argument, the following existence of optimal controls can be established. In particular, utilizing the weak-weak continuity of the control-to-state operator $\mathcal{T} : \boldsymbol{U} \to \mathcal{W}^2(\Omega_T)$ as presented in Theorem 4.1, the following theorem can be shown. As this is now standard in the field, we do not provide the details here and refer the reader to [66] on how to establish this theorem. Here, we are mainly interested in the regularity of the optimal solutions.

Theorem 6.1. *The optimal control problem* (1.16) *has a solution, that is, there is at least one global minimizer* $\boldsymbol{u} \in \boldsymbol{U}$ *of* $\mathcal{J}$*. Moreover, we have* $\boldsymbol{u} = -\lambda_q^{-1}\boldsymbol{y}$ *almost everywhere in* Ω_T*.*

Given $s \geq 0$, we define $\partial_t : L^2(I; W^{-s,2}(\Omega)) \to W_0^{-1,2}(I; W^{-s,2}(\Omega))$ by duality, that is,

$$\langle \partial_t h, g\rangle_{W_0^{-1,2}(W^{-s,2}),W_0^{1,2}(W^{s,2})} := -\int_I \langle h, \partial_t g\rangle_{W^{-s,2},W^{s,2}}\,\mathrm{d}t,$$

for $(h,g) \in L^2(I; W^{-s,2}(\Omega)) \times W_0^{1,2}(I; W^{s,2}(\Omega))$. By abuse of notation, we also use the same notation for the distributional time-derivative when $W^{s,2}(\Omega)$ is replaced by $W_{\boldsymbol{n}}^{s,2}(\Omega)$, $\boldsymbol{W}_{0,\sigma}^{s,2}(\Omega)$, or $\mathbb{W}_{\boldsymbol{n},\mathrm{s}}^{s,2}(\Omega)$.

With respect to the reduced cost functional $\mathcal{J}$ defined above, the right-hand sides of the adjoint problem (5.7) are given by

$$\begin{aligned}
g_{\mathrm{do}} &= \lambda_{\mathrm{do},4}(\Delta^4\phi - \Delta^2\phi_4) - \lambda_{\mathrm{do},5}\partial_t(\partial_t\phi - \phi_5)\\
g_{\mathrm{c}} &= \lambda_{\mathrm{dc},2}(\Delta^2\mu - \Delta\mu_2)\\
g_{\mathrm{dh}} &= \lambda_{\mathrm{dh},4}(\Delta^4\theta - \Delta^2\theta_4) - \lambda_{\mathrm{dh},5}\partial_t(\partial_t\theta - \theta_5)\\
&\quad + \lambda_{\mathrm{dh},6}\partial_t(\Delta\partial_t\theta - \nabla\cdot\boldsymbol{\theta}_6) - \lambda_{\mathrm{dh},7}\partial_t(\Delta^2\partial_t\theta - \Delta\theta_7)\\
\boldsymbol{g}_{\mathrm{dv}} &= \lambda_{\mathrm{dv},2}(\Delta^2\boldsymbol{v} - \Delta\boldsymbol{v}_2) - \lambda_{\mathrm{dv},3}\partial_t(\partial_t\boldsymbol{v} - \boldsymbol{v}_3)\\
\mathbb{G}_{ds} &= \lambda_{ds,2}(\Delta^2\mathbb{S} - \Delta\mathbb{S}_2) - \lambda_{ds,3}\partial_t(\partial_t\mathbb{S} - \mathbb{S}_3).
\end{aligned}$$

On the other hand, the terminal data for (5.7) are given by

$$\begin{aligned}
\varphi_T &= \lambda_{T\mathrm{o},2}(\Delta^2\phi(T) - \Delta\phi_{T,2})\\
&\quad + \lambda_{T\mathrm{c},0}[F''(\phi(T))(\mu(T) - \mu_T) - \alpha(\Delta\mu(T) - \Delta\mu_T)]
\end{aligned}$$

$\vartheta_T = -\lambda_{T\mathrm{h},3}(\Delta^4\theta(T) - \Delta\nabla\cdot\boldsymbol{\theta}_{T,3})$, $\boldsymbol{y}_T = -\lambda_{T\mathrm{v},1}(\Delta\boldsymbol{v}(T) - \nabla\cdot\mathbb{V}_{T,1})$, and $\mathbb{Y}_T = -\lambda_{T\mathrm{s},1}(\Delta\mathbb{S}(T) - \nabla\cdot\mathfrak{S}_{T,1})$. Here, the second term in φ_T is due to $\xi = -\alpha\Delta\psi + F''(\phi)\psi$ and the following computation:

$$\begin{aligned}
&\int_{\Omega_T} \lambda_{T\mathrm{c},0}(\mu(T) - \mu_T)\xi(T)\,\mathrm{d}\omega\\
&\quad = \int_{\Omega_T} \lambda_{T\mathrm{c},0}(\mu(T) - \mu_T)(-\alpha\Delta\psi(T) + F''(\phi(T))\psi(T))\,\mathrm{d}\omega\\
&\quad = \int_{\Omega_T} \lambda_{T\mathrm{c},0}F''(\phi(T))(\mu(T) - \mu_T)\psi(T)\,\mathrm{d}\omega\\
&\qquad - \langle \lambda_{T\mathrm{c},0}\alpha(\Delta\mu(T) - \Delta\mu_T), \psi(T)\rangle_{W_{\boldsymbol{n}}^{-2,2},W_{\boldsymbol{n}}^{2,2}}.
\end{aligned}$$

In particular, by the chain rule, we have

$$\begin{aligned}
&\mathcal{G}'(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S})(\psi,\xi,\eta,\boldsymbol{w},\mathbb{T}) = \langle (g_{\mathrm{do}}, g_{\mathrm{c}}, g_{\mathrm{dh}}, \boldsymbol{g}_{\mathrm{dv}}, \mathbb{G}_{\mathrm{ds}}), (\psi,\xi,\eta,\boldsymbol{w},\mathbb{T})\rangle_{(\mathcal{V}_0^2)^*,\mathcal{V}_0^2}\\
&\quad + \langle (\varphi_T, \vartheta_T, \boldsymbol{y}_T, \mathbb{Y}_T), (\psi(T), \eta(T), \boldsymbol{w}(T), \mathbb{T}(T))\rangle_{(\mathcal{D}^2)^*,\mathcal{D}^2}.
\end{aligned}$$

Thanks to the regularity of the solution to the state equation (see Theorem 3.1), we have $g_{\mathrm{do}} \in L^2(I; W_{\boldsymbol{n}}^{-4,2}(\Omega)) + W_0^{-1,2}(I; L^2(\Omega))$, $g_{\mathrm{c}} \in L^2(I; W_{\boldsymbol{n}}^{-2,2}(\Omega))$, $\boldsymbol{g}_{\mathrm{dv}} \in L^2(I; \boldsymbol{W}_{0,\sigma}^{-2,2}(\Omega)) + W_0^{-1,2}(I; \boldsymbol{L}_\sigma^2(\Omega))$, and $\mathbb{G}_{ds} \in L^2(I; \mathbb{W}_{\boldsymbol{n},\mathrm{s}}^{-2,2}(\Omega)) + W_0^{-1,2}(I; \mathbb{L}_{\mathrm{s}}^2(\Omega))$.

Likewise, we have $g_{dh} \in L^2(I; W_{\boldsymbol{n}}^{-4,2}(\Omega)) + W_0^{-1,2}(I; W_{\boldsymbol{n}}^{\ell,2}(\Omega))$, where

$$\ell := \begin{cases} -2 & \text{if } \lambda_{dh,7} > 0, \\ -1 & \text{if } \lambda_{dh,7} = 0, \lambda_{dh,6} > 0, \\ 0 & \text{if } \lambda_{dh,7} = 0, \lambda_{dh,6} > 0, \nabla \cdot \boldsymbol{\theta}_6 \in L^2(I; L^2(\Omega)) \\ & \quad \text{or } \lambda_{dh,7} = \lambda_{dh,6} = 0, \lambda_{dh,5} > 0, \\ 1 & \text{if } \lambda_{dh,7} = \lambda_{dh,6} = 0, \lambda_{dh,5} > 0, \theta_5 \in L^2(I; W^{1,2}(\Omega)), \\ 2 & \text{if } \lambda_{dh,7} = \lambda_{dh,6} = 0, \lambda_{dh,5} > 0, \theta_5 \in L^2(I; W_{\boldsymbol{n}}^{2,2}(\Omega)). \end{cases}$$

For the terminal data, we have $(\varphi_T, \vartheta_T, \boldsymbol{y}_T, \mathbb{Y}_T) \in \mathcal{D}^2(\Omega)^*$. Thus, it follows that the variational solution to the adjoint problem (5.7) satisfies the conclusion of Theorem 5.2, with the following function spaces:

$$\begin{aligned} X_{\mathrm{o}}^2(\Omega_T) &= L^2(I; W_{\boldsymbol{n}}^{4,2}(\Omega)) \cap W_0^{1,2}(I; L^2(\Omega)) \\ X_{\mathrm{h}}^2(\Omega_T) &= L^2(I; W_{\boldsymbol{n}}^{4,2}(\Omega)) \cap W_0^{1,2}(I; W_{\boldsymbol{n}}^{-\ell,2}(\Omega)) \\ \boldsymbol{X}_{\mathrm{v}}^2(\Omega_T) &= L^2(I; \boldsymbol{W}_{0,\sigma}^{2,2}(\Omega)) \cap W_0^{1,2}(I; \boldsymbol{L}_\sigma^2(\Omega)) \\ \mathbb{X}_{\mathrm{s}}^2(\Omega_T) &= L^2(I; \mathbb{W}_{\boldsymbol{n},\mathrm{s}}^{2,2}(\Omega)) \cap W_0^{1,2}(I; \mathbb{L}_{\mathrm{s}}^2(\Omega)). \end{aligned}$$

In particular, it holds that $\mathcal{V}_0^2(\Omega_T) \hookrightarrow \mathcal{X}^2(\Omega_T)$ from (5.9) and

$$(B_N X_{\mathrm{h}}^2(\Omega_T))^* = L^2(I; W_{\boldsymbol{n}}^{-2,2}(\Omega)) + W_0^{-1,2}(I; W_{\boldsymbol{n}}^{\ell+2,2}(\Omega)).$$

If the coefficients involving the time derivatives vanish, that is, $\lambda_{do,5} = \lambda_{dh,5} = \lambda_{dh,6} = \lambda_{dh,7} = \lambda_{dv,3} = \lambda_{ds,3} = 0$, then Remark 5.3 applies. As a result, we have $\boldsymbol{u} \in W^{1,2,2}(I; \boldsymbol{L}_\sigma^2(\Omega), \boldsymbol{W}_{0,\sigma}^{-2,2}(\Omega)) \hookrightarrow C(\bar{I}; \boldsymbol{W}_{0,\sigma}^{-1,2}(\Omega))$ by Theorem 6.1. Thus, the required time-regularity and initial-value of the control $\boldsymbol{u}$ given in Theorem A.4 for $k = 0$ holds. If the other conditions for the sources and initial conditions for the state system are satisfied as well for $k = 0$ in Theorem A.4, then (A.145) for $k = 0$ is attained by the optimal state. In particular, $\phi \in W^{1,2,2}(I; W^{4,2}(\Omega), W^{2,2}(\Omega)) \hookrightarrow L^\infty(I; W^{3,2}(\Omega)) \hookrightarrow L^\infty(I; W^{1,\infty}(\Omega))$. Recall that this is the additional regularity for ϕ needed in the proof of Theorem 5.9.

6.2. Cost Functionals with Low Spatial Derivatives. Now, let us consider the following cost functionals without the temporal derivatives and with lower spatial derivatives for the states:

$$\begin{aligned} J_{\mathrm{o},\Omega_T}(\phi) &:= \frac{1}{2} \int_{\Omega_T} \lambda_{do,0} |\phi - \phi^0|^2 + \lambda_{do,1} |\nabla \phi - \boldsymbol{\phi}_1|^2 \, \mathrm{d}\omega \\ &\quad + \frac{1}{2} \int_{\Omega_T} \lambda_{do,2} |\Delta \phi - \phi_2|^2 + \lambda_{do,3} |\nabla \Delta \phi - \boldsymbol{\phi}_3|^2 \, \mathrm{d}\omega \\ J_{\mathrm{c},\Omega_T}(\mu) &:= \frac{1}{2} \int_{\Omega_T} \lambda_{dc,0} |\mu - \mu^0|^2 + \lambda_{dc,1} |\nabla \mu - \boldsymbol{\mu}_1|^2 \, \mathrm{d}\omega \\ J_{\mathrm{h},\Omega_T}(\theta) &:= \frac{1}{2} \int_{\Omega_T} \lambda_{dh,0} |\theta - \theta^0|^2 + \lambda_{dh,1} |\nabla \theta - \boldsymbol{\theta}_1|^2 \, \mathrm{d}\omega \\ &\quad + \frac{1}{2} \int_{\Omega_T} \lambda_{dh,2} |\Delta \theta - \theta_2|^2 + \lambda_{dh,3} |\nabla \Delta \theta - \boldsymbol{\theta}_3|^2 \, \mathrm{d}\omega \end{aligned}$$

$$J_{\mathrm{v},\Omega_T}(\boldsymbol{v}) := \frac{1}{2}\int_{\Omega_T} \lambda_{dv,0}|\boldsymbol{v}-\boldsymbol{v}^0|^2 + \lambda_{dv,1}|\nabla\boldsymbol{v}-\mathbb{V}_1|^2 \,\mathrm{d}\omega$$
$$J_{\mathrm{s},\Omega_T}(\mathbb{S}) := \frac{1}{2}\int_{\Omega_T} \lambda_{ds,0}|\mathbb{S}-\mathbb{S}^0|^2 + \lambda_{ds,1}|\nabla\mathbb{S}-\mathfrak{S}_1|^2 \,\mathrm{d}\omega.$$

At the terminal time, we set $J_{\mathrm{c},\Omega}(\mu) := 0$ and

$$\begin{aligned}
J_{\mathrm{o},\Omega}(\phi) &:= \frac{1}{2}\int_{\Omega} \lambda_{T\mathrm{o},0}|\phi(T)-\phi_{T,0}|^2 + \lambda_{T\mathrm{o},1}|\nabla\phi(T)-\boldsymbol{\phi}_{T,1}|^2 \,\mathrm{d}x\\
J_{\mathrm{h},\Omega}(\theta) &:= \frac{1}{2}\int_{\Omega} \lambda_{T\mathrm{h},0}|\theta(T)-\theta_{T,0}|^2 + \lambda_{T\mathrm{h},1}|\nabla\theta(T)-\boldsymbol{\theta}_{T,1}|^2 \,\mathrm{d}x\\
&\quad + \frac{1}{2}\int_{\Omega} \lambda_{T\mathrm{h},2}|\Delta\theta(T)-\theta_{T,2}|^2 \,\mathrm{d}x\\
J_{\mathrm{v},\Omega}(\boldsymbol{v}) &:= \frac{1}{2}\int_{\Omega} \lambda_{T\mathrm{v},0}|\boldsymbol{v}(T)-\boldsymbol{v}_{T,0}|^2 \,\mathrm{d}x\\
J_{\mathrm{s},\Omega}(\mathbb{S}) &:= \frac{1}{2}\int_{\Omega} \lambda_{T\mathrm{s},0}|\mathbb{S}(T)-\mathbb{S}_{T,0}|^2 \,\mathrm{d}x.
\end{aligned}$$

Once again, the λ parameters are assumed to be positive and the target states are at least square integrable, unless stated otherwise. The discussion below can be easily adjusted to the case where time-derivatives appear in the cost functionals as in the previous subsection.

Theorem 6.1 also holds in the case of the above cost functionals. Here, the right-hand sides of the adjoint system (5.7) are given by

$$\begin{aligned}
g_{d\mathrm{o}} &= \lambda_{d\mathrm{o},0}(\phi-\phi^0) - \lambda_{d\mathrm{o},1}(\Delta\phi - \nabla\cdot\boldsymbol{\phi}_1)\\
&\quad + \lambda_{d\mathrm{o},2}(\Delta^2\phi - \Delta\phi_2) - \lambda_{d\mathrm{o},3}(\Delta^3\phi - \Delta\nabla\cdot\boldsymbol{\phi}_3)\\
g_{\mathrm{c}} &= \lambda_{d\mathrm{c},0}(\mu-\mu^0) - \lambda_{d\mathrm{c},1}(\Delta\mu - \nabla\cdot\boldsymbol{\mu}_1)\\
g_{d\mathrm{h}} &= \lambda_{d\mathrm{h},0}(\theta-\theta^0) - \lambda_{d\mathrm{h},1}(\Delta\theta - \nabla\cdot\boldsymbol{\theta}_1)\\
&\quad + \lambda_{d\mathrm{h},2}(\Delta^2\theta - \Delta\theta_2) - \lambda_{d\mathrm{h},3}(\Delta^3\theta - \Delta\nabla\cdot\boldsymbol{\theta}_3)\\
\boldsymbol{g}_{d\mathrm{v}} &= \lambda_{d\mathrm{v},0}(\boldsymbol{v}-\boldsymbol{v}^0) - \lambda_{d\mathrm{v},1}(\Delta\boldsymbol{v} - \nabla\cdot\mathbb{V}_1)\\
\mathbb{G}_{ds} &= \lambda_{ds,0}(\mathbb{S}-\mathbb{S}^0) - \lambda_{ds,1}(\Delta\mathbb{S} - \nabla\cdot\mathfrak{S}_1)
\end{aligned}$$

while the terminal data are as follows:

$$\begin{aligned}
\varphi_T &= \lambda_{T\mathrm{o},0}(\phi(T)-\phi_{T,0}) - \lambda_{T\mathrm{o},1}(\Delta\phi(T) - \nabla\cdot\boldsymbol{\phi}_{T,1})\\
\vartheta_T &= \lambda_{T\mathrm{h},0}(\theta(T)-\theta_{T,0}) - \lambda_{T\mathrm{h},1}(\Delta\theta(T) - \nabla\cdot\boldsymbol{\theta}_{T,1}) + \lambda_{T\mathrm{h},2}(\Delta^2\theta(T) - \Delta\theta_{T,2})\\
\boldsymbol{y}_T &= \lambda_{T\mathrm{v},0}(\boldsymbol{v}(T)-\boldsymbol{v}_{T,0}), \qquad \mathbb{Y}_T = \lambda_{T\mathrm{s},0}(\mathbb{S}(T)-\mathbb{S}_{T,0}).
\end{aligned}$$

It holds that $(\varphi_T, \vartheta_T, \boldsymbol{y}_T, \mathbb{Y}_T) \in \mathcal{D}^1(\Omega)^*$, so that the conclusion of Remark 5.5 is satisfied where we have $X^1_{\mathrm{o}}(\Omega_T)^* = L^2(I; W^{-3,2}_{\boldsymbol{n}}(\Omega))$, $(B_N X^1_{\mathrm{h}}(\Omega_T))^* = L^2(I; W^{-1,2}(\Omega))$, $\boldsymbol{X}^1_{\mathrm{v}}(\Omega_T)^* = L^2(I; \boldsymbol{W}^{-1,2}_{0,\sigma}(\Omega))$, and $\mathbb{X}^1_{\mathrm{s}}(\Omega_T)^* = L^2(I; \mathbb{W}^{-1,2}_{\mathrm{s}}(\Omega))$. Thus, for the optimal control, one has $\boldsymbol{u} \in W^{1,2,2}(I; \boldsymbol{W}^{1,2}_{0,\sigma}(\Omega), \boldsymbol{W}^{-1,2}_{0,\sigma}(\Omega)) \hookrightarrow C(\bar{I}; \boldsymbol{L}^2_\sigma(\Omega))$. The time-regularity and initial-value of the control $\boldsymbol{u}$ required by Theorem A.4 is verified for $k = 1$. As a result, (A.145) also holds for $k = 1$, provided that all assumptions on the sources and the initial data in the state system are also achieved.

If $\nabla\cdot\boldsymbol{\phi}_3, \nabla\cdot\boldsymbol{\mu}_1$, $\nabla\cdot\boldsymbol{\theta}_3 \in L^2(I;L^2(\Omega))$, $\boldsymbol{v}^0, \nabla\cdot\mathbb{V}_1 \in L^2(I;\boldsymbol{L}^2(\Omega))$, $\mathbb{S}^0, \nabla\cdot\mathfrak{G}_1 \in L^2(I;\mathbb{L}^2_{\rm s}(\Omega))$, $\nabla\cdot\boldsymbol{\phi}_{T,1} \in L^2(\Omega)$, $\Delta\theta_{T,2} \in W^{-1,2}(\Omega)$, $\boldsymbol{v}_{T,0} \in \boldsymbol{W}^{1,2}_{0,\sigma}(\Omega)$, and $\mathbb{S}_{T,0} \in \mathbb{W}^{1,2}_{\rm s}(\Omega)$, then $(\varphi_T, \vartheta_T, \boldsymbol{y}_T, \mathbb{Y}_T) \in \mathcal{D}^0(\Omega)^*$. Hence, Remark 5.7 applies to the solution of the adjoint system. Here, $X^0_{\rm o}(\Omega_T)^* = L^2(I;W^{-2,2}_{\boldsymbol{n}}(\Omega))$, $(B_N X^0_{\rm h}(\Omega_T))^* = L^2(I;L^2(\Omega))$, $\boldsymbol{X}^0_{\rm v}(\Omega_T)^* = L^2(I;\boldsymbol{L}^2(\Omega))$, and $\mathbb{X}^0_{\rm s}(\Omega_T)^* = L^2(I;\mathbb{L}^2_{\rm s}(\Omega))$. Therefore, $\boldsymbol{u} \in W^{1,2,2}(I;\boldsymbol{W}^{2,2}_{0,\sigma}(\Omega), \boldsymbol{L}^2(\Omega)) \hookrightarrow C(\bar{I};\boldsymbol{W}^{1,2}_{0,\sigma}(\Omega))$. Once again, (A.145) also holds for $k=2$ as long as the conditions for the sources and the initial data in the state system are fulfilled.

Suppose that $\lambda_{do,3} = \lambda_{dc,1} = \lambda_{To,1} = 0$, $\nabla\cdot\boldsymbol{\phi}_1, \Delta\phi_2 \in L^2(I;W^{k-1,2}(\Omega))$, $\mu^0 \in L^2(I;W^{k+1,2}(\Omega))$, and $\phi_{T,0} \in W^{k+1,2}_{\boldsymbol{n}}(\Omega)$ for $k=0,1$. Then, Theorem 5.9 is valid; see also the last statement of the previous subsection. If, in addition, $\lambda_{dh,3} = \lambda_{Th,2} = 0$, $\nabla\cdot\boldsymbol{\theta}_1, \Delta\theta_2 \in L^2(I;W^{k-1}(\Omega))$, and $\theta_{T,0}, \nabla\cdot\boldsymbol{\theta}_{T,1} \in W^{k,2}(\Omega)$ for $k=0,1$, then the result of Theorem 5.12 is attained. Finally, if $\mu^0 \in W^{1,2,2}(I;W^{2,2}(\Omega), W^{-2,2}_{\boldsymbol{n}}(\Omega))$, then Corollary 5.10 applies.

Remark 6.2. Let us mention the cases where the control is present in either of the evolution equation governing the concentration ϕ or the temperature θ. If $f_{\rm o}$ is replaced by $f_{\rm o}+u$ in (1.15) with $\boldsymbol{u}=\boldsymbol{0}$ and $u \in L^2(\Omega_T)$, then the optimal control is given by $u = -\lambda_q^{-1}\varphi$. Similarly, in the case where $f_{\rm h}$ is replaced by $f_{\rm h}+u$, the optimal control satisfies $u = -\lambda_q^{-1}\vartheta$. In either case, the regularity of the control is the same as that of the dual of the order parameter or the dual temperature.

A. Solutions to the Linearized System

We give a comprehensive proof of Theorem 4.2 on the existence, uniqueness, and stability of weak solutions (Appendix A.1), very weak solutions (Appendix A.2), and strong solutions (Appendix A.3) to the linearized system (4.2). The type of solutions depends on the regularity of the source terms and initial data in (4.2). We then apply Theorem 4.2 to deduce solutions with higher time-differentiability for the nonlinear system (1.15) under suitable smoothness conditions on the source terms and initial data in Appendix A.4. To establish Theorem 4.2, we will first address weak solutions, followed by very weak solutions using a density argument. The case of strong solutions follows a similar approach as for the nonlinear system (1.15).

A.1. Weak Solutions to the Linearized System. Given source functions and initial data

$$(h_{\rm o}, h_{\rm c}, h_{\rm h}, \boldsymbol{h}_{\rm v}, \mathbb{H}_{\rm s}) \in \mathcal{U}^1(\Omega_T)^*, \qquad (\psi_0, \eta_0, \boldsymbol{w}_0, \mathbb{T}_0) \in \mathcal{D}^1(\Omega), \tag{A.1}$$

a quintuple $(\psi, \xi, \eta, \boldsymbol{w}, \mathbb{T}) \in \mathcal{V}^1(\Omega_T)$ is called a *weak solution* to (4.2) if the following variational equations are satisfied

$$\begin{aligned} &\langle \partial_t\psi, \varphi\rangle_{W^{-1,2},W^{1,2}} + \int_\Omega (\boldsymbol{w}\cdot\nabla\phi + \boldsymbol{v}\cdot\nabla\psi)\varphi\,\mathrm{d}x \\ &\quad + \int_\Omega [m'(\phi,\theta)(\psi,\eta)\nabla\mu + m(\phi,\theta)\nabla\xi]\cdot\nabla\varphi\,\mathrm{d}x = \langle h_{\rm o}, \varphi\rangle_{W^{-1,2},W^{1,2}} \end{aligned} \tag{A.2}$$

$$\begin{aligned}
&\int_\Omega (\partial_t \eta)\vartheta + \tau \nabla \partial_t \eta \cdot \nabla \vartheta \,\mathrm{d}x + \int_\Omega (\boldsymbol{w} \cdot \nabla \theta + \boldsymbol{v} \cdot \nabla \eta)\vartheta \,\mathrm{d}x \\
&\quad + \int_\Omega [\chi'(\phi,\theta)(\psi,\eta)\nabla\theta + \chi(\phi,\theta)\nabla\eta] \cdot \nabla\vartheta \,\mathrm{d}x - \int_\Omega b\nabla\Delta\eta \cdot \nabla\vartheta \,\mathrm{d}x \\
&\quad - \int_\Omega (\mathbb{T} : \mathbb{D}\boldsymbol{v} + \mathbb{S} : \mathbb{D}\boldsymbol{w})\vartheta \,\mathrm{d}x - \int_\Omega a_0 \mathbf{g} \cdot \boldsymbol{w}\vartheta \,\mathrm{d}x = \langle h_{\mathrm{h}}, \vartheta\rangle_{W^{-1,2},W^{1,2}} && \text{(A.3)}\\
&\langle \partial_t \boldsymbol{w}, \boldsymbol{y}\rangle_{\boldsymbol{W}^{-1,2}_{0,\sigma},\boldsymbol{W}^{1,2}_{0,\sigma}} + \int_\Omega [(\boldsymbol{w}\cdot\nabla)\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{w}] \cdot \boldsymbol{y} \,\mathrm{d}x \\
&\quad + \int_\Omega [2\nu'(\phi,\theta)(\psi,\eta)\mathbb{D}\boldsymbol{v} + 2\nu(\phi,\theta)\mathbb{D}\boldsymbol{w}] : \mathbb{D}\boldsymbol{y} \,\mathrm{d}x \\
&\quad + \int_\Omega (\sigma\eta\mathbb{S} + \mathbb{M}_{\mathbb{S}}(\theta,\mathbb{S})\mathbb{T}) : \mathbb{D}\boldsymbol{y} \,\mathrm{d}x - \int_\Omega \kappa(\xi\nabla\phi + \mu\nabla\psi) \cdot \boldsymbol{y} \,\mathrm{d}x \\
&\quad - \int_\Omega (b_{\mathrm{o}}\psi + b_{\mathrm{h}}\eta)\mathbf{g} \cdot \boldsymbol{y} \,\mathrm{d}x = \langle \boldsymbol{h}_{\mathrm{v}}, \boldsymbol{y}\rangle_{\boldsymbol{W}^{-1,2}_{0,\sigma},\boldsymbol{W}^{1,2}_{0,\sigma}} && \text{(A.4)}\\
&\langle \partial_t \mathbb{T}, \mathbb{Y}\rangle_{\mathbb{W}^{-1,2}_{\mathrm{s}},\mathbb{W}^{1,2}_{\mathrm{s}}} + \int_\Omega [(\boldsymbol{w}\cdot\nabla)\mathbb{S} + (\boldsymbol{v}\cdot\nabla)\mathbb{T}] : \mathbb{Y} \,\mathrm{d}x \\
&\quad + \int_\Omega [\varepsilon'(\phi,\theta)(\psi,\eta)\nabla\mathbb{S} + \varepsilon(\phi,\theta)\nabla\mathbb{T}] \therefore \nabla\mathbb{Y} \,\mathrm{d}x \\
&\quad + \int_\Omega [\mathbb{J}(\boldsymbol{w},\mathbb{S}) + \mathbb{J}(\boldsymbol{v},\mathbb{T})] : \mathbb{Y} \,\mathrm{d}x - \int_\Omega [\lambda\mathbb{D}\boldsymbol{w} + \mathbb{P}'(\mathbb{S})\mathbb{T}] : \mathbb{Y} \,\mathrm{d}x \\
&\quad = \langle \mathbb{H}_{\mathrm{s}}, \mathbb{Y}\rangle_{\mathbb{W}^{-1,2}_{\mathrm{s}},\mathbb{W}^{1,2}_{\mathrm{s}}} && \text{(A.5)}
\end{aligned}$$

almost everywhere in I for every test function $(\varphi, \vartheta, \boldsymbol{y}, \mathbb{Y}) \in W^{1,2}(\Omega) \times W^{1,2}(\Omega) \times \boldsymbol{W}^{1,2}_{0,\sigma}(\Omega) \times \mathbb{W}^{1,2}(\Omega)$, the equation $\xi + \alpha\Delta\psi - F''(\phi)\psi = h_{\mathrm{c}}$ holds in $L^2(I; W^{1,2}(\Omega))$, and the initial conditions in (4.2) are satisfied in $\mathcal{D}^1(\Omega)$.

The existence of solutions will follow once we derive a priori estimates for the Faedo–Galerkin approximations. Hence, it suffices to formally derive the a priori estimates necessary for weak sequential arguments. We split the derivation in several steps. We proceed in such a way that some of the steps can be carried out in the very weak formulation of the linearized system. To simplify the final estimates in each step, we denote by $K_j : I \to [0, \infty)$ a generic function depending on the spatial norms of a strong solution $(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S})$ such that $K_j = K_j(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S}) \in L^1(I)$ and

$$\|K_j\|_{L^1(I)} \le C(\|(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S})\|_{\mathcal{W}^2}). \tag{A.6}$$

In each appearance of K_j below, this integrability condition can be verified with the help of the continuous embeddings (3.1)–(3.10).

Let us now derive the a priori estimates needed for the existence of weak solutions.

Estimates for ψ in $L^\infty(L^2) \cap L^2(W^{2,2}_{\boldsymbol{n}})$ and ξ in $L^2(L^2)$. Testing (A.2) with $\varphi = \psi$ yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |\psi|^2 \,\mathrm{d}x + \int_\Omega (\boldsymbol{w}\cdot\nabla\phi)\psi \,\mathrm{d}x + \int_\Omega m'(\phi,\theta)(\psi,\eta)\nabla\mu\cdot\nabla\psi \,\mathrm{d}x \\
+ \int_\Omega m(\phi,\theta)\nabla\xi\cdot\nabla\psi \,\mathrm{d}x = \langle h_{\mathrm{o}}, \psi\rangle_{W^{-1,2},W^{1,2}}.
\end{aligned} \tag{A.7}$$

Utilizing the Hölder and Young inequalities to the second and third integrals, the Cauchy–Schwarz inequality to the right-hand side, and using the estimate (2.3), one has

$$|\langle h_{\mathrm{o}}, \psi\rangle_{W^{-1,2},W^{1,2}}| \le c(\|h_{\mathrm{o}}\|_{W^{-1,2}}^2 + \|\psi\|_{W^{1,2}}^2) \tag{A.8}$$

$$\int_\Omega |(\boldsymbol{w}\cdot\nabla\phi)\psi|\,\mathrm{d}x \le \delta_0\|\boldsymbol{w}\|_{\boldsymbol{L}^2_\sigma}^2 + c_{\delta_0}\|\nabla\phi\|_{\boldsymbol{L}^\infty}^2\|\psi\|_{L^2}^2 \tag{A.9}$$

$$\begin{aligned}\int_\Omega |m'(\phi,\theta)(\psi,\eta)\nabla\mu\cdot\nabla\psi|\,\mathrm{d}x &\le (|m_\phi|_\infty\|\psi\|_{L^2} + |m_\theta|_\infty\|\eta\|_{L^2})\|\nabla\mu\|_{\boldsymbol{L}^4}\|\nabla\psi\|_{\boldsymbol{L}^4}\\ &\le \delta\|\Delta\psi\|_{L^2}^2 + c_\delta\|\nabla\mu\|_{\boldsymbol{L}^4}^2(\|\psi\|_{L^2}^2 + \|\eta\|_{L^2}^2).\end{aligned} \tag{A.10}$$

Integrating by parts and using the equation for ξ in (4.2), the fourth integral in (A.7) can be written as

$$\begin{aligned}\int_\Omega m(\phi,\theta)\nabla\xi\cdot\nabla\psi\,\mathrm{d}x &= -\int_\Omega \xi\,\nabla\cdot(m(\phi,\theta)\nabla\psi)\,\mathrm{d}x\\ &= -\int_\Omega \xi[(m_\phi(\phi,\theta)\nabla\phi + m_\theta(\phi,\theta)\nabla\theta)\cdot\nabla\psi + m\Delta\psi]\,\mathrm{d}x\\ &= -\int_\Omega \xi(m_\phi(\phi,\theta)\nabla\phi + m_\theta(\phi,\theta)\nabla\theta)\cdot\nabla\psi\,\mathrm{d}x\\ &\quad + \int_\Omega m(\phi,\theta)(\alpha\Delta\psi - F''(\phi)\psi - h_{\mathrm{c}})\Delta\psi\,\mathrm{d}x.\end{aligned} \tag{A.11}$$

We have the following estimates for the terms arising from the second integral on the right-hand side

$$\int_\Omega \alpha m(\phi,\theta)\Delta\psi\Delta\psi\,\mathrm{d}x \ge \alpha m_0\|\Delta\psi\|_{L^2}^2 \tag{A.12}$$

$$\int_\Omega |m(\phi,\theta)h_{\mathrm{c}}\Delta\psi|\,\mathrm{d}x \le \delta\|\Delta\psi\|_{L^2}^2 + c_\delta|m|_\infty^2\|h_{\mathrm{c}}\|_{L^2}^2 \tag{A.13}$$

$$\int_\Omega |m(\phi,\theta)F''(\phi)\psi\Delta\psi|\,\mathrm{d}x \le \delta\|\Delta\psi\|_{L^2}^2 + c_\delta|m|_\infty^2\|F''(\phi)\|_{L^\infty}^2\|\psi\|_{L^2}^2. \tag{A.14}$$

From (2.16) and $\phi\in L^\infty(I;W^{2,2}_{\boldsymbol{n}}(\Omega))$, it follows that $F''(\phi)\in L^\infty(I;L^\infty(\Omega))$.

Applying the Hölder inequality and (2.12), we have

$$\begin{aligned}&\int_\Omega |\xi(m_\phi(\phi,\theta)\nabla\phi + m_\theta(\phi,\theta)\nabla\theta)\cdot\nabla\psi|\,\mathrm{d}x\\ &\quad\le \|\xi\|_{L^2}(|m_\phi|_\infty\|\nabla\phi\|_{\boldsymbol{L}^4} + |m_\theta|_\infty\|\nabla\theta\|_{\boldsymbol{L}^4})\|\psi\|_{L^2}^{1/2}\|\Delta\psi\|_{L^2}^{1/2}\\ &\quad\le \delta^2\|\xi\|_{L^2}^2 + \delta^2\|\Delta\psi\|_{L^2}^2 + c_\delta(\|\nabla\phi\|_{\boldsymbol{L}^4}^4 + \|\nabla\theta\|_{\boldsymbol{L}^4}^4)\|\psi\|_{L^2}^2.\end{aligned} \tag{A.15}$$

Finally, the equation for the linearized chemical potential ξ gives us

$$\delta\|\xi\|_{L^2}^2 - 2\alpha^2\delta\|\Delta\psi\|_{L^2}^2 \le c\delta(\|F''(\phi)\|_{L^\infty}^2\|\psi\|_{L^2}^2 + \|h_{\mathrm{c}}\|_{L^2}^2). \tag{A.16}$$

We use the estimates (A.12)–(A.15) in (A.11), apply the obtained inequality along with (A.8)–(A.10) to (A.7), and take the sum with (A.16). Choosing $0<\delta<1$ small enough in such a way that $\alpha m_0 - (3+2\alpha^2)\delta - \delta^2 > 0$, we can see that for some constant $c = c_{\delta,m_0,\alpha} > 0$ and $K_1 = K_1(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S}) \in L^1(I)$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\psi\|_{L^2}^2 + \frac{1}{c}\|\Delta\psi\|_{L^2}^2 + \frac{1}{c}\|\xi\|_{L^2}^2 \\ &\quad \le \delta_0\|\boldsymbol{w}\|_{\boldsymbol{L}^2_\sigma}^2 + c_{\delta_0}[K_1(\|\psi\|_{W^{1,2}}^2 + \|\eta\|_{L^2}^2) + \|h_{\mathrm{o}}\|_{W^{-1,2}}^2 + \|h_{\mathrm{c}}\|_{L^2}^2]. \qquad \text{(A.17)}\end{aligned}$$

Estimates for ψ in $L^\infty(W^{1,2}) \cap L^2(W^{3,2}_{\boldsymbol{n}})$ and ξ in $L^2(W^{1,2})$. Taking the test function $\varphi = -\Delta\psi$ in (A.2) yields the equation

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\psi|^2 \,\mathrm{d}x - \int_\Omega (\boldsymbol{w}\cdot\nabla\phi)\Delta\psi \,\mathrm{d}x - \int_\Omega (\boldsymbol{v}\cdot\nabla\psi)\Delta\psi\,\mathrm{d}x \\ &\quad - \int_\Omega m'(\phi,\theta)(\psi,\eta)\nabla\mu\cdot\nabla\Delta\psi\,\mathrm{d}x - \int_\Omega m(\phi,\theta)\nabla\xi\cdot\nabla\Delta\psi\,\mathrm{d}x \\ &\quad = -\langle h_{\mathrm{o}}, \Delta\psi\rangle_{W^{-1,2},W^{1,2}}. \qquad \text{(A.18)}\end{aligned}$$

By Hölder and Young inequalities, the embedding $W^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$, and (2.5), we obtain the following estimates for the right-hand side and the second, third, and fourth integrals on the left-hand side

$$|\langle h_{\mathrm{o}}, \Delta\psi\rangle_{W^{-1,2},W^{1,2}}| \le \delta\|\nabla\Delta\psi\|_{\boldsymbol{L}^2}^2 + c_\delta(\|h_{\mathrm{o}}\|_{W^{-1,2}}^2 + \|\psi\|_{L^2}^2) \qquad \text{(A.19)}$$

$$\int_\Omega |(\boldsymbol{w}\cdot\nabla\phi)\Delta\psi|\,\mathrm{d}x \le \delta\|\nabla\Delta\psi\|_{\boldsymbol{L}^2}^2 + c_\delta\|\nabla\phi\|_{\boldsymbol{L}^4}^2\|\boldsymbol{w}\|_{\boldsymbol{L}^2_\sigma}^2 \qquad \text{(A.20)}$$

$$\int_\Omega |(\boldsymbol{v}\cdot\nabla\psi)\Delta\psi|\,\mathrm{d}x \le \delta\|\nabla\Delta\psi\|_{\boldsymbol{L}^2}^2 + c_\delta\|\boldsymbol{v}\|_{\boldsymbol{L}^4_\sigma}^2\|\nabla\psi\|_{\boldsymbol{L}^2}^2 \qquad \text{(A.21)}$$

$$\begin{aligned}&\int_\Omega |m'(\phi,\theta)(\psi,\eta)\nabla\mu\cdot\nabla\Delta\psi|\,\mathrm{d}x \\ &\quad \le \delta\|\nabla\Delta\psi\|_{\boldsymbol{L}^2}^2 + c_\delta\|\nabla\mu\|_{\boldsymbol{L}^4}^2(|m_\phi|_\infty^2\|\psi\|_{W^{1,2}}^2 + |m_\theta|_\infty^2\|\eta\|_{W^{1,2}}^2). \qquad \text{(A.22)}\end{aligned}$$

The fifth integral in (A.18) can be bounded from below as follows:

$$\begin{aligned}&-\int_\Omega m(\phi,\theta)\nabla\xi\cdot\nabla\Delta\psi\,\mathrm{d}x \\ &\quad = \int_\Omega m(\phi,\theta)[\alpha\nabla\Delta\psi - \nabla(F''(\phi)\psi) - \nabla h_{\mathrm{c}}]\cdot\nabla\Delta\psi\,\mathrm{d}x \\ &\quad \ge \frac{m_0\alpha}{2}\|\nabla\Delta\psi\|_{\boldsymbol{L}^2}^2 - c|m|_\infty^2(\|\nabla(F''(\phi)\psi)\|_{\boldsymbol{L}^2}^2 + \|\nabla h_{\mathrm{c}}\|_{\boldsymbol{L}^2}^2). \qquad \text{(A.23)}\end{aligned}$$

From the chain rule, we obtain $\nabla(F''(\phi)\psi) = F'''(\phi)\psi\nabla\phi + F''(\phi)\nabla\psi$, and hence,

$$\begin{aligned}\|\nabla(F''(\phi)\psi)\|_{\boldsymbol{L}^2}^2 &\le c(\|F'''(\phi)\|_{L^\infty}^2\|\nabla\phi\|_{\boldsymbol{L}^4}^2\|\psi\|_{L^4}^2 + \|F''(\phi)\|_{L^\infty}^2\|\nabla\psi\|_{\boldsymbol{L}^2}^2) \\ &\le c(\|F'''(\phi)\|_{L^\infty}^2\|\nabla\phi\|_{\boldsymbol{L}^4}^2 + \|F''(\phi)\|_{L^\infty}^2)\|\psi\|_{W^{1,2}}^2. \qquad \text{(A.24)}\end{aligned}$$

The gradient of the linearized potential is given by $\nabla\xi = -\alpha\nabla\Delta\psi + \nabla(F''(\phi)\psi) + \nabla h_{\mathrm{c}}$. Thus, by the triangle inequality, we obtain after rearrangement the following:

$$\delta\|\nabla\xi\|_{\boldsymbol{L}^2}^2 - 2\alpha^2\delta\|\nabla\Delta\psi\|_{\boldsymbol{L}^2}^2 \le c\delta(\|\nabla(F''(\phi)\psi)\|_{\boldsymbol{L}^2}^2 + \|\nabla h_{\mathrm{c}}\|_{\boldsymbol{L}^2}^2). \qquad \text{(A.25)}$$

Plugging the estimates (A.19)–(A.23), taking the sum with (A.25), applying (A.24), and then taking $\delta > 0$ sufficiently small so that $\frac{m_0\alpha}{2} - (4 + 2\alpha^2)\delta > 0$, we obtain a constant $c = c_{\delta,m_0,\alpha} > 0$ and $K_2 = K_2(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S}) \in L^1(I)$ such that

$$\frac{1}{2}\frac{d}{dt}\|\nabla\psi\|_{\boldsymbol{L}^2}^2 + \frac{1}{c}\|\nabla\Delta\psi\|_{\boldsymbol{L}^2}^2 + \frac{1}{c}\|\nabla\xi\|_{\boldsymbol{L}^2}^2$$
$$\leq c[K_2(\|\psi\|_{W^{1,2}}^2 + \|\eta\|_{W^{1,2}}^2 + \|\boldsymbol{w}\|_{\boldsymbol{L}^2_\sigma}^2) + \|h_{\rm o}\|_{W^{-1,2}}^2 + \|\nabla h_{\rm c}\|_{\boldsymbol{L}^2}^2]. \tag{A.26}$$

Estimate for η in $L^\infty(W^{1,2}) \cap L^2(W^{2,2}_{\boldsymbol{n}})$. Using the test function $\vartheta = \eta$ in (A.3) and then integrating by parts, one has

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\eta|^2 + \tau|\nabla\eta|^2\,\mathrm{d}x + \int_\Omega (\boldsymbol{w}\cdot\nabla\theta)\eta\,\mathrm{d}x + \int_\Omega \chi'(\phi,\theta)(\psi,\eta)\nabla\theta\cdot\nabla\eta\,\mathrm{d}x$$
$$+ \int_\Omega \chi(\phi,\theta)\nabla\eta\cdot\nabla\eta\,\mathrm{d}x + \int_\Omega b|\Delta\eta|^2\,\mathrm{d}x - \int_\Omega \mathbb{T} : \mathbb{D}\boldsymbol{v}\eta\,\mathrm{d}x$$
$$- \int_\Omega \mathbb{S} : \mathbb{D}\boldsymbol{w}\eta\,\mathrm{d}x - \int_\Omega a_0\mathbf{g}\cdot\boldsymbol{w}\eta\,\mathrm{d}x = \langle h_{\rm h},\eta\rangle_{W^{-1,2},W^{1,2}}. \tag{A.27}$$

The integral terms except for the seventh integral on the left-hand side can be bounded from above as follows:

$$|\langle h_{\rm h},\eta\rangle_{W^{-1,2},W^{1,2}}| \leq c(\|h_{\rm h}\|_{W^{-1,2}}^2 + \|\eta\|_{W^{1,2}}^2) \tag{A.28}$$
$$\int_\Omega |(\boldsymbol{w}\cdot\nabla\theta)\eta|\,\mathrm{d}x \leq \frac{\delta_0}{3}\|\boldsymbol{w}\|_{\boldsymbol{L}^2_\sigma}^2 + c_{\delta_0}\|\nabla\theta\|_{\boldsymbol{L}^\infty}^2\|\eta\|_{L^2}^2 \tag{A.29}$$
$$\int_\Omega |\chi'(\phi,\theta)(\psi,\eta)\nabla\theta\cdot\nabla\eta|\,\mathrm{d}x$$
$$\leq c\|\nabla\eta\|_{\boldsymbol{L}^2}^2 + c\|\nabla\theta\|_{\boldsymbol{L}^\infty}^2(|\chi_\phi|_\infty^2\|\psi\|_{L^2}^2 + |\chi_\theta|_\infty^2\|\eta\|_{L^2}^2) \tag{A.30}$$
$$\int_\Omega |\chi(\phi,\theta)\nabla\eta\cdot\nabla\eta|\,\mathrm{d}x \leq |\chi|_\infty\|\nabla\eta\|_{\boldsymbol{L}^2}^2 \tag{A.31}$$
$$\int_\Omega |\mathbb{T} : \mathbb{D}\boldsymbol{v}\eta|\,\mathrm{d}x \leq \delta_0\|\mathbb{T}\|_{\mathbb{L}^2_{\rm s}}^2 + c_{\delta_0}\|\nabla\boldsymbol{v}\|_{\mathbb{L}^4}^2\|\eta\|_{W^{1,2}}^2 \tag{A.32}$$
$$\int_\Omega |a_0\mathbf{g}\cdot\boldsymbol{w}\eta|\,\mathrm{d}x \leq \frac{\delta_0}{3}\|\boldsymbol{w}\|_{\boldsymbol{L}^2_\sigma}^2 + c_{\delta_0}|a_0\mathbf{g}|^2\|\eta\|_{L^2}^2. \tag{A.33}$$

Using $\mathbb{S} : \mathbb{D}\boldsymbol{w} = \mathbb{S} : \nabla\boldsymbol{w}$, $\nabla\cdot(\eta\mathbb{S}) = \mathbb{S}\nabla\eta + \eta\,\nabla\cdot\mathbb{S}$, and by-parts integration, we obtain

$$\int_\Omega \mathbb{S} : \mathbb{D}\boldsymbol{w}\eta\,\mathrm{d}x = -\int_\Omega \boldsymbol{w}\cdot(\mathbb{S}\nabla\eta + \eta\,\nabla\cdot\mathbb{S})\,\mathrm{d}x$$
$$\leq \frac{\delta_0}{3}\|\boldsymbol{w}\|_{\boldsymbol{L}^2_\sigma}^2 + c_{\delta_0}(\|\mathbb{S}\|_{\mathbb{L}^\infty_{\rm s}}^2\|\nabla\eta\|_{\boldsymbol{L}^2}^2 + \|\nabla\mathbb{S}\|_{(\mathbb{L}^4_{\rm s})^2}^2\|\eta\|_{W^{1,2}}^2). \tag{A.34}$$

Utilizing the inequalities (A.28)–(A.34) in (A.27), we get for some $K_3 = K_3(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S}) \in L^1(I)$ that

$$\frac{1}{2}\frac{d}{dt}(\|\eta\|_{L^2}^2 + \tau\|\nabla\eta\|_{\boldsymbol{L}^2}^2) + b\|\Delta\eta\|_{L^2}^2 - \delta_0\|\boldsymbol{w}\|_{\boldsymbol{L}^2_\sigma}^2 - \delta_0\|\mathbb{T}\|_{\mathbb{L}^2_{\rm s}}^2$$
$$\leq c_{\delta_0}[K_3(\|\psi\|_{L^2}^2 + \|\eta\|_{W^{1,2}}^2) + \|h_{\rm h}\|_{W^{-1,2}}^2]. \tag{A.35}$$

Estimate for η in $L^\infty(W^{2,2}_{\boldsymbol{n}})\cap L^2(W^{3,2}_{\boldsymbol{n}})$. With the test function $\vartheta = -\Delta\eta$ in (A.3), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\eta|^2 + \tau|\Delta\eta|^2 \,\mathrm{d}x - \int_\Omega (\boldsymbol{w}\cdot\nabla\theta)\Delta\eta\,\mathrm{d}x - \int_\Omega (\boldsymbol{v}\cdot\nabla\eta)\Delta\eta\,\mathrm{d}x \\
&\quad - \int_\Omega \chi'(\phi,\theta)(\psi,\eta)\nabla\theta\cdot\nabla\Delta\eta\,\mathrm{d}x - \int_\Omega \chi(\phi,\theta)\nabla\eta\cdot\nabla\Delta\eta\,\mathrm{d}x \\
&\quad + \int_\Omega b|\nabla\Delta\eta|^2\,\mathrm{d}x + \int_\Omega \mathbb{T}:\mathbb{D}\boldsymbol{v}\Delta\eta\,\mathrm{d}x + \int_\Omega \mathbb{S}:\mathbb{D}\boldsymbol{w}\Delta\eta\,\mathrm{d}x \\
&\quad + \int_\Omega a_0\mathbf{g}\cdot\boldsymbol{w}\Delta\eta\,\mathrm{d}x = -\langle h_{\mathrm{h}},\Delta\eta\rangle_{W^{-1,2},W^{1,2}}. \qquad \text{(A.36)}
\end{aligned}$$

Similar to the case for the linearized order parameter ψ, the succeeding estimates can be derived

$$|\langle h_{\mathrm{h}},\Delta\eta\rangle_{W^{-1,2},W^{1,2}}| \le \delta\|\nabla\Delta\eta\|^2_{\boldsymbol{L}^2} + c_\delta(\|h_{\mathrm{h}}\|^2_{W^{-1,2}} + \|\eta\|^2_{L^2}) \qquad \text{(A.37)}$$

$$\int_\Omega |(\boldsymbol{w}\cdot\nabla\theta)\Delta\eta|\,\mathrm{d}x \le \delta\|\nabla\Delta\eta\|^2_{\boldsymbol{L}^2} + c_\delta\|\nabla\theta\|^2_{\boldsymbol{L}^4}\|\boldsymbol{w}\|^2_{\boldsymbol{L}^2_\sigma} \qquad \text{(A.38)}$$

$$\int_\Omega |(\boldsymbol{v}\cdot\nabla\eta)\Delta\eta|\,\mathrm{d}x \le \delta\|\nabla\Delta\eta\|^2_{\boldsymbol{L}^2} + c_\delta\|\boldsymbol{v}\|^2_{\boldsymbol{L}^4_\sigma}\|\nabla\eta\|^2_{\boldsymbol{L}^2} \qquad \text{(A.39)}$$

$$\begin{aligned}
&\int_\Omega |\chi'(\phi,\theta)(\psi,\eta)\nabla\theta\cdot\nabla\Delta\eta|\,\mathrm{d}x \\
&\quad \le \delta\|\nabla\Delta\eta\|^2_{\boldsymbol{L}^2} + c_\delta\|\nabla\theta\|^2_{\boldsymbol{L}^4}(|\chi_\phi|^2_\infty\|\psi\|^2_{W^{1,2}} + |\chi_\theta|^2_\infty\|\eta\|^2_{W^{1,2}}) \qquad \text{(A.40)}
\end{aligned}$$

$$\int_\Omega |\chi(\phi,\theta)\nabla\eta\cdot\nabla\Delta\eta|\,\mathrm{d}x \le \delta\|\nabla\Delta\eta\|^2_{\boldsymbol{L}^2} + c_\delta|\chi|^2_\infty\|\nabla\eta\|^2_{\boldsymbol{L}^2}. \qquad \text{(A.41)}$$

In addition to these, the remaining integrals in (A.36) satisfy the following inequalities:

$$\int_\Omega |\mathbb{T}:\mathbb{D}\boldsymbol{v}\Delta\eta|\,\mathrm{d}x \le \delta\|\nabla\Delta\eta\|^2_{\boldsymbol{L}^2} + c_\delta\|\nabla\boldsymbol{v}\|^2_{\mathbb{L}^4}\|\mathbb{T}\|^2_{\mathbb{L}^2_s} \qquad \text{(A.42)}$$

$$\int_\Omega |\mathbb{S}:\mathbb{D}\boldsymbol{w}\Delta\eta|\,\mathrm{d}x \le \delta_0\|\nabla\boldsymbol{w}\|^2_{\mathbb{L}^2} + c_{\delta_0}\|\mathbb{S}\|^2_{\mathbb{L}^\infty_s}\|\Delta\eta\|^2_{L^2} \qquad \text{(A.43)}$$

$$\int_\Omega |a_0\mathbf{g}\cdot\boldsymbol{w}\Delta\eta|\,\mathrm{d}x \le c(\|\Delta\eta\|^2_{L^2} + \|\boldsymbol{w}\|^2_{\boldsymbol{L}^2_\sigma}). \qquad \text{(A.44)}$$

Invoking the estimates (A.37)–(A.44) in the energy identity (A.36) and then choosing $0 < \delta < \frac{b}{6}$, one can obtain a constant $c = c_{\delta,b} > 0$ and a function $K_4 = K_4(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S}) \in L^1(I)$ such that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla\eta\|^2_{\boldsymbol{L}^2} + \tau\|\Delta\eta\|^2_{L^2}) + \frac{1}{c}\|\nabla\Delta\eta\|^2_{\boldsymbol{L}^2} - \delta_0\|\nabla\boldsymbol{w}\|^2_{\mathbb{L}^2} \\
&\quad \le c_{\delta_0}[K_4(\|\psi\|^2_{W^{1,2}} + \|\eta\|^2_{W^{2,2}_{\boldsymbol{n}}} + \|\boldsymbol{w}\|^2_{\boldsymbol{L}^2_\sigma} + \|\mathbb{T}\|^2_{\mathbb{L}^2_s}) + \|h_{\mathrm{h}}\|^2_{W^{-1,2}}]. \qquad \text{(A.45)}
\end{aligned}$$

Estimate for $\boldsymbol{w}$ in $L^\infty(\boldsymbol{L}^2_\sigma)\cap L^2(\boldsymbol{W}^{1,2}_{0,\sigma})$. Let us deal with the linearized Navier–Stokes equation. Applying the test function $\boldsymbol{y} = \boldsymbol{w}$ in (A.4), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{w}|^2\,\mathrm{d}x + \int_\Omega (\boldsymbol{w}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}\,\mathrm{d}x$$

$$+\int_\Omega 2\nu'(\phi,\theta)(\psi,\eta)\mathbb{D}\boldsymbol{v}:\mathbb{D}\boldsymbol{w}\,\mathrm{d}x+\int_\Omega 2\nu(\phi,\theta)\mathbb{D}\boldsymbol{w}:\mathbb{D}\boldsymbol{w}\,\mathrm{d}x$$
$$+\int_\Omega \sigma\eta\mathbb{S}:\nabla\boldsymbol{w}\,\mathrm{d}x+\int_\Omega \mathbb{M}_{\mathbb{S}}(\theta,\mathbb{S})\mathbb{T}:\nabla\boldsymbol{w}\,\mathrm{d}x-\int_\Omega \kappa\xi\nabla\phi\cdot\boldsymbol{w}\,\mathrm{d}x$$
$$-\int_\Omega \kappa\mu\nabla\psi\cdot\boldsymbol{w}\,\mathrm{d}x-\int_\Omega (b_{\mathrm{o}}\psi+b_{\mathrm{h}}\eta)\mathbf{g}\cdot\boldsymbol{w}\,\mathrm{d}x=\langle\boldsymbol{h}_{\mathrm{v}},\boldsymbol{w}\rangle_{\boldsymbol{W}^{-1,2}_{0,\sigma},\boldsymbol{W}^{1,2}_{0,\sigma}}. \tag{A.46}$$

The Korn, Gagliardo–Nirenberg, Hölder, and Young inequalities provide us the following estimates for each integral terms, except for the third one,

$$|\langle\boldsymbol{h}_{\mathrm{v}},\boldsymbol{w}\rangle_{\boldsymbol{W}^{-1,2}_{0,\sigma},\boldsymbol{W}^{1,2}_{0,\sigma}}|\le\delta\|\nabla\boldsymbol{w}\|^2_{\mathbb{L}^2}+c_\delta\|\boldsymbol{h}_{\mathrm{v}}\|^2_{\boldsymbol{W}^{-1,2}_{0,\sigma}} \tag{A.47}$$
$$\int_\Omega 2\nu(\phi,\theta)\mathbb{D}\boldsymbol{w}:\mathbb{D}\boldsymbol{w}\,\mathrm{d}x\ge\nu_0\|\nabla\boldsymbol{w}\|^2_{\mathbb{L}^2} \tag{A.48}$$
$$\int_\Omega |(\boldsymbol{w}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w}|\,\mathrm{d}x\le\delta\|\nabla\boldsymbol{w}\|^2_{\mathbb{L}^2}+c_\delta\|\nabla\boldsymbol{v}\|^2_{\mathbb{L}^4}\|\boldsymbol{w}\|^2_{\boldsymbol{L}^2_\sigma} \tag{A.49}$$
$$\int_\Omega |\sigma\eta\mathbb{S}:\nabla\boldsymbol{w}|\,\mathrm{d}x\le\delta\|\nabla\boldsymbol{w}\|^2_{\mathbb{L}^2}+c_\delta\|\mathbb{S}\|^2_{\mathbb{L}^\infty_{\mathrm{s}}}\|\eta\|^2_{L^2} \tag{A.50}$$
$$\int_\Omega |\mathbb{M}_{\mathbb{S}}(\theta,\mathbb{S})\mathbb{T}:\nabla\boldsymbol{w}|\,\mathrm{d}x\le\delta\|\nabla\boldsymbol{w}\|^2_{\mathbb{L}^2}+c_\delta(\|\theta\|^2_{L^\infty}+\|\mathbb{S}\|^2_{\mathbb{L}^\infty_{\mathrm{s}}}+1)\|\mathbb{T}\|^2_{\mathbb{L}^2_{\mathrm{s}}} \tag{A.51}$$
$$\int_\Omega |\kappa\xi\nabla\phi\cdot\boldsymbol{w}|\,\mathrm{d}x\le\delta_0\|\xi\|^2_{L^2}+c_{\delta_0}\|\nabla\phi\|^2_{\boldsymbol{L}^\infty}\|\boldsymbol{w}\|^2_{\boldsymbol{L}^2_\sigma} \tag{A.52}$$
$$\int_\Omega |\kappa\mu\nabla\psi\cdot\boldsymbol{w}|\,\mathrm{d}x\le\delta\|\nabla\boldsymbol{w}\|^2_{\mathbb{L}^2}+c_\delta\|\mu\|^2_{L^4}\|\nabla\psi\|^2_{\boldsymbol{L}^2} \tag{A.53}$$
$$\int_\Omega |(b_{\mathrm{o}}\psi+b_{\mathrm{h}}\eta)\mathbf{g}\cdot\boldsymbol{w}|\,\mathrm{d}x\le c[(\|\psi\|^2_{L^2}+\|\eta\|^2_{L^2})|\mathbf{g}|^2+\|\boldsymbol{w}\|^2_{\boldsymbol{L}^2_\sigma}]. \tag{A.54}$$

For the third integral in (A.46), thanks to the Young inequality, we have

$$\begin{aligned}\int_\Omega &|2\nu'(\phi,\theta)(\psi,\eta)\mathbb{D}\boldsymbol{v}:\mathbb{D}\boldsymbol{w}|\,\mathrm{d}x\\ &\le c(|\nu_\phi|_\infty\|\psi\|_{L^4}+|\nu_\theta|_\infty\|\eta\|_{L^4})\|\nabla\boldsymbol{v}\|_{\mathbb{L}^4}\|\nabla\boldsymbol{w}\|_{\mathbb{L}^2}\\ &\le\delta\|\nabla\boldsymbol{w}\|^2_{\mathbb{L}^2}+c_\delta\|\nabla\boldsymbol{v}\|^2_{\mathbb{L}^4}(|\nu_\phi|^2_\infty\|\psi\|^2_{W^{1,2}}+|\nu_\theta|^2_\infty\|\eta\|^2_{W^{1,2}}).\end{aligned} \tag{A.55}$$

Using the estimates (A.47)–(A.55) in the equation (A.46) and then choosing $0<\delta<\frac{\nu_0}{6}$ leads to, for some $c=c_{\delta,\nu_0}>0$ and $K_5=K_5(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S})\in L^1(I)$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|^2_{\boldsymbol{L}^2_\sigma}+\frac{1}{c}\|\nabla\boldsymbol{w}\|^2_{\mathbb{L}^2}-\delta_0\|\xi\|^2_{L^2}\\ &\qquad\le c_{\delta_0}[K_5(\|\psi\|^2_{W^{1,2}}+\|\eta\|^2_{W^{1,2}}+\|\boldsymbol{w}\|^2_{\boldsymbol{L}^2_\sigma}+\|\mathbb{T}\|^2_{\mathbb{L}^2_{\mathrm{s}}})+\|\boldsymbol{h}_{\mathrm{v}}\|^2_{\boldsymbol{W}^{-1,2}_{0,\sigma}}].\end{aligned} \tag{A.56}$$

Estimate for $\mathbb{T}$ in $L^\infty(\mathbb{L}^2_{\mathrm{s}})\cap L^2(\mathbb{W}^{1,2}_{\mathrm{s}})$. Our last estimate is concerned with the linearized viscoelastic stress tensor. Applying the test function $\mathbb{Y}=\mathbb{T}$ in (A.5) yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\mathbb{T}|^2\,\mathrm{d}x+\int_\Omega(\boldsymbol{w}\cdot\nabla)\mathbb{S}:\mathbb{T}\,\mathrm{d}x+\int_\Omega\mathbb{J}(\boldsymbol{w},\mathbb{S}):\mathbb{T}\,\mathrm{d}x+\int_\Omega\mathbb{J}(\boldsymbol{v},\mathbb{T}):\mathbb{T}\,\mathrm{d}x$$

$$+\int_\Omega \varepsilon'(\phi,\theta)(\psi,\eta)\nabla\mathbb{S} \therefore \nabla\mathbb{T}\,\mathrm{d}x + \int_\Omega \varepsilon(\phi,\theta)\nabla\mathbb{T}\therefore\nabla\mathbb{T}\,\mathrm{d}x - \int_\Omega \mathbb{P}'(\mathbb{S})\mathbb{T}:\mathbb{T}\,\mathrm{d}x$$
$$-\int_\Omega \lambda\mathbb{D}\boldsymbol{w}:\mathbb{T}\,\mathrm{d}x = \langle \mathbb{H}_\mathrm{s},\mathbb{T}\rangle_{\mathbb{W}_\mathrm{s}^{-1,2},\mathbb{W}_\mathrm{s}^{1,2}}. \tag{A.57}$$

With regard to the source, convection, commutator, and anti-commutator terms in the latter equation, we apply the Hölder and Young inequalities to deduce the following:

$$|\langle \mathbb{H}_\mathrm{s},\mathbb{T}\rangle_{\mathbb{W}_\mathrm{s}^{-1,2},\mathbb{W}_\mathrm{s}^{1,2}}| \le \delta\|\nabla\mathbb{T}\|^2_{(\mathbb{L}^2_\mathrm{s})^2} + c_\delta(\|\mathbb{H}_\mathrm{s}\|^2_{\mathbb{W}_\mathrm{s}^{-1,2}} + \|\mathbb{T}\|^2_{\mathbb{L}^2_\mathrm{s}}) \tag{A.58}$$
$$\int_\Omega |(\boldsymbol{w}\cdot\nabla)\mathbb{S}:\mathbb{T}|\,\mathrm{d}x \le \delta\|\nabla\mathbb{T}\|^2_{(\mathbb{L}^2_\mathrm{s})^2} + c_\delta\|\mathbb{S}\|^2_{\mathbb{L}^\infty_\mathrm{s}}\|\boldsymbol{w}\|^2_{\boldsymbol{L}^2_\sigma} \tag{A.59}$$
$$\int_\Omega |\mathbb{J}(\boldsymbol{w},\mathbb{S}):\mathbb{T}|\,\mathrm{d}x \le \delta_0\|\nabla\boldsymbol{w}\|^2_{\mathbb{L}^2} + c_{\delta_0}\|\mathbb{S}\|^2_{\mathbb{L}^\infty_\mathrm{s}}\|\mathbb{T}\|^2_{\mathbb{L}^2_\mathrm{s}} \tag{A.60}$$
$$\int_\Omega |\mathbb{J}(\boldsymbol{v},\mathbb{T}):\mathbb{T}|\,\mathrm{d}x \le \delta\|\nabla\mathbb{T}\|^2_{(\mathbb{L}^2_\mathrm{s})^2} + c_\delta(\|\nabla\boldsymbol{v}\|^2_{\mathbb{L}^2}+1)\|\mathbb{T}\|^2_{\mathbb{L}^2_\mathrm{s}}. \tag{A.61}$$

In (A.59), we utilized the anti-symmetry of the trilinear form corresponding to the left-hand side with respect to the second and third arguments. With regard to the terms involving stress diffusion, we get

$$\int_\Omega \varepsilon(\phi,\theta)\nabla\mathbb{T}\therefore\nabla\mathbb{T}\,\mathrm{d}x \ge \varepsilon_0\|\nabla\mathbb{T}\|^2_{(\mathbb{L}^2_\mathrm{s})^2} \tag{A.62}$$
$$\int_\Omega |\varepsilon'(\phi,\theta)(\psi,\eta)\nabla\mathbb{S}\therefore\nabla\mathbb{T}|\,\mathrm{d}x$$
$$\le \delta\|\nabla\mathbb{T}\|^2_{(\mathbb{L}^2_\mathrm{s})^2} + c_\delta\|\nabla\mathbb{S}\|^2_{(\mathbb{L}^4_\mathrm{s})^2}(|\varepsilon_\phi|^2_\infty\|\psi\|^2_{W^{1,2}} + |\varepsilon_\theta|^2_\infty\|\eta\|^2_{W^{1,2}}). \tag{A.63}$$

Finally, we have the following estimates:

$$\int_\Omega |\lambda\mathbb{D}\boldsymbol{w}:\mathbb{T}|\,\mathrm{d}x \le \delta_0\|\nabla\boldsymbol{w}\|^2_{\mathbb{L}^2} + c_{\delta_0}\|\mathbb{T}\|^2_{\mathbb{L}^2_\mathrm{s}} \tag{A.64}$$
$$\int_\Omega |\mathbb{P}'(\mathbb{S})\mathbb{T}:\mathbb{T}|\,\mathrm{d}x \le \delta\|\nabla\mathbb{T}\|^2_{(\mathbb{L}^2_\mathrm{s})^2} + c_\delta(\|\mathbb{S}\|^4_{\mathbb{L}^4_\mathrm{s}}+1)\|\mathbb{T}\|^2_{\mathbb{L}^2_\mathrm{s}}. \tag{A.65}$$

Therefore, by utilizing (A.58)–(A.65) in (A.57) and taking $0<\delta<\frac{\varepsilon_0}{5}$, it can be seen that there exist $c = c_{\delta,\varepsilon_0} > 0$ and $K_6 = K_6(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S}) \in L^1(I)$ such that

$$\frac{1}{2}\frac{d}{dt}\|\mathbb{T}\|^2_{\mathbb{L}^2_\mathrm{s}} + \frac{1}{c}\|\nabla\mathbb{T}\|^2_{(\mathbb{L}^2_\mathrm{s})^2} - 2\delta_0\|\nabla\boldsymbol{w}\|^2_{\mathbb{L}^2}$$
$$\le c_{\delta_0}[K_6(\|\psi\|^2_{W^{1,2}} + \|\eta\|^2_{W^{1,2}} + \|\boldsymbol{w}\|^2_{\boldsymbol{L}^2_\sigma} + \|\mathbb{T}\|^2_{\mathbb{L}^2_\mathrm{s}}) + \|\mathbb{H}_\mathrm{s}\|^2_{\mathbb{W}_\mathrm{s}^{-1,2}}]. \tag{A.66}$$

Let us now combine the a priori estimates obtained from (A.17) and (A.26) for ψ and ξ, (A.35) and (A.45) for η, (A.56) for $\boldsymbol{w}$, and (A.66) for $\mathbb{T}$. Taking the sum of these and choosing $\delta_0 > 0$ small enough so that all coefficients on the left-hand side of the resulting inequality are positive, we deduce the following energy inequality:

$$\frac{1}{2}\frac{d}{dt}E + \frac{1}{c}D \le c(S + KE) \quad \text{in } I, \tag{A.67}$$

where $c > 0$, $K = K(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S}) := K_1 + K_2 + \cdots + K_6 \in L^1(I)$, and $E, D, S : I \to [0, \infty)$ are given as follows:

$$
\begin{aligned}
S &:= \|(h_{\mathrm{o}}, h_{\mathrm{c}}, h_{\mathrm{h}}, \boldsymbol{h}_{\mathrm{v}}, \mathbb{H}_{\mathrm{s}})\|^2_{(\mathcal{U}^1)^*} \\
E &:= \|\psi\|^2_{W^{1,2}} + \|\eta\|^2_{W^{1,2}} + \tau(\|\nabla\eta\|^2_{\boldsymbol{L}^2} + \|\Delta\eta\|^2_{L^2}) + \|\boldsymbol{w}\|^2_{\boldsymbol{L}^2_\sigma} + \|\mathbb{T}\|^2_{\mathbb{L}^2_{\mathrm{s}}} \\
D &:= \|\nabla\Delta\psi\|^2_{\boldsymbol{L}^2} + \|\nabla\xi\|^2_{\boldsymbol{L}^2} + \|\nabla\Delta\eta\|^2_{\boldsymbol{L}^2} + \|\nabla\boldsymbol{w}\|^2_{\mathbb{L}^2} + \|\nabla\mathbb{T}\|^2_{(\mathbb{L}^2_{\mathrm{s}})^2}.
\end{aligned}
$$

Applying Grönwall Lemma to (A.67) and adapting the same process for the estimation of the time derivatives as in the fourth step of the proof of Theorem 3.1, we can conclude that there exists a continuous function $\mathfrak{C} = \mathfrak{C}(\|(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S})\|_{\mathcal{W}^2}) : [0, \infty) \to [0, \infty)$ such that

$$
\|(\psi, \xi, \eta, \boldsymbol{w}, \mathbb{T})\|_{\mathcal{V}^1} \le \mathfrak{C}[\|(h_{\mathrm{o}}, h_{\mathrm{c}}, h_{\mathrm{h}}, \boldsymbol{h}_{\mathrm{v}}, \mathbb{H}_{\mathrm{s}})\|_{(\mathcal{U}^1)^*} + \|(\psi_0, \eta_0, \boldsymbol{w}_0, \mathbb{T}_0)\|_{\mathcal{D}^1}]. \tag{A.68}
$$

With a standard Faedo–Galerkin method, this estimate implies the following theorem.

Theorem A.1. *Let* (A1)$_3$ *and* (A2)$_1$ *be satisfied. Given sources and initial data as in* (A.1)*, the linearized system* (4.2) *admits a unique weak solution satisfying the a priori estimate* (A.68)*. If in addition, $h_{\mathrm{c}} \in W^{1,2,2}(I; W^{1,2}(\Omega), W^{-3,2}_{\boldsymbol{n}}(\Omega))$, then $(\psi, \xi, \eta, \boldsymbol{w}, \mathbb{T}) \in \mathcal{W}^1(\Omega_T)$ and* (A.68) *holds with $\mathcal{V}^1(\Omega_T)$ and $\mathcal{U}^1(\Omega_T)^*$ replaced by $\mathcal{W}^1(\Omega_T)$ and $\mathcal{Y}^1(\Omega_T)^*$, respectively.*

Proof. Based on what have been discussed above, we only need to prove the second statement. For this, let us note the following equation in the sense of distributions:

$$
\partial_t\xi = -\alpha\Delta\partial_t\psi + F'''(\phi)\psi\partial_t\phi + F''(\phi)\partial_t\psi + \partial_t h_{\mathrm{c}}. \tag{A.69}
$$

We have $\Delta\partial_t\psi, \partial_t h_{\mathrm{c}} \in L^2(I; W^{-3,2}_{\boldsymbol{n}}(\Omega))$ since $\|\Delta\partial_t\psi\|_{L^2(W^{-3,2}_{\boldsymbol{n}})} \le \|\partial_t\psi\|_{L^2(W^{-1,2})}$ and $\|\partial_t h_{\mathrm{c}}\|_{L^2(W^{-3,2}_{\boldsymbol{n}})} \le \|h_{\mathrm{c}}\|_{W^{1,2,2}(W^{1,2}, W^{-3,2}_{\boldsymbol{n}})}$. For each $\varphi \in L^2(I; W^{1,2}(\Omega))$, it holds that

$$
\begin{aligned}
\int_I\int_\Omega |\varphi F'''(\phi)\psi\partial_t\phi| \,\mathrm{d}x\,\mathrm{d}t &\le \|F'''(\phi)\|_{L^\infty(L^\infty)}\|\psi\|_{L^\infty(L^4)}\|\partial_t\phi\|_{L^2(L^2)}\|\varphi\|_{L^2(L^4)} \\
&\le c_\phi\|\psi\|_{L^\infty(W^{1,2})}\|\varphi\|_{L^2(W^{1,2})}
\end{aligned}
$$

for some generic constant $c_\phi = c(\|\phi\|_{W^{1,2,2}(W^{4,2}_{\boldsymbol{n}}, L^2)}) > 0$. Thus, we have $F'''(\phi)\psi\partial_t\phi \in L^2(I; W^{-1,2}(\Omega))$ and

$$
\|F'''(\phi)\psi\partial_t\phi\|_{L^2(W^{-1,2})} \le c_\phi\|\psi\|_{L^\infty(W^{1,2})} \le c_\phi\|\psi\|_{W^{1,2,2}(W^{3,2}_{\boldsymbol{n}}, W^{-1,2})}.
$$

Analogous to (A.24), we obtain $F'''(\phi)\varphi \in L^\infty(I; W^{1,2}(\Omega))$ and

$$
\begin{aligned}
\|F'''(\phi)\varphi\|_{L^2(W^{1,2})} &\le c(\|F''(\phi)\|_{L^\infty(L^\infty)}\|\varphi\|_{L^2(W^{1,2})} \\
&\quad + \|F''(\phi)\|_{L^\infty(L^\infty)}\|\varphi\|_{L^2(L^4)}\|\nabla\phi\|_{L^\infty(\boldsymbol{L}^4)}) \le c_\phi\|\varphi\|_{L^2(W^{1,2})}.
\end{aligned}
$$

By duality, we deduce that $F''(\phi)\partial_t\psi \in L^2(I; W^{-1,2}(\Omega))$ and $\|F''(\phi)\partial_t\psi\|_{L^2(W^{-1,2})} \le c_\phi\|\partial_t\psi\|_{L^2(W^{-1,2})}$. Since $L^2(I; W^{-1,2}(\Omega)) \hookrightarrow L^2(I; W^{-3,2}_{\boldsymbol{n}}(\Omega))$, we have $\partial_t\xi \in L^2(I; W^{-3,2}_{\boldsymbol{n}}(\Omega))$. The estimate for $\partial_t\xi$ can be deduce from the above inequalities and (A.68). □

A.2. Very Weak Solutions to the Linearized System. In this subsection, we consider source functions and initial data

$$(h_{\mathrm{o}}, h_{\mathrm{c}}, h_{\mathrm{h}}, \boldsymbol{h}_{\mathrm{v}}, \mathbb{H}_{\mathrm{s}}) \in \mathcal{U}^2(\Omega_T)^*, \qquad (\psi_0, \eta_0, \boldsymbol{w}_0, \mathbb{T}_0) \in \mathcal{D}^0(\Omega). \tag{A.70}$$

We shall say that $(\psi, \xi, \eta, \boldsymbol{w}, \mathbb{T}) \in \mathcal{V}^0(\Omega_T)$ is a *very weak solution* to (4.2) if for every test function $(\varphi, \vartheta, \boldsymbol{y}, \mathbb{Y}) \in W^{2,2}_{\boldsymbol{n}}(\Omega) \times W^{2,2}_{\boldsymbol{n}}(\Omega) \times \boldsymbol{W}^{2,2}_{0,\sigma}(\Omega) \times \mathbb{W}^{2,2}_{\boldsymbol{n},\mathrm{s}}(\Omega)$ the following variational equations are satisfied:

$$\begin{aligned} &\langle \partial_t \psi, \varphi\rangle_{W^{-2,2}_{\boldsymbol{n}}, W^{2,2}_{\boldsymbol{n}}} + \int_\Omega (\boldsymbol{w}\cdot\nabla\phi + \boldsymbol{v}\cdot\nabla\psi)\varphi \,\mathrm{d}x \\ &\quad + \int_\Omega [m'(\phi,\theta)(\psi,\eta)\nabla\mu\cdot\nabla\varphi - \xi\,\nabla\cdot(m(\phi,\theta)\nabla\varphi)]\,\mathrm{d}x \\ &\quad = \langle h_{\mathrm{o}}, \varphi\rangle_{W^{-2,2}_{\boldsymbol{n}}, W^{2,2}_{\boldsymbol{n}}} \end{aligned} \tag{A.71}$$

$$\begin{aligned} &\int_\Omega \partial_t\eta(\vartheta - \tau\Delta\vartheta)\,\mathrm{d}x + \int_\Omega (\boldsymbol{w}\cdot\nabla\theta + \boldsymbol{v}\cdot\nabla\eta)\vartheta\,\mathrm{d}x \\ &\quad + \int_\Omega [\chi'(\phi,\theta)(\psi,\eta)\nabla\theta + \chi(\phi,\theta)\nabla\eta]\cdot\nabla\vartheta\,\mathrm{d}x + \int_\Omega b\Delta\eta\Delta\vartheta\,\mathrm{d}x \\ &\quad - \int_\Omega \mathbb{T} : \mathbb{D}\boldsymbol{v}\vartheta\,\mathrm{d}x + \int_\Omega \boldsymbol{w}\cdot(\nabla\cdot(\vartheta\mathbb{S}))\,\mathrm{d}x - \int_\Omega a_0\mathbf{g}\cdot\boldsymbol{w}\vartheta\,\mathrm{d}x \\ &\quad = \langle h_{\mathrm{h}}, \vartheta\rangle_{W^{-2,2}_{\boldsymbol{n}}, W^{2,2}_{\boldsymbol{n}}} \end{aligned} \tag{A.72}$$

$$\begin{aligned} &\langle \partial_t\boldsymbol{w}, \boldsymbol{y}\rangle_{\boldsymbol{W}^{-2,2}_{0,\sigma}, \boldsymbol{W}^{2,2}_{0,\sigma}} + \int_\Omega [((\boldsymbol{w}\cdot\nabla)\boldsymbol{v})\cdot\boldsymbol{y} - ((\boldsymbol{v}\cdot\nabla)\boldsymbol{y})\cdot\boldsymbol{w}]\,\mathrm{d}x \\ &\quad + \int_\Omega [2\nu'(\phi,\theta)(\psi,\eta)\mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{y} - 2\boldsymbol{w}\cdot(\nabla\cdot(\nu(\phi,\theta)\mathbb{D}\boldsymbol{y}))]\,\mathrm{d}x \\ &\quad + \int_\Omega (\sigma\eta\mathbb{S} + \mathbb{M}_{\mathbb{S}}(\theta,\mathbb{S})\mathbb{T}) : \mathbb{D}\boldsymbol{y}\,\mathrm{d}x - \int_\Omega \kappa(\xi\nabla\phi + \mu\nabla\psi)\cdot\boldsymbol{y}\,\mathrm{d}x \\ &\quad - \int_\Omega (b_{\mathrm{o}}\psi + b_{\mathrm{h}}\eta)\mathbf{g}\cdot\boldsymbol{y}\,\mathrm{d}x = \langle \boldsymbol{h}_{\mathrm{v}}, \boldsymbol{y}\rangle_{\boldsymbol{W}^{-2,2}_{0,\sigma}, \boldsymbol{W}^{2,2}_{0,\sigma}} \end{aligned} \tag{A.73}$$

$$\begin{aligned} &\langle \partial_t\mathbb{T}, \mathbb{Y}\rangle_{\mathbb{W}^{-2,2}_{\boldsymbol{n},\mathrm{s}}, \mathbb{W}^{2,2}_{\boldsymbol{n},\mathrm{s}}} + \int_\Omega [((\boldsymbol{w}\cdot\nabla)\mathbb{S}) : \mathbb{Y} - ((\boldsymbol{v}\cdot\nabla)\mathbb{Y}) : \mathbb{T}]\,\mathrm{d}x \\ &\quad - \int_\Omega \boldsymbol{w}\cdot(\nabla\cdot([\mathbb{S},\mathbb{Y}] - a\{\mathbb{S},\mathbb{Y}\}))\,\mathrm{d}x + \int_\Omega \mathbb{J}(\boldsymbol{v},\mathbb{T}) : \mathbb{Y}\,\mathrm{d}x \\ &\quad + \int_\Omega [\varepsilon'(\phi,\theta)(\psi,\eta)\nabla\mathbb{S} \,\therefore\, \nabla\mathbb{Y} - \mathbb{T} : \nabla\cdot(\varepsilon(\phi,\theta)\nabla\mathbb{Y}) + \boldsymbol{w}\cdot(\nabla\cdot(\lambda\mathbb{Y}))]\,\mathrm{d}x \\ &\quad - \int_\Omega \mathbb{P}'(\mathbb{S})\mathbb{T} : \mathbb{Y}\,\mathrm{d}x = \langle \mathbb{H}_{\mathrm{s}}, \mathbb{Y}\rangle_{\mathbb{W}^{-2,2}_{\boldsymbol{n},\mathrm{s}}, \mathbb{W}^{2,2}_{\boldsymbol{n},\mathrm{s}}} \end{aligned} \tag{A.74}$$

almost everywhere in I, the equation $\xi + \alpha\Delta\psi - F''(\phi)\psi = h_{\mathrm{c}}$ holds in $L^2(I; L^2(\Omega))$, and the initial condition is satisfied in $\mathcal{D}^0(\Omega)$. In what follows, we obtain the a priori estimates required for the existence of very weak solutions.

Estimates for ψ in $L^\infty(L^2) \cap L^2(W^{2,2}_{\boldsymbol{n}})$ and for ξ in $L^2(L^2)$. We test (A.71) with $\varphi = \psi$ and adapt the same strategy as in the case of weak solutions. Now, instead of (A.8), we estimate h_{o} according to

$$|\langle h_{\mathrm{o}}, \psi\rangle_{W_{\boldsymbol{n}}^{-2,2}, W_{\boldsymbol{n}}^{2,2}}| \le \delta\|\Delta\psi\|_{L^2}^2 + c_\delta(\|h_{\mathrm{o}}\|_{W_{\boldsymbol{n}}^{-2,2}}^2 + \|\psi\|_{L^2}^2).$$

Thus, instead of (A.17) we have the following

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\psi\|_{L^2}^2 + \frac{1}{c}\|\Delta\psi\|_{L^2}^2 + \frac{1}{c}\|\xi\|_{L^2}^2 - \delta_0\|\boldsymbol{w}\|_{\boldsymbol{L}_\sigma^2}^2\\ &\quad\le c_{\delta_0}[K_1(\|\psi\|_{L^2}^2 + \|\eta\|_{L^2}^2) + \|h_{\mathrm{o}}\|_{W_{\boldsymbol{n}}^{-2,2}}^2 + \|h_{\mathrm{c}}\|_{L^2}^2].\end{aligned} \tag{A.75}$$

Estimate for η in $L^\infty(W^{1,2}) \cap L^2(W_{\boldsymbol{n}}^{2,2})$. Using the test function $\vartheta = \eta$ in (A.72) and with a similar procedure as above, but now applied to h_{h}, the estimate (A.35) turns into

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\eta\|_{L^2}^2 + \tau\|\nabla\eta\|_{\boldsymbol{L}^2}^2) + \frac{1}{c}\|\Delta\eta\|_{L^2}^2 - \delta_0\|\boldsymbol{w}\|_{\boldsymbol{L}_\sigma^2}^2 - \delta_0\|\mathbb{T}\|_{\mathbb{L}_s^2}^2\\ &\quad\le c_{\delta_0}[K_3(\|\psi\|_{L^2}^2 + \|\eta\|_{W^{1,2}}^2) + \|h_{\mathrm{h}}\|_{W_{\boldsymbol{n}}^{-2,2}}^2].\end{aligned} \tag{A.76}$$

Estimate for $\boldsymbol{w}$ in $L^\infty(\boldsymbol{W}_{0,\sigma}^{-1,2}) \cap L^2(\boldsymbol{L}_\sigma^2)$. We apply the test function $\boldsymbol{y} = \boldsymbol{A}_S^{-1}\boldsymbol{w}$ in the variational equation (A.73) and estimate the integral terms. First, recall that under appropriate scaling of the usual norm of $\boldsymbol{W}_{0,\sigma}^{2,2}(\Omega)$, the Stokes operator $\boldsymbol{A}_S : \boldsymbol{W}_{0,\sigma}^{2,2}(\Omega) \to \boldsymbol{L}_\sigma^2(\Omega)$ is a unitary operator and admits a unique extension $\boldsymbol{A}_S : \boldsymbol{W}_{0,\sigma}^{1,2}(\Omega) \to \boldsymbol{W}_{0,\sigma}^{-1,2}(\Omega)$ that is also an unitary, see for instance, [65, Theorem III.2.1.1] and [68, Proposition 3.4.5]. Hence,

$$\|\boldsymbol{A}_S^{-1}\boldsymbol{y}\|_{\boldsymbol{W}_{0,\sigma}^{2,2}} = \|\boldsymbol{y}\|_{\boldsymbol{L}_\sigma^2}, \quad \|\boldsymbol{A}_S^{-1}\boldsymbol{z}\|_{\boldsymbol{W}_{0,\sigma}^{1,2}} = \|\boldsymbol{z}\|_{\boldsymbol{W}_{0,\sigma}^{-1,2}} \tag{A.77}$$

for all $(\boldsymbol{y}, \boldsymbol{z}) \in \boldsymbol{L}_\sigma^2(\Omega) \times \boldsymbol{W}_{0,\sigma}^{-1,2}(\Omega)$. Thus, for the term involving the source we have

$$|\langle \boldsymbol{h}_{\mathrm{v}}, \boldsymbol{A}_S^{-1}\boldsymbol{w}\rangle_{\boldsymbol{W}_{0,\sigma}^{-2,2}, \boldsymbol{W}_{0,\sigma}^{2,2}}| \le \delta\|\boldsymbol{w}\|_{\boldsymbol{L}_\sigma^2}^2 + c_\delta\|\boldsymbol{h}_{\mathrm{v}}\|_{\boldsymbol{W}_{0,\sigma}^{-2,2}}^2. \tag{A.78}$$

Next, for the terms involving diffusion we apply $2\nabla\cdot\mathbb{D} = -\boldsymbol{A}_S$, the Hölder and Young inequalities, and (A.77) to deduce the following estimates:

$$\begin{aligned}&-\int_\Omega 2\boldsymbol{w}\cdot(\nabla\cdot(\nu(\phi,\theta)\mathbb{D}\boldsymbol{A}_S^{-1}\boldsymbol{w}))\,\mathrm{d}x\\ &\quad= \int_\Omega \nu(\phi,\theta)|\boldsymbol{w}|^2\,\mathrm{d}x - \int_\Omega 2\boldsymbol{w}\cdot\mathbb{D}A_S^{-1}\boldsymbol{w}(\nu_\phi(\phi,\theta)\nabla\phi + \nu_\theta(\phi,\theta)\nabla\theta)\,\mathrm{d}x\\ &\quad\ge \frac{\nu_0}{2}\|\boldsymbol{w}\|_{\boldsymbol{L}_\sigma^2}^2 - c(|\nu_\phi|_\infty^2\|\nabla\phi\|_{\boldsymbol{L}^\infty}^2 + |\nu_\theta|_\infty^2\|\nabla\theta\|_{\boldsymbol{L}^\infty}^2)\|\boldsymbol{w}\|_{\boldsymbol{W}_{0,\sigma}^{-1,2}}^2\end{aligned} \tag{A.79}$$

$$\begin{aligned}&\int_\Omega |2\nu'(\phi,\theta)(\psi,\eta)\mathbb{D}\boldsymbol{v} : \mathbb{D}\boldsymbol{A}_S^{-1}\boldsymbol{w}|\,\mathrm{d}x\\ &\quad\le \delta\|\boldsymbol{w}\|_{\boldsymbol{L}_\sigma^2}^2 + c_\delta\|\nabla\boldsymbol{v}\|_{\mathbb{L}^4}^2(|\nu_\phi|_\infty^2\|\psi\|_{L^2}^2 + |\nu_\theta|_\infty^2\|\eta\|_{L^2}^2).\end{aligned} \tag{A.80}$$

The terms arising from convection and viscoelastic stress are bounded from above according to

$$\int_\Omega |((\boldsymbol{w}\cdot\nabla)\boldsymbol{v})\cdot\boldsymbol{A}_S^{-1}\boldsymbol{w}|\,\mathrm{d}x \le \delta\|\boldsymbol{w}\|_{\boldsymbol{L}_\sigma^2}^2 + c_\delta\|\nabla\boldsymbol{v}\|_{\mathbb{L}^4}^2\|\boldsymbol{w}\|_{\boldsymbol{W}_{0,\sigma}^{-1,2}}^2 \tag{A.81}$$

$$\int_\Omega |((\boldsymbol{v}\cdot\nabla)\boldsymbol{A}_S^{-1}\boldsymbol{w})\cdot\boldsymbol{w}|\,\mathrm{d}x \le \delta\|\boldsymbol{w}\|_{\boldsymbol{L}_\sigma^2}^2 + c_\delta\|\boldsymbol{v}\|_{\boldsymbol{L}^\infty}^2\|\boldsymbol{w}\|_{\boldsymbol{W}_{0,\sigma}^{-1,2}}^2 \tag{A.82}$$

$$\int_\Omega |\sigma\eta\mathbb{S} : \mathbb{D}\boldsymbol{A}_S^{-1}\boldsymbol{w}|\,\mathrm{d}x \le c(\|\mathbb{S}\|_{\mathbb{L}^\infty_{\mathrm{s}}}^2\|\eta\|_{L^2}^2 + \|\boldsymbol{w}\|^2_{\boldsymbol{W}^{-1,2}_{0,\sigma}}) \tag{A.83}$$

$$\begin{aligned}&\int_\Omega |\mathbb{M}_{\mathbb{S}}(\theta,\mathbb{S})\mathbb{T} : \mathbb{D}\boldsymbol{A}_S^{-1}\boldsymbol{w}|\,\mathrm{d}x\\ &\quad\le \delta_0\|\mathbb{T}\|^2_{\mathbb{L}^2_{\mathrm{s}}} + c_{\delta_0}(\|\theta\|^2_{L^\infty} + \|\mathbb{S}\|^2_{\mathbb{L}^\infty_{\mathrm{s}}} + 1)\|\boldsymbol{w}\|^2_{\boldsymbol{W}^{-1,2}_{0,\sigma}}.\end{aligned} \tag{A.84}$$

Finally, for the integrals arising from surface tension and gravity, we obtain

$$\int_\Omega |\kappa\xi\nabla\phi\cdot\boldsymbol{A}_S^{-1}\boldsymbol{w}|\,\mathrm{d}x \le \delta_0\|\xi\|^2_{L^2} + c_{\delta_0}\|\nabla\phi\|^2_{\boldsymbol{L}^4}\|\boldsymbol{w}\|^2_{\boldsymbol{W}^{-1,2}_{0,\sigma}} \tag{A.85}$$

$$\int_\Omega |\kappa\mu\nabla\psi\cdot\boldsymbol{A}_S^{-1}\boldsymbol{w}|\,\mathrm{d}x \le \delta_0\|\Delta\psi\|^2_{L^2} + c_{\delta_0}\|\mu\|^2_{L^2}\|\boldsymbol{w}\|^2_{\boldsymbol{W}^{-1,2}_{0,\sigma}} \tag{A.86}$$

$$\int_\Omega |(b_{\mathrm{o}}\psi + b_{\mathrm{h}}\eta)\mathbf{g}\cdot\boldsymbol{A}_S^{-1}\boldsymbol{w}|\,\mathrm{d}x \le c[(\|\psi\|^2_{L^2} + \|\eta\|^2_{L^2})|\mathbf{g}|^2 + \|\boldsymbol{w}\|^2_{\boldsymbol{W}^{-1,2}_{0,\sigma}}]. \tag{A.87}$$

Applying the estimates (A.78)–(A.87) to the variational equation (A.73) with $\boldsymbol{y} = \boldsymbol{A}_S^{-1}\boldsymbol{w}$ and choosing $0 < \delta < \frac{\nu_0}{8}$, we obtain a constant $c = c_{\delta,\nu_0} > 0$ and $K_6 = K_6(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S}) \in L^1(I)$ so that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|^2_{\boldsymbol{W}^{-1,2}_{0,\sigma}} + \frac{1}{c}\|\boldsymbol{w}\|^2_{\boldsymbol{L}^2_\sigma} - \delta_0\|\Delta\psi\|^2_{L^2} - \delta_0\|\xi\|^2_{L^2} - \delta_0\|\mathbb{T}\|^2_{\mathbb{L}^2_{\mathrm{s}}}\\ &\qquad\le c_{\delta_0}[K_6(\|\psi\|^2_{L^2} + \|\eta\|^2_{L^2} + \|\boldsymbol{w}\|^2_{\boldsymbol{W}^{-1,2}_{0,\sigma}}) + \|\boldsymbol{h}_{\mathrm{v}}\|^2_{\boldsymbol{W}^{-2,2}_{0,\sigma}}].\end{aligned} \tag{A.88}$$

Estimate for $\mathbb{T}$ *in* $L^\infty(\mathbb{W}^{-1,2}_{\mathrm{s}})\cap L^2(\mathbb{L}^2_{\mathrm{s}})$. The linear map $\mathbb{B}_N := I + \mathbb{A}_N : \mathbb{W}^{2,2}_{\boldsymbol{n},\mathrm{s}}(\Omega) \to \mathbb{L}^2_{\mathrm{s}}(\Omega)$ is unitary and admits a unique extension as a map from $\mathbb{W}^{1,2}_{\mathrm{s}}(\Omega)$ onto $\mathbb{W}^{-1,2}_{\mathrm{s}}(\Omega)$ that is also unitary. Similar to (A.77), one has

$$\|\mathbb{B}_N^{-1}\mathbb{Y}\|_{\mathbb{W}^{2,2}_{\boldsymbol{n},\mathrm{s}}} = \|\mathbb{Y}\|_{\mathbb{L}^2_{\mathrm{s}}}, \quad \|\mathbb{B}_N^{-1}\mathbb{X}\|_{\mathbb{W}^{1,2}_{\mathrm{s}}} = \|\mathbb{X}\|_{\mathbb{W}^{-1,2}_{\mathrm{s}}} \tag{A.89}$$

for all $(\mathbb{Y},\mathbb{X}) \in \mathbb{L}^2_{\mathrm{s}}(\Omega)\times\mathbb{W}^{-1,2}_{\mathrm{s}}(\Omega)$. Let us use the test function $\mathbb{Y} = \mathbb{B}_N^{-1}\mathbb{T}$ in (A.74) and derive estimates for the resulting terms. By (A.89) and Hölder and Young inequalities,

$$|\langle\mathbb{H}_{\mathrm{s}},\mathbb{B}_N^{-1}\mathbb{T}\rangle_{\mathbb{W}^{-2,2}_{\boldsymbol{n},\mathrm{s}},\mathbb{W}^{2,2}_{\boldsymbol{n},\mathrm{s}}}| \le \delta\|\mathbb{T}\|^2_{\mathbb{L}^2_{\mathrm{s}}} + c_\delta\|\mathbb{H}_{\mathrm{s}}\|^2_{\mathbb{W}^{-2,2}_{\boldsymbol{n},\mathrm{s}}} \tag{A.90}$$

$$\int_\Omega |((\boldsymbol{w}\cdot\nabla)\mathbb{S}) : \mathbb{B}_N^{-1}\mathbb{T}|\,\mathrm{d}x \le \delta_0\|\boldsymbol{w}\|^2_{\boldsymbol{L}^2_\sigma} + c_{\delta_0}\|\nabla\mathbb{S}\|^2_{(\mathbb{L}^4_{\mathrm{s}})^2}\|\mathbb{T}\|^2_{\mathbb{W}^{-1,2}_{\mathrm{s}}} \tag{A.91}$$

$$\int_\Omega |((\boldsymbol{v}\cdot\nabla)\mathbb{B}_N^{-1}\mathbb{T}) : \mathbb{T}|\,\mathrm{d}x \le \delta\|\mathbb{T}\|^2_{\mathbb{L}^2_{\mathrm{s}}} + c_\delta\|\boldsymbol{v}\|^2_{\boldsymbol{L}^\infty}\|\mathbb{T}\|^2_{\mathbb{W}^{-1,2}_{\mathrm{s}}} \tag{A.92}$$

$$\int_\Omega |\mathbb{P}'(\mathbb{S})\mathbb{T} : \mathbb{B}_N^{-1}\mathbb{T}|\,\mathrm{d}x \le \delta\|\mathbb{T}\|^2_{\mathbb{L}^2_{\mathrm{s}}} + c_\delta(1 + \|\mathbb{S}\|^4_{\mathbb{L}^8_{\mathrm{s}}})\|\mathbb{T}\|^2_{\mathbb{W}^{-1,2}_{\mathrm{s}}}. \tag{A.93}$$

With regard to the commutator and anti-commutator terms, we have

$$\begin{aligned}&\int_\Omega |\boldsymbol{w}\cdot(\nabla\cdot([\mathbb{S},\mathbb{B}_N^{-1}\mathbb{T}] - a\{\mathbb{S},\mathbb{B}_N^{-1}\mathbb{T}\}))|\,\mathrm{d}x\\ &\qquad\le \delta_0\|\boldsymbol{w}\|^2_{\boldsymbol{L}^2_\sigma} + c_{\delta_0}(\|\nabla\mathbb{S}\|^2_{(\mathbb{L}^4_{\mathrm{s}})^2} + \|\mathbb{S}\|^2_{\mathbb{L}^\infty_{\mathrm{s}}})\|\mathbb{T}\|^2_{\mathbb{W}^{-1,2}_{\mathrm{s}}}\end{aligned} \tag{A.94}$$

$$\int_\Omega |\mathbb{J}(\boldsymbol{v},\mathbb{T}) : \mathbb{B}_N^{-1}\mathbb{T}|\,\mathrm{d}x \le \delta\|\mathbb{T}\|^2_{\mathbb{L}^2_{\mathrm{s}}} + c_\delta\|\nabla\boldsymbol{v}\|^2_{\mathbb{L}^4}\|\mathbb{T}\|^2_{\mathbb{W}^{-1,2}_{\mathrm{s}}}. \tag{A.95}$$

For the stress diffusion terms, by performing similar procedure as in the case of the velocity, we obtain

$$\begin{aligned}
&\int_\Omega |\varepsilon'(\phi,\theta)(\psi,\eta)\nabla\mathbb{S} \therefore \nabla\mathbb{B}_N^{-1}\mathbb{T}| \,\mathrm{d}x \\
&\quad\leq \delta\|\mathbb{T}\|_{\mathbb{L}^2_{\mathrm{s}}}^2 + c_\delta\|\nabla\mathbb{S}\|_{(\mathbb{L}^4_{\mathrm{s}})^2}^2(|\varepsilon_\phi|_\infty^2\|\psi\|_{L^2}^2 + |\varepsilon_\theta|_\infty^2\|\eta\|_{L^2}^2) \qquad \text{(A.96)}\\
&\int_\Omega |\mathbb{T} : \nabla\cdot(\varepsilon(\phi,\theta)\nabla\mathbb{B}_N^{-1}\mathbb{T})| \,\mathrm{d}x \\
&\quad\geq \frac{\varepsilon_0}{2}\|\mathbb{T}\|_{\mathbb{L}^2_{\mathrm{s}}}^2 - c(|\varepsilon_\phi|_\infty^2\|\nabla\phi\|_{\boldsymbol{L}^\infty}^2 + |\varepsilon_\theta|_\infty^2\|\nabla\theta\|_{\boldsymbol{L}^\infty}^2)\|\mathbb{T}\|_{\mathbb{W}_{\mathrm{s}}^{-1,2}}^2. \qquad \text{(A.97)}
\end{aligned}$$

Finally, by performing the divergence operator

$$\int_\Omega |\boldsymbol{w}\cdot(\nabla\cdot(\lambda\mathbb{B}_N^{-1}\mathbb{T}))| \,\mathrm{d}x \leq \delta_0\|\boldsymbol{w}\|_{\boldsymbol{L}^2_\sigma}^2 + c_{\delta_0}\lambda^2\|\mathbb{T}\|_{\mathbb{W}_{\mathrm{s}}^{-1,2}}^2. \qquad \text{(A.98)}$$

Thus, upon plugging the estimates (A.90)–(A.98) in the equation (A.74) with $\mathbb{Y} = \mathbb{B}_N^{-1}\mathbb{T}$ and taking $0 < \delta < \frac{\varepsilon_0}{10}$, we obtain a constant $c = c_{\delta,\varepsilon_0} > 0$ and $K_7 = K_7(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S}) \in L^1(I)$ such that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbb{T}\|_{\mathbb{W}_{\mathrm{s}}^{-1,2}}^2 + \frac{1}{c}\|\mathbb{T}\|_{\mathbb{L}^2_{\mathrm{s}}}^2 - 3\delta_0\|\boldsymbol{w}\|_{\boldsymbol{L}^2_\sigma}^2 \\
&\quad\leq c_{\delta_0}[K_7(\|\psi\|_{L^2}^2 + \|\eta\|_{L^2}^2 + \|\mathbb{T}\|_{\mathbb{W}_{\mathrm{s}}^{-1,2}}^2) + \|\mathbb{H}_{\mathrm{s}}\|_{\mathbb{W}_{\boldsymbol{n},\mathrm{s}}^{-2,2}}^2]. \qquad \text{(A.99)}
\end{aligned}$$

Now, we take the sum of (A.75), (A.76), (A.88), and (A.99) and choose $\delta_0 > 0$ small enough to obtain the a priori estimate (A.67) with $K := K_1 + K_3 + K_6 + K_7 \in L^1(I)$ and $E, D, S : I \to [0,\infty)$ are given by

$$\begin{aligned}
S &:= \|(h_{\mathrm{o}}, h_{\mathrm{c}}, h_{\mathrm{h}}, \boldsymbol{h}_{\mathrm{v}}, \mathbb{H}_{\mathrm{s}})\|_{(\mathcal{U}^2)^*}^2 \\
E &:= \|\psi\|_{L^2}^2 + \|\eta\|_{L^2}^2 + \tau\|\nabla\eta\|_{\boldsymbol{L}^2}^2 + \|\boldsymbol{w}\|_{\boldsymbol{W}_{0,\sigma}^{-1,2}}^2 + \|\mathbb{T}\|_{\mathbb{W}_{\mathrm{s}}^{-1,2}}^2 \\
D &:= \|\Delta\psi\|_{L^2}^2 + \|\xi\|_{L^2}^2 + \|\Delta\eta\|_{L^2}^2 + \|\boldsymbol{w}\|_{\boldsymbol{L}^2_\sigma}^2 + \|\mathbb{T}\|_{\mathbb{L}^2_{\mathrm{s}}}^2.
\end{aligned}$$

Applying the Grönwall inequality to the differential inequality and estimating the time derivatives will lead to the inequality

$$\|(\psi,\xi,\eta,\boldsymbol{w},\mathbb{T})\|_{\mathcal{V}^0} \leq \mathfrak{C}[\|(h_{\mathrm{o}}, h_{\mathrm{c}}, h_{\mathrm{h}}, \boldsymbol{h}_{\mathrm{v}}, \mathbb{H}_{\mathrm{s}})\|_{(\mathcal{U}^2)^*} + \|(\psi_0,\eta_0,\boldsymbol{w}_0,\mathbb{T}_0)\|_{\mathcal{D}^0}] \qquad \text{(A.100)}$$

for some continuous function $\mathfrak{C} = \mathfrak{C}(\|(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S})\|_{\mathcal{W}^2}) : [0,\infty) \to [0,\infty)$.

Theorem A.2. *Suppose that* (A1)$_3$ *and* (A2)$_1$ *hold. Consider sources and initial data as in* (A.70)*. Then, the linearized system* (4.2) *has a unique very weak solution satisfying the a priori estimate* (A.100)*. Also, if* $h_{\mathrm{c}} \in W^{1,2,2}(I; L^2(\Omega), W_{\boldsymbol{n}}^{-4,2}(\Omega))$*, then* $(\psi,\xi,\eta,\boldsymbol{w},\mathbb{T}) \in \mathcal{W}^0(\Omega_T)$ *and the estimate* (A.100) *is valid but with* $\mathcal{V}^0(\Omega_T)$ *and* $\mathcal{U}^2(\Omega_T)^*$ *replaced by* $\mathcal{W}^0(\Omega_T)$ *and* $\mathcal{Y}^2(\Omega_T)^*$*, respectively.*

Proof. The estimate (A.100) along with a density argument applied to the weak formulation will establish the existence, uniqueness, and stability of very weak solutions to the linearized system. More precisely, given initial data $(\psi_0,\eta_0,\boldsymbol{w}_0,\mathbb{T}_0) \in \mathcal{D}^0(\Omega)$ and source functions $(h_{\mathrm{o}}, h_{\mathrm{c}}, h_{\mathrm{h}}, \boldsymbol{h}_{\mathrm{v}}, \mathbb{H}_{\mathrm{s}}) \in \mathcal{U}^2(\Omega_T)^*$, we shall take approximation sequences $\{(\psi_{0,k},\eta_{0,k},\boldsymbol{w}_{0,k},\mathbb{T}_{0,k})\}_{k=1}^\infty \subset \mathcal{D}^1(\Omega)$ for the data and $\{(h_{\mathrm{o},k}, h_{\mathrm{c},k}, h_{\mathrm{h},k}, \boldsymbol{h}_{\mathrm{v},k}, \mathbb{H}_{\mathrm{s},k})\}_{k=1}^\infty \subset \mathcal{U}^1(\Omega_T)^*$ for the sources such that

$$(\psi_{0,k},\eta_{0,k},\boldsymbol{w}_{0,k},\mathbb{T}_{0,k}) \to (\psi_0,\eta_0,\boldsymbol{w}_0,\mathbb{T}_0) \text{ in } \mathcal{D}^0(\Omega)$$

$$(h_{\mathrm{o},k}, h_{\mathrm{c},k}, h_{\mathrm{h},k}, \boldsymbol{h}_{\mathrm{v},k}, \mathbb{H}_{\mathrm{s},k}) \to (h_{\mathrm{o}}, h_{\mathrm{c}}, h_{\mathrm{h}}, \boldsymbol{h}_{\mathrm{v}}, \mathbb{H}_{\mathrm{s}}) \text{ in } \mathcal{U}^2(\Omega_T)^*$$

as $k \to \infty$. By Theorem A.1, there exists a unique $(\psi_k, \xi_k, \eta_k, \boldsymbol{w}_k, \mathbb{T}_k) \in \mathcal{V}^1(\Omega_T)$ such that

$$\begin{aligned}\mathcal{A}(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S})&(\psi_k, \xi_k, \eta_k, \boldsymbol{w}_k, \mathbb{T}_k)\\ &= ((h_{\mathrm{o},k}, h_{\mathrm{c},k}, h_{\mathrm{h},k}, \boldsymbol{h}_{\mathrm{v},k}, \mathbb{H}_{\mathrm{s},k}), (\psi_{0,k}, \eta_{0,k}, \boldsymbol{w}_{0,k}, \mathbb{T}_{0,k}))\end{aligned} \tag{A.101}$$

and (A.100) holds with indices k. Take note that the function $\mathfrak{C} > 0$ in this inequality is independent of k.

Thanks to linearity, we deduce that $\{(\psi_k, \xi_k, \eta_k, \boldsymbol{w}_k, \mathbb{T}_k)\}_{k=1}^{\infty}$ is Cauchy in $\mathcal{V}^0(\Omega_T)$. Let us denote by $(\psi, \xi, \eta, \boldsymbol{w}, \mathbb{T}) \in \mathcal{V}^0(\Omega_T)$ the limit of this sequence. We claim that this is the very weak solution of (4.2). Indeed, recall that (A.101) is equivalent to the variational equations (A.2)–(A.5). Taking test functions $(\varphi, \vartheta, \boldsymbol{y}, \mathbb{Y}) \in W^{2,2}_{\boldsymbol{n}}(\Omega) \times W^{2,2}_{\boldsymbol{n}}(\Omega) \times \boldsymbol{W}^{2,2}_{0,\sigma}(\Omega) \times \mathbb{W}^{2,2}_{\boldsymbol{n},\mathrm{s}}(\Omega)$ in these equations, integrating by parts, and passing to the limit $k \to \infty$, we obtain (A.71)–(A.74). Also, it is not difficult to verify that $\xi + \alpha\Delta\psi - F''(\phi)\psi = h_{\mathrm{c}}$ in $L^2(I; L^2(\Omega))$ and that the initial condition holds in $\mathcal{D}^0(\Omega)$. Finally, the a priori estimate (A.100) follows by passing to the limit to the one satisfied by the approximating sequences. Uniqueness of the very weak solution follows from standard arguments.

The second statement of the theorem can be shown using the same argument as in Theorem A.1. Indeed, observe that $\Delta\partial_t\psi, \partial_t h_{\mathrm{c}} \in L^2(I; W^{-4,2}_{\boldsymbol{n}}(\Omega))$. For each $\varphi \in L^2(I; W^{2,2}_{\boldsymbol{n}}(\Omega))$, one has

$$\begin{aligned}\int_I\int_\Omega |\varphi F'''(\phi)\psi\partial_t\phi| \,\mathrm{d}x\,\mathrm{d}t &\le \|F'''(\phi)\|_{L^\infty(L^\infty)}\|\psi\|_{L^\infty(L^2)}\|\partial_t\phi\|_{L^2(L^2)}\|\varphi\|_{L^2(L^\infty)}\\ &\le c_\phi\|\psi\|_{L^\infty(L^2)}\|\varphi\|_{L^2(W^{2,2}_{\boldsymbol{n}})}.\end{aligned}$$

This estimate implies that $F'''(\phi)\psi\partial_t\phi \in L^2(I; W^{-2,2}_{\boldsymbol{n}}(\Omega))$. Now, using $\partial_{\boldsymbol{n}}(F''(\phi)\varphi) = F'''(\phi)\varphi\partial_{\boldsymbol{n}}\phi + F''(\phi)\partial_{\boldsymbol{n}}\varphi = 0$ on Γ_T and (A.111) in the next subsection, we deduce that

$$\begin{aligned}\|F''(\phi)\varphi\|_{L^2(W^{2,2}_{\boldsymbol{n}})} &\le c_\phi(\|\varphi\|_{L^2(L^2)} + \|\nabla\phi\|^2_{L^\infty(\boldsymbol{L}^4)}\|\varphi\|_{L^2(L^\infty)}\\ &\quad + \|\nabla\phi\|_{L^\infty(\boldsymbol{L}^4)}\|\nabla\varphi\|_{L^2(\boldsymbol{L}^4)} + \|\Delta\phi\|_{L^\infty(L^2)}\|\varphi\|_{L^2(L^\infty)} + \|\Delta\varphi\|_{L^2(L^2)})\\ &\le c_\phi\|\varphi\|_{L^2(W^{2,2}_{\boldsymbol{n}})}.\end{aligned}$$

By duality, this yields $F''(\phi)\varphi \in L^2(I; W^{-2,2}_{\boldsymbol{n}}(\Omega))$. Therefore, we have $\partial_t\xi \in L^2(I; W^{-4,2}_{\boldsymbol{n}}(\Omega))$ □

A.3. Strong Solutions to the Linearized System. A weak solution to (4.2) satisfying $(\psi, \xi, \eta, \boldsymbol{w}, \mathbb{T}) \in \mathcal{V}^2(\Omega_T)$ will be called a *strong solution*. For this, we consider sources and initial data satisfying

$$(h_{\mathrm{o}}, h_{\mathrm{c}}, h_{\mathrm{h}}, \boldsymbol{h}_{\mathrm{v}}, \mathbb{H}_{\mathrm{s}}) \in \mathcal{U}^0(\Omega_T)^*, \qquad (\psi_0, \eta_0, \boldsymbol{w}_0, \mathbb{T}_0) \in \mathcal{D}^2(\Omega). \tag{A.102}$$

Since $\mathcal{D}^2(\Omega) \subset \mathcal{D}^1(\Omega)$ and $\mathcal{U}^0(\Omega_T)^* \subset \mathcal{U}^1(\Omega_T)^*$, Theorem A.1 gives as the a priori regularity $(\psi, \xi, \eta, \boldsymbol{w}, \mathbb{T}) \in \mathcal{V}^1(\Omega_T)$. In the succeeding computations, we again formally derive the necessary a priori estimates, and for this reason, we shall use directly the strong formulation (4.2) of the linearized system.

Estimates for ψ in $L^\infty(W^{2,2}_{\boldsymbol{n}}) \cap L^2(W^{4,2}_{\boldsymbol{n}})$ and ξ in $L^2(W^{2,2}_{\boldsymbol{n}})$. We apply the test function $\Delta^2\psi$ in the first equation of (4.2) and estimate each term in the resulting equation. For the right-hand side and the convection terms, we obtain

$$\int_\Omega |h_{\mathrm{o}}\Delta^2\psi| \,\mathrm{d}x \le \delta\|\Delta^2\psi\|^2_{L^2} + c_\delta\|h_{\mathrm{o}}\|^2_{L^2} \tag{A.103}$$

$$\int_\Omega |(\boldsymbol{w}\cdot\nabla\phi)\Delta^2\psi| \,\mathrm{d}x \le \delta\|\Delta^2\psi\|^2_{L^2} + c_\delta\|\nabla\phi\|^2_{\boldsymbol{L}^\infty}\|\boldsymbol{w}\|^2_{\boldsymbol{L}^2_\sigma} \tag{A.104}$$

$$\int_\Omega |(\boldsymbol{v}\cdot\nabla\psi)\Delta^2\psi| \,\mathrm{d}x \le \delta\|\Delta^2\psi\|^2_{L^2} + c_\delta\|\boldsymbol{v}\|^2_{\boldsymbol{L}^4}\|\Delta\psi\|^2_{L^2}. \tag{A.105}$$

Performing the divergence in the mobility term, we have

$$\begin{aligned}
-\int_\Omega \nabla\cdot(m'(\phi,\theta)(\psi,\eta)\nabla\mu)\Delta^2\psi\,\mathrm{d}x &= -\int_\Omega m'(\phi,\theta)(\psi,\eta)\Delta\mu\Delta^2\psi\,\mathrm{d}x \\
&\quad -\int_\Omega (m_\phi(\phi,\theta)\nabla\psi + m_\theta(\phi,\theta)\nabla\eta)\cdot\nabla\mu\Delta^2\psi\,\mathrm{d}x \\
&\quad -\int_\Omega [m'_\phi(\phi,\theta)(\psi,\eta)\nabla\phi + m'_\theta(\phi,\theta)(\psi,\eta)\nabla\theta]\cdot\nabla\mu\Delta^2\psi\,\mathrm{d}x.
\end{aligned}$$

In what follows, unlike in the previous discussions, we will not emphasize the norms of the coefficient functions and their derivatives for brevity. With this convention, it follows that

$$\begin{aligned}
&\int_\Omega |m'(\phi,\theta)(\psi,\eta)\Delta\mu\Delta^2\psi|\,\mathrm{d}x \\
&\qquad \le \delta\|\Delta^2\psi\|^2_{L^2} + c_\delta\|\Delta\mu\|^2_{L^2}(\|\psi\|^2_{W^{2,2}_{\boldsymbol{n}}} + \|\eta\|^2_{W^{2,2}_{\boldsymbol{n}}})
\end{aligned} \tag{A.106}$$

$$\begin{aligned}
&\int_\Omega |(m_\phi(\phi,\theta)\nabla\psi + m_\theta(\phi,\theta)\nabla\eta)\cdot\nabla\mu\Delta^2\psi|\,\mathrm{d}x \\
&\qquad \le \delta\|\Delta^2\psi\|^2_{L^2} + c_\delta\|\nabla\mu\|^2_{\boldsymbol{L}^4}(\|\Delta\psi\|^2_{L^2} + \|\Delta\eta\|^2_{L^2})
\end{aligned} \tag{A.107}$$

$$\begin{aligned}
&\int_\Omega |[m'_\phi(\phi,\theta)(\psi,\eta)\nabla\phi + m'_\theta(\psi,\eta)\nabla\theta]\cdot\nabla\mu\Delta^2\psi|\,\mathrm{d}x \\
&\qquad \le \delta\|\Delta^2\psi\|^2_{L^2} + c_\delta J_{\phi,\theta}\|\nabla\mu\|^2_{\boldsymbol{L}^4}(\|\psi\|^2_{W^{2,2}_{\boldsymbol{n}}} + \|\eta\|^2_{W^{2,2}_{\boldsymbol{n}}})
\end{aligned} \tag{A.108}$$

where $J_{\phi,\theta} := \|\nabla\phi\|^2_{\boldsymbol{L}^4} + \|\nabla\theta\|^2_{\boldsymbol{L}^4} \in L^\infty(I)$. Also, note that

$$\begin{aligned}
-\int_\Omega \nabla\cdot(m(\phi,\theta)\nabla\xi)\Delta^2\psi\,\mathrm{d}x &= -\int_\Omega m(\phi,\theta)\Delta\xi\Delta^2\psi\,\mathrm{d}x \\
&\quad -\int_\Omega (m_\phi(\phi,\theta)\nabla\phi + m_\theta(\phi,\theta)\nabla\theta)\cdot\nabla\xi\Delta^2\psi\,\mathrm{d}x.
\end{aligned}$$

The right-hand sides can be bounded from above using the Gagliardo–Nirenberg inequality as follows:

$$\begin{aligned}
&\int_\Omega |(m_\phi(\phi,\theta)\nabla\phi + m_\theta(\phi,\theta)\nabla\theta)\cdot\nabla\xi\Delta^2\psi|\,\mathrm{d}x \\
&\qquad \le \delta\|\Delta^2\psi\|^2_{L^2} + \delta^2\|\Delta\xi\|^2_{L^2} + c_\delta J^2_{\phi,\theta}\|\nabla\xi\|^2_{\boldsymbol{L}^2}
\end{aligned} \tag{A.109}$$

$$-\int_\Omega m(\phi,\theta)\Delta\xi\Delta^2\psi\,\mathrm{d}x = \int_\Omega m(\phi,\theta)(\alpha\Delta^2\psi - \Delta(F''(\phi)\psi) - \Delta h_{\mathrm{c}})\Delta^2\psi\,\mathrm{d}x$$

$$\geq \frac{m_0\alpha}{2}\|\Delta^2\psi\|_{L^2}^2 - c(\|\Delta(F''(\phi)\psi)\|_{L^2}^2 + \|\Delta h_{\mathrm{c}}\|_{L^2}^2). \tag{A.110}$$

From $\Delta(F''(\phi)\psi) = F^{(4)}(\phi)|\nabla\phi|^2\psi + 2F'''(\phi)\nabla\phi\cdot\nabla\psi + F'''(\phi)\psi\Delta\phi + F''(\phi)\Delta\psi$, we have

$$\begin{aligned}\|\Delta(F''(\phi)\psi)\|_{L^2}^2 &\leq c[\|F^{(4)}(\phi)\|_{L^\infty}^2\|\nabla\phi\|_{\boldsymbol{L}^4}^4\|\psi\|_{L^\infty}^2 + \|F'''(\phi)\|_{L^\infty}^2\|\nabla\phi\|_{\boldsymbol{L}^4}^2\|\nabla\psi\|_{\boldsymbol{L}^4}^2 \\ &\quad + \|F'''(\phi)\|_{L^\infty}^2\|\Delta\phi\|_{L^2}^2\|\psi\|_{L^\infty}^2 + \|F''(\phi)\|_{L^\infty}^2\|\Delta\psi\|_{L^2}^2].\end{aligned} \tag{A.111}$$

Finally, the Laplacian of ξ satisfies

$$\delta\|\Delta\xi\|_{L^2}^2 - 2\alpha^2\delta\|\Delta^2\psi\|_{L^2}^2 \leq c\delta(\|\Delta(F''(\phi)\psi)\|_{L^2}^2 + \|\Delta h_{\mathrm{c}}\|_{L^2}^2). \tag{A.112}$$

Applying (A.111) in (A.110) and (A.112), using these together with (A.103)–(A.109) and taking $0 < \delta < 1$ small enough so that $\frac{m_0\alpha}{2} - 7\delta - 2\alpha^2\delta > 0$, we obtain for some constant $c > 0$ and $K_8 = K_8(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S}) \in L^1(I)$ the following

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta\psi\|_{L^2}^2 + \frac{1}{c}\|\Delta^2\psi\|_{L^2}^2 + \frac{1}{c}\|\Delta\xi\|_{L^2}^2 &\leq c[K_8(\|\psi\|_{W_{\boldsymbol{n}}^{2,2}}^2 + \|\eta\|_{W_{\boldsymbol{n}}^{2,2}}^2 + \|\boldsymbol{w}\|_{\boldsymbol{L}_\sigma^2}^2) \\ &\quad + \|h_{\mathrm{o}}\|_{L^2}^2 + \|h_{\mathrm{c}}\|_{W_{\boldsymbol{n}}^{2,2}}^2 + J_{\phi,\theta}^2\|\nabla\xi\|_{\boldsymbol{L}^2}^2].\end{aligned} \tag{A.113}$$

Estimate for η in $L^\infty(W_{\boldsymbol{n}}^{3,2}) \cap L^2(W_{\boldsymbol{n}}^{4,2})$. We shall apply the test function $\Delta^2\eta$ to the third equation in (4.2). Similar to (A.103)–(A.105) we have

$$\int_\Omega |h_{\mathrm{h}}\Delta^2\eta|\,\mathrm{d}x \leq \delta\|\Delta^2\eta\|_{L^2}^2 + c_\delta\|h_{\mathrm{h}}\|_{L^2}^2 \tag{A.114}$$

$$\int_\Omega |(\boldsymbol{w}\cdot\nabla\theta)\Delta^2\eta|\,\mathrm{d}x \leq \delta\|\Delta^2\eta\|_{L^2}^2 + c_\delta\|\nabla\theta\|_{\boldsymbol{L}^\infty}^2\|\boldsymbol{w}\|_{\boldsymbol{L}_\sigma^2}^2 \tag{A.115}$$

$$\int_\Omega |(\boldsymbol{v}\cdot\nabla\eta)\Delta^2\eta|\,\mathrm{d}x \leq \delta\|\Delta^2\eta\|_{L^2}^2 + c_\delta\|\boldsymbol{v}\|_{\boldsymbol{L}^4}^2\|\Delta\eta\|_{L^2}^2. \tag{A.116}$$

Similar to (A.106)–(A.110), we deduce the following

$$\begin{aligned}&\int_\Omega |\nabla\cdot(\chi'(\phi,\theta)(\psi,\eta)\nabla\theta)\Delta^2\eta|\,\mathrm{d}x \\ &\qquad \leq \delta\|\Delta^2\eta\|_{L^2}^2 + c_\delta(J_{\phi,\theta}+1)\|\Delta\theta\|_{L^2}^2(\|\psi\|_{W_{\boldsymbol{n}}^{2,2}}^2 + \|\eta\|_{W_{\boldsymbol{n}}^{2,2}}^2)\end{aligned} \tag{A.117}$$

$$\int_\Omega |\nabla\cdot(\chi(\phi,\theta)\nabla\eta)\Delta^2\eta|\,\mathrm{d}x \leq \delta\|\Delta^2\eta\|_{L^2}^2 + c_\delta\|\Delta\eta\|_{L^2}^2 + c_\delta J_{\phi,\theta}^2\|\nabla\eta\|_{\boldsymbol{L}^2}^2. \tag{A.118}$$

For the terms involving the gravity and viscoelastic stress, one has

$$\int_\Omega |a_0\mathbf{g}\cdot\boldsymbol{w}\Delta^2\eta|\,\mathrm{d}x \leq \delta\|\Delta^2\eta\|_{L^2}^2 + c_\delta|a_0\mathbf{g}|^2\|\boldsymbol{w}\|_{\boldsymbol{L}_\sigma^2}^2 \tag{A.119}$$

$$\int_\Omega |\mathbb{T}:\mathbb{D}\boldsymbol{v}\Delta^2\eta|\,\mathrm{d}x \leq \delta\|\Delta^2\eta\|_{L^2}^2 + c_\delta\|\nabla\boldsymbol{v}\|_{\mathbb{L}^4}^2\|\mathbb{T}\|_{\mathbb{W}_{\mathrm{s}}^{1,2}}^2 \tag{A.120}$$

$$\int_\Omega |\mathbb{S}:\mathbb{D}\boldsymbol{w}\Delta^2\eta|\,\mathrm{d}x \leq \delta\|\Delta^2\eta\|_{L^2}^2 + c_\delta\|\mathbb{S}\|_{\mathbb{L}_{\mathrm{s}}^\infty}^2\|\nabla\boldsymbol{w}\|_{\mathbb{L}^2}^2. \tag{A.121}$$

Therefore, by choosing $0 < \delta < \frac{b}{8}$, we see that there exist $c > 0$ and $K_9 = K_9(\phi, \mu, \theta, \boldsymbol{v}, \mathbb{S}) \in L^1(I)$ for which

$$\frac{1}{2}\frac{d}{dt}(\|\Delta\eta\|_{L^2}^2 + \tau\|\nabla\Delta\eta\|_{\boldsymbol{L}^2}^2) + \frac{1}{c}\|\Delta^2\eta\|_{L^2}^2 \leq c[K_9(\|\psi\|_{W_{\boldsymbol{n}}^{2,2}}^2 + \|\eta\|_{W_{\boldsymbol{n}}^{2,2}}^2$$

$$+ \|\nabla \boldsymbol{w}\|_{\mathbb{L}^2}^2 + \|\mathbb{T}\|_{\mathbb{W}_{\rm s}^{1,2}}^2) + \|h_{\rm h}\|_{L^2}^2 + J_{\phi,\theta}^2\|\nabla\eta\|_{\boldsymbol{L}^2}^2]. \tag{A.122}$$

Estimate for $\boldsymbol{w}$ in $L^\infty(\boldsymbol{W}_{0,\sigma}^{1,2}) \cap L^2(\boldsymbol{W}_{0,\sigma}^{2,2})$. We take the test function $-\Delta\boldsymbol{w}$ for the linearized Navier–Stokes equation in (4.2) and estimate each terms. First, we have

$$\int_\Omega |\boldsymbol{h}_{\rm v}\cdot\Delta\boldsymbol{w}|\,{\rm d}x \le \delta\|\Delta\boldsymbol{w}\|_{\boldsymbol{L}^2}^2 + c_\delta\|\boldsymbol{h}_{\rm v}\|_{\boldsymbol{L}_\sigma^2}^2 \tag{A.123}$$

$$\int_\Omega |(b_{\rm o}\psi + b_{\rm h}\eta)\mathbf{g}\cdot\Delta\boldsymbol{w}|\,{\rm d}x \le \delta\|\Delta\boldsymbol{w}\|_{\boldsymbol{L}^2}^2 + c_\delta(\|\psi\|_{L^2}^2 + \|\eta\|_{L^2}^2)|\mathbf{g}|^2 \tag{A.124}$$

$$\int_\Omega |(\boldsymbol{w}\cdot\nabla)\boldsymbol{v}\cdot\Delta\boldsymbol{w}|\,{\rm d}x \le \delta\|\Delta\boldsymbol{w}\|_{\boldsymbol{L}^2}^2 + c_\delta\|\nabla\boldsymbol{v}\|_{\mathbb{L}^4}^2\|\nabla\boldsymbol{w}\|_{\mathbb{L}^2}^2 \tag{A.125}$$

$$\int_\Omega |(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}\cdot\Delta\boldsymbol{w}|\,{\rm d}x \le \delta\|\Delta\boldsymbol{w}\|_{\boldsymbol{L}^2}^2 + c_\delta\|\boldsymbol{v}\|_{\boldsymbol{L}^4}^2\|\nabla\boldsymbol{w}\|_{\mathbb{L}^2}^2. \tag{A.126}$$

For the viscosity term, the following bounds can be shown as before:

$$\begin{aligned}&\int_\Omega |\nabla\cdot(2\nu'(\phi,\theta)(\psi,\eta)\mathbb{D}\boldsymbol{v})\cdot\Delta\boldsymbol{w}|\,{\rm d}x\\ &\qquad\le \delta\|\Delta\boldsymbol{w}\|_{\boldsymbol{L}^2}^2 + c_\delta(J_{\phi,\theta}+1)\|\Delta\boldsymbol{v}\|_{\boldsymbol{L}^2}^2(\|\psi\|_{W_{\boldsymbol{n}}^{2,2}}^2 + \|\eta\|_{W_{\boldsymbol{n}}^{2,2}}^2)\end{aligned} \tag{A.127}$$

$$\begin{aligned}&\int_\Omega |\nabla\cdot(2\nu(\phi,\theta)\mathbb{D}\boldsymbol{w})\cdot\Delta\boldsymbol{w}|\,{\rm d}x\\ &\qquad\ge \frac{\nu_0}{2}\|\Delta\boldsymbol{w}\|_{\boldsymbol{L}^2}^2 - c_\delta(\|\nabla\phi\|_{\boldsymbol{L}^\infty}^2 + \|\nabla\theta\|_{\boldsymbol{L}^\infty}^2)\|\nabla\boldsymbol{w}\|_{\mathbb{L}^2}^2.\end{aligned} \tag{A.128}$$

Furthermore, we have

$$\int_\Omega |\nabla\cdot(\sigma\eta\mathbb{S})\cdot\Delta\boldsymbol{w}|\,{\rm d}x \le \delta\|\Delta\boldsymbol{w}\|_{\boldsymbol{L}^2}^2 + c_\delta\|\mathbb{S}\|_{\mathbb{W}_{\boldsymbol{n},{\rm s}}^{2,2}}^2\|\eta\|_{W^{1,2}}^2 \tag{A.129}$$

$$\begin{aligned}&\int_\Omega |\nabla\cdot(\mathbb{M}_{\mathbb{S}}(\theta,\mathbb{S})\mathbb{T})\cdot\Delta\boldsymbol{w}|\,{\rm d}x\\ &\qquad\le \delta\|\Delta\boldsymbol{w}\|_{\boldsymbol{L}^2}^2 + c_\delta(\|\theta\|_{W_{\boldsymbol{n}}^{2,2}}^2 + \|\mathbb{S}\|_{\mathbb{W}_{\boldsymbol{n},{\rm s}}^{2,2}}^2 + 1)\|\mathbb{T}\|_{\mathbb{W}_{\rm s}^{1,2}}^2\end{aligned} \tag{A.130}$$

$$\int_\Omega |\kappa\xi\nabla\phi\cdot\Delta\boldsymbol{w}|\,{\rm d}x \le \delta\|\Delta\boldsymbol{w}\|_{\boldsymbol{L}^2}^2 + c_\delta\|\nabla\phi\|_{\boldsymbol{L}^4}^2\|\xi\|_{L^4}^2 \tag{A.131}$$

$$\int_\Omega |\kappa\mu\nabla\psi\cdot\Delta\boldsymbol{w}|\,{\rm d}x \le \delta\|\Delta\boldsymbol{w}\|_{\boldsymbol{L}^2}^2 + c_\delta\|\mu\|_{L^4}^2\|\psi\|_{W_{\boldsymbol{n}}^{2,2}}^2. \tag{A.132}$$

Applying (A.123)–(A.132) to the equation obtained by testing the fourth equation of (4.2) with $-\Delta\boldsymbol{w}$ and choosing $0 < \delta < \frac{\nu_0}{18}$, one obtains

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{w}\|_{\mathbb{L}^2}^2 + \frac{1}{c}\|\Delta\boldsymbol{w}\|_{\boldsymbol{L}^2}^2 &\le cK_{10}(\|\psi\|_{W_{\boldsymbol{n}}^{2,2}}^2 + \|\eta\|_{W_{\boldsymbol{n}}^{2,2}}^2 + \|\nabla\boldsymbol{w}\|_{\mathbb{L}^2}^2 + \|\mathbb{T}\|_{\mathbb{W}_{\rm s}^{1,2}}^2)\\ &\quad + c(\|\boldsymbol{h}_{\rm v}\|_{\boldsymbol{L}_\sigma^2}^2 + J_{\phi,\theta}\|\xi\|_{W^{1,2}}^2)\end{aligned} \tag{A.133}$$

for some $c > 0$ and $K_{10} = K_{10}(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S}) \in L^1(I)$.

Estimate for $\mathbb{T}$ in $L^\infty(\mathbb{W}^{1,2}_{\rm s}) \cap L^2(\mathbb{W}^{2,2}_{\boldsymbol{n},\rm s})$. We apply the test function $-\Delta\mathbb{T}$ to the fifth equation in (4.2). As usual, we obtain the following

$$\int_\Omega |\mathbb{H}_{\rm s} : \Delta\mathbb{T}|\,{\rm d}x \le \delta\|\Delta\mathbb{T}\|^2_{\mathbb{L}^2_{\rm s}} + c_\delta\|\mathbb{H}_{\rm s}\|^2_{\mathbb{L}^2_{\rm s}} \tag{A.134}$$

$$\int_\Omega |(\boldsymbol{w}\cdot\nabla)\mathbb{S} : \Delta\mathbb{T}|\,{\rm d}x \le \delta\|\Delta\mathbb{T}\|^2_{\mathbb{L}^2_{\rm s}} + c_\delta\|\nabla\mathbb{S}\|^2_{(\mathbb{L}^4_{\rm s})^2}\|\nabla\boldsymbol{w}\|^2_{\mathbb{L}^2} \tag{A.135}$$

$$\int_\Omega |(\boldsymbol{v}\cdot\nabla)\mathbb{T} : \Delta\mathbb{T}|\,{\rm d}x \le \delta\|\Delta\mathbb{T}\|^2_{\mathbb{L}^2_{\rm s}} + c_\delta\|\boldsymbol{v}\|^2_{\boldsymbol{L}^\infty}\|\nabla\mathbb{T}\|^2_{(\mathbb{L}^2_{\rm s})^2}. \tag{A.136}$$

With regard to the commutator and anti-commutator terms, we get

$$\int_\Omega |\mathbb{J}(\boldsymbol{w},\mathbb{S}) : \Delta\mathbb{T}|\,{\rm d}x \le \delta\|\Delta\mathbb{T}\|^2_{\mathbb{L}^2_{\rm s}} + c_\delta\|\mathbb{S}\|^2_{\mathbb{L}^\infty_{\rm s}}\|\nabla\boldsymbol{w}\|^2_{\mathbb{L}^2} \tag{A.137}$$

$$\int_\Omega |\mathbb{J}(\boldsymbol{v},\mathbb{T}) : \Delta\mathbb{T}|\,{\rm d}x \le \delta\|\Delta\mathbb{T}\|^2_{\mathbb{L}^2_{\rm s}} + c_\delta\|\nabla\boldsymbol{v}\|^2_{\mathbb{L}^4}\|\mathbb{T}\|^2_{\mathbb{W}^{1,2}_{\rm s}}. \tag{A.138}$$

For the diffusion terms, similar to (A.127) and (A.128), we have

$$\begin{aligned}&\int_\Omega |\nabla\cdot(\varepsilon'(\phi,\theta)(\psi,\eta)\nabla\mathbb{S}) : \Delta\mathbb{T}|\,{\rm d}x\\ &\qquad\le \delta\|\Delta\mathbb{T}\|^2_{\mathbb{L}^2_{\rm s}} + c_\delta(J_{\phi,\theta}+1)\|\Delta\mathbb{S}\|^2_{\mathbb{L}^2_{\rm s}}(\|\psi\|^2_{W^{2,2}_{\boldsymbol{n}}} + \|\eta\|^2_{W^{2,2}_{\boldsymbol{n}}})\end{aligned} \tag{A.139}$$

$$\begin{aligned}&\int_\Omega |\nabla\cdot(\varepsilon(\phi,\theta)\nabla\mathbb{T}) \mathbin{\therefore} \Delta\mathbb{T}|\,{\rm d}x\\ &\qquad\ge \frac{\varepsilon_0}{2}\|\Delta\mathbb{T}\|^2_{\mathbb{L}^2_{\rm s}} - c_\delta(\|\nabla\phi\|^2_{\boldsymbol{L}^\infty} + \|\nabla\theta\|^2_{\boldsymbol{L}^\infty})\|\mathbb{T}\|^2_{\mathbb{W}^{1,2}_{\rm s}}.\end{aligned} \tag{A.140}$$

Finally, we have the following inequalities

$$\int_\Omega |\lambda\mathbb{D}\boldsymbol{w} : \Delta\mathbb{T}|\,{\rm d}x \le \delta\|\Delta\mathbb{T}\|^2_{\mathbb{L}^2_{\rm s}} + c_\delta\|\nabla\boldsymbol{w}\|^2_{\mathbb{L}^2} \tag{A.141}$$

$$\int_\Omega |\mathbb{P}'(\mathbb{S})\mathbb{T} : \Delta\mathbb{T}|\,{\rm d}x \le \delta\|\Delta\mathbb{T}\|^2_{\mathbb{L}^2_{\rm s}} + c_\delta(1+\|\mathbb{S}\|^4_{\mathbb{L}^8_{\rm s}})\|\mathbb{T}\|^2_{\mathbb{W}^{1,2}_{\rm s}}. \tag{A.142}$$

Choosing $0<\delta<\frac{\varepsilon_0}{16}$, the estimates (A.134)–(A.142) when applied to the equation derived by testing the fifth equation of (4.2) with $-\Delta\mathbb{T}$ show

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\mathbb{T}\|^2_{(\mathbb{L}^2_{\rm s})^2} + \frac{1}{c}\|\Delta\mathbb{T}\|^2_{\mathbb{L}^2_{\rm s}}\\ &\quad\le c[K_{11}(\|\psi\|^2_{W^{2,2}_{\boldsymbol{n}}} + \|\eta\|^2_{W^{2,2}_{\boldsymbol{n}}} + \|\nabla\boldsymbol{w}\|^2_{\mathbb{L}^2} + \|\mathbb{T}\|^2_{\mathbb{W}^{1,2}_{\rm s}}) + \|\mathbb{H}_{\rm s}\|^2_{\mathbb{L}^2_{\rm s}}]\end{aligned} \tag{A.143}$$

for some $c>0$ and $K_{11} = K_{11}(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S}) \in L^1(I)$.

Therefore, we have the energy estimate (A.67) with $K := K_8+K_9+K_{10}+K_{11} \in L^1(I)$ and $E, D, S : I \to [0,\infty)$ are given by

$$\begin{aligned}S &:= \|(h_{\rm o}, h_{\rm c}, h_{\rm h}, \boldsymbol{h}_{\rm v}, \mathbb{H}_{\rm s})\|^2_{(\mathcal{U}^0)^*} + (J^2_{\phi,\theta} + J_{\phi,\theta})\|\xi\|^2_{W^{1,2}} + J^2_{\phi,\theta}\|\eta\|^2_{W^{1,2}}\\ E &:= \|\Delta\psi\|^2_{L^2} + \|\Delta\eta\|^2_{L^2} + \tau\|\nabla\Delta\eta\|^2_{\boldsymbol{L}^2} + \|\nabla\boldsymbol{w}\|^2_{\mathbb{L}^2} + \|\nabla\mathbb{T}\|^2_{(\mathbb{L}^2_{\rm s})^2}\\ D &:= \|\Delta^2\psi\|^2_{L^2} + \|\Delta\xi\|^2_{L^2} + \|\Delta^2\eta\|^2_{L^2} + \|\Delta\boldsymbol{w}\|^2_{\boldsymbol{L}^2} + \|\Delta\mathbb{T}\|^2_{\mathbb{L}^2_{\rm s}}.\end{aligned}$$

Combining the above a priori estimates, as well as those that can be derived for the time-derivatives, we obtain for some continuous function $\mathfrak{C} = \mathfrak{C}(\|(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S})\|_{\mathcal{W}^2})$:

$[0,\infty) \to [0,\infty)$ the estimate

$$\|(\psi,\xi,\eta,\boldsymbol{w},\mathbb{T})\|_{\mathcal{V}^2} \le \mathfrak{C}[\|(h_{\mathrm{o}},h_{\mathrm{c}},h_{\mathrm{h}},\boldsymbol{h}_{\mathrm{v}},\mathbb{H}_{\mathrm{s}})\|_{(\mathcal{U}^0)^*} + \|(\psi_0,\eta_0,\boldsymbol{w}_0,\mathbb{T}_0)\|_{\mathcal{D}^2}]. \tag{A.144}$$

This leads to the following theorem.

Theorem A.3. *Let* $(\mathrm{A1})_4$ *and* $(\mathrm{A2})_2$ *be satisfied. Given sources and initial data as in* (A.102)*, the linearized system* (4.2) *admits a unique strong solution satisfying the a priori estimate* (A.144)*. If in addition,* $h_{\mathrm{c}} \in W^{1,2,2}(I; W^{2,2}_{\boldsymbol{n}}(\Omega), W^{-2,2}_{\boldsymbol{n}}(\Omega))$*, then* $(\psi,\xi,\eta,\boldsymbol{w},\mathbb{T}) \in \mathcal{W}^2(\Omega_T)$ *and* (A.144) *holds with* $\mathcal{V}^2(\Omega_T)$ *and* $\mathcal{U}^0(\Omega_T)^*$ *replaced by* $\mathcal{W}^2(\Omega_T)$ *and* $\mathcal{Y}^0(\Omega_T)^*$*, respectively.*

Proof. The a priori estimate (A.144) is the crucial ingredient in the Faedo–Galerkin method for showing that the weak solution lies in $\mathcal{V}^2(\Omega_T)$. Take note that for the second statement, it is enough to observe that $\Delta\partial_t\psi, \partial_t h_{\mathrm{c}} \in L^2(I; W^{-2,2}_{\boldsymbol{n}}(\Omega))$, $F'''(\phi)\psi\partial_t\phi, F''(\phi)\partial_t\psi \in L^2(I; L^2(\Omega))$, and follow the same argument as in the proof of Theorem A.1. □

A.4. Time-Regular Solutions of the State System. In this subsection, we provide strong solutions with additional time-regularity under smooth enough initial data and source functions in the state system (1.15). By bootstrapping, such results will lead to additional smoothness of the optimal solution to the control problem (see Section 6).

Theorem A.4. *Let* $k = 0,1,2$ *and suppose that* $(\mathrm{A1})_{3+k}$ *and* $(\mathrm{A2})_{1+k}$ *hold. In addition to the assumptions stated in Theorem 3.1, let us suppose that* $(\partial_t f_{\mathrm{o}}, 0, \partial_t f_{\mathrm{h}}, \partial_t \boldsymbol{f}_{\mathrm{v}} + \partial_t \boldsymbol{u}, \partial_t \mathbb{F}_{\mathrm{s}}) \in \mathcal{Y}^{2-k}(\Omega_T)^*$*,* $f_{\mathrm{o}}(0) \in W^{k,2}_{\boldsymbol{n}}(\Omega)$*,* $f_{\mathrm{h}}(0) \in W^{k-1,2}(\Omega)$*,* $\boldsymbol{f}_{\mathrm{v}}(0) + \boldsymbol{u}(0) \in \boldsymbol{W}^{k-1,2}_{0,\sigma}(\Omega)$*,* $\mathbb{F}_{\mathrm{s}}(0) \in \mathbb{W}^{k-1,2}_{\mathrm{s}}(\Omega)$*,* $\phi_0 \in W^{k+4,2}_{\boldsymbol{n}}(\Omega)$*,* $\theta_0 \in W^{k+3,2}_{\boldsymbol{n}}(\Omega)$*,* $\boldsymbol{v}_0 \in \boldsymbol{W}^{k+1,2}_{0,\sigma}(\Omega)$*, and* $\mathbb{S}_0 \in \mathbb{W}^{k+1,2}_{\boldsymbol{n},\mathrm{s}}(\Omega)$*. Then, the strong solution of* (1.15) *satisfies*

$$(\partial_t\phi, \partial_t\mu, \partial_t\theta, \partial_t\boldsymbol{v}, \partial_t\mathbb{S}) \in \mathcal{W}^k(\Omega_T) \tag{A.145}$$

and there is a monotone increasing and continuous function $\mathfrak{C} : [0,\infty) \to [0,\infty)$ *for which*

$$\begin{aligned}\|(\partial_t\phi, \partial_t\mu, \partial_t\theta, \partial_t\boldsymbol{v}, \partial_t\mathbb{S})\|_{\mathcal{W}^k} &\le \mathfrak{C}(\|(\partial_t f_{\mathrm{o}}, 0, \partial_t f_{\mathrm{h}}, \partial_t \boldsymbol{f}_{\mathrm{v}} + \partial_t \boldsymbol{u}, \partial_t \mathbb{F}_{\mathrm{s}})\|_{(\mathcal{Y}^{2-k})^*} \\ &\quad + \|(f_{\mathrm{o}}(0), f_{\mathrm{h}}(0), \boldsymbol{f}_{\mathrm{v}}(0) + \boldsymbol{u}(0), \mathbb{F}_{\mathrm{s}}(0))\|_{W^{k,2}_{\boldsymbol{n}} \times W^{k-1,2} \times \boldsymbol{W}^{k-1,2}_{0,\sigma} \times \mathbb{W}^{k-1,2}_{\mathrm{s}}} \\ &\quad + \|(\phi_0, \theta_0, \boldsymbol{v}_0, \mathbb{S}_0)\|_{W^{k+4,2}_{\boldsymbol{n}} \times W^{k+3,2}_{\boldsymbol{n}} \times \boldsymbol{W}^{k+1,2}_{0,\sigma} \times \mathbb{W}^{k+1,2}_{\boldsymbol{n},\mathrm{s}}}).\end{aligned} \tag{A.146}$$

Proof. Taking formally the time derivatives of the first five equations in (1.15), we obtain the linearized system

$$\begin{aligned}&\mathcal{A}(\phi,\mu,\theta,\boldsymbol{v},\mathbb{S})(\partial_t\phi, \partial_t\mu, \partial_t\theta, \partial_t\boldsymbol{v}, \partial_t\mathbb{S}) \\ &\quad = ((\partial_t f_{\mathrm{o}}, 0, \partial_t f_{\mathrm{h}}, \partial_t \boldsymbol{f}_{\mathrm{v}} + \partial_t \boldsymbol{u}, \partial_t \mathbb{F}_{\mathrm{s}}), (\partial_t\phi(0), \partial_t\theta(0), \partial_t\boldsymbol{v}(0), \partial_t\mathbb{S}(0))).\end{aligned}$$

Hence, according to Theorem 4.2, it is enough to show that $(\partial_t\phi(0), \partial_t\theta(0), \partial_t\boldsymbol{v}(0), \partial_t\mathbb{S}(0)) \in \mathcal{D}^k(\Omega)$ for $k = 0,1,2$. Although the following arguments utilize the strong formulation of the differentiated system, these can be made rigorous by a classical Faedo–Galerkin approach.

Estimates for $\partial_t\phi(0)$ and $\mu_0 := \mu(0)$. We evaluate at $t = 0$ the first and second equations in (1.15) to obtain

$$\begin{aligned}&\partial_t\phi(0) + \boldsymbol{v}_0 \cdot \nabla\phi_0 - \nabla \cdot (m(\phi_0, \theta_0)\nabla\mu_0) = f_{\mathrm{o}}(0),\\ &\mu_0 = -\alpha\Delta\phi_0 + F'(\phi_0).\end{aligned}$$

Using the Hölder inequality and the Sobolev embedding theorem, we obtain

$$\|\boldsymbol{v}_0 \cdot \nabla\phi_0\|_{W^{k,2}} \le c\|\boldsymbol{v}_0\|_{\boldsymbol{W}^{k+1,2}_{0,\sigma}}\|\phi_0\|_{W^{k+2,2}_{\boldsymbol{n}}}.$$

The initial diffusive mobility term can be written as

$$\nabla \cdot (m(\phi_0, \theta_0)\nabla\mu_0) = (m_\phi(\phi_0, \theta_0)\nabla\phi_0 + m_\theta(\phi_0, \theta_0)\nabla\theta_0) \cdot \nabla\mu_0 + m(\phi_0, \theta_0)\Delta\mu_0.$$

Thus, it follows that

$$\|\nabla \cdot (m(\phi_0, \theta_0)\nabla\mu_0)\|_{W^{k,2}} \le \mathfrak{C}(\|\phi_0\|_{W^{k+2,2}_{\boldsymbol{n}}} + \|\theta_0\|_{W^{k+2,2}_{\boldsymbol{n}}})\|\mu_0\|_{W^{k+2,2}},$$

where $\mathfrak{C} : [0,\infty) \to [0,\infty)$ denotes a generic continuous function that is monotonically increasing. Following the proof of Lemma 2.3, it is not difficult to see that $F'(\phi_0) \in W^{k+2,2}(\varOmega)$ whenever $\phi_0 \in W^{k+2,2}_{\boldsymbol{n}}(\varOmega)$ and

$$\|F'(\phi_0)\|_{W^{k+2,2}} \le \mathfrak{C}(\|\phi_0\|_{W^{k+2,2}_{\boldsymbol{n}}}).$$

Thus, by taking $t = 0$ in the second equation of (1.15), the initial chemical potential can be estimated by

$$\|\mu_0\|_{W^{k+2,2}} \le c[\|\phi_0\|_{W^{k+4,2}_{\boldsymbol{n}}} + \mathfrak{C}(\|\phi_0\|_{W^{k+2,2}_{\boldsymbol{n}}})].$$

When applied with the Faedo–Galerkin method, the above estimates will lead to $\partial_t\phi(0) \in W^{k,2}_{\boldsymbol{n}}(\varOmega)$. Note that $\partial_{\boldsymbol{n}}\partial_t\phi(0) = 0$ on $\varGamma$ if $k = 0, 1$ and $\partial_{\boldsymbol{n}}\Delta\partial_t\phi(0) = 0$ on $\varGamma$ if $k = 2$ are consequences of the fact that we have these properties at the level of Faedo-Galerkin approximations. This observation will be applied as well to $\partial_{\boldsymbol{n}}\partial_t\theta(0)$ and $\partial_{\boldsymbol{n}}\partial_t\mathbb{S}(0)$ below.

Estimate for $\partial_t\theta(0)$. We apply $B_N^{-1} := (I + \tau A_N)^{-1} \in \mathcal{L}(W^{k-1,2}(\varOmega), W^{k+1,2}(\varOmega))$ to third equation of (1.15) and evaluate at $t = 0$ to obtain

$$\partial_t\theta(0) + B_N^{-1}[\boldsymbol{v}_0 \cdot \nabla\theta_0 - \nabla \cdot (\chi(\phi_0, \theta_0)\nabla\theta_0) + b\Delta^2\theta_0] = B_N^{-1}[a_0\mathbf{g} \cdot \boldsymbol{v}_0 + \mathbb{S}_0 : \mathbb{D}\boldsymbol{v}_0 + f_{\mathrm{h}}(0)].$$

First, the convection and diffusion terms can be estimated as follows:

$$\begin{aligned}&\|B_N^{-1}(\boldsymbol{v}_0 \cdot \nabla\theta_0)\|_{W^{k+1,2}} \le c\|\boldsymbol{v}_0 \cdot \nabla\theta_0\|_{W^{k-1,2}} \le c\|\boldsymbol{v}_0\|_{\boldsymbol{W}^{k+1,2}_{0,\sigma}}\|\theta_0\|_{W^{k+1,2}_{\boldsymbol{n}}},\\ &\|B_N^{-1}\nabla \cdot (\chi(\phi_0, \theta_0)\nabla\theta_0)\|_{W^{k+1,2}} \le \mathfrak{C}(\|\phi_0\|_{W^{k+2,2}_{\boldsymbol{n}}} + \|\theta_0\|_{W^{k+2,2}_{\boldsymbol{n}}})\|\theta_0\|_{W^{k+1,2}_{\boldsymbol{n}}}.\end{aligned}$$

For the bi-Laplacian term, one has

$$\|B_N^{-1}(b\Delta^2\theta_0)\|_{W^{k+1,2}} \le c\|\Delta^2\theta_0\|_{W^{k-1,2}} \le c\|\theta_0\|_{W^{k+3,2}_{\boldsymbol{n}}}.$$

Moreover, we have the following estimates appearing on the right-hand sides

$$\begin{aligned}&\|B_N^{-1}(a_0\mathbf{g} \cdot \boldsymbol{v}_0)\|_{W^{k+1,2}} \le c\|\boldsymbol{v}_0\|_{\boldsymbol{W}^{k-1,2}_{0,\sigma}}\\ &\|B_N^{-1}(\mathbb{S}_0 : \mathbb{D}\boldsymbol{v}_0)\|_{W^{k+1,2}} \le c\|\mathbb{S}_0 : \mathbb{D}\boldsymbol{v}_0\|_{W^{k-1,2}} \le c\|\boldsymbol{v}_0\|_{\boldsymbol{W}^{k+1,2}_{0,\sigma}}\|\mathbb{S}_0\|_{\mathbb{W}^{k+1,2}_{\boldsymbol{n},\mathrm{s}}}.\end{aligned}$$

Therefore, from these inequalities, we can see that $\partial_t\theta(0) \in W^{k+1,2}_{\boldsymbol{n}}(\varOmega)$.

Estimate for $\partial_t \boldsymbol{v}(0)$. Let us evaluate the extended Navier–Stokes equation in (1.15) at $t = 0$ to get

$$\begin{aligned}\partial_t \boldsymbol{v}(0) &+ \boldsymbol{P}_\sigma[(\boldsymbol{v}_0 \cdot \nabla)\boldsymbol{v}_0 - \nabla \cdot (2\nu(\phi_0, \theta_0)\mathbb{D}\boldsymbol{v}_0)] \\ &= \boldsymbol{P}_\sigma[\nabla \cdot \mathbb{M}(\theta_0, \mathbb{S}_0) + \kappa\mu_0 \nabla\phi_0 + \rho(\phi_0, \theta_0)\mathbf{g} + \boldsymbol{f}_{\mathrm{v}}(0) + \boldsymbol{u}(0)].\end{aligned}$$

Analogous to the situation above, we have

$$\begin{aligned}&\|(\boldsymbol{v}_0 \cdot \nabla)\boldsymbol{v}_0\|_{\boldsymbol{W}^{k-1,2}} \le c\|\boldsymbol{v}_0\|^2_{\boldsymbol{W}^{k+1,2}_{0,\sigma}} \\ &\|\nabla \cdot (2\nu(\phi_0, \theta_0)\mathbb{D}\boldsymbol{v}_0)\|_{\boldsymbol{W}^{k-1,2}} \le \mathfrak{C}(\|\phi_0\|_{W^{k+2,2}_{\boldsymbol{n}}} + \|\theta_0\|_{W^{k+2,2}_{\boldsymbol{n}}})\|\boldsymbol{v}_0\|_{\boldsymbol{W}^{k+1,2}_{0,\sigma}}.\end{aligned}$$

Also, the terms appearing on the right-hand side can be bounded from above according to

$$\begin{aligned}&\|\mu_0 \nabla\phi_0\|_{\boldsymbol{W}^{k-1,2}} \le c\|\mu_0\|_{W^{k+1,2}}\|\phi_0\|_{W^{k+1,2}_{\boldsymbol{n}}} \\ &\|\nabla \cdot \mathbb{M}(\theta_0, \mathbb{S}_0)\|_{\boldsymbol{W}^{k-1,2}} \le c(\|\mathbb{S}_0\|_{\mathbb{W}^{k+1,2}_{\boldsymbol{n},\mathrm{s}}} + \|\theta_0\|_{W^{k+1,2}_{\boldsymbol{n}}} + 1)\|\mathbb{S}_0\|_{\mathbb{W}^{k+1,2}_{\boldsymbol{n},\mathrm{s}}} \\ &\|\rho(\phi_0, \theta_0)\mathbf{g}\|_{\boldsymbol{W}^{k-1,2}} \le c(1 + \|\phi_0\|_{W^{k-1,2}_{\boldsymbol{n}}} + \|\theta_0\|_{W^{k-1,2}_{\boldsymbol{n}}}).\end{aligned}$$

These will imply $\partial_t \boldsymbol{v}(0) \in \boldsymbol{W}^{k-1,2}_{0,\sigma}(\Omega)$ after applying the Faedo–Galerkin method. Again, we note that $\partial_t \boldsymbol{v}(0) = \mathbf{0}$ on Γ when $k = 2$ by using the same reasoning mentioned above for $\partial_t \phi(0)$.

Estimate for $\partial_t \mathbb{S}(0)$. Finally, we take $t = 0$ in the fifth equation of (1.15) so that

$$\partial_t \mathbb{S}(0) + (\boldsymbol{v}_0 \cdot \nabla)\mathbb{S}_0 + \mathbb{J}(\boldsymbol{v}_0, \mathbb{S}_0) - \nabla \cdot (\varepsilon(\phi_0, \theta_0)\nabla\mathbb{S}_0) = \lambda\mathbb{D}\boldsymbol{v}_0 + \mathbb{P}(\mathbb{S}_0) + \mathbb{F}_{\mathrm{s}}(0).$$

By utilizing the following estimates

$$\begin{aligned}&\|(\boldsymbol{v}_0 \cdot \nabla)\mathbb{S}_0 + \mathbb{J}(\boldsymbol{v}_0, \mathbb{S}_0)\|_{\mathbb{W}^{k-1,2}} \le c\|\boldsymbol{v}_0\|_{\boldsymbol{W}^{k+1,2}_{0,\sigma}}\|\mathbb{S}_0\|_{\mathbb{W}^{k+1,2}_{\boldsymbol{n},\mathrm{s}}} \\ &\|\nabla \cdot (\varepsilon(\phi_0, \theta_0)\nabla\mathbb{S}_0)\|_{\mathbb{W}^{k-1,2}} \le \mathfrak{C}(\|\phi_0\|_{W^{k+2,2}_{\boldsymbol{n}}} + \|\theta_0\|_{W^{k+2,2}_{\boldsymbol{n}}})\|\mathbb{S}_0\|_{\mathbb{W}^{k+1,2}_{\boldsymbol{n},\mathrm{s}}} \\ &\|\lambda\mathbb{D}\boldsymbol{v}_0 + \mathbb{P}(\mathbb{S}_0)\|_{\mathbb{W}^{k-1,2}} \le c\|\boldsymbol{v}_0\|_{\boldsymbol{W}^{k,2}_{0,\sigma}} + \mathfrak{C}(\|\mathbb{S}_0\|_{\mathbb{W}^{k+1,2}_{\boldsymbol{n},\mathrm{s}}})\end{aligned}$$

it can be deduced that $\partial_t \mathbb{S}(0) \in \mathbb{W}^{k-1,2}_{\boldsymbol{n},\mathrm{s}}(\Omega)$.

Hence, the above estimates imply that $(\partial_t \phi(0), \partial_t \theta(0), \partial_t \boldsymbol{v}(0), \partial_t \mathbb{S}(0)) \in \mathcal{D}^k(\Omega)$ for $k = 0, 1, 2$ whenever the sources and initial data satisfy the conditions as stated by the theorem. Therefore, (A.146) holds thanks to Theorem 4.2. □

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