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Partially Dissipative Viscous System of Balance Laws and Application to Kuznetsov–Westervelt Equation

PARTIALLY DISSIPATIVE VISCOUS SYSTEM OF BALANCE LAWS AND APPLICATION TO KUZNETSOV–WESTERVELT EQUATION

GILBERT PERALTA

Abstract.
We provide the well-posedness for a partially dissipative viscous system of balance laws in smooth Sobolev spaces under the same assumptions as in the case of inviscid balance laws. A priori estimates for coupled hyperbolic-parabolic linear systems with coefficients having limited regularity are derived using Friedrichs regularization and Moser-type estimates. Local existence for nonlinear systems will be established using the results of the linear theory and a suitable iteration scheme. The local existence theory is then applied to the Kuznetsov–Westervelt equation with damping for nonlinear wave acoustic propagation. Existence of global solutions for small data and their asymptotic stability are established.

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Parabolic-hyperbolic systems, Kuznetsov–Westervelt equation, asymptotic stability, energy method.

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1. Introduction

Partial viscous systems of conservation laws can be used to describe many physical phenomena. These are coupled systems of partial differential equations consisting of first-order hyperbolic equations and second-order parabolic equations. The hyperbolic part describes the evolution of the conservative quantities in the system, while the parabolic part constitutes those that posses dissipative mechanisms, for instance, viscosity and thermal conductivity. Several equations in fluid dynamics, magnetohydrodynamics, viscoelasticity, thermoelasticity, heat conduction with finite speed propagation, and nonlinear acoustics can be written in the form of such systems.

In this paper, we study the well-posedness of a partially dissipative viscous system of conservation laws with source terms that depend on the states. In this situation, the systems are called balance laws instead of conservation laws. More precisely, we consider the following coupled quasilinear hyperbolic-parabolic system of balance laws in non-divergence form

$$\begin{cases} A_0(w,z)\partial_t w + \displaystyle\sum_{j=1}^{d} A_j(w,z)\partial_j w = f(w,z,\nabla z) & \text{in } (0,\infty)\times\mathbb{R}^d, \\ B_0(w,z)\partial_t z - \displaystyle\sum_{j,k=1}^{d} B_{jk}(w,z)\partial_k\partial_j z = g(w,z,\nabla w,\nabla z) & \text{in } (0,\infty)\times\mathbb{R}^d, \\ w\big|_{t=0} = w_0, \qquad z\big|_{t=0} = z_0 & \text{in } \mathbb{R}^d. \end{cases} \tag{1.1}$$

The unknown state variables are $w : (0,\infty)\times\mathbb{R}^d \to \mathbb{R}^p$ and $z : (0,\infty)\times\mathbb{R}^d \to \mathbb{R}^{n-p}$. Here, n is a positive integer representing the size of the system, $0 \leq p \leq n$ is an integer representing the size of the hyperbolic part, while w_0 and z_0 denote the given initial data for the hyperbolic and parabolic components, respectively. The case $p = 0$ corresponds to a purely parabolic system, while in the case $p = n$ we have a plain hyperbolic system. Here, the functions A_j, B_{jk}, f, and g encode the type of the physics modelled by the system (1.1). Precise regularity assumptions and other conditions for the coefficient matrices and on the inhomogeneities will be given below.

The coupled system (1.1) has been studied by Kawashima in his seminal work [25]. For general data in the Sobolev space $H^m(\mathbb{R}^d)$, where $m > \frac{d}{2} + 2$ is an integer, local-in-time existence and uniqueness of solutions has been established and energy estimates for the solutions in terms of the initial data have been provided. In addition, existence of global-in-time solutions for small amplitude data and their asymptotic stability has been proved as long as $m > \frac{d}{2} + 3$ and the so-called Shizuta–Kawashima (SK) condition is satisfied.

Local existence and uniqueness of solutions for the corresponding system of conservation laws, that is when f and g vanish, has been improved by Serre to $m > \frac{d}{2} + 1$ in [33]. However, energy estimates were not explicitly stated. This result means that the well-posedness theory for the viscid case is valid under the same assumptions as those in the inviscid case, for example those provided by Majda [27], Benzoni–Gavage and Serre [2], Rauch [32], and Metiviér [29]. The methodology presented in [33] is classical, one linearizes the system and then proceed with an iteration scheme. With regards to the linear system with smooth coefficients, the existence and uniqueness of solutions to the Cauchy problem for smooth data is obtained through a vanishing viscosity method, that is, by adding artificial viscosity to the whole system and passing to the limit as the artificial viscosity parameter tends to zero. For the iteration scheme, a priori estimates through nonlinear Moser-type inequalities for Sobolev spaces is the main ingredient. These a priori estimates provide a high-norm boundedness and low-norm contraction of the iterates, thereby providing existence of solutions for nonlinear systems. We would like to point out that in [25], the extra additional regularity is needed in the application of commutator estimates in Matsumura [28] and Mizohata [30].

The paper [33] has been extended by the author in the study of singular limits for a family of viscous conservation laws in [35]. Under certain stability criterion on the diffusion tensors, the uniform boundedness and convergence of solutions for the associated Cauchy problems have been established.

Our aim is to prove that the result in [33] is also valid in the non-homogenous case for $m > \frac{d}{2} + 1$, that is, for the system (1.1) originally considered in [25]. The new feature of the paper in comparison to the references mentioned above is the derivation of an a priori estimate for coupled hyperbolic-parabolic systems with limited regularity as well as the well-posedness of such systems, which in some sense are in the same spirit as in the inviscid case. We recall that for linear systems, a priori estimates provided in [25] involve only purely hyperbolic or purely parabolic cases (see [25, Proposition 2.7]). For uniqueness of solutions, we shall follow the method of Kawashima using mollifiers combined with Friedrichs regularization and Moser-type estimates to bypass the additional needed regularity.

Let us mention some of the previous works related to this manuscript. In [36], Vol'pert and Hudjaev considered a coupled nonlinear hyperbolic-parabolic system with initial data in $H^m(\mathbb{R}^d)$, where $m > \frac{d}{2} + 3$ in the general case and $m > \frac{d}{2} + 2$ in the quasilinear case. Using mollifiers, an auxiliary PDE system was studied and the local-in-time existence of solutions were established through the Schauder's fixed point theorem. The results were applied to a model for viscous compressible non-isothermal fluids.

Exploiting the result of [36] and using an entropy functional for symmetrization, Giovangigli and Massot applied the abstract result to multicomponent reactive flows for dilute polyatomic gas mixtures in [11]. Using the results and methods in [25], the corresponding asymptotic stability to constant equilibrium states was established in [12] under the condition $m \geq [\frac{d}{2}] + 2$, where $[\cdot]$ denotes the integral part of a real number. Moreover, the global-in-time existence and asymptotic stability for general systems were demonstrated in [13] when $m \geq [\frac{d}{2}] + 3$. The full vibrational non-equilibrium scenario was considered under a stronger assumption, $m > \frac{d}{2} + 3$, again using [36].

Regarding applications to ambipolar ionized gas mixtures where electric and magnetic fields were ignored, Giovangigli and Graille [10] considered entropic variables to symmetrize the system. Global-in-time existence for small data were proved when $m \geq [\frac{d}{2}] + 2$ and rational decay estimates were shown when $m \geq [\frac{d}{2}] + 3$ and an additional integrability of the deviation between the solutions and the steady states is satisfied. The extension to magnetized gas mixtures using partial symmetrization techniques and [36] were considered in [16]. Also, applications of hyperbolic-parabolic systems to compressible non-isothermal diffuse interface fluids can be found in [9].

Giovangigli and Yong studied the asymptotic stability with respect to the Chapman–Enskog expansions in [17] for coupled hyperbolic-parabolic systems. Asymptotic stability analysis of multicomponent reactive fluids under fast chemistry was proved by the same authors in [18] when $m \geq [\frac{d}{2}] + 2$. Second-order accurate reduced systems with respect to the relaxation parameter for small diffusion and stiff terms were studied in [15].

Common to most of the papers mentioned above are the decoupling of the parabolic and hyperbolic components of the system via linearization, application of Friedrichs regularization, a range condition for the initial data, demonstration of high-norm boundedness using commutator estimates, and the convergence of successive approximates (Picard iteration). These are also the same techniques utilized in this paper, however, as alluded earlier, the main difference is the condition $m > \frac{d}{2} + 1$, which was adapted in the unforced case as in the works of Serre in [33, 34, 35]. Besides, the application we have in mind deals with a nonlinear wave equation acoustic propagation model.

Recently, Crin-Barat and Danchin considered quasilinear hyperbolic systems that are symmetrizable and with partial dissipation in *hybrid* Besov-type spaces in [6] for the one-dimensional case and in [5] for the multi-dimensional case. Here, the SK condition is utilized, and Paley–Littlewood decompositions were used in the analysis of low and high frequencies with respect to the Fourier variables. Extension to the case where the solutions possess L^p-type boundedness for the low frequencies, where $p > 2$, is given in [7]. These were applied to isentropic compressible Euler systems with velocity damping (friction term) or with relaxation. The results of the current paper may be improved if one decouples the hyperbolic and parabolic parts and proceeds as in the analysis given in the purely hyperbolic case. Such adaptation is not covered in this paper, as it is outside of the scope.

All throughout this work, we shall consider the following assumptions as in [25]. The phase space O is assumed to be an open and convex subset of $\mathbb{R}^n$, and up to translation, we may take without restriction that it contains the origin. We use the notation $Q_T = [0,T] \times \mathbb{R}^d$ for the time-space domain, while $M_{m\times n}(\mathbb{R})$ denotes the set of matrices with real entries and size $m \times n$.

(A1) The functions $A_0 \in C^\infty(O; M_{p\times p}(\mathbb{R}))$ and $B_0 \in C^\infty(O; M_{(n-p)\times(n-p)}(\mathbb{R}))$ are symmetric and uniformly positive-definite on compact subsets of O, that is, for every compact subset K of O there exists $\varrho_K > 0$ such that $A_0(u)w \cdot w \geq \varrho_K |w|^2$ and $B_0(u)z \cdot z \geq \varrho_K |z|^2$ for every $u \in K$, $w \in \mathbb{R}^p$ and $z \in \mathbb{R}^{n-p}$.

(A2) For every $j = 1, \dots, d$, $A_j \in C^\infty(O; M_{p\times p}(\mathbb{R}))$ is symmetric.

(A3) The functions $B_{jk} \in C^\infty(O; M_{(n-p)\times(n-p)}(\mathbb{R}))$ are symmetric and satisfies $B_{jk} = B_{kj}$ for every $j, k = 1, \dots, d$. Moreover, for every compact subset K of O there exists $\zeta_K > 0$ such that

$$\sum_{j,k=1}^{d} B_{jk}(u) z_j z_k \geq \zeta_K |z|^2 \tag{1.2}$$

for every $u \in K$ and $z = (z_1, \dots, z_d) \in \mathbb{R}^d$.

(A4) We have $f \in C^\infty(O \times \mathbb{R}^{d\times(n-p)}; \mathbb{R}^p)$, $g \in C^\infty(O \times \mathbb{R}^{d\times n}; \mathbb{R}^{n-p})$, $f(0) = 0$, and $g(0) = 0$.

Notice that the coefficient matrices A_0, B_0, and B_{jk} are only locally uniformly positive-definite in the phase space. We limit ourselves to the case where the hyperbolic part is symmetric. Almost every model, if not all, in continuum physics can be put in a symmetric form. For example, systems that admit strongly convex entropies for which there is an available symmetrizer for the system. Also, the assumption that the diffusion tensors are symmetric is reasonable after a possible change of coordinates. This symmetric property is an example of the reciprocity relations of Onsager [31]. For more details, we refer the reader to [33, 34].

Various examples of physical models that can be written in the form of (1.1) can be found in [25]. Another example is the quasilinear strongly damped wave equation arising in nonlinear acoustics

$$\begin{aligned}(1 - 2\kappa u)\partial_{tt} u - c^2 \Delta u - b\Delta\partial_t u \\ = 2\kappa(\partial_t u)^2 + \sigma(|\nabla u|^2 + \mathcal{I}\nabla u \cdot \nabla \partial_t u) \text{ in } (0,\infty) \times \mathbb{R}^d\end{aligned} \tag{1.3}$$

where $\mathcal{I}$ is the integral operator

$$(\mathcal{I}v)(t,x) = \int_0^t v(\tau, x)\, \mathrm{d}\tau. \tag{1.4}$$

This model is called the Kuznetsov–Westervelt equation, where $u : (0,T) \times \mathbb{R}^d \to \mathbb{R}$ represents the acoustic pressure fluctuation, $c > 0$ is the speed of sound, $b > 0$ is the diffusivity of sound, and κ and σ are certain constants. By introducing the state variables $w = (u, \mathcal{I}u, \nabla u)$ and $z = \partial_t u$, one can easily check that (1.3) can be recasted in the form of (1.1). Further information will be given in Section 6.

The plan of this paper is as follows. In Section 2, we briefly recall the theory for purely hyperbolic and purely parabolic problems. Derivation of a priori estimates for the coupled linearized system having frozen coefficients with limited regularity will be the focus of Section 3. The well-posedness of linear and nonlinear systems will be given in Section 4 and Section 5, respectively. Section 6 deals with a modified version of the Kuznetsov–Westervelt equation incorporating additional damping terms. Finally, we recall in the Appendix the Friedrichs regularization and classical commutator and Moser-type estimates, which play important roles in the derivation of the a priori estimates.

2. Linear Hyperbolic and Parabolic Systems

The well-posedness of the hyperbolic-parabolic system (1.1) will be developed based on the results of Kawashima [25] for hyperbolic and parabolic systems with variable coefficients. For this purpose, consider the hyperbolic system

$$\begin{cases} A_0(v)\partial_t w + \displaystyle\sum_{j=1}^{d} A_j(v)\partial_j w = f & \text{in } (0,T)\times\mathbb{R}^d, \\ w|_{t=0} = w_0 & \text{in } \mathbb{R}^d, \end{cases} \tag{2.1}$$

with the unknown $w : (0,T)\times\mathbb{R}^d \to \mathbb{R}^p$ and the parabolic system

$$\begin{cases} B_0(v)\partial_t z - \displaystyle\sum_{j,k=1}^{d} B_{jk}(v)\partial_k\partial_j w = g & \text{in } (0,T)\times\mathbb{R}^d, \\ z|_{t=0} = z_0 & \text{in } \mathbb{R}^d, \end{cases} \tag{2.2}$$

with the unknown $z : (0,T)\times\mathbb{R}^d \to \mathbb{R}^{n-p}$. The frozen coefficient $v : (0,T)\times\mathbb{R}^d \to \mathbb{R}^n$, initial data $w_0 : \mathbb{R}^d \to \mathbb{R}^p$ and $z_0 : \mathbb{R}^d \to \mathbb{R}^{n-p}$, and source terms $f : (0,T)\times\mathbb{R}^d \to \mathbb{R}^p$ and $g : (0,T)\times\mathbb{R}^d \to \mathbb{R}^{n-p}$ are given functions. In the succeeding discussions, it will be useful to decompose the vector-valued function v as $[w_v\ z_v]^\top$ where w_v and z_v have p and $(n-p)$ components, respectively, and the superscript $^\top$ denotes transposition. Likewise, we write $u = [w\ z]^\top$.

We shall abbreviate the notation for the Sobolev space $H^m(\mathbb{R}^d)$ and Lebesgue space $L^p(\mathbb{R}^d)$ into H^m and L^p, respectively.

Theorem 2.1. [25, Proposition 2.7] *Suppose that the assumptions* (A1)–(A4) *are satisfied and let* $m > \frac{d}{2} + 1$ *be an integer. Assume that* $v \in C(0,T;H^{m+1})$*,* $w_v \in C^1(0,T;H^m)$*, and* $z_v \in C^1(0,T;H^{m-1}) \cap H^1(0,T;H^m)$*.*

(a) *Let* $1 \le \ell \le m+1$ *be an integer. If* $f \in C(0,T;H^{\ell-1}) \cap L^2(0,T;H^\ell)$ *and* $w_0 \in H^\ell$*, then the hyperbolic system* (2.1) *has a unique solution*

$$w \in C(0,T;H^\ell) \cap C^1(0,T;H^{\ell-1}).$$

(b) *Let $2 \leq \ell \leq m+1$ be an integer. If $g \in C(0,T;H^{\ell-1})$ and $z_0 \in H^\ell$, then the parabolic system* (2.2) *has a unique solution*
$$z \in C(0,T;H^\ell) \cap C^1(0,T;H^{\ell-2}) \cap L^2(0,T;H^{\ell+1}).$$

Energy estimates for the solutions are also available, however, they will not be utilized and hence we omit them in this paper. The proof of Theorem 2.1 is based on a priori estimates and Kato's theory for evolution equations.

For the hyperbolic PDE (2.1), there are other formulation of existence and uniqueness of solutions depending on the regularity of v and f. In the Sobolev space setting, we mention [2, 27, 29, 32] for such alternative formulations. On the other hand, for the parabolic PDE (2.2) with infinitely differentiable v and $g = 0$, the result in [33] is applicable.

3. A Priori Estimates for Coupled Linear Hyperbolic-Parabolic Systems

In this section, we will derive a priori estimates for coupled linear hyperbolic-parabolic systems. We consider the diffusion term in non-conservative form and include lower-order terms both in the hyperbolic and parabolic components. Define the following linear operators with variable coefficients

$$L_h[v](w,z) := A_0(v)\partial_t w + \sum_{j=1}^{d}(A_j(v)\partial_j w + C_j(v)\partial_j z) + D_1(v)w + D_2(v)z \tag{3.1}$$

$$L_p[v](w,z) := B_0(v)\partial_t z - \sum_{j,k=1}^{d} B_{jk}(v)\partial_j\partial_k z + \sum_{j=1}^{d}(E_j(v)\partial_j w + F_j(v)\partial_j z) + G_1(v)w + G_2(v)z, \tag{3.2}$$

where $v \in L^\infty(0,T;H^m) \cap H^1(0,T;H^{m-1})$ is given and $m > \frac{d}{2}+1$ is a fixed integer.

With regards to the coefficients for the lower-order terms, we consider the following hypothesis.

(A5) It holds that $C_j, D_2 \in C^\infty(O; M_{p\times(n-p)}(\mathbb{R}))$, $E_j, G_1 \in C^\infty(O; M_{(n-p)\times p}(\mathbb{R}))$, $F_j, G_2 \in C^\infty(O; M_{(n-p)\times(n-p)}(\mathbb{R}))$, and $D_1 \in C^\infty(O; M_{p\times p}(\mathbb{R}))$ for every $j = 1,\ldots,d$.

In certain situations, it is convenient to rewrite the two operators in (3.1) and (3.2) as a single operator. To do this, we introduce the block matrix-valued functions

$$S_0 := \begin{bmatrix} A_0 & 0 \\ 0 & B_0 \end{bmatrix}, \quad S_j := \begin{bmatrix} A_j & C_j \\ E_j & F_j \end{bmatrix},$$
$$R := \begin{bmatrix} D_1 & D_2 \\ G_1 & G_2 \end{bmatrix}, \quad Z_{jk} := \begin{bmatrix} 0 & 0 \\ 0 & B_{jk} \end{bmatrix}.$$

Define the following linear differential operator with variable coefficients

$$L[v]u := S_0(v)\partial_t u + \sum_{j=1}^{d} S_j(v)\partial_j u - \sum_{j,k=1}^{d} Z_{jk}(v)\partial_j\partial_k u + R(v)u.$$

Then, it is easy to see that we have

$$L[v]u = [L_h(v)w \ L_p(v)z]^\top.$$

Later, we will use the decomposition of $S_j = S_j^1 + S_j^2$ where

$$S_j^1 := \begin{bmatrix} A_j & 0 \\ 0 & 0 \end{bmatrix}, \qquad S_j^2 := \begin{bmatrix} 0 & C_j \\ E_j & F_j \end{bmatrix}, \tag{3.3}$$

and notice that S_j^1 is symmetric. Alternatively, one can write S_j as a sum of its symmetric and anti-symmetric parts, however, the above decomposition is sufficient for our purposes.

Let $\widetilde{L}$ be the operator corresponding to the conservative form of L in the second-order terms. More precisely, $\widetilde{L}$ is given by

$$\widetilde{L}[v] := S_0(v)\partial_t + \sum_{j=1}^d S_j(v)\partial_j - \sum_{j,k=1}^d \partial_j(Z_{jk}(v)\partial_k) + R(v).$$

Theorem 3.1. *Suppose that $v \in L^\infty(0,T;H^m) \cap H^1(0,T;H^{m-1})$ and $m > \frac{d}{2}+1$ is an integer. Assume that the range of v is contained in a compact subset K_0 of O and it contains the origin. Let K be another compact subset in O whose interior contains K_0. Then, for every $u = [w \ z]^\top \in H^1(0,T;H^m) \cap L^2(0,T;H^{m+1})$, there exists a constant $C > 0$ that depends only on d, m, K and the coefficients of $L[v]$ such that for every $\nu \in (0,1)$, we have*

$$\begin{aligned} &\sup_{t\in[0,T]} \|u(t)\|_{H^m}^2 + \int_0^T \|\nabla z(t)\|_{H^m}^2 \, \mathrm{d}t \\ &\le Ce^{C\mathfrak{p}_2(T,v)} \left(\|u(0)\|_{H^m}^2 + \int_0^T \{T^\nu \|L_h[v]u(t)\|_{H^m}^2 + \|L_p[v]u(t)\|_{H^{m-1}}^2\} \, \mathrm{d}t \right) \end{aligned} \tag{3.4}$$

where $\mathfrak{p}_2(T,v) := (1+T^{-\nu})\{T\mathfrak{p}_1(\|v\|_{L^\infty(0,T;H^m)}) + \sqrt{T}\|\partial_t v\|_{L^2(0,T;H^{m-1})}\}$ and $\mathfrak{p}_1(r) := 1 + r + r^2 + r^4$.

Proof. By a standard density argument, we may assume without loss of generality that u is infinitely differentiable and vanishes outside a compact subset of $\mathbb{R}^d$. Let α be a multi-index with length at most m. Taking the αth derivative of $S_0^{-1}L[v]u$, multiplying by S_0, taking the inner product with $\partial^\alpha u$ and then rearranging the terms yield

$$\begin{aligned} \widetilde{L}[v]\partial^\alpha u \cdot \partial^\alpha u &= S_0[\partial^\alpha, S_0^{-1}]L[v]u \cdot \partial^\alpha u + \partial^\alpha L[v]u \cdot \partial^\alpha u \\ &\quad - \sum_{j=1}^d S_0[\partial^\alpha, S_0^{-1}S_j\partial_j]u \cdot \partial^\alpha u - S_0[\partial^\alpha, S_0^{-1}R]u \cdot \partial^\alpha u \\ &\quad + \sum_{j,k=1}^d \{B_0[\partial^\alpha, B_0^{-1}B_{jk}]\partial_j\partial_k z - \mathrm{d}B_{jk}[\partial_j v]\partial_k\partial^\alpha z\} \cdot \partial^\alpha z \end{aligned} \tag{3.5}$$

where $[L_1, L_2] := L_1L_2 - L_2L_1$ is the commutator between two operators L_1 and L_2. The double sum in (3.5) involves only the component z due to the structure of the diffusion matrices Z_{jk}. Here and below, we do not explicitly write the dependence of the coefficient matrices on the frozen coefficient v. Also, $\mathrm{d}A$ denotes the first-order differential of A viewed as a linear form.

The first step is to bound from below the L^1-norm of the left-hand side of (3.5). To do this, we rewrite the said term as follows

$$\begin{aligned}\widetilde{L}[v]\partial^\alpha u\cdot\partial^\alpha u &= \frac{1}{2}\frac{d}{dt}(S_0\partial^\alpha u\cdot\partial^\alpha u)-\frac{1}{2}\,\mathrm{d}S_0[\partial_t v]\partial^\alpha u\cdot\partial^\alpha u\\ &\quad+\frac{1}{2}\sum_{j=1}^d\{\partial_j(S_j^1\partial^\alpha u\cdot\partial^\alpha u)-\mathrm{d}S_j^1[\partial_j v]\partial^\alpha u\cdot\partial^\alpha u+2S_j^2\partial_j\partial^\alpha u\cdot\partial^\alpha u\}\\ &\quad+\sum_{j,k=1}^d\{B_{jk}\partial_k\partial^\alpha z\cdot\partial_j\partial^\alpha z-\partial_j(B_{jk}\partial_k\partial^\alpha z\cdot\partial^\alpha z)\}+R\partial^\alpha u\cdot\partial^\alpha u. \qquad (3.6)\end{aligned}$$

Here, we used the symmetry of S_0 and S_j^1 for $j=1,\dots,d$. We integrate both sides of this equation over $\mathbb{R}^d$. Notice that the integral of the first term in the single sum and the second term in the double sum both vanish according to the divergence theorem and the fact that u vanishes at infinity. In the following estimates, C will denote generic positive constants that depend only on m, d, K, and the coefficients appearing in the operator $L[v]$.

Applying Hölder's inequality and the Sobolev embedding $H^{m-1}\subset L^\infty$ to the second term on the right-hand side of (3.6), we obtain the estimate

$$\begin{aligned}\|\mathrm{d}S_0[\partial_t v]\partial^\alpha u\cdot\partial^\alpha u\|_{L^1}&\le\|\,\mathrm{d}S_0\|_{L^\infty}\|\partial_t v\|_{L^\infty}\|\partial^\alpha u\|_{L^2}^2\\ &\le C\|\partial_t v\|_{H^{m-1}}\|\partial^\alpha u\|_{L^2}^2. \qquad (3.7)\end{aligned}$$

A similar process yields the inequality

$$\|\mathrm{d}S_j^1[\partial_j v]\partial^\alpha u\cdot\partial^\alpha u\|_{L^1}+\|R\partial^\alpha u\cdot\partial^\alpha u\|_{L^1}\le C(1+\|v\|_{H^m})\|\partial^\alpha u\|_{L^2}^2. \qquad (3.8)$$

Next, we estimate the last term in the single sum of (3.6). By using the definition of S_j^2 in (3.3) and then integrating by parts, see Proposition 7.6, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}S_j^2\partial_j\partial^\alpha u\cdot\partial^\alpha u\,\mathrm{d}x&=\int_{\mathbb{R}^d}\{C_j\partial_j\partial^\alpha z\cdot\partial^\alpha w+F_j\partial_j\partial^\alpha z\cdot\partial^\alpha z+E_j^T\partial^\alpha z\cdot\partial_j\partial^\alpha w\}\,\mathrm{d}x\\ &=\int_{\mathbb{R}^d}(C_j-E_j^T)\partial_j\partial^\alpha z\cdot\partial^\alpha w\,\mathrm{d}x-\int_{\mathbb{R}^d}\mathrm{d}E_j^T[\partial_j v]\partial^\alpha z\cdot\partial^\alpha w\,\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^d}F_j\partial_j\partial^\alpha z\cdot\partial^\alpha z\,\mathrm{d}x. \qquad (3.9)\end{aligned}$$

Applying Young's inequality to the first integral in (3.9), we have

$$\begin{aligned}\|(C_j-E_j^T)\partial_j\partial^\alpha z\cdot\partial^\alpha w\|_{L^1}&\le C_\eta\|(C_j^T-E_j)\partial^\alpha w\|_{L^2}^2+\eta\|\partial_j\partial^\alpha z\|_{L^2}^2\\ &\le C_\eta\|\partial^\alpha w\|_{L^2}^2+\eta\|\partial_j\partial^\alpha z\|_{L^2}^2\\ &\le C_\eta\|\partial^\alpha u\|_{L^2}^2+\eta\|\partial_j\partial^\alpha z\|_{L^2}^2 \qquad (3.10)\end{aligned}$$

where $\eta>0$ is a positive constant to be chosen later. In a similar manner, we obtain

$$\begin{aligned}\|F_j\partial_j\partial^\alpha z\cdot\partial^\alpha z\|_{L^1}&\le C_\eta\|\partial^\alpha z\|_{L^2}^2+\eta\|\partial_j\partial^\alpha z\|_{L^2}^2\\ &\le C_\eta\|\partial^\alpha u\|_{L^2}^2+\eta\|\partial_j\partial^\alpha z\|_{L^2}^2. \qquad (3.11)\end{aligned}$$

On the other hand, the second term in (3.9) can be estimated from above in the same way as in (3.7) to obtain

$$\begin{aligned}\|\,\mathrm{d}E_j^T[\partial_j v]\partial^\alpha z\cdot\partial^\alpha w\|_{L^1} &\le C\|v\|_{H^m}\|\partial^\alpha z\|_{L^2}\|\partial^\alpha w\|_{L^2} && (3.12)\\ &\le C\|v\|_{H^m}\|\partial^\alpha u\|_{L^2}^2. && (3.13)\end{aligned}$$

Combining the inequalities (3.10)–(3.13) yields

$$\|S_j^2\partial_j\partial^\alpha u\cdot\partial^\alpha u\|_{L^1}\le C_\eta(1+\|v\|_{H^m})\|\partial^\alpha u\|_{L^2}^2+2\eta\|\partial_j\partial^\alpha z\|_{L^2}^2. \qquad (3.14)$$

With regards to the first term in the double sum of (3.6), assumption (1.2) provides us the following estimate from below

$$\sum_{j,k=1}^d\int_{\mathbb{R}^d}B_{jk}\partial_k\partial^\alpha z\cdot\partial_j\partial^\alpha z\,\mathrm{d}x\ge\zeta_K\|\nabla\partial^\alpha z\|_{L^2}^2. \qquad (3.15)$$

By utilizing (3.7), (3.8), (3.14), and (3.15), we achieve the following estimate

$$\begin{aligned}\int_{\mathbb{R}^d}\widetilde{L}[v]\partial^\alpha u\cdot\partial^\alpha u\,\mathrm{d}x &\ge\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}S_0\partial^\alpha u\cdot\partial^\alpha u\,\mathrm{d}x+(\zeta_K-2\eta)\|\nabla\partial^\alpha z\|_{L^2}^2\\ &\quad -C_\eta(1+\|v\|_{H^m}+\|\partial_t v\|_{H^{m-1}})\|\partial^\alpha u\|_{L^2}^2. && (3.16)\end{aligned}$$

The next step is to integrate each term on the right-hand side of (3.5) and derive estimates from above. First, notice that $u(t), S_j(v(t))\in H^m$ for almost every $t\in[0,T]$ and for every j. From the Cauchy–Schwarz inequality, the commutator estimate (7.5) of Proposition 7.2, and Proposition 7.3, we get

$$\begin{aligned}\|S_0[\partial^\alpha,S_0^{-1}S_j\partial_j]u\cdot\partial^\alpha u\|_{L^1}&\le C\|S_0\|_{L^\infty}\|S_0^{-1}S_j\|_{H^m}\|u\|_{H^m}\|\partial^\alpha u\|_{L^2}\\ &\le C(\|v\|_{L^\infty})\|v\|_{H^m}\|u\|_{H^m}^2. && (3.17)\end{aligned}$$

If $|\alpha|=0$ then the commutators in (3.5) vanishes. Now, suppose that $|\alpha|\ge1$ so that by Leibniz rule

$$[\partial^\alpha,S_0^{-1}R]u\;=\sum_{\beta+\gamma=\alpha,\,|\beta|\ge1}d_{\beta\gamma}\partial^{\beta-\mu_\beta}(\partial^{\mu_\beta}(S_0^{-1}R))\partial^\gamma u \qquad (3.18)$$

where $d_{\beta\gamma}$ are constants and μ_β is a multi-index of length one and $\beta-\mu_\beta$ is a nonnegative multi-index. Since $|\beta-\mu_\beta|+|\gamma|=|\alpha|-1\le m-1$ and $\partial^{\mu_\beta}(S_0^{-1}R)(t), u(t)\in H^{m-1}$ for almost every $t\in[0,T]$, we can apply (7.3) to obtain

$$\begin{aligned}\|\partial^{\beta-\mu_\beta}(\partial^{\mu_\beta}(S_0^{-1}R))\partial^\gamma u\|_{L^2}&\le C\|\partial^{\mu_\beta}(S_0^{-1}R)\|_{H^{m-1}}\|u\|_{H^{m-1}}\\ &\le C(\|v\|_{L^\infty})\|v\|_{H^m}^2\|u\|_{H^{m-1}}. && (3.19)\end{aligned}$$

Thus, using the Cauchy–Schwarz inequality in (3.18) and then summing up yields

$$\|[\partial^\alpha,S_0^{-1}R]u\cdot\partial^\alpha u\|_{L^1}\le C(\|v\|_{L^\infty})\|v\|_{H^m}^2\|u\|_{H^m}^2. \qquad (3.20)$$

According to Young's inequality, for each j and k there holds

$$\|\,\mathrm{d}B_{jk}[\partial_j v]\partial_k\partial^\alpha z\cdot\partial^\alpha z\|_{L^1}\le\eta\|\partial_k\partial^\alpha z\|_{L^2}^2+C_\eta(\|v\|_{L^\infty})\|v\|_{H^m}^2\|\partial^\alpha z\|_{L^2}^2. \qquad (3.21)$$

Also, we expand the commutator $[\partial^\alpha,B_0^{-1}B_{jk}]z$ in virtue of the Leibniz rule so that

$$[\partial^\alpha,B_0^{-1}B_{jk}]z\;=\sum_{\beta+\gamma=\alpha,\,|\beta|\ge1}d_{\beta\gamma}\partial^{\beta-\mu_\beta}(\partial^{\mu_\beta}(B_0^{-1}B_{jk}))\partial^\gamma\partial_j\partial_k z.$$

Employing Young's inequality together with the same procedure as in the derivation of the estimate (3.19), we get

$$\|[\partial^\alpha, B_0^{-1}B_{jk}]z\cdot\partial^\alpha z\|_{L^1} \le \eta\|\partial_k z\|_{H^m}^2 + C_\eta(\|v\|_{L^\infty})\|v\|_{H^m}^4\|\partial^\alpha z\|_{L^2}^2. \tag{3.22}$$

Recall that $\partial^\alpha L[v]u\cdot\partial^\alpha u = \partial^\alpha L_h[v]u\cdot\partial^\alpha w + \partial^\alpha L_p[v]u\cdot\partial^\alpha z$. For the first term on the right-hand side, we utilize Young's inequality. With regards to the second term, we integrate by parts to pass one derivative of $L_p[v]u$ to $\partial^\alpha z$. Through these, we obtain the following estimate for each $\nu > 0$

$$\begin{aligned}\|\partial^\alpha L[v]u\cdot\partial^\alpha u\|_{L^1} &\le T^\nu\|\partial^\alpha L_h[v]u\|_{L^2}^2 + CT^{-\nu}\|\partial^\alpha w\|_{L^2}^2\\ &\quad + C_\eta\|L_p[v]u\|_{H^{m-1}}^2 + \eta\|\nabla\partial^\alpha z\|_{L^2}^2\end{aligned} \tag{3.23}$$

By Leibniz rule and using the same argument as in (3.19) and (3.20), the first commutator on the right-hand side of (3.5) can be estimated according to

$$\begin{aligned}&\|S_0[\partial^\alpha, S_0^{-1}]L[v]u\cdot\partial^\alpha u\|_{L^1}\\ &\le C(\|v\|_{L^\infty})\|v\|_{H^m}\|L[v]u\|_{H^{m-1}}\|\partial^\alpha u\|_{L^2}\\ &\le T^\nu\|L_h[v]u\|_{H^{m-1}}^2 + \|L_p[v]u\|_{H^{m-1}}^2 + C(\|v\|_{L^\infty})(1+T^{-\nu})\|v\|_{H^m}^2\|u\|_{H^m}^2.\end{aligned} \tag{3.24}$$

Combining the estimates (3.7), (3.8), (3.14), (3.16), (3.17), (3.20), and (3.22)–(3.24), and then taking the sum over all d-tuples α with length at most m, we obtain that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{S_0,m}^2 + (\zeta_K - c_d\eta)\|\nabla z\|_{H^m}^2\\ &\qquad\le C_\eta(\|v\|_{L^\infty})(1+T^{-\nu})(\mathfrak{p}_1(\|v\|_{H^m}) + \|\partial_t v\|_{H^{m-1}})\|u\|_{H^m}^2\\ &\qquad\quad + C(\|v\|_{L^\infty})T^\nu\|L_h[v]u\|_{H^m}^2 + C_\eta(\|v\|_{L^\infty})\|L_p[v]u\|_{H^{m-1}}^2\end{aligned} \tag{3.25}$$

where

$$\|u\|_{S_0,m}^2 := \sum_{|\alpha|\le m}\int_{\mathbb{R}^d} S_0\partial^\alpha u\cdot\partial^\alpha u\,\mathrm{d}x, \tag{3.26}$$

c_d is a constant independent of η and $\mathfrak{p}_1(r) := 1 + r + r^2 + r^4$.

Recall that the range of v lies in K and so $\|v\|_{L^\infty(Q_T)} \le \mathrm{diam}(K)$, hence we can replace the constant $C(\|v\|_{L^\infty(Q_T)})$ by a constant C depending on K. Moreover, $\|\cdot\|_{S_0,m}$ is equivalent to the H^m-norm. More precisely, there exist constants $c_{1,K}, c_{2,K} > 0$ such that for every $u \in H^m$

$$c_{1,K}\|u\|_{H^m} \le \|u\|_{S_0,m} \le c_{2,K}\|u\|_{H^m}.$$

Choosing $0 < \eta < \frac{\zeta_K}{c_d}$ and then using Gronwall's inequality in (3.25), we obtain the estimate (3.4) of the theorem. □

4. Well-Posedness for Linear Hyperbolic-Parabolic Systems

The current section is devoted to the existence and uniqueness of solutions for coupled parabolic-hyperbolic systems with limited regularity that is analogous to Theorem 2.1. However, our results are valid only for integers $m > \frac{d}{2} + 1$. This stems from the use of the energy estimate in Theorem 3.1. Nevertheless, such results are similar to the inviscid case.

Introduce the Banach spaces

$$X_T^m := L^\infty(0,T;H^m) \cap H^1(0,T;H^{m-1})$$
$$Y_T^m := \{u \in C(0,T;H^m) : z \in L^2(0,T;H^{m+1})\}$$

equipped with the norms

$$\|u\|_{X_T^m} := (\|u\|^2_{L^\infty(0,T;H^m)} + \|\partial_t u\|^2_{L^2(0,T;H^{m-1})})^{\frac{1}{2}}$$
$$\|u\|_{Y_T^m} := (\|u\|^2_{C(0,T;H^m)} + \|\nabla z\|^2_{L^2(0,T;H^m)})^{\frac{1}{2}}.$$

Recall that $u = [w\ z]^\top$ where w and z have p and $n-p$ components, respectively.

First, we establish the following lower-order estimates that will be utilize in the case of nonlinear systems.

Lemma 4.1. *Suppose that $m > \frac{d}{2}+1$ is an integer and $v_1, v_2 \in X_T^m$ satisfy a range condition as in Theorem 3.1. Let $u_j = [w_j\ z_j]^\top \in X_T^m$ with $z_j \in L^2(0,T;H^{m+1})$ for $j=1,2$. Assume that $f_j \in L^2(0,T;H^m)$, $g_j \in L^2(0,T;H^{m-1})$ and let $F_j = [f_j\ g_j]^\top$ for $j = 1,2$. If u_1 and u_2 satisfy the linear systems $L[v_1]u_1 = F_1$ and $L[v_2]u_2 = F_2$, respectively, and*

$$\max_{j=1,2}\{\|v_j\|_{X_T^m}, \|u_j\|_{X_T^m}, \|z_j\|_{L^2(0,T;H^{m+1})}\} \le M,$$

then there exists a constant $C = C(d,m,K,M) > 0$, depending continuously on its arguments, such that for every $\nu \in (0,1)$ we have

$$\sup_{t\in[0,T]} \|u(t)\|^2_{L^2} + \int_0^T \|\nabla z(t)\|^2_{L^2}\,\mathrm{d}t$$
$$\le Ce^{\mathfrak{p}_3(T)}\left(\|u(0)\|^2_{L^2} + \int_0^T \{T^\nu\|f(t)\|^2_{L^2} + \|g(t)\|^2_{L^2} + \|v(t)\|^2_{L^2}\}\,\mathrm{d}t\right) \tag{4.1}$$

where $u := u_1 - u_2$, $z := z_1 - z_2$, $v := v_1 - v_2$, $f := f_1 - f_2$, $g := g_1 - g_2$, and $\mathfrak{p}_3(T) := CT(1+T+T^{-\nu})$.

Proof. Let $u := u_2 - u_1$, $v := v_2 - v_1$, and $F := F_2 - F_1$. Taking the inner product of the equation $L[v_1]u_1 - L[v_2]u_2 = F$ with u and then rearranging the terms we have

$$\widetilde{L}[v_1]u\cdot u + \sum_{j,k=1}^d \mathrm{d}B_{jk}(v_1)[\partial_j v_1]\partial_k z\cdot z$$
$$= F\cdot u + \{S_0(v_1) - S_0(v_2)\}\partial_t u_2\cdot u + \{R(v_1) - R(v_2)\}u_2\cdot u$$
$$+ \sum_{j=1}^d \{S_j(v_1) - S_j(v_2)\}\partial_j u_2\cdot u + \sum_{j,k=1}^d \{B_{jk}(v_1) - B_{jk}(v_2)\}\partial_k z_2\cdot\partial_j z.$$

Let us denote the right-hand side by $Q_1 + Q_2$ where Q_2 represents the double sum. On one hand, by the same process as in the proof of the inequality (3.16), we have

$$\int_{\mathbb{R}^d} \widetilde{L}[v_1]u\cdot u + \sum_{j,k=1}^d \mathrm{d}B_{jk}(v_1)[\partial_j v_1]\partial_k z\cdot z\,\mathrm{d}x \tag{4.2}$$

$$\geq \frac{1}{2}\frac{d}{dt}\|u\|_{S_0,0}^2 + \frac{\zeta_K}{2}\|\nabla z\|_{L^2}^2 - C(1+\|v\|_{H^m}+\|\partial_t v\|_{H^{m-1}})\|u\|_{L^2}^2$$

for some constant $C = C(d, m, K, M) > 0$, where $\|\cdot\|_{S_0,0}$ is given by (3.26).

On the other hand, using a first-order Taylor expansion and Hölder's inequality, we have the following estimate for Q_1

$$\begin{aligned}\int_{\mathbb{R}^d} Q_1 \,\mathrm{d}x &\leq \|F\|_{L^2}\|u\|_{L^2} + C(\|\partial_t u_2\|_{L^\infty} + \|\nabla u_2\|_{L^\infty})\|u\|_{L^2}\|v\|_{L^2} \\ &\leq C(M)(T^\nu\|f\|_{L^2}^2 + \|g\|_{L^2}^2 + \|v\|_{L^2}^2 + (1 + T^{-\nu} + \|\partial_t u_2\|_{H^{m-1}}^2)\|u\|_{L^2}^2). \end{aligned} \tag{4.3}$$

For the second term Q_2, we apply the same argument together with the embedding $H^{m-1} \subset L^\infty$ to obtain

$$\begin{aligned}\int_{\mathbb{R}^d} Q_2 \,\mathrm{d}x &\leq C\|\nabla z_2\|_{L^\infty}\|v\|_{L^2}\|\nabla z\|_{L^2} \\ &\leq \frac{\zeta_K}{4}\|\nabla z\|_{L^2}^2 + C\|\nabla z_2\|_{H^m}^2\|v\|_{L^2}^2. \end{aligned} \tag{4.4}$$

Combining (4.2)–(4.4) and then applying Gronwall's Lemma, we obtain (4.1). □

The succeeding lemma is similar to the regularity result in [25, Lemma 2.6].

Lemma 4.2. *Assume that $m > \frac{d}{2} + 1$ is an integer and $v \in X_T^m$ satisfy a range condition as in Theorem 3.1. Let $u = [w\ z]^\top \in X_T^m$ be such that $z \in L^2(0,T;H^{m+1})$. Suppose that $u_0 \in H^m$, $f \in L^2(0,T;H^m)$, $g \in L^2(0,T;H^{m-1})$, and let $F = [f\ g]^\top$. If u satisfies the linear hyperbolic-parabolic system*

$$L[v]u = F, \qquad u|_{t=0} = u_0 \tag{4.5}$$

then $u \in C(0,T;H^m)$.

Proof. The main idea is to take the convolution of the equation with a Friedrichs mollifier. Let α be a multi-index with $|\alpha| \leq m$. Taking the αth derivative of $S_0^{-1}L[v]u$ and then applying S_0R_ε, where R_ε is the regularization operator defined in the Appendix, we obtain

$$\widetilde{L}[v]\partial^\alpha u_\varepsilon = \partial^\alpha F_\varepsilon + S_0R_\varepsilon G^\alpha + J_\varepsilon^\alpha + \sum_{j,k=1}^d \mathrm{d}Z_{jk}[\partial_j v]\partial_k\partial^\alpha u_\varepsilon$$

where $u_\varepsilon := R_\varepsilon u$, $F_\varepsilon := R_\varepsilon F$,

$$G^\alpha := -\sum_{j=1}^d [\partial^\alpha, S_0^{-1}S_j]u + \sum_{j,k=1}^d [\partial^\alpha, S_0^{-1}Z_{jk}]\partial_j\partial_k u - [\partial^\alpha, S_0^{-1}R]u + [\partial^\alpha, S_0^{-1}]F,$$

$$\begin{aligned} J_\varepsilon^\alpha := {} & S_0[R_\varepsilon, S_0^{-1}]\partial^\alpha F - \sum_{j=1}^d S_0[R_\varepsilon, S_0^{-1}S_j]\partial_j\partial^\alpha u + \sum_{j,k=1}^d S_0[R_\varepsilon, S_0^{-1}Z_{jk}]\partial_j\partial_k\partial^\alpha u \\ & - S_0[R_\varepsilon, S_0^{-1}R]\partial^\alpha u. \end{aligned}$$

We have $u_\varepsilon \in H^1(0,T;H^\infty) \subset C(0,T;H^\infty)$. Also, $u_\varepsilon \to u$ in $L^2(0,T;H^m)$, $z_\varepsilon \to z$ in $L^2(0,T;H^{m+1})$, $f_\varepsilon \to f$ in $L^2(0,T;H^m)$, and $g_\varepsilon \to g$ in $L^2(0,T;H^{m-1})$ according to Proposition 7.5 and the Lebesgue dominated convergence theorem.

Since $\partial^\alpha u \in L^2(0,T;L^2)$ and $\partial^\alpha F, \partial_j\partial_k\partial^\alpha z, \partial_j\partial^\alpha u \in L^2(0,T;H^{-1})$, it follows from Corollary 7.8 and the Lebesgue dominated convergence theorem once more that $J_\varepsilon^\alpha \to 0$ in $L^2(0,T;L^2)$. Because the commutators in G^α are of order at most $m-1$, we can see that $G^\alpha \in L^2(0,T;L^2)$. Therefore, $S_0R_\varepsilon G^\alpha \to S_0G^\alpha$ in $L^2(0,T;L^2)$ by Proposition 7.5. Using the same argument as in (3.16), we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \widetilde{L}[v]\partial^\alpha(u_\varepsilon - u_\delta)\cdot\partial^\alpha(u_\varepsilon - u_\delta)\,\mathrm{d}x \\
&\geq \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} S_0\partial^\alpha(u_\varepsilon - u_\delta)\cdot\partial^\alpha(u_\varepsilon - u_\delta)\,\mathrm{d}x + (\zeta_K - c_d\eta)\|\nabla\partial^\alpha(z_\varepsilon - z_\delta)\|_{L^2}^2 \\
&\quad - C_\eta(1 + \|v\|_{H^m} + \|\partial_t v\|_{H^{m-1}})\|\partial^\alpha(u_\varepsilon - u_\delta)\|_{L^2}^2. \qquad (4.6)
\end{aligned}$$

Similarly, for some constant C_η depending only on $\eta > 0$, the compact set K and the norm of v in $L^\infty(0,T;H^m)\cap H^1(0,T;H^{m-1})$, we have the following estimates

$$\begin{aligned}
&\int_{\mathbb{R}^d} \partial^\alpha(F_\varepsilon - F_\delta)\cdot\partial^\alpha(u_\varepsilon - u_\delta)\,\mathrm{d}x \qquad (4.7)\\
&\quad\leq \frac{1}{2}\|f_\varepsilon - f_\delta\|_{H^m}^2 + \frac{1}{2}\|u_\varepsilon - u_\delta\|_{H^m}^2 + C_\eta\|g_\varepsilon - g_\delta\|_{H^{m-1}}^2 + \eta\|\nabla(z_\varepsilon - z_\delta)\|_{H^m}^2 \\
&\int_{\mathbb{R}^d} \{S_0R_\varepsilon G^\alpha - S_0R_\delta G^\alpha + J_\varepsilon^\alpha - J_\delta^\alpha\}\cdot\partial^\alpha(u_\varepsilon - u_\delta)\,\mathrm{d}x \qquad (4.8)\\
&\quad\leq \frac{1}{2}\|S_0R_\varepsilon G^\alpha - S_0R_\delta G^\alpha\|_{L^2}^2 + \frac{1}{2}\|J_\varepsilon^\alpha - J_\delta^\alpha\|_{L^2}^2 + \|u_\varepsilon - u_\delta\|_{H^m}^2 \\
&\int_{\mathbb{R}^d} \mathrm{d}Z_{jk}[\partial_j v]\partial_k\partial^\alpha(u_\varepsilon - u_\delta)\cdot\partial^\alpha(u_\varepsilon - u_\delta)\,\mathrm{d}x \qquad (4.9)\\
&\quad\leq \eta\|\nabla(z_\varepsilon - z_\delta)\|_{H^m}^2 + C_\eta\|z_\varepsilon - z_\delta\|_{H^m}^2.
\end{aligned}$$

Combining (4.6)–(4.9), taking the sum over all multi-indices α of length at most m, choosing $\eta > 0$ small enough, and then applying Gronwall's inequality yield

$$\begin{aligned}
&\|u_\varepsilon - u_\delta\|_{C(0,T;H^m)}^2 + \int_0^T \|\nabla z_\varepsilon(t) - \nabla z_\delta(t)\|_{H^m}^2\,\mathrm{d}t \\
&\leq C\bigg(\|u_\varepsilon(0) - u_\delta(0)\|_{H^m}^2 + \int_0^T \|f_\varepsilon(t) - f_\delta(t)\|_{H^m}^2 + \|g_\varepsilon(t) - g_\delta(t)\|_{H^{m-1}}^2\,\mathrm{d}t \\
&\qquad + \sum_{|\alpha|\leq m}\int_0^T \|S_0R_\varepsilon G^\alpha(t) - S_0R_\delta G^\alpha(t)\|_{L^2}^2 + \|J_\varepsilon^\alpha(t) - J_\delta^\alpha(t)\|_{L^2}^2\,\mathrm{d}t\bigg)
\end{aligned}$$

where C depends only on v and T. Note that for each $\varepsilon > 0$ we have $u_\varepsilon(0) = R_\varepsilon u_0$ and hence $u_\varepsilon(0) \to u_0$ in H^m. Therefore, $(u_\varepsilon)_\varepsilon$ and $(z_\varepsilon)_\varepsilon$ are Cauchy sequences in $C(0,T;H^m)$ and $L^2(0,T;H^{m+1})$, respectively. By uniqueness of limits, these sequences converge to u and z respectively, thereby obtaining the desired regularities $u \in C(0,T;H^m)$ and $z \in L^2(0,T;H^{m+1})$. □

We now establish the well-posedness of the linear system (4.5).

Theorem 4.3. *Let $m > \frac{d}{2} + 1$ be an integer and $v \in X_T^m \cap C^1(0,T;L^2)$ satisfies a range condition as in Theorem 3.1. Suppose that $u_0 \in H^m$, $f \in C(0,T;L^2) \cap L^2(0,T;H^m)$, and $g \in C(0,T;L^2)\cap L^2(0,T;H^{m-1})$. The linear system (4.5) has a* *unique solution* $u \in C(0,T;H^m) \cap H^1(0,T;H^{m-1})$ *with* $z \in L^2(0,T;H^{m+1})$*. Moreover, there exists a constant* $C = C(d,m,K,M) > 0$ *such that for every* $\nu \in (0,1)$*, there holds*

$$\sup_{t\in[0,T]} \|u(t)\|_{H^m}^2 + \int_0^T \|\nabla z(t)\|_{H^m}^2 \,\mathrm{d}t$$
$$\leq Ce^{C\mathfrak{p}_2(T,v)} \left(\|u_0\|_{H^m}^2 + \int_0^T T^\nu \|f(t)\|_{H^m}^2 + \|g(t)\|_{H^{m-1}}^2 \,\mathrm{d}t \right) \tag{4.10}$$

where $\mathfrak{p}_2(T,v)$ *is of the form given by Theorem 3.1.*

Proof. *Step 1.* First, we consider the simple case where $D_1, D_2, E_j, C_j, F_j, G_1, G_2$ vanish and follow the methods in [25]. In this case, the hyperbolic and parabolic systems are uncoupled. We mollify the initial data, the frozen coefficient and the right-hand sides, that is, let $u_{0\varepsilon} := R_\varepsilon u_0$, $v_\varepsilon := R_\varepsilon v$, and $F_\varepsilon := R_\varepsilon F$. It follows that $u_{0\varepsilon} \in H^\infty$, $F_\varepsilon \in L^2(0,T;H^\infty) \cap C(0,T;H^\infty)$ and $v_\varepsilon \in C(0,T;H^\infty) \cap C^1(0,T;H^\infty)$, and moreover, we have $u_{0\varepsilon} \to u_0$ in H^m, $f_\varepsilon \to f$ in $L^2(0,T;H^m)$, $g_\varepsilon \to g$ in $L^2(0,T;H^{m-1})$, and $v_\varepsilon \to v$ in $L^2(0,T;H^m) \cap H^1(0,T;H^{m-1})$.

Consider the approximate linear hyperbolic system

$$L_h[v_\varepsilon]u^\varepsilon = f_\varepsilon, \qquad w^\varepsilon|_{t=0} = w_{0\varepsilon}, \tag{4.11}$$

and the approximate linear parabolic system

$$L_p[v_\varepsilon]u^\varepsilon = g_\varepsilon, \qquad z^\varepsilon|_{t=0} = z_{0\varepsilon}. \tag{4.12}$$

Theorem 2.1 implies that the Cauchy problems (4.11) and (4.12) admit solutions such that $w^\varepsilon, z^\varepsilon \in C(0,T;H^\infty) \cap C^1(0,T;H^\infty)$. Therefore, we can apply Theorem 3.1 and have the energy estimate

$$\sup_{t\in[0,T]} \|u^\varepsilon(t)\|_{H^m}^2 + \int_0^T \|\nabla z^\varepsilon(t)\|_{H^m}^2 \,\mathrm{d}t$$
$$\leq Ce^{C\mathfrak{p}_2(T,v_\varepsilon)} \left(\|u_{0\varepsilon}\|_{H^m}^2 + \int_0^T T^\nu \|f_\varepsilon(t)\|_{H^m}^2 + \|g_\varepsilon(t)\|_{H^{m-1}}^2 \,\mathrm{d}t \right). \tag{4.13}$$

From Proposition 7.5, the right-hand side of this estimate can be bounded above by the same term where $(u_{0\varepsilon}, v_\varepsilon, f_\varepsilon, g_\varepsilon)$ is replaced by (u_0, v, f, g). This implies that $(u_\varepsilon)_\varepsilon$ and $(z_\varepsilon)_\varepsilon$ are bounded in $L^\infty(0,T;H^m)$ and $L^2(0,T;H^{m+1})$, respectively. Therefore, u_ε converges weakly-star to u in $L^\infty(0,T;H^m)$ and z_ε converges weakly in $L^2(0,T;H^{m+1})$ to the component z of u. Moreover, by the lower semi-continuity of the norm with respect to the weak and weak-star topologies, we have

$$\liminf_{\varepsilon\to 0} \|u^\varepsilon\|_{L^\infty(0,T;H^m)} \leq \|u\|_{L^\infty(0,T;H^m)}$$
$$\liminf_{\varepsilon\to 0} \|\nabla z^\varepsilon\|_{L^2(0,T;H^m)} \leq \|\nabla z\|_{L^2(0,T;H^m)}.$$

Let $u_{\varepsilon,\delta} := u^\varepsilon - u^\delta$ and adapt similar definitions for $v_{\varepsilon,\delta}$, $f_{\varepsilon,\delta}$ and $g_{\varepsilon,\delta}$. Note that

$$\partial_t u^\varepsilon = S_0(v_\varepsilon)^{-1}\Big\{ -\sum_{j=1}^d S_j(v_\varepsilon)\partial_j u^\varepsilon + \sum_{j,k=1}^d Z_{jk}(v_\varepsilon)\partial_j\partial_k u^\varepsilon + F_\varepsilon \Big\}$$

and from this, we can see that $\partial_t u_\varepsilon$ is uniformly bounded in $L^2(0,T;H^{m-1})$, and hence converges weakly necessarily to $\partial_t u$ in this space. From Lemma 4.1, there exists a constant $C_T > 0$ independent of ε and δ such that

$$\begin{aligned}&\sup_{t\in[0,T]} \|u_{\varepsilon,\delta}(t)\|_{L^2}^2 + \int_0^T \|\nabla z_{\varepsilon,\delta}(t)\|_{L^2}^2 \,\mathrm{d}t \\ &\le C_T \left(\|u_{\varepsilon 0} - u_{\delta 0}\|_{L^2}^2 + \int_0^T \{\|f_{\varepsilon,\delta}(t)\|_{L^2}^2 + \|g_{\varepsilon,\delta}(t)\|_{L^2}^2 + \|v_{\varepsilon,\delta}(t)\|_{L^2}^2\} \,\mathrm{d}t \right).\end{aligned}$$

As a result, $(u^\varepsilon)_\varepsilon$ and $(z^\varepsilon)_\varepsilon$ are Cauchy sequences in $C(0,T;L^2)$ and $L^2(0,T;H^1)$, and they converge to u and z, respectively.

By interpolation, $u_\varepsilon \to u$ in $C(0,T;H^s)$ and $z_\varepsilon \to z$ in $L^2(0,T;H^{s+1})$ for every real number $s < m$. Choose s such that $\frac{d}{2} + 1 < s < m$. The Sobolev space $H^{s-1} \subset L^\infty$ is a Banach algebra, and in particular, $v_\varepsilon \to v$ almost everywhere on $[0,T] \times \mathbb{R}^d$. Passing to the limit on the approximate systems (4.11) and (4.12) in $L^2(0,T;H^{s-1})$ yields that u satisfies the linear system (4.5), and hence, $\partial_t u \in L^2(0,T;H^{m-1})$. Moreover, we have the regularity $u \in C(0,T;H^m)$ according to Lemma 4.2. The energy estimate (4.10) can be obtained by passing to the limit inferior in the estimate (4.13) for u^ε.

Step 2. Now we consider the general case where the coefficient matrices associated with the lower-order terms in the operators L_h and L_p do not vanish. This will be done through a standard fixed point argument. In this direction, we consider the map $\mathfrak{F}u = \widetilde{u}$ defined as follows: Given a sufficiently smooth $u = [w\ z]^\top$, let $\widetilde{u} = [\widetilde{w}\ \widetilde{z}]^\top$ be the solution of the uncoupled linear system

$$\begin{cases} A_0(v)\partial_t \widetilde{w} + \displaystyle\sum_{j=1}^d A_j(v)\partial_j \widetilde{w} = J_h[v]u + f, \\ B_0(v)\partial_t \widetilde{z} - \displaystyle\sum_{j,k=1}^d B_{jk}(v)\partial_j\partial_k \widetilde{z} = J_p[v]u + g, \\ \widetilde{w}|_{t=0} = w_0, \quad \widetilde{z}|_{t=0} = z_0, \end{cases} \tag{4.14}$$

where $J_h[v]$ and $J_p[v]$ are the linear operators

$$\begin{aligned} J_h[v]u &:= -\sum_{j=1}^d C_j(v)\partial_j z - D_1(v)w - D_2(v)z, \\ J_p[v]u &:= -\sum_{j=1}^d \{E_j(v)\partial_j w + F_j(v)\partial_j z\} - G_1(v)w - G_2(v)z. \end{aligned}$$

If $u \in Y_T^m$, then $J_h[v]u \in C(0,T;H^{m-1}) \cap L^2(0,T;H^m)$ and $J_p[v]u \in C(0,T;H^{m-1})$. From what we have shown in Step 1, the linear system (4.14) has a unique solution $\widetilde{u} \in Y_T^m$. This means that the linear map $\mathfrak{F} : Y_T^m \to Y_T^m$ given by $\mathfrak{F}u = \widetilde{w}$ is well-defined. For simplicity, we denote the z component of $\mathfrak{F}^k u$ by $\mathfrak{F}^k z$ for positive integers k, where $\mathfrak{F}^k$ denotes the composition of k-copies of $\mathfrak{F}$.

According to the energy estimates in Step 1, for every $u_1, u_2 \in Y_T^m$ and $\tau \in (0,T]$, if $u := u_1 - u_2$, then

$$\begin{aligned}
&\sup_{t\in[0,\tau]} \|\mathfrak{F}u(t)\|_{H^m}^2 + \int_0^\tau \|\nabla \mathfrak{F}z(t)\|_{H^m}^2 \,\mathrm{d}t \\
&\quad \le Ce^{C\mathfrak{p}_2(T,v)} \left(\int_0^\tau \tau^\nu \|J_h[v]u(t)\|_{H^m}^2 + \|J_p[v](t)\|_{H^{m-1}}^2 \,\mathrm{d}t \right) \\
&\quad \le Ce^{C\mathfrak{p}_2(T,v)} (1+\tau)^\nu \left(\int_0^\tau \|J_h[v]u(t)\|_{H^m}^2 + \|J_p[v](t)\|_{H^{m-1}}^2 \,\mathrm{d}t \right).
\end{aligned}$$

Applying Propositions 7.1–7.3, it is not hard to see that

$$\begin{aligned}
&\int_0^\tau \|J_h[v]u(t)\|_{H^m}^2 + \|J_p[v]u(t)\|_{H^{m-1}}^2 \,\mathrm{d}t \\
&\quad \le C_1(1+T) \left(\sup_{t\in[0,\tau]} \|u(t)\|_{H^m}^2 + \int_0^\tau \|\nabla z(t)\|_{H^m}^2 \,\mathrm{d}t \right)
\end{aligned}$$

for some constant $C_1 > 0$ depending only on d, m, K and v. Therefore, it holds that

$$\begin{aligned}
&\sup_{t\in[0,\tau]} \|\mathfrak{F}u(t)\|_{H^m}^2 + \int_0^\tau \|\nabla \mathfrak{F}z(t)\|_{H^m}^2 \,\mathrm{d}t \\
&\quad \le \kappa (1+\tau)^\nu \left(\sup_{t\in[0,\tau]} \|u(t)\|_{H^m}^2 + \int_0^\tau \|\nabla z(t)\|_{H^m}^2 \,\mathrm{d}t \right)
\end{aligned}$$

where $\kappa := CC_1 e^{C\mathfrak{p}_2(T,v)}(1+T)$.

An induction argument shows that for every positive integer k and $\tau \in (0,T]$,

$$\begin{aligned}
&\sup_{t\in[0,\tau]} \|\mathfrak{F}^k u(t)\|_{H^m}^2 + \int_0^\tau \|\nabla \mathfrak{F}^k z(t)\|_{H^m}^2 \,\mathrm{d}t \\
&\quad \le \frac{\kappa^k}{(k-1)!}(1+\tau)^{\nu+k-1} \left(\sup_{t\in[0,\tau]} \|u(t)\|_{H^m}^2 + \int_0^\tau \|\nabla z(t)\|_{H^m}^2 \,\mathrm{d}t \right).
\end{aligned}$$

Choosing $\tau = T$ and noting that $0 < \nu < 1$, we have

$$\|\mathfrak{F}^k u\|_{Y_T^m}^2 \le \frac{(\kappa(1+T))^k}{(k-1)!} \|u\|_{Y_T^m}^2.$$

For sufficiently large k the constant on the right-hand side becomes arbitrary small. Hence, $\mathfrak{F}^k$ will be a contraction and in virtue of the Banach Fixed Point Theorem, $\mathfrak{F}$ has a fixed point which corresponds to a solution of the linear system (4.5).

To prove that the energy estimate (4.10) holds as well, we proceed as in the previous step by approximating the frozen coefficient and the initial data using mollifiers. Hence, the a priori estimate in Theorem 3.1 is applicable. The approximated solution converges to the solution of the original problem and passing to the limit inferior for the energy estimates of the approximations, we obtain the energy estimate (4.10) for the solution of the general linear system (4.5). □

5. LOCAL WELL-POSEDNESS FOR NONLINEAR HYPERBOLIC-PARABOLIC SYSTEMS

We are now in position to prove the local-in-time well-posedness of the nonlinear hyperbolic-parabolic system (1.1).

Theorem 5.1. *Suppose that $m > \frac{d}{2} + 1$ is an integer and $u_0 = [w_0\ z_0]^\top \in H^m$ with range in a compact and convex subset K_0 of O containing the origin. Let K be another compact and convex subset of O whose interior contains K_0 and $\|u_0\|_{H^m} < M_0$. Then, the nonlinear system (1.1) has a unique solution such that $u = [w\ z]^\top \in C(0, T; H^m)$ with $w \in C^1(0, T; H^{m-1})$ and $z \in C^1(0, T; H^{m-2}) \cap L^2(0, T; H^{m+1})$ for some $T = T(d, m, K, M_0) > 0$. Moreover, there is a constant $C = C(d, m, K, M_0) > 0$ such that*

$$\sup_{t\in[0,T]} \|u(t)\|_{H^m}^2 + \int_0^T \|\nabla z(t)\|_{H^m}^2 \,\mathrm{d}t \le C\|u_0\|_{H^m}^2. \tag{5.1}$$

Proof. Given constants $M_1 > M_0$, $M_2 > M_0$ and $T > 0$, define the following space

$$W_T^m := \{u \in X_T^m \cap Y_T^m : \|u\|_{Y_T^m} \le M_1, \|\partial_t u\|_{L^2(0,T;H^{m-1})} \le M_2, \mathrm{Ran}\, u \subset K\}.$$

For a given $u^{n-1} = [w^{n-1}\ z^{n-1}]^\top \in W_T^m$ consider the uncoupled linearized system for $u^n = [w^n\ z^n]^\top \in W_T^m$

$$\begin{cases} A_0(u^{n-1})\partial_t w^n + \displaystyle\sum_{j=1}^d A_j(u^{n-1})\partial_j w^n = f(w^{n-1}, z^{n-1}, \nabla z^{n-1}) \\ B_0(u^{n-1})\partial_t z^n - \displaystyle\sum_{j,k=1}^d B_{jk}(u^{n-1})\partial_k\partial_j z^n = g(w^{n-1}, z^{n-1}, \nabla w^{n-1}, \nabla z^{n-1}) \\ w^n|_{t=0} = w_0, \quad z^n|_{t=0} = z_0 \end{cases} \tag{5.2}$$

with initial iterate $u^0 := [w_0\ z_0]^\top$. From the hypotheses of the theorem, $u^0 \in W_T^m$. In the following argument, we will reduce T so that we have $u^n \in W_T^m$ for every n. If $u^{n-1} \in W_T^m$, then $f(u^{n-1}, \nabla z^{n-1}) \in C(0, T; H^{m-1}) \cap L^2(0, T; H^m)$ and $g(u^{n-1}, \nabla u^{n-1}) \in C(0, T; H^{m-1})$. According to Theorem 4.3, the system (5.2) has a unique solution $u^n \in C(0, T; H^m)$, and moreover, $w^n \in C^1(0, T; H^{m-1})$ and $z^n \in C^1(0, T; H^{m-2}) \cap L^2(0, T; H^{m+1})$.

Range Condition. Let K_ρ be a ρ-neighborhood of K_0 that is contained in K, that is, $K_\rho = \{x \in \mathbb{R}^n : \mathrm{dist}(x, K_0) < \rho\} \subset K$. If $u \in W_T^m$, then

$$|u^n(t, x) - u_0(x)| \le \varrho_{d,m-1} \int_0^t \|\partial_t u^n(s)\|_{H^{m-1}} \,\mathrm{d}s \le \varrho_{d,m-1} M_2 T^{\frac{1}{2}}$$

for every $(t, x) \in [0, T] \times \mathbb{R}^d$, where $\varrho_{d,m-1}$ is the constant from the embedding $H^{m-1} \subset L^\infty$. Therefore, we have $\mathrm{Ran}\, u \subset K_\rho$ provided that $T > 0$ satisfies $\varrho_{d,m-1} M_2 T^{\frac{1}{2}} < \rho$.

High-Norm Boundedness. If $u^{n-1} \in W_T^m$, then from the energy estimate for the solution of the linear system (5.2), we have

$$\|u^n\|_{C(0,T;H^m)}^2 + \int_0^T \|\nabla z^n(t)\|_{H^m}^2 \,\mathrm{d}t \tag{5.3}$$

$$\leq \widetilde{C}(T)\left(M_0^2 + \int_0^T T^\nu \|f_{n-1}(t)\|_{H^m}^2 + \|g_{n-1}(t)\|_{H^{m-1}}^2 \,\mathrm{d}t\right)$$

where we have set $\widetilde{C}(T) := Ce^{C(1+T^{-\nu})(T\mathfrak{p}_1(M_1)+\sqrt{T}M_2)}$, $f_{n-1} := f(w^{n-1}, z^{n-1}, \nabla z^{n-1})$, and $g_{n-1} := g(w^{n-1}, z^{n-1}, \nabla z^{n-1}, \nabla z^{n-1})$. Here, we can take $C \geq 1$ without loss of generality. Applying Proposition 7.3,

$$\int_0^T \|f_{n-1}(t)\|_{H^m}^2 \,\mathrm{d}t \leq C\int_0^T \|u^{n-1}(t)\|_{H^m}^2 + \|\nabla z^{n-1}(t)\|_{H^m}^2 \,\mathrm{d}t \leq C(1+T)M_1^2 \tag{5.4}$$

$$\int_0^T \|g_{n-1}(t)\|_{H^{m-1}}^2 \,\mathrm{d}t \leq C\int_0^T \|u^{n-1}(t)\|_{H^m}^2 \,\mathrm{d}t \leq CTM_1^2. \tag{5.5}$$

Combining (5.3)–(5.5) provides us the estimate

$$\|u^n\|_{C(0,T;H^m)}^2 + \int_0^T \|\nabla z^n(t)\|_{H^m}^2 \,\mathrm{d}t \leq \widetilde{C}(T)\{M_0^2 + T^\nu(1+T)M_1^2 + TM_1^2\}.$$

Let us choose $M_1 > 1$ so that $M_1^2 > CM_0^2$. For a fix $\nu \in (0, \frac{1}{2})$, we can take $T > 0$ small enough so that $\widetilde{C}(T)\{M_0^2 + T^\nu(1+T)M_1^2 + TM_1^2\} < M_1^2$. From the differential equation

$$\partial_t u^n = S_0(u^{n-1})^{-1}\Big\{-\sum_{j=1}^d S_j(u^{n-1})\partial_j u^n + \sum_{j,k=1}^d Z_{jk}(u^{n-1})\partial_j\partial_k u^n + F_{n-1}\Big\}$$

where $F_{n-1} := [f_{n-1}\ g_{n-1}]^\top$, we have the estimate

$$\|\partial_t u^n\|_{H^{m-1}} \leq CM_1^2(M_1 + \|\nabla z^n\|_{H^{m+1}}).$$

Squaring this inequality and integrating with respect to time, we obtain

$$\|\partial_t u^n\|_{L^2(0,T;H^{m-1})}^2 \leq C(1+T)M_1^6.$$

By taking M_2 so that $M_2^2 > CM_1^6$ and then by making T small enough, we can now conclude that $u^n \in W_T^m$ whenever $u^{n-1} \in W_T^m$.

Low-Norm Contraction. Let us define $v^n := u^{n+1} - u^n$, $\zeta^n := z^{n+1} - z^n$, $f^n := f_n - f_{n-1}$, and $g^n := g_n - g_{n-1}$. Modifying the proof of Lemma 4.1, it can be deduced that

$$\begin{aligned}\mathscr{E}_n &:= \|v^n\|_{C(0,T;L^2)}^2 + \int_0^T \|\zeta^n(t)\|_{H^1}^2 \,\mathrm{d}t \\ &\leq Ce^{\mathfrak{p}_3(T)}\int_0^T \{T^\nu\|f^n(t)\|_{L^2}^2 + \|g^n(t)\|_{H^{-1}}^2 + \|v^{n-1}(t)\|_{L^2}^2\}\,\mathrm{d}t.\end{aligned} \tag{5.6}$$

Let us estimate the terms on the right-hand side. By the mean-value theorem

$$\int_0^T \|f^n(t)\|_{L^2}^2 \,\mathrm{d}t \leq CT\|v^{n-1}\|_{C(0,T;L^2)}^2 + \int_0^T \|\zeta^{n-1}(t)\|_{H^1}^2 \,\mathrm{d}t. \tag{5.7}$$

Theorem 7.4 with $s = m-1$ and $t = r = -1$ is applicable since $m \geq 2$, hence

$$\begin{aligned}\|g^n\|_{H^{-1}} &\leq \int_0^1 \|\,\mathrm{d}g(u_\theta^n, \nabla u_\theta^n)[v^{n-1}\ \nabla v^{n-1}]^\top\|_{H^{-1}} \,\mathrm{d}\theta \\ &\leq C\|\,\mathrm{d}g(u_\theta^n, \nabla u_\theta^n)\|_{H^{m-1}}(\|v^{n-1}\|_{H^{-1}} + \|\nabla v^{n-1}\|_{H^{-1}}) \leq C\|v^{n-1}\|_{L^2},\end{aligned}$$

where $u_\theta^n = \theta u^n + (1-\theta)u^{n-1}$ and $\theta \in [0,1]$. As a consequence, we have

$$\int_0^T \|g^n(t)\|_{H^{-1}}^2 \, \mathrm{d}t \leq CT\|v^{n-1}\|_{C(0,T;L^2)}^2. \tag{5.8}$$

Plugging (5.7) and (5.8) in (5.6), we obtain that

$$\mathscr{E}_n \leq Ce^{\mathfrak{p}_3(T)}(T + T^\nu + T^{\nu+1})\mathscr{E}_{n-1}.$$

Observe that $\mathfrak{p}_3$ is bounded on the compact interval $[0,T]$ since $\nu \in (0, \frac{1}{2})$. Let $T > 0$ be sufficiently small so that $0 < Ce^{\mathfrak{p}_3(T)}(T + T^\nu + T^{\nu+1}) < 1$. Then, $(u^n)_n$ and $(z^n)_n$ are Cauchy sequences in $C(0,T;L^2)$ and $L^2(0,T;H^1)$, respectively.

By interpolation one can obtain a solution of the nonlinear system by doing the same arguments as those that were provided in the linear case. The energy estimate for the solution of the nonlinear system follows from the estimate on the linear system where the frozen coefficient is equal to the solution u and by reducing $T > 0$ if necessary. The uniqueness of solutions for the nonlinear system follows from a similar argument as in the low-norm contraction part. Finally, $w \in C^1(0,T;H^{m-1})$ and $z \in C^1(0,T;H^{m-2}) \cap L^2(0,T;H^{m+1})$ follows from the differential equations and Proposition 7.4. □

6. The Kuznetsov–Westervelt Equation

As an application of the local existence theorem in the previous section, we consider a quasilinear strongly damped wave equation arising in nonlinear acoustics. The problem will be posed in the entire space and the main result of this section is the global-in-time existence and asymptotic stability of solutions. The Kuznetsov equation, which is a second order approximation for viscous, thermally conducting, inert fluids, and its simplified form known as the Westervelt equation, are the two most frequently used models in nonlinear acoustic wave propagation. Medical and industrial applications of these models include high-intensity focused ultrasound, such as, lithotripsy, sonochemistry, thermotherapy, and ultrasound cleaning, see [24].

An outline for the derivation of the model is presented for the sake of the reader. Starting from the continuity and the Navier–Stokes equations and assuming that the acoustic velocity is irrotational, one can derive the following model

$$u_{tt} - c^2\Delta u - b\Delta u_t = \frac{1}{\varrho_0 c^2}\frac{B}{2A}(u^2)_{tt} + \varrho_0(v \cdot v)_{tt}, \tag{6.1}$$

called the Kuznetsov equation. Here, $u = u(t,x)$ is the acoustic pressure fluctuation at time t and position x, while $v = v(t,x)$ is the acoustic particle velocity and it satisfies the continuity equation

$$\varrho_0 v_t + \nabla u = 0 \tag{6.2}$$

where ϱ_0 is the constant mass density. The positive constants c, b, and B/A represent the speed of sound, diffusivity of sound, and the parameter of nonlinearity.

A simplified model is obtained by ignoring local nonlinear effects. In this way, the term $u^2 - (\varrho_0 c)^2 v \cdot v$ is assumed to be small, and as a result (6.1) turns into

$$u_{tt} - c^2\Delta u - b\Delta u_t = \frac{1}{\varrho_0 c^2}\left(1 + \frac{B}{2A}\right)(u^2)_{tt}, \tag{6.3}$$

known as the Westervelt equation. This model describes the propagation of plane waves and is appropriate when the distance of propagation exceeds that of the wavelength. However, this approximate model is not suitable for standing waves, we refer to [24] for more details.

Using (6.2), both equations (6.1) and (6.3) can be written in the form of

$$(1 - 2\kappa u)u_{tt} - c^2\Delta u - b\Delta u_t = 2\kappa u_t^2 + \sigma(|\nabla u|^2 + \mathcal{I}\nabla u \cdot \nabla u_t), \tag{6.4}$$

where $\mathcal{I}$ is defined by (1.4). For the Kuznetsov equation we have $\kappa = B/(2A\varrho_0 c^2)$ and $\sigma = 0$ while $\kappa = (1+B/2A)\varrho_0 c^2$ and $\sigma = 2/\varrho_0$ in the case of the Westervelt equation. We consider a slightly modified model that includes friction and transversal acoustic pressure terms, namely,

$$(1 - 2\kappa u)u_{tt} - c^2\Delta u - b\Delta u_t + eu_t + du = 2\kappa u_t^2 + \sigma(|\nabla u|^2 + \mathcal{I}\nabla u \cdot \nabla u_t), \tag{6.5}$$

where e and d are positive constants.

Both the Kuznetsov and Westervelt equations have been studied in bounded domains under various boundary conditions. For example, Dirichlet boundary conditions has been considered in [21] for the homogeneous case and in [23] for the non-homogenous case, see also [22]. The common feature of these works are the local existence, global existence, and exponential stability of either strong or weak solutions. One of the main tools is semigroup theory for strongly damped wave equations. In fact, the evolution of the linearized system is described by an analytic semigroup. Appropriate energy estimates for the non-autonomous and non-homogeneous linearized systems are obtained using the variation of parameters formula and a fixed point argument. The case of non-homogeneous Dirichlet boundary condition is more delicate and an appropriate lifting of the boundary data into the interior is needed in the analysis. Exponential stability for the nonlinear models were obtained through barrier's method. For bounded domains having fractal boundaries, we refer the reader to the recent work [8].

Finite time horizon optimal control problems have been considered for the Westervelt and Kuznetsov equations in [4]. The boundary conditions are of mixed type where a Neumann condition is imposed on a part of the boundary where the control is applied and an absorbing boundary condition on the remaining part of the boundary. The purpose of such absorbing conditions is to avoid reflections on the artificial boundary of the computational domain. Functionals of tracking type were treated and first-order necessary conditions were derived.

The paper [20] focuses on a Westervelt-type model in bounded domains of spatial dimension at most 3 that takes the form

$$f'(u_t)u_{tt} - \Delta u = 0 \tag{6.6}$$

where f is sufficiently smooth satisfying $f(0) = 0$ and $f'(0) > 0$. For Dirichlet boundary conditions, the local existence and uniqueness of solutions for the nonlinear wave equation (6.6) was established while global existence and exponential decay was obtained for absorbing boundary conditions. The latter is possible thanks to the presence of boundary dissipation, for which observability estimates can be derived.

6.1. LOCAL EXISTENCE AND REGULARITY OF SOLUTIONS. In what follows, we shall rewrite the following quasilinear strongly damped wave equation

$$\begin{cases} (1-2\kappa u)u_{tt} - c^2\Delta u - b\Delta u_t + eu_t + du = 2\kappa u_t^2 + \sigma(|\nabla u|^2 + \mathcal{I}\nabla u \cdot \nabla u_t), \\ u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1, \end{cases} \tag{6.7}$$

in $(0,T)\times\mathbb{R}^d$, where $\mathcal{I}$ is the nonlocal-in-time operator given in (1.4) and $d \geq 2$. We assume that the coefficients are constants, and moreover, $c, b > 0$ and $e, d, \kappa, \sigma \in \mathbb{R}$. Notice that if κ is nonzero, then (6.7) has a potential degeneracy in the sense that the second order time derivative of u may vanish. Our goal is to prove the existence of solutions for initial data $(u_0, u_1) \in H^{s+1} \times H^s$ with integer $s > \frac{d}{2} + 1$ and u_0 taking values in the open ball

$$R_\kappa := \{x \in \mathbb{R} : |x| < \rho_\kappa\} \tag{6.8}$$

where $\rho_k = \frac{1}{2|\kappa|}$ for $\kappa \neq 0$ and $\rho_\kappa = \infty$ if $\kappa = 0$. For global existence of solutions, we shall impose the condition $e, d > 0$ and the smallness of the initial data.

The equation (6.7) can be formulated as a coupled hyperbolic-parabolic system, and thus, we can apply the local existence theory in Section 5. For this direction, we introduce the state variables $w := (w_1, w_2, w_3)^\top := (\nabla u, u, \mathcal{I}u)^\top$ and $z := u_t$. Then we have $w_{3t} = u = w_2$ and so $w_t = (\nabla z, z, w_2)^\top$. As a consequence, the time-derivative of z is given by

$$(1-2\kappa w_2)z_t - b\Delta z = c^2\nabla \cdot w_1 - dw_2 - ez + 2\kappa z^2 + \sigma(|w_1|^2 + \nabla w_3 \cdot \nabla z).$$

For each $(w, z) \in \mathbb{R}^{d+2} \times \mathbb{R}$, let $A_0(w, z)$ be the $(d+2)\times(d+2)$ identity matrix, $A_j(w, z)$ be the $(d+2)\times(d+2)$ zero matrix for $j = 1, 2, \ldots, d$, $B_0(w, z) = 1 - 2\kappa w_2$, and $B_{jk}(w, z) = b\delta_{jk}$ for $j, k = 1, 2, \ldots, d$, where δ_{jk} is the Kronecker delta symbol. Likewise, for $(w, z, y) \in \mathbb{R}^{d+2} \times \mathbb{R} \times \mathbb{R}^d$ and $\omega \in \mathbb{R}^{(d+2)\times d}$, we define the following functions

$$\begin{aligned} f(w, z, y) &:= (y, z, w_2)^\top \\ g(w, z, \omega, y) &:= c^2\mathrm{Tr}\,\omega_1 - dw_2 - ez + 2\kappa z^2 + \sigma(|w_1|^2 + \omega_3 \cdot y) \end{aligned}$$

where we decomposed $\omega = (\omega_1, \omega_2, \omega_3)^\top$, with ω_1 having size $d \times d$, while ω_2 and ω_3 both have size $d \times 1$. Here, Tr denotes the trace of a matrix. With these functions, (6.7) can now be written in the form of (1.1) with $w_0 = (\nabla u_0, u_0, 0)$ and $z_0 = u_1$.

Let $\mathcal{O}_\kappa := \{(w, z) \in \mathbb{R}^{d+2} \times \mathbb{R} : w_2 \in R_\kappa\}$ where R_κ is given by (6.8). According to the local existence theorem in Section 5, given $\delta_0 > 0$ and integer $s > \frac{d}{2} + 1$, for every initial data $(w_0, z_0) \in H^s \times H^s$ whose range lies compactly inside $\mathcal{O}_\kappa$ and satisfying $\|w_0\|_{H^s}^2 + \|z_0\|_{H^s}^2 \leq \delta_0$, the system (1.1) admits a unique solution on $[0, T_0]$ for some $T_0 = T_0(s, \delta_0) > 0$ such that the range of (w, z) lies in $\mathcal{O}_\kappa$ and

$$\begin{aligned} &w \in C(0, T_0; H^s) \cap C^1(0, T_0; H^{s-1}) \\ &z \in C(0, T_0; H^s) \cap C^1(0, T_0; H^{s-2}) \cap L^2(0, T; H^{s+1}). \end{aligned}$$

Moreover, there exists a constant $C = C(\delta_0, s) > 0$ such that

$$\sup_{0\le t\le T_0} (\|w(t)\|_{H^s}^2 + \|z(t)\|_{H^s}^2) + \int_0^{T_0} \|\nabla z(t)\|_{H^s}^2 \,\mathrm{d}t \le C(\|w_0\|_{H^s}^2 + \|z_0\|_{H^s}^2).$$

Translating these results to the Kuznetsov–Westervelt equation (6.7), we obtain a local existence theorem. Due to the structure of the nonlinear terms in (6.7), one can obtain additional regularity for the higher-order time derivatives of the solution. We remark that these additional regularity will play a crucial role in determining the appropriate energy and dissipation functionals that will be utilized in deriving a priori estimates for the nonlinear system. In order to prove this further regularity, we shall use the following classical Moser-type estimate for the product of suitable Sobolev functions in [26, Proposition B.4]. The information about the constant can be seen in the proof of the said proposition.

Theorem 6.1. *Let $s_1 > \frac{d}{2}$, $s_2 \ge -s_1$, and $c > 0$. Suppose that $f \in H^{s_2}$, $g \in H^{s_1} \cap H^{s_2}$, and $1 + g(x) \ge c$ for every $x \in \mathbb{R}^d$. Then, $(1+g)^{-1} f \in H^{s_1}$ and there exists a constant $C > 0$ which is increasing with respect to the second argument such that*

$$\left\|(1+g)^{-1} f\right\|_{H^{s_2}} \le C(c^{-1}, \|g\|_{H^{s_1}}) N(f, g)$$

where

$$N(f,g) = \begin{cases} \|f\|_{H^{s_2}}, & s_2 \le s_1 \\ \|f\|_{H^{s_2}} + \|f\|_{H^{s_1}} \|g\|_{H^{s_1}}, & s_2 > s_1. \end{cases}$$

A consequence of this result for space-time dependent functions is given below.

Corollary 6.2. *Suppose that $s_1 > \frac{d}{2}$, $|s_2| \le s_1$, $c \in (0,1)$, and $1 \le p \le \infty$. If $u \in L^p(0,T;H^{s_1})$ and $v \in L^\infty(0,T;H^{s_1})$ with $\|v\|_{L^\infty((0,T)\times\mathbb{R}^d)} \le c$, then $(1+v)^{-1}u \in L^p(0,T;H^{s_2})$ and there exists $C > 0$ such that*

$$\left\|(1+v)^{-1} u\right\|_{L^p(0,T;H^{s_2})} \le C((1-c)^{-1}, \|v\|_{L^\infty(0,T;H^{s_1})}) \|u\|_{L^p(0,T;H^{s_1})}.$$

For each real number s and $0 < T < \infty$, define the following space

$$X^{s,2}(0,T) := C(0,T;H^{s+1}) \cap C^1(0,T;H^s) \cap C^2(0,T;H^{s-2}).$$

Theorem 6.3. *Let $s > \frac{d}{2} + 1$ be an integer and $\delta_0 > 0$. For every initial data $u_0 \in H^{s+1}$ and $u_1 \in H^s$ such that the range of u_0 lies on a compact set of R_κ and $\|u_0\|_{H^{s+1}}^2 + \|u_1\|_{H^s}^2 \le \delta_0$, the equation (6.7) has a unique solution u such that $u(t,x) \in R_\kappa$ of every $(t,x) \in [0,T_0] \times \mathbb{R}^d$ and*

$$u \in X^{s,2}(0,T_0), \quad u_t \in L^2(0,T_0;H^{s+1}).$$

The solution u satisfies the energy estimate

$$\sup_{0\le t\le T_0} (\|u(t)\|_{H^{s+1}}^2 + \|u_t(t)\|_{H^s}^2 + \|(\mathcal{I}u)(t)\|_{H^s}^2) + \int_0^{T_0} \|\nabla u_t(t)\|_{H^s}^2 \,\mathrm{d}t \le C(\|u_0\|_{H^{s+1}}^2 + \|u_1\|_{H^s}^2) \tag{6.9}$$

for some constant $C = C(\delta_0, s) > 0$, and moreover, we have

$$u_{tt} \in L^2(0,T_0;H^{s-1}), \; u_{ttt} \in L^2(0,T_0;H^{s-3}). \tag{6.10}$$

Proof. The local existence of solutions including its stated regularity, range property and stability estimate (6.9) is a consequence of the discussion presented above. To see that the solution satisfies (6.10), we introduce the following operators

$$Pu := c^2\Delta u + b\Delta u_t - eu_t - du \tag{6.11}$$

$$S(u) := 2\kappa u_t^2 + \sigma(|\nabla u|^2 + (\nabla \mathcal{I}u)\cdot\nabla u_t) \tag{6.12}$$

so that we have

$$u_{tt} = (1-2\kappa u)^{-1}(Pu + S(u)). \tag{6.13}$$

First, observe that $Pu \in L^2(0,T_0;H^{s-1})$. Since the range of u lies on a compact set in R_κ, there exists $0 < \widetilde{\rho}_\kappa < \rho_\kappa$ such that $|u(t,x)| \leq \widetilde{\rho}_\kappa$ for every $(t,x) \leq [0,T_0]\times\mathbb{R}^d$. From the regularity $u \in C(0,T_0;H^{s+1})$ and taking $s_1 = s+1$ and $s_2 = s-1$ in Corollary 6.2, we have $(1-2\kappa u)^{-1}Pu \in L^2(0,T_0;H^{s-1})$. On the other hand, since $u_t, \nabla u, \nabla\mathcal{I}u \in C(0,T_0;H^s)$, we have $S(u) \in C(0,T_0;H^{s-1})$ and it follows that $(1-2\kappa u)^{-1}S(u) \in L^2(0,T_0;H^{s-1})$. As a result, we have $u_{tt} \in L^2(0,T_0;H^{s-1})$.

The product and chain rules applied to (6.13) give us the following equation

$$u_{ttt} = (1-2\kappa u)^{-1}(Pu_t + S(u)_t) + 2\kappa(1-2\kappa u)^{-2}u_t(Pu + S(u)).$$

In virtue of $Pu_t \in L^2(0,T_0;H^{s-3})$, taking $s_1 = s+1$ and $s_2 = s-3$ in Corollary 6.2, we get $(1-2\kappa u)Pu_t \in L^2(0,T_0;H^{s-3})$. Because $(\mathcal{I}u)_t = u$, we have

$$S(u)_t = 4\kappa u_t u_{tt} + 3\sigma\nabla u\cdot\nabla u_t + \sigma\nabla\mathcal{I}u\cdot\nabla u_{tt}. \tag{6.14}$$

Taking $s_1 = s$, $s_2 = s_3 = s-2$ in Theorem 7.4, we can see that $S(u)_t \in L^2(0,T_0;H^{s-2})$. Also, it is not difficult to see that $(1-2\kappa u)^{-2}u_t(Pu+S(u)) \in L^2(0,T_0;H^{s-1})$, and therefore, we have $u_{ttt} \in L^2(0,T_0;H^{s-3})$. This completes the proof of (6.10). □

For $\kappa \neq 0$, the range condition for u_0 stated in the previous theorem is satisfied as soon as $\delta_0 < (\rho_\kappa/\varrho_{d,s-1})^2$ where $\varrho_{d,s-1}$ is the constant of the embedding $H^{s-1} \subset L^\infty$. Indeed, for $x \in \mathbb{R}^d$, one has

$$|u_0(x)| \leq \varrho_{d,s-1}\|u_0\|_{H^{s-1}} \leq \varrho_{d,s-1}\sqrt{\delta_0} < \rho_\kappa.$$

For each real number s and $0 < T < \infty$, we define the space

$$\begin{aligned} Y^s(0,T) := \{u \in X^{s,2}(0,T) : u_t \in L^2(0,T;H^{s+1}),\\ u_{tt} \in L^2(0,T;H^{s-1}),\ u_{ttt} \in L^2(0,T;H^{s-3})\}. \end{aligned}$$

6.2. Energy Identities for the Linearized System. The goal of this subsection is to provide energy identities for the linearized equation corresponding to (6.7). To be precise, we shall derive energy identities for the linear differential operator

$$Lu := u_{tt} - c^2\Delta u - b\Delta u_t + eu_t + du = u_{tt} - Pu \tag{6.15}$$

for a fix $T > 0$ and sufficiently smooth u, where P is the operator defined by (6.11). The energy identities will be utilized in the proof of global existence of solutions. All throughout this section, $s \geq 2$ is an integer. If $u \in Y^s(0,T)$, then one can easily check that we have $Lu \in L^2(0,T;H^{s-1})$ and $Lu_t \in L^2(0,T;H^{s-3})$. For simplicity, let

$$v := \mathcal{I}u$$

so that $v_t = u$.

For an integer $\varsigma > 0$, we define for $t \in [0,T]$

$$F_\varsigma(w,z)(t) := \int_0^t (w(\tau), z(\tau))_{H^{\varsigma-1}}\, \mathrm{d}\tau + \sum_{|\alpha|=\varsigma} \int_0^t \langle \partial^\alpha w(\tau), \partial^\alpha z(\tau)\rangle\, \mathrm{d}\tau,$$

for $w \in L^2(0,T;H^{\varsigma-1})$ and $z \in L^2(0,T;H^{\varsigma+1})$. Here, the brackets denote duality pairing between H^{-1} and H^1. Note that if in addition $w \in L^2(0,T;H^\varsigma)$, then using the Fourier transform this duality pairing can be replaced by the inner product in L^2. Likewise, for $w \in L^2(0,T;H^{-1})$ and $z \in L^2(0,T;H^1)$, we set

$$F_0(w,z)(t) := \int_0^t \langle w(\tau), z(\tau)\rangle\, \mathrm{d}\tau \qquad \text{for } t \in [0,T].$$

Lemma 6.4. *Suppose that $s \geq 2$ is an integer and $u \in Y^s(0,T)$. For $t \in [0,T]$, let*

$$F_1(t) := F_s(Lu, u_t)(t)$$
$$F_2(t) := F_{s-2}(Lu_t, u_{tt})(t).$$

Then for $k = 1,2$ and $t \in [0,T]$ we have

$$E_k(t) + 2D_k(t) = E_k(0) + 2F_k(t)$$

where

$$\begin{aligned}
E_1(t) &:= \|u_t(t)\|_{H^s}^2 + c^2\|\nabla u(t)\|_{H^s}^2 + d\|u(t)\|_{H^s}\\
D_1(t) &:= \int_0^t (b\|\nabla u_t(\tau)\|_{H^s}^2 + e\|u_t(\tau)\|_{H^s}^2)\, \mathrm{d}\tau\\
E_2(t) &:= \|u_{tt}(t)\|_{H^{s-2}}^2 + c^2\|\nabla u_t(t)\|_{H^{s-2}}^2 + d\|u_t(t)\|_{H^{s-1}}^2\\
D_2(t) &:= \int_0^t (b\|\nabla u_{tt}(\tau)\|_{H^{s-2}}^2 + e\|u_{tt}(\tau)\|_{H^{s-2}}^2)\, \mathrm{d}\tau.
\end{aligned}$$

Proof. Using a standard density argument, it is enough to consider the case where $u \in \mathscr{C}_0^\infty([0,T]\times\mathbb{R}^d)$, that is, u is infinitely differentiable and has compact support in $[0,T]\times\mathbb{R}^d$. Consequently, $Lu \in \mathscr{C}_0^\infty([0,T]\times\mathbb{R}^d)$. Given $w \in \mathscr{C}_0^\infty([0,T]\times\mathbb{R}^d)$, a direct calculation gives us the equation

$$\begin{aligned}
(Lw)w_t = \frac{1}{2}\frac{d}{dt}(|w_t|^2 + c^2|\nabla w|^2 + d|w|^2) - \mathrm{div}(c^2 w_t \Delta u + b w_t \Delta w_t)\\
+ b|\nabla w_t|^2 + e|w_t|^2. \qquad (6.16)
\end{aligned}$$

Since $\partial^\alpha u \in \mathscr{C}_0^\infty([0,T]\times\mathbb{R}^d)$ for every multi-index α, taking $w = \partial^\alpha u$ in (6.16), integrating over $[0,T]\times\mathbb{R}^d$ and then getting the sum of the resulting terms over all $|\alpha| = s$, we obtain $E_1(t) + 2D_1(t) = E_1(0) + 2F_1(t)$. The case $k = 2$ can be obtained in a similar way by replacing u by u_t and s by $s-2$. □

Lemma 6.5. *Let $s \geq 2$ be integer and $u \in Y^s(0,T)$. For $t \in [0,T]$, let*

$$F_3(t) := F_s(Lu, u)(t)$$
$$F_4(t) := F_{s-1}(Lu, v)(t).$$

Then for $k = 3, 4$ and $t \in [0, T]$ there holds

$$E_k(t) + 2D_k(t) = E_k(0) + 2F_k(t)$$

where

$$\begin{aligned}
E_3(t) &:= 2(u_t(t), u(t))_{H^s} + b\|\nabla u(t)\|_{H^s}^2 + e\|u(t)\|_{H^s}^2 \\
D_3(t) &:= \int_0^t (d\|u(\tau)\|_{H^s}^2 + c^2\|\nabla u(\tau)\|_{H^s}^2 - \|u_t(\tau)\|_{H^s}^2)\,\mathrm{d}\tau \\
E_4(t) &:= 2(u_t(t), v(t))_{H^{s-1}} - \|u(t)\|_{H^{s-1}}^2 + c^2\|\nabla v(t)\|_{H^{s-1}}^2 + d\|v(t)\|_{H^{s-1}}^2 \\
&\qquad + b(\nabla u(t), \nabla v(t))_{H^{s-1}} + e(u(t), v(t))_{H^s} \\
D_4(t) &:= -\int_0^t (b\|\nabla u(\tau)\|_{H^{s-1}}^2 + e\|u(\tau)\|_{H^{s-1}}^2)\,\mathrm{d}\tau.
\end{aligned}$$

Proof. The energy identities can be deduced as in the proof of the previous lemma thanks to the following equations

$$\begin{aligned}
(Lw)w &= \frac{1}{2}\frac{d}{dt}(2w_t w + b|\nabla w|^2 + e|w|^2) - \operatorname{div}(bw\nabla w_t + c^2 w\nabla w) \\
&\quad - \ |w_t|^2 + c^2|\nabla w|^2 + d|w|^2 \\
(Lw)\mathcal{I}w &= \frac{1}{2}\frac{d}{dt}(2w_t(\mathcal{I}w) - |w|^2 + c^2|\nabla\mathcal{I}w|^2 + d|\mathcal{I}w|^2 + b\nabla w\cdot\nabla\mathcal{I}w + ew\mathcal{I}w) \\
&\quad - \ (b|\nabla w|^2 + e|w|^2)
\end{aligned}$$

which can be easily verified for $w \in \mathscr{C}_0^\infty([0, T] \times \mathbb{R}^d)$. □

6.3. ENERGY ESTIMATES FOR NONLINEAR TERMS. With the operator S defined in (6.12) and the linear wave operator L given by (6.15), the Kuznetsov–Westervelt equation (6.7) can be rewritten as

$$Lu = 2\kappa u u_{tt} + S(u). \tag{6.17}$$

In what follows, we shall use this equation to derive estimates for the functionals F_k for $k = 1, 2, 3, 4$ defined in Lemmas 6.4 and 6.5. We introduce the following energy and dissipation functionals for $t \geq 0$

$$\begin{aligned}
E(t)^2 &:= \sup_{0\leq\tau\leq t} (\|u(\tau)\|_{H^s}^2 + \|\nabla u(\tau)\|_{H^s}^2 + \|u_t(t)\|_{H^s}^2 + \|u_{tt}(t)\|_{H^{s-2}}^2 + \|v(\tau)\|_{H^s}^2) \\
D(t)^2 &:= \int_0^t (\|u(\tau)\|_{H^{s+1}}^2 + \|u_t(\tau)\|_{H^{s+1}}^2 + \|u_{tt}(\tau)\|_{H^{s-1}}^2)\,\mathrm{d}\tau
\end{aligned}$$

where we recall that $v = \mathcal{I}u$. Moreover, we define the *total* energy functional

$$N(t)^2 := E(t)^2 + D(t)^2,$$

for $t \geq 0$. Notice that E, D, and N are increasing functions of time and $D(0) = 0$.

We need to transform the energy estimates for the local solution in terms of N. In the situation of Theorem 6.3, one can easily see that there exists a constant $C_{\delta_0,T_0} > 0$ such that

$$N(t)^2 \leq C_{\delta_0,T_0} N(0)^2$$

for every $t \in [0, T_0]$. Indeed, let $\widetilde{E}$ and $\widetilde{D}$ be the functionals E and D without the terms involving u_{tt}, respectively. One can easily check from (6.9) that for some $C_{\delta_0,T_0} > 0$, we have

$$\widetilde{E}(T_0)^2 + \widetilde{D}(T_0)^2 \le C_{\delta_0,T_0} N(0)^2. \tag{6.18}$$

Thus, it remains to verify the estimate for the term involving u_{tt}. This follows from (6.13), for which we have the estimate

$$\sup_{0\le t\le T_0} \|u_{tt}(t)\|_{H^{s-2}}^2 + \int_0^{T_0} \|u_{tt}(t)\|_{H^{s-1}}^2 \,\mathrm{d}t \le C(\theta_{\delta_0}, \widetilde{E}(T_0))\widetilde{E}(T_0)^2\widetilde{D}(T_0)^2$$

where $\theta_{\delta_0} = (1 - 2|\kappa|\varrho_{d,s-1}\sqrt{\delta_0})^{-1}$, and then applying (6.18).

To obtain bounds for the nonlinear terms, the following auxiliary Moser-type trilinear estimate will be useful.

Lemma 6.6. *Let $s > \frac{d}{2} + 1$, $0 \le \varsigma \le s$, and $m \ge 0$ be integers. If $\mu \in L^\infty(0, T; H^{s-1})^m$, $\nu \in L^2(0, T; H^{\varsigma^*})^m$, and $\lambda \in L^2(0, T; H^{\varsigma+1})$, where $\varsigma^* = \min(0, \varsigma - 1)$, then there exists a constant $C > 0$ independent of μ, ν, λ, and T such that*

$$|F_\varsigma(\mu\cdot\nu, \lambda)(T)| \le C\|\mu\|_{L^\infty(0,T;H^{s-1})^m} \int_0^T (\|\nu(t)\|_{(H^{\varsigma^*})^m}^2 + \|\lambda(t)\|_{H^{\varsigma+1}}^2)\,\mathrm{d}t.$$

Proof. By considering each component, we may take without loss of generality that $m = 1$. Suppose $\varsigma = 0$. In this case, the duality pairing becomes an inner product in L^2, and thus, we have

$$F_0(\mu\nu, \lambda)(T) = \int_0^T\int_{\mathbb{R}^d} (\mu(t)\nu(t))\lambda(t)\,\mathrm{d}t.$$

By Hölder's inequality and the embedding $H^{s-1} \subset L^\infty$, we obtain

$$|F_0(\mu\nu, \lambda)(T)| \le \varrho_{d,s-1}\|\mu\|_{L^\infty(0,T;H^{s-1})} \int_0^T (\|\nu(t)\|_{L^2}^2 + \|\lambda(t)\|_{L^2}^2)\,\mathrm{d}t. \tag{6.19}$$

Suppose $1 \le \varsigma \le s$ and let α be a multi-index such that $|\alpha| \le \varsigma$. We can rewrite $F_\varsigma(\mu\nu, \lambda)$ as follows

$$F_\varsigma(\mu\nu, \lambda)(T) = F_0(\mu\nu, \lambda)(T) + \sum_{1\le|\alpha|\le\varsigma} \int_0^T\int_{\mathbb{R}^d} \partial^\alpha(\mu(t)\nu(t))\partial^\alpha\lambda(t)\,\mathrm{d}x\,\mathrm{d}t.$$

Of course, in this representation, if $|\alpha| = \varsigma$ then the integral is viewed as a duality pairing between H^{-1} and H^1. The first term on the right-hand side has been already estimated in (6.19). Suppose $1 \le |\alpha| \le \varsigma$. Decompose α as $\alpha = \alpha_0 + \alpha_1$ where $|\alpha_1| = 1$. Integrating by parts yields

$$\int_0^T\int_{\mathbb{R}^d} \partial^\alpha(\mu(t)\nu(t))\partial^\alpha\lambda(t)\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T\int_{\mathbb{R}^d} \partial^{\alpha_0}(\mu(t)\nu(t))\partial^{\alpha+\alpha_1}\lambda(t)\,\mathrm{d}x\,\mathrm{d}t$$

By definition, we have $|\alpha_0| \le \varsigma - 1 \le s - 1$ and $|\alpha + \alpha_1| \le \varsigma + 1$. Applying Theorem 7.4 with $s_1 = s - 1$ and $s_2 = s_3 = \varsigma - 1$ and the Cauchy-Schwartz inequality, we obtain

$$\left|\int_0^T\int_{\mathbb{R}^d} \partial^\alpha(\mu(t)\nu(t))\partial^\alpha\lambda(t)\,\mathrm{d}x\,\mathrm{d}t\right| \le \int_0^T\int_{\mathbb{R}^d} |\partial^{\alpha_0}(\mu(t)\nu(t))\partial^{\alpha+\alpha_1}\lambda(t)|\,\mathrm{d}x\,\mathrm{d}t$$

$$\le C\int_0^T \|\mu(t)\|_{H^{s-1}}\|\nu(t)\|_{H^{\varsigma-1}}\|\lambda(t)\|_{H^{\varsigma+1}}\,\mathrm{d}t$$
$$\le C\|\mu\|_{L^\infty(0,T;H^{s-1})}\int_0^T (\|\nu(t)\|^2_{H^{\varsigma-1}} + \|\lambda(t)\|^2_{H^{\varsigma+1}})\,\mathrm{d}t.$$

Taking the sum proves the desired estimate since $\varsigma^* = \varsigma - 1$. □

A similar procedure proves the following estimate.

Lemma 6.7. *Let $s > \frac{d}{2} + 1$, $0 \le \varsigma \le s$, and $m \ge 0$ be integers. If $\mu, \nu \in L^\infty(0,T;H^{s-1})^m$ and $\lambda \in L^\infty(0,T;H^{\varsigma+1})$, then*

$$|F_\varsigma(\mu\cdot\nu,\lambda)(T)| \le C\|\lambda\|_{L^\infty(0,T;H^{\varsigma+1})^m}\int_0^T (\|\mu(t)\|^2_{(H^{s-1})^m} + \|\nu(t)\|^2_{(H^{s-1})^m})\,\mathrm{d}t.$$

Now we are ready to prove the estimates for the functionals F_k introduced in Lemmas 6.4 and 6.5. First, we consider the case where $k = 1, 3, 4$.

Lemma 6.8. *Let $s > \frac{d}{2} + 1$ be an integer, $T > 0$, and $u \in Y^s(0,T)$ satisfy (6.17). There exists a constant $C > 0$ independent of u and T such that for each $k = 1, 3, 4$ there holds*

$$|F_k(T)| \le CE(T)D(T)^2. \tag{6.20}$$

Proof. First, let us consider the case where $k = 1$. Let $\mu := (2\kappa u, 2\kappa u_t, \sigma\nabla u, \sigma\nabla v)^\top$, $\nu := (u_{tt}, \kappa u_t, \nabla u, \nabla u_t)^\top$, and $\lambda := u_t$. It follows from (6.17) that $Lu = \mu\cdot\nu$. Because $u \in Y^s(0,T)$, we have $\mu \in C(0,T;H^s)^{2d+2}$, $\nu \in L^2(0,T;H^{s-1})^{2d+2}$, and $\lambda \in L^2(0,T;H^{s+1})$. Moreover, the norms of μ and ν can be estimated as follows

$$\sup_{0\le t\le T}\|\mu(t)\|_{(H^{s-1})^{2d+2}} \le C\sup_{0\le t\le T}(\|u_t(t)\|_{H^{s-1}} + \|u(t)\|_{H^s} + \|v(t)\|_{H^s}) \le CE(T)$$
$$\int_0^T \|\nu(t)\|^2_{(H^{s-1})^{2d+2}}\,\mathrm{d}t \le C\int_0^T (\|u_t(t)\|^2_{H^s} + \|u_{tt}(t)\|^2_{H^{s-1}} + \|u(t)\|^2_{H^s})\,\mathrm{d}t \le CD(T)^2$$
$$\int_0^T \|\lambda(t)\|^2_{H^s}\,\mathrm{d}t = \int_0^T \|u_t(t)\|^2_{H^s}\,\mathrm{d}t \le CD(T)^2.$$

Hence, (6.20) in the case of $k = 1$ follows from these estimates along with the trilinear estimate in Lemma 6.6 with $\varsigma = s$.

The proof of (6.20) for $k = 3$ can be done in the same way by choosing μ and ν as above and $\lambda := u$. For $k = 4$, we again choose μ and ν as above, take $\lambda := v \in L^\infty(0,T;H^s)$ and apply Lemma 6.7 with $\varsigma = s - 1$. □

Lemma 6.9. *Let $T > 0$, $s > \frac{d}{2}+1$ be an integer, and $u \in Y^s(0,T)$ satisfy (6.17) and $\|u\|_{L^\infty([0,T]\times\mathbb{R}^d)} \le \gamma$ for some $\gamma \in (0,\rho_\kappa)$. There exists a constant $C > 0$ depending continuously on its arguments such that*

$$|F_2(T)| \le C(\vartheta_\gamma, E(T))E(T)(1 + E(T)^2)D(T)^2$$

where $\vartheta_\gamma := (1 - 2|\kappa|\gamma)^{-1}$.

Proof. First, let us recall that $F_2(T) = F_{s-2}(Lu_t, u_{tt})(T)$. We shall use the representation (6.13) so that

$$Lu = u_{tt} - Pu = 2\kappa(1-2\kappa u)^{-1}uPu + (1-2\kappa u)^{-1}S(u)$$

and by the rules of differentiation, we have

$$\begin{aligned} Lu_t &= 2\kappa(1-2\kappa u)^{-1}u_tPu + 2\kappa(1-2\kappa u)^{-2}u_t(2\kappa uPu + S(u)) \\ &\quad + (1-2\kappa u)^{-1}(2\kappa uPu_t + S(u)_t). \end{aligned}$$

Here, we shall use the equation (6.14) for $S(u)_t$.

Since $d \geq 2$, we have $s \geq 3$. Note that we can write $F_2(T) = F_{s-2}(\mu\cdot\nu, \lambda)$, where

$$\begin{aligned} &\lambda := u_{tt} \\ &\mu := (1-2\kappa u)^{-1}(2\kappa u_t, 2\kappa u_t u, 2\kappa u_t, 2\kappa u, 4\kappa u_t, 3\sigma\nabla u, \sigma\nabla v)^\top, \\ &\nu := \left(Pu, (1-2\kappa u)^{-1}2\kappa Pu, (1-2\kappa u)^{-1}S(u), Pu_t, u_{tt}, \nabla u_t, \nabla u_{tt}\right)^\top. \end{aligned}$$

We have that $\mu \in C(0,T;H^{s-1})^{2d+5}$, $\nu \in L^2(0,T;H^{s-3})^{2d+5}$, and $\lambda \in L^2(0,T;H^{s-2})$ with

$$\int_0^T \|\lambda(t)\|_{H^{s-2}}^2 \,\mathrm{d}t \leq CD(T)^2.$$

By using Corollary 6.2 and H^{s-1} being a Banach algebra, the norm of μ can be estimated from above as follows

$$\begin{aligned} &\|\mu\|_{L^\infty(0,T;H^{s-1})^{2d+5}} \\ &\quad \leq C(\vartheta_\gamma, \|u\|_{L^\infty(0,T;H^{s-1})})(\|u_t\|_{L^\infty(0,T;H^{s-1})} + \|u\|_{L^\infty(0,T;H^s)} + \|v\|_{L^\infty(0,T;H^s)}) \\ &\quad \leq C(\vartheta_\gamma, E(T))E(T). \end{aligned}$$

Using the same corollary with $s_1 = s-1$ and $s_2 = s-3$, we have

$$\begin{aligned} &\left\|(1-2\kappa u)^{-1}(Pu, S(u))\right\|_{L^2(0,T;H^{s-3})^2}^2 \\ &\quad \leq C(\vartheta_\gamma, \|u\|_{L^\infty(0,T;H^{s-1})})(\|Pu\|_{L^2(0,T;H^{s-3})}^2 + \|S(u)\|_{L^2(0,T;H^{s-3})}^2) \end{aligned}$$

where the norms of Pu and $S(u)$ can be bounded from above as follows

$$\begin{aligned} \|Pu\|_{L^2(0,T;H^{s-3})}^2 &\leq C(\|u_t\|_{L^2(0,T;H^{s-1})}^2 + \|u\|_{L^2(0,T;H^{s-1})}^2) \leq CD(T)^2 \\ \|S(u)\|_{L^2(0,T;H^{s-3})}^2 &\leq C(\|u_t^2\|_{L^2(0,T;H^{s-1})}^2 + \||\nabla u|^2\|_{L^2(0,T;H^{s-1})}^2 + \|\nabla v\cdot\nabla u_t\|_{L^2(0,T;H^{s-1})}^2). \end{aligned}$$

Here, we utilized the continuous embedding $H^{s-3} \hookrightarrow H^{s-1}$ in the case of $S(u)$.

Using the fact that H^{s-1} is a Banach algebra and applying an L^∞-L^2 Hölder-type estimate with respect to time, we obtain

$$\begin{aligned} \|u_t^2\|_{L^2(0,T;H^{s-1})}^2 &\leq C\|u_t\|_{L^\infty(0,T;H^{s-1})}^2\|u_t\|_{L^2(0,T;H^{s-1})}^2 \leq CE(T)^2D(T)^2 \\ \||\nabla u|^2\|_{L^2(0,T;H^{s-1})}^2 &\leq C\|u\|_{L^\infty(0,T;H^s)}^2\|u\|_{L^2(0,T;H^s)}^2 \leq CE(T)^2D(T)^2 \\ \|\nabla v\cdot\nabla u_t\|_{L^2(0,T;H^{s-1})}^2 &\leq C\|v\|_{L^\infty(0,T;H^s)}^2\|u_t\|_{L^2(0,T;H^s)}^2 \leq CE(T)^2D(T)^2. \end{aligned}$$

The remaining terms in ν can be estimated as

$$\|(Pu_t, u_{tt}, \nabla u_t, \nabla u_{tt})\|_{L^2(0,T;H^{s-3})^{2d+2}}^2 \leq C(\|u_{tt}\|_{L^2(0,T;H^{s-1})}^2 + \|u_t\|_{L^2(0,T;H^{s-1})}^2) \leq CD(T)^2.$$

Combining the above estimates, we deduce that

$$\int_0^T \|\nu(t)\|^2_{(H^{s-3})^{2d+5}}\,\mathrm{d}t \leq C(\vartheta_\gamma, E(T))(1+E(T)^2)D(T)^2$$

Applying the trilinear Moser-type estimate Lemma 6.6 with $\varsigma = s-2 \geq 1$ and looking at the definitions of E and D, we obtain the desired estimate for $F_2(T)$. □

Now, we are in position to prove an a priori estimate independent of T, for which global existence of solutions and its asymptotic stability will follow.

Lemma 6.10. *Suppose that $u \in Y^s(0,T)$ satisfies the Kuznetsov–Westervelt equation (6.7). Then, there exists $\delta_1 > 0$ and a constant $C_{\delta_1} > 0$ independent of T such that if $N(T)^2 \leq \delta_1$, then*

$$N(t)^2 \leq C_{\delta_1} N(0)^2 \tag{6.21}$$

for every $t \in [0,T]$.

Proof. For $\varepsilon > 0$, we define $\mathcal{E}(t) := E_1(t)+E_2(t)+\varepsilon(E_3(t)+E_4(t))$ and $\mathcal{D}(t) := D_1(t)+D_2(t)+\varepsilon(D_3(t)+D_4(t))$, see Lemma 6.4 and Lemma 6.5 for the definitions of E_k and D_k for $k = 1,2,3,4$. By the Cauchy–Schwarz inequality

$$E_3(t) \geq b\|\nabla u(t)\|^2_{H^s} + (e-1)\|u(t)\|^2_{H^s} - \|u_t(t)\|^2_{H^s}$$

and for every $\eta > 0$ there exists $C_\eta > 0$ such that

$$E_4(t) \geq (c^2-\eta)\|\nabla v(t)\|^2_{H^{s-1}} + (d-\eta)\|v(t)\|^2_{H^{s-1}} - C_\eta(\|u_t(t)\|^2_{H^{s-1}} + \|u(t)\|^2_{H^s}).$$

Choosing η small enough and then ε small enough, we can see that $\mathcal{E}$ is equivalent to E and $\mathcal{D}$ is equivalent to D. In other words, there exist positive constants $c_1 = c_1(\eta,\varepsilon)$ and $c_2 = c_2(\eta,\varepsilon)$ both independent of t such that $c_1E(t) \leq \mathcal{E}(t) \leq c_2E(t)$ and $c_1D(t) \leq \mathcal{D}(t) \leq c_2D(t)$. Therefore, we have

$$E(T)^2 + D(T)^2 \leq CN(0)^2 + C(|F_1(T)| + |F_2(T)| + |F_3(T)| + |F_4(T)|).$$

Using the estimates on F_k and the fact that $E(T) \leq N(T)$, it follows from Lemma 6.8 and Lemma 6.9 that

$$E(T)^2 + D(T)^2 \leq CN(0)^2 + C(\vartheta_\lambda, N(T))N(T)(1+N(T)^2)D(T)^2$$

where λ is a positive constant such that $|u(t,x)| \leq \lambda$ for every $(t,x) \in [0,T]\times\mathbb{R}^d$. Fix a positive number $\delta_* < (\varrho_{d,s-1}/\rho_\kappa)^2$ and let $0 < \delta_1 < \delta_*$. If $N(T)^2 \leq \delta_1$, then

$$E(T)^2 + \{1 - C(\vartheta_{\delta_*}, \sqrt{\delta_*})\delta_1(1+\delta_1^2)\}D(T)^2 \leq CN(0)^2$$

where $\vartheta_{\delta_*} := (1-2|\kappa|\varrho_{d,s-1}\sqrt{\delta_*})^{-1}$. Hence, we choose δ_1 small enough so that the coefficient of $D(T)^2$ is positive. The conclusion now follows from the fact that N is monotonically increasing. □

Theorem 6.11. *There exists $\delta_2 > 0$ such that if $N(0)^2 < \delta_2$ then (6.7) has a unique global solution $u \in Y^s(0,\infty)$, and for some constant $C_{\delta_2} > 0$ independent of T, we have*

$$\sup_{0\leq t\leq T}(\|u(t)\|^2_{H^s} + \|\nabla u(t)\|^2_{H^s} + \|u_t(t)\|^2_{H^s} + \|(\mathcal{I}u)(t)\|^2_{H^s}) \tag{6.22}$$

$$+\int_0^T (\|u(t)\|_{H^{s+1}}^2 + \|u_t(t)\|_{H^{s+1}}^2 + \|u_{tt}(t)\|_{H^{s-1}}^2)\, dt \leq C_{\delta_2}(\|u_0\|_{H^{s+1}}^2 + \|u_1\|_{H^s}^2)$$

for every $T > 0$. Moreover, it holds that $\|u(t)\|_{W^{2,\infty}} + \|u_t(t)\|_{L^\infty} \to 0$ as $t \to \infty$.

Proof. Taking $\delta_0 = \delta_1/2$ in Theorem 6.3, there exists $T_0 = T(\delta_1) > 0$ such that equation (6.7) has a unique solution $u \in Y^s(0, T_0)$ and we have $N(T_0)^2 \leq \widetilde{C}_{\delta_1} N(0)^2$ for some constant $\widetilde{C}_{\delta_1} > 0$. Choose

$$\delta_2 := \min\{\delta_1/2, \delta_1/\widetilde{C}_{\delta_1}, \delta_1/(2C_{\delta_1}), \delta_1/(\widetilde{C}_{\delta_1} C_{\delta_1})\}$$

where C_{δ_1} is the constant in Lemma 6.10. If $N(0)^2 < \delta_2$, then we have a solution u on $[0, T_0]$, and moreover,

$$N(T_0)^2 \leq \widetilde{C}_{\delta_1} N(0)^2 < \widetilde{C}_{\delta_1}\delta_2 \leq \delta_1.$$

According to the a priori estimate of Lemma 6.10 with $T = T_0$, we have

$$N(t)^2 \leq C_{\delta_1} N(0)^2 < C_{\delta_1}\delta_2 < \delta_1/2$$

for every $t \in [0, T_0]$. Thus, we can use $t = T_0$ as the initial time and apply the local existence once more to have a solution on $[T_0, 2T_0]$. Then, we have

$$N(t)^2 \leq \widetilde{C}_{\delta_1} N(T_0)^2 < \widetilde{C}_{\delta_1} C_{\delta_1}\delta_2 \leq \delta_1$$

for every $t \in [T_0, 2T_0]$. Therefore, it follows that $N(t)^2 \leq \delta_1$ for every $t \in [0, 2T_0]$, and consequently by Lemma 6.10, we have $N(t)^2 \leq C_{\delta_1} N(0)^2$ for all $t \in [0, 2T_0]$. By induction, we have a solution $u \in Y^s(0, nT_0)$ for every positive integer n and $N(t) \leq C_{\delta_1} N(0)^2$ for every $t \in [0, nT_0]$. This proves the existence of a global solution and the energy estimate. The asymptotic stability follows from the fact that $|u|^2 \in W^{1,1}(0, \infty; H^{s+1})$, $|u_t|^2 \in W^{1,1}(0, \infty; H^{s-1})$, and the embeddings $H^{s+1} \subset W^{2,\infty}$ and $H^{s-1} \subset L^\infty$. □

7. Appendix

7.1. Moser and Commutator Estimates. Here, we recall some standard estimates involving Sobolev spaces. For the proofs of the following propositions, we refer the reader to [1, 2, 19] for example.

Proposition 7.1. *Let $s \geq 0$ and $u, v \in L^\infty \cap H^s$. Then, there exists a constant $C > 0$ that depends only on s and d such that*

$$\|uv\|_{H^s} \leq C(\|u\|_{L^\infty}\|v\|_{H^s} + \|v\|_{L^\infty}\|u\|_{H^s}) \tag{7.1}$$

and for every d-tuples α and β such that $|\alpha| + |\beta| = s$, we have

$$\|(\partial^\alpha u)(\partial^\beta v)\|_{L^2} \leq C(\|u\|_{L^\infty}\|v\|_{H^s} + \|v\|_{L^\infty}\|u\|_{H^s}). \tag{7.2}$$

If $s > \frac{d}{2}$ then the above proposition and the embedding $H^s \subset L^\infty$ imply that H^s is a Banach algebra and we have $\|uv\|_{H^s} \leq C\|u\|_{H^s}\|v\|_{H^s}$ for every $u, v \in H^s$. In addition, if $|\alpha| + |\beta| \leq s$, then from (7.2) we obtain an alternative estimate

$$\|(\partial^\alpha u)(\partial^\beta v)\|_{L^2} \leq C\|u\|_{H^s}\|v\|_{H^s}. \tag{7.3}$$

Proposition 7.2. *Let $s > 1$ and $u, v \in H^s$ with $\nabla u, \nabla v \in L^\infty$. There is a constant $C > 0$ depending only on s and d such that for every d-tuple α with $|\alpha| \le s$ we have*

$$\|[\partial^\alpha, v\nabla]u\|_{L^2} \le C(\|\nabla u\|_{L^\infty}\|v\|_{H^s} + \|\nabla v\|_{L^\infty}\|u\|_{H^s}). \tag{7.4}$$

In particular, if $s > \frac{d}{2} + 1$ then for every $u, v \in H^s$ and $|\alpha| \le s$, we have

$$\|[\partial^\alpha, v\nabla]u\|_{L^2} \le C\|u\|_{H^s}\|v\|_{H^s}. \tag{7.5}$$

Proposition 7.3. *Let $s > \frac{d}{2}$, $u \in H^s$ and $F \in C^\infty(\mathbb{R})$ satisfies $F(0) = 0$. Then, $F(u) \in H^s$ and there are continuous functions $C : (0, \infty) \to (0, \infty)$ and $\widetilde{C} : (0, \infty) \to (0, \infty)$ depending on F, s, and d as parameters such that*

$$\|F(u)\|_{H^s} \le C(\|u\|_{L^\infty})\|u\|_{H^s} \tag{7.6}$$

and for every $u, v \in H^s$, there holds

$$\|F(u) - F(v)\|_{H^s} \le \widetilde{C}(\max\{\|u\|_{L^\infty}, \|v\|_{L^\infty}\})\|u - v\|_{H^s}. \tag{7.7}$$

Another useful fact is the following multiplication property for Sobolev functions.

Theorem 7.4. *Let $s_1, s_2, s_3 \in \mathbb{R}$ be such that $s_1 + s_2 \ge 0$, $s_3 \le s_1$, $s_3 \le s_2$, and $s_3 + \frac{d}{2} < s_1 + s_2$. If $f \in H^{s_1}$ and $g \in H^{s_2}$, then $fg \in H^{s_3}$ and there exists a constant $C = C_{s_1,s_2,s_3,d} > 0$ such that*

$$\|fg\|_{H^{s_3}} \le C\|f\|_{H^{s_1}}\|g\|_{H^{s_2}}.$$

7.2. Mollifiers and Cut-Off Functions. Let $\rho \in C_0^\infty(\mathbb{R}^d)$ be a nonnegative function such that $\int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x = 1$. For each $\varepsilon > 0$, let $\rho_\varepsilon(x) := \varepsilon^{-d}\rho(x/\varepsilon)$ and define the convolution operator R_ε by $R_\varepsilon u := \rho_\varepsilon * u$. It is well-known that $R_\varepsilon u \in H^\infty$ for every $u \in H^r$, $r \in \mathbb{R}$, and $\varepsilon > 0$. Note that if $\rho(-x) = \rho(x)$ then R_ε is self-adjoint in L^2.

Define also the smooth cut-off function $\chi \in C_0^\infty(\mathbb{R}^d)$ which is identically 1 on a neighborhood of the origin and for each $\varepsilon > 0$ let $\chi_\varepsilon(x) := \chi(\varepsilon x)$. The proof of the following proposition can be seen for example in [3].

Proposition 7.5. *Let $s \in \mathbb{R}$. There exists a constant $C > 0$ depending only on s, ρ and χ such that for every $\varepsilon \in (0, 1)$ and $u \in H^s$ we have*

$$\|R_\varepsilon u\|_{H^s} \le C\|u\|_{H^s} \quad \text{and} \quad \|\chi_\varepsilon u\|_{H^s} \le C\|u\|_{H^s}. \tag{7.8}$$

Moreover, we have $R_\varepsilon u \to u$ and $\chi_\varepsilon u \to u$ in H^s.

As a consequence, we have the following integration by parts formula.

Proposition 7.6. *Suppose that $A \in W^{1,\infty}$ and $u, v \in H^1$. Then,*

$$\int_{\mathbb{R}^d} A\partial_j u \cdot v \,\mathrm{d}x = -\int_{\mathbb{R}^d} (\partial_j A)u \cdot v \,\mathrm{d}x - \int_{\mathbb{R}^d} Au \cdot \partial_j v \,\mathrm{d}x.$$

Proof. Note that $\chi_\varepsilon u \to u$ and $\chi_\varepsilon v \to v$ in H^1 according to Proposition 7.5. Because χ_ε has a compact support, we can integrate by parts to obtain

$$\begin{aligned}\int_{\mathbb{R}^d} A\partial_j u \cdot u \,\mathrm{d}x &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} \partial_j(\chi_\varepsilon u) \cdot A^T(\chi_\varepsilon v)\,\mathrm{d}x \\ &= -\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} \{(\chi_\varepsilon u) \cdot (\partial_j A^T)(\chi_\varepsilon v) + (\chi_\varepsilon u) \cdot A^T\partial_j(\chi_\varepsilon v)\}\,\mathrm{d}x\end{aligned}$$

$$= -\int_{\mathbb{R}^d} (\partial_j A)u \cdot v \, \mathrm{d}x - \int_{\mathbb{R}^d} Au \cdot \partial_j v \, \mathrm{d}x$$

and the proposition is proved. □

Proposition 7.7. *Let $a \in W^{1,\infty}$ and $u \in L^2$. There exists $C > 0$ independent of u, a and $\varepsilon \in (0,1)$ such that for every $j = 1, \dots, d$ we have*

$$\|[R_\varepsilon, a\partial_j]u\|_{L^2} \le C\|a\|_{W^{1,\infty}}\|u\|_{L^2}.$$

Furthermore, $[R_\varepsilon, a\partial_j]u \to 0$ in L^2 as $\varepsilon \to 0$

Corollary 7.8. *Let $a \in W^{1,\infty}$ and $u \in H^{-1}$. Then there exists $C > 0$ independent of u, a, and $\varepsilon \in (0,1)$ such that*

$$\|[R_\varepsilon, a]u\|_{L^2} \le C\|a\|_{W^{1,\infty}}\|u\|_{H^{-1}}.$$

Moreover, $[R_\varepsilon, a]u \to 0$ in L^2 as $\varepsilon \to 0$.

Proof. According to a well-known result, there exist $u_j \in L^2$ for $j = 0, 1, \dots, d$ such that $u = u_0 + \partial_1 u_1 + \cdots + \partial_d u_d$. From Proposition 7.5, it is easy to see that $\|[R_\varepsilon, a]u_0\|_{L^2} \le C\|a\|_{L^\infty}\|u_0\|_{L^2}$ and $[R_\varepsilon, a]u_0 \to 0$ in L^2. On the other hand, for each j the commutator $[R_\varepsilon, a]\partial_j u_j$ can be written as

$$[R_\varepsilon, a]\partial_j u_j = [R_\varepsilon, a\partial_j]u_j - a[R_\varepsilon, \partial_j]u_j.$$

From Proposition 7.7 we have $\|[R_\varepsilon, a]\partial_j u_j\|_{L^2} \le C\|a\|_{W^{1,\infty}}\|u_j\|_{L^2}$ and as $\varepsilon \to 0$, we obtain $[R_\varepsilon, a]\partial_j u_j \to 0$ in L^2. Taking the sum proves that $[R_\varepsilon, a]u \to 0$ in L^2 as $\varepsilon \to 0$, and then taking the infimum both sides over all possible representations of u as sums of the derivatives of L^2-functions, we obtain the desired estimate. □

Corollary 7.9. *Suppose that $a \in W^{1,\infty}$ and $u \in L^2$. Then, for some constant $C > 0$ independent of a, u, and $\varepsilon \in (0,1)$, we have*

$$\|[R_\varepsilon, a]u\|_{H^1} \le C\|a\|_{W^{1,\infty}}\|u\|_{L^2}.$$

Moreover, $[R_\varepsilon, a]u \to 0$ in H^1 as $\varepsilon \to 0$.

Proof. From Corollary 7.8, we only need to consider the term $\partial_j[R_\varepsilon, a]u$. We can rewrite this term as follows

$$\partial_j[R_\varepsilon, a]u = R_\varepsilon((\partial_j a)u) - (\partial_j a)R_\varepsilon u + [R_\varepsilon, a\partial_j]u \to 0$$

as $\varepsilon \to 0$ in L^2. The estimate of the corollary follows from Propositions 7.5 and 7.7. □

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