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Weak and Very Weak Solutions to the Viscous Cahn–Hilliard–Oberbeck–Boussinesq Phase-Field System on Two-Dimensional Bounded Domains

WEAK AND VERY WEAK SOLUTIONS TO THE VISCOUS CAHN–HILLIARD–OBERBECK–BOUSSINESQ PHASE-FIELD SYSTEM ON TWO-DIMENSIONAL BOUNDED DOMAINS

GILBERT PERALTA

Abstract.
In this paper, we consider weak and very weak solutions to the viscous Cahn–Hilliard–Oberbeck–Boussinesq system for non-isothermal, viscous and incompressible binary fluid flows in two-dimensional bounded domains. The source functions have low spatial regularities and the initial data belong to some interpolation spaces. The essential tools employed in the analysis are the extended maximal parabolic regularity for the associated linearized system and the well-posedness of the nonlinear part with the solution of the linearized dynamics as the frozen coefficients. We resolve the linear system by decomposition into the viscous biharmonic heat, Stokes, and heat equations. A spectral Faedo–Galerkin framework shall be pursued for the nonlinear part. Higher integrability with respect to time will be established using interpolation and compactness methods.

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Cahn–Hilliard equation, Oberbeck–Boussinesq system, binary fluids, weak solutions, very weak solutions, interpolation spaces, maximal parabolic regularity, Faedo–Galerkin method.

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Table of Contents

1. Introduction

Consider an open, bounded and connected domain $\Omega \subset \mathbb{R}^2$ with a sufficiently smooth boundary Γ. Let $0 < T < \infty$ be a given final time, $I := (0, T)$ be the temporal domain, $Q := I \times \Omega$ the space-time domain and $\Sigma := I \times \Gamma$ the lateral boundary of Q. This paper will investigate the following system of nonlinear partial differential equations modeling the dynamics of non-isothermal, viscous and incompressible binary fluids:

$$\begin{cases} \partial_t\phi + \operatorname{div}(\phi\boldsymbol{u}) - m\Delta\mu = \sigma & \text{in } Q,\\ \mu = \tau\partial_t\phi - \epsilon\Delta\phi + F(\phi) + l_{\mathrm{c}}\theta + \lambda & \text{in } Q,\\ \partial_t\theta - l_{\mathrm{h}}\partial_t\phi + \operatorname{div}((\theta - l_{\mathrm{h}}\phi)\boldsymbol{u}) - \kappa\Delta\theta = \alpha\boldsymbol{g}\cdot\boldsymbol{u} + h & \text{in } Q,\\ \partial_t\boldsymbol{u} + \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) - \nu\Delta u + \nabla\mathfrak{p} = \mathcal{K}(\mu - l_{\mathrm{c}}\theta)\nabla\phi + \ell(\phi,\theta)\boldsymbol{g} + \boldsymbol{f} & \text{in } Q,\\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } Q,\\ \phi = \Delta\phi = 0, \quad \theta = 0, \quad \boldsymbol{u} = \boldsymbol{0} & \text{on } \Sigma,\\ \phi(0) = \phi_0, \quad \theta(0) = \theta_0, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

The unknown state variables are $\phi : Q \to \mathbb{R}$, $\mu : Q \to \mathbb{R}$, $\theta : Q \to \mathbb{R}$, $\boldsymbol{u} : Q \to \mathbb{R}^2$ and $\mathfrak{p} : Q \to \mathbb{R}$. These represent the order parameter for the normalized fractional part of a binary fluid mixture, chemical potential, temperature deviation with respect to some critical value, mean velocity and pressure, respectively. In (1.1), $F(\phi) = \beta_0\phi^3 - \beta_1\phi$ is the derivative of a polynomial approximation of the Landau–Ginzburg–Wilson free energy functional and $\ell(\phi,\theta) = \alpha_0 + \alpha_1\phi + \alpha_2\theta$ is a linearized equation of state for the density, with constant coefficients $\beta_0, \beta_1 > 0$ and $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$. The other constant parameters are the diffusive mobility $m > 0$, viscosity coefficient $\tau > 0$, interfacial thickness $\epsilon > 0$, thermal conductivity $\kappa > 0$, kinematic viscosity $\nu > 0$, capillarity stress coefficient $\mathcal{K} > 0$, and gravitational force $\boldsymbol{g} \in \mathbb{R}^2$. Moreover, $l_{\mathrm{c}}, l_{\mathrm{h}} > 0$ are constants related to the latent heat and $\alpha \in \mathbb{R}$ for linearized adiabatic heat. The initial concentration, temperature and velocity are $\phi_0 : \Omega \to \mathbb{R}$, $\theta_0 : \Omega \to \mathbb{R}$ and $\boldsymbol{u}_0 : \Omega \to \mathbb{R}^2$, respectively.

The order parameter ϕ describes the concentration of the binary fluid mixture, for instance, $\phi = 1$ signifies a pure single-phase while $\phi = -1$ represents the other phase when $\beta_0 = \beta_1 = 1$. System (1.1) is a coupling of the Cahn–Hilliard system [16] for non-equilibrium phase transitions and the Oberbeck–Boussinesq system [13, 52] in thermohydraulics that accounts for surface tension due to capillary action. Here, the coupling between the order parameter and the temperature is of phase-field-type [15]. For a derivation of the system (1.1) in the absence of the term $\tau\partial_t\phi$ and other relevant references, we refer to [19, 24, 46, 54]. Concerning the Cahn–Hilliard equation and the Cahn–Hilliard–Navier–Stokes system with more general potentials that include the physically meaningful logarithmic energy potentials, one may consult the papers [1, 2, 3, 4, 5, 33, 35, 36, 42, 43] and the references therein.

The additional viscous term serves as a regularization to the Cahn–Hilliard system and it plays a crucial role in the attainment of suitable a priori estimates leading to the well-posedness of (1.1) with source terms of low regularity. Such a viscous term has been introduced by Grinfeld and Novick-Cohen [44] to model phase separation in polymer systems. Recent works related to that model are the boundary optimal control for the viscous Cahn–Hilliard system in [21] and a finite element scheme for the viscous Cahn–Hilliard–Navier–Stokes system with dynamic boundary conditions and its convergence analysis in [20]. An analysis for the long-term dynamics with respect to the viscous parameter τ can be found in [25].

In the nonlinear system (1.1), the functions $\boldsymbol{f} : Q \to \mathbb{R}^2$, $h : Q \to \mathbb{R}$, $\sigma : Q \to \mathbb{R}$ and $\lambda : Q \to \mathbb{R}$ correspond to external body forces, heat source or sink, concentration-source and micro-forces, respectively, see [45] for the latter.

In this work, we aim to establish the existence and uniqueness of weak solutions to (1.1) having the source functions $\sigma \in L^r(I; W^{-1,q}(\Omega))$, $\lambda \in L^r(I; W_0^{1,q}(\Omega))$, $h \in L^r(I; W^{-1,s}(\Omega))$ and $\boldsymbol{f} \in L^r(I; \boldsymbol{W}^{-1,p}(\Omega))$, with suitable range of values for p, q, s and r. In the case $p = q = s = r = 2$, a Hilbert space framework can be utilized to prove the existence and uniqueness of weak solutions. We shall also provide the existence and uniqueness of very weak solutions with source terms having less regularity, namely, $\sigma \in L^r(I; [W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)]')$, $\lambda \in L^r(I; L^q(\Omega))$, $h \in L^r(I; [W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega)]')$ and $\boldsymbol{f} \in L^r(I; [\boldsymbol{W}^{2,p}(\Omega) \cap \boldsymbol{W}_0^{1,p}(\Omega)]')$, with prime denoting duality. To be more precise, we shall consider these source functions to lie in sums of reflexive Banach spaces containing the above function spaces.

In the context of weak solutions, our main interest in this study is the case where $1 < q < 2$, $\frac{4}{3} \le s < 2$, $\frac{4}{3} \le p < 2$, $q \le s$ and $4 \le r < \infty$ that will cover later the situation of measure-valued sources (Theorem 4.12 and Section 4.5). This type of problem has been studied by Casas and Kunisch for the two-dimensional evolutionary Navier–Stokes equation in [18] and its application to sparse optimal control in [17]. The analysis of the state equation relies on an extended maximal parabolic regularity (MPR) for the Stokes equation, see also [58] for a different approach but with a more regular source function.

Our goal is to develop the corresponding well-posedness theory for non-isothermal, incompressible and viscous binary flows. Moreover, we shall consider the notion of very weak solutions with $\frac{4}{3} < q < \infty$, $4 \le p, s, r < \infty$ and $q \le s$ (Theorem 4.11). The definitions will be in such a way that weak solutions are also very weak solutions. Note that this is not always the case in previous literature for the Navier–Stokes equation. The lower bounds for these parameters are imposed so that a Faedo–Galerkin approach for the nonlinear part is possible. In contrast to [18], note here that the parameter r for time-integrability is independent of the other parameters p, s, and q related to spatial regularity. This is due to the fact that the convection terms have been expressed in divergence form. Also, this will provide a unified treatment for the weak and very weak formulations. For the Navier–Stokes equation in (1.1), the solutions we consider here belong to the Serrin's class, hence uniqueness are to be expected.

Some studies on the maximal parabolic regularity for the Cahn–Hilliard equation and phase-field systems can be found in [55, 56, 66]. For very weak solutions of the Navier–Stokes equation we refer to [49] for the stationary case and to [8, 10, 11, 29, 30, 31] for the time-dependent case. With respect to (1.1), we consider the MPR theory for the system that couples the biharmonic heat, Stokes and heat equations. Our strategy is to study each equation separately, where available known results are applicable, treat the coupling terms as sources in each component and then apply suitable embedding theorems. The latter will be done at first in the Hilbertian case, thanks to the analyticity of the underlying semigroup. It is well-known that this is enough to obtain maximal parabolic regularity in the case of Hilbert spaces [22].

Let us present the main strategy in the study of (1.1) under the scenarios described above. In principle, by eliminating the chemical potential μ, this system can be put as an abstract semi-linear parabolic problem

$$\begin{bmatrix} \partial_t \boldsymbol{\mathcal{B}} \boldsymbol{Z} + \boldsymbol{\mathcal{A}}(\boldsymbol{Z}, \mathfrak{p}) + \boldsymbol{\mathcal{N}}(\boldsymbol{Z}) = \boldsymbol{F} & \text{in } I, \\ \boldsymbol{Z}(0) = \boldsymbol{Z}_0, \end{bmatrix} \tag{1.2}$$

where $\boldsymbol{Z} = (\phi, \theta, \boldsymbol{u})$ and $\boldsymbol{F} = (\sigma + m\Delta\lambda, h, \boldsymbol{f})$, with an appropriate linear elliptic operator $\boldsymbol{\mathcal{A}}$, a nonlinear operator $\boldsymbol{\mathcal{N}}$ and the linear operator $\boldsymbol{\mathcal{B}}$ is given by $\boldsymbol{\mathcal{B}}\boldsymbol{Z} = (\phi - m\tau\Delta\phi, \theta - l_{\mathrm{h}}\phi, \boldsymbol{u})$.

Suppose that $\boldsymbol{Z}_0 = \boldsymbol{Z}_{0L} + \boldsymbol{Z}_{0N}$ and $\boldsymbol{F} = \boldsymbol{F}_L + \boldsymbol{F}_N$, where $\boldsymbol{Z}_{0L}$ and $\boldsymbol{F}_L$ lie in some reflexive Banach spaces while $\boldsymbol{Z}_{0N}$ and $\boldsymbol{F}_N$ belong to some Hilbert spaces. We shall decompose the solution of (1.2) as a sum of a suitable weak or very weak solution of the linear system

$$\begin{bmatrix} \partial_t \boldsymbol{\mathcal{B}} \boldsymbol{Z}_L + \boldsymbol{\mathcal{A}}(\boldsymbol{Z}_L, \mathfrak{p}_L) = \boldsymbol{F}_L & \text{in } I, \\ \boldsymbol{Z}_L(0) = \boldsymbol{Z}_{0L}, \end{bmatrix} \tag{1.3}$$

and an appropriate weak solution of the nonlinear system

$$\begin{bmatrix} \partial_t \boldsymbol{\mathcal{B}} \boldsymbol{Z}_N + \boldsymbol{\mathcal{A}}(\boldsymbol{Z}_N, \mathfrak{p}_N) + \boldsymbol{\mathcal{N}}(\boldsymbol{Z}_L + \boldsymbol{Z}_N) = \boldsymbol{F}_N & \text{in } I, \\ \boldsymbol{Z}_N(0) = \boldsymbol{Z}_{0N}. \end{bmatrix} \tag{1.4}$$

Then $\boldsymbol{Z} = \boldsymbol{Z}_L + \boldsymbol{Z}_N$ with the associated pressure $\mathfrak{p} = \mathfrak{p}_L + \mathfrak{p}_N$ would be a weak or very weak solution of the semi-linear equation (1.2). For such a decomposition, extended MPR theorems will be used for the linear system (1.3) and a classical spectral Faedo–Galerkin method will be pursued for the nonlinear system (1.4). We would like to point out that these are in fact the main ideas that were utilized in [18] for the in-stationary Navier–Stokes equation, see also [49] for the stationary case and [50] for the stationary Boussinesq system with inhomogeneous Dirichlet boundary conditions.

The main challenge in the derivation of the priori estimates involving the term $\mu\nabla\phi$ is the low regularity of the chemical potential μ. Nevertheless, this is compensated by the viscous term $\tau\partial_t\phi$ in the equation for the chemical potential, leading to a better regularity for the time-derivative of the order parameter ϕ, and as a result for that ϕ. Also, due to the low spatial regularity of the sources, we need to impose the stronger integrability condition $r \geq 4$ compared to the typical Hilbert space framework.

We point out that the results of this paper can be specialized to various situations. For instance, these are the viscous convective Cahn–Hilliard equation (constant $\boldsymbol{u}$ and θ), the coupled viscous isothermal Cahn–Hilliard–Navier–Stokes system (constant θ), the Oberbeck–Boussinesq system (constant ϕ) and the non-isothermal viscous Cahn–Hilliard system (constant $\boldsymbol{u}$).

The structure of this paper is organized as follows: We recall some function spaces needed in the analysis as well as the precise formulations of (1.3) and (1.4) in Section 2. Extended maximal parabolic regularity theorems for the linearized system will be presented in Section 3 and the well-posedness of the nonlinear system will be discussed in Section 4. In Section 5, we will establish the differentiability of the operator that maps the source functions and initial data to the very weak or weak solutions. Finally, we present solutions with higher integrability with respect to time under additional conditions on the source functions and the initial data in Section 6.

2. Notation and Orientation

Let us introduce the notation for the function spaces and operators to be employed in this work. The last part of this section deals with the precise formulations of (1.3) and (1.4) when applied to (1.1).

2.1. Interpolation Spaces. We denote a continuous embedding by $\hookrightarrow$ and a compact embedding by $\hookrightarrow\hookrightarrow$. Suppose that X and Y are Banach spaces such that $X \hookrightarrow Z$ and $Y \hookrightarrow Z$ for some Hausdorff topological vector space Z. The sum $X + Y := \{u + v : u \in X, v \in Y\}$ is also a Banach space when endowed with the norm

$$\|z\|_{X+Y} := \inf_{\substack{z=x+y \\ x\in X,\, y\in Y}} \{\|x\|_X + \|y\|_Y\} \quad \forall z \in X + Y.$$

The notation := means that the expression on the left is defined by the expression on the right. The intersection $X \cap Y$ is also a Banach space when equipped with the norm

$$\|v\|_{X\cap Y} := \max\{\|v\|_X, \|v\|_Y\} \quad \forall v \in X \cap Y.$$

Given two Banach spaces X and Y described above, $0 < \theta < 1$ and $1 \le p < \infty$, we consider the real interpolation space $(X, Y)_{\theta,p}$ to be the Banach space of all elements $z \in X + Y$ such that the following norm is finite

$$\|z\|_{(X,Y)_{\theta,p}} := \left(\int_0^\infty K(t, z)^p t^{-\theta p} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}}, \quad K(t, z) := \inf_{\substack{z=x+y \\ x\in X,\, y\in Y}} \{\|x\|_X + t\|y\|_Y\}.$$

Here, we follow the definition based on Petree's K-method. It follows that if X_0 and Y_0 are Banach spaces with $X_0 \hookrightarrow X$ and $Y_0 \hookrightarrow Y$, then $(X_0, Y_0)_{\theta,p} \hookrightarrow (X, Y)_{\theta,p}$.

The space of linear and bounded operators from X into Y will be denoted by $\mathcal{L}(X, Y)$ and $\mathcal{L}(X) := \mathcal{L}(X, X)$. All throughout in this paper, by an isomorphism we mean a topological one. If $A \in \mathcal{L}(X + Y, X_1 + Y_1)$ is an isomorphism such that the restrictions $A|_X \in \mathcal{L}(X, X_1)$ and $A|_Y \in \mathcal{L}(Y, Y_1)$ are also isomorphisms, then $A(X, Y)_{\theta,p} = (X_1, Y_1)_{\theta,p}$.

A prime will denote duality. More precisely, X' is the space of all linear and continuous functionals in a Banach space X, while $p' = \frac{p}{p-1}$ for a real number $1 < p < \infty$. If $X\cap Y$ is dense in X and Y, then $(X\cap Y)' = X'+Y'$, $(X+Y)' = X'\cap Y'$ and $(X, Y)'_{\theta,p} = (X', Y')_{\theta,p'}$ for every $0 < \theta < 1$ and $1 < p < \infty$, see [12, Theorem 2.7.1] and [64, Theorem 1.11.2]. If in addition, X and Y are reflexive, then an immediate consequence of these equations is that $X \cap Y$, $X + Y$ and $(X, Y)_{\theta,p}$ are also reflexive. With these, the function spaces for the sources, initial data, and weak or very weak solutions we consider here will be reflexive, with the exception of those source functions in Section 4.5.

For further details on interpolation theory, we refer the reader to the standard texts [7, 12, 51, 64] on this subject.

2.2. Lebesgue, Sobolev, and Solenoidal Function Spaces. For $1 \leq p \leq \infty$ and $s \geq 0$, $L^p(\Omega)$ and $W^{s,p}(\Omega)$ indicate the classical Lebesgue and Sobolev spaces [6]. The subspace of $W^{s,p}(\Omega)$ having elements that vanish on the boundary Γ in the sense of traces will be denoted by $W_0^{s,p}(\Omega)$ and its dual by $W^{-s,p'}(\Omega) := W_0^{s,p}(\Omega)'$ when $1 < p < \infty$. For the vector-valued case, we set $\boldsymbol{L}^p(\Omega) := L^p(\Omega) \times L^p(\Omega)$, $\boldsymbol{W}^{s,p}(\Omega) := W^{s,p}(\Omega) \times W^{s,p}(\Omega)$ and $\boldsymbol{W}_0^{s,p}(\Omega) := W_0^{s,p}(\Omega) \times W_0^{s,p}(\Omega)$.

Denote by $\boldsymbol{L}_\sigma^p(\Omega)$ the closure of the set of all divergence-free vector fields in $C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ with respect to the norm of $\boldsymbol{L}^p(\Omega)$. We use the notation $\widehat{L}^p(\Omega) := \{\pi \in L^p(\Omega) : \int_\Omega \pi \,\mathrm{d}x = 0\}$ for the closed subspace of $L^p(\Omega)$ with elements having zero averages over Ω. Likewise, we set $\widehat{W}_0^{s,p}(\Omega) := W_0^{s,p}(\Omega) \cap \widehat{L}^p(\Omega)$ and $\widehat{W}^{-s,p'}(\Omega) := \widehat{W}_0^{s,p}(\Omega)'$.

Given $1 < q < \infty$, we consider the Dirichlet Laplacian $A_q = -\Delta : D(A_q) \subset L^q(\Omega) \to L^q(\Omega)$ with domain $D(A_q) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$. Note that there exist constants $c_1, c_2 > 0$ such that $c_1\|\phi\|_{W^{2,q}(\Omega)} \leq \|A_q\phi\|_{L^q(\Omega)} \leq c_2\|\phi\|_{W^{2,q}(\Omega)}$ for every $\phi \in D(A_q)$, see [41, Lemma 9.17]. For each $s \geq 0$ we let

$$X^{s,q}(\Omega) := D(A_q^{s/2}), \qquad X^{-s,q'}(\Omega) := X^{s,q}(\Omega)'$$

with $\|\phi\|_{X^{s,q}(\Omega)} := \|A_q^{s/2}\phi\|_{L^q(\Omega)}$ for $\phi \in X^{s,q}(\Omega)$. In particular, $X^{0,q}(\Omega) = L^q(\Omega)$, $X^{1,q}(\Omega) = W_0^{1,q}(\Omega)$, $X^{2,q}(\Omega) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$, $X^{3,q}(\Omega) = \{\phi \in W^{3,q}(\Omega) : \phi = \Delta\phi = 0 \text{ on } \Gamma\}$ and $X^{4,q}(\Omega) = W^{4,q}(\Omega) \cap X^{3,q}(\Omega)$. As usual, we again set $\boldsymbol{X}^{s,q}(\Omega) := X^{s,q}(\Omega) \times X^{s,q}(\Omega)$.

Let us consider the Stokes operator $\boldsymbol{A}_p = -\boldsymbol{P}_p\Delta : D(\boldsymbol{A}_p) \subset \boldsymbol{L}_\sigma^p(\Omega) \to \boldsymbol{L}_\sigma^p(\Omega)$ for $1 < p < \infty$. Here, $\boldsymbol{P}_p : \boldsymbol{L}^p(\Omega) \to \boldsymbol{L}_\sigma^p(\Omega)$ is the Leray–Helmholtz projector for which $\boldsymbol{P}_p\boldsymbol{v} + \nabla\pi_{\boldsymbol{v}} = \boldsymbol{v}$, where $\pi_{\boldsymbol{v}} \in \widehat{W}^{1,p}(\Omega)$ is the weak solution to the boundary value problem

$$\left[\begin{array}{ll} \Delta\pi_{\boldsymbol{v}} = \operatorname{div}\boldsymbol{v} & \text{in } \Omega, \\ (\nabla\pi_{\boldsymbol{v}} - \boldsymbol{v}) \cdot \boldsymbol{n} = 0 & \text{on } \Gamma, \end{array}\right.$$

with $\boldsymbol{n}$ being the unit normal vector outward to Γ, see [34]. It holds that $\boldsymbol{P}_p^2 = \boldsymbol{P}_p$ and $\boldsymbol{P}_p' = \boldsymbol{P}_{p'}$ for the dual operator. We have $D(\boldsymbol{A}_p) = \boldsymbol{X}^{2,p}(\Omega) \cap \boldsymbol{L}_\sigma^p(\Omega)$ and there exist positive constants c_3 and c_4 such that $c_3\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq \|\boldsymbol{A}_p\boldsymbol{u}\|_{\boldsymbol{L}_\sigma^p(\Omega)} \leq c_4\|\boldsymbol{A}_p\|_{\boldsymbol{W}^{2,p}(\Omega)}$ for every $\boldsymbol{u} \in D(\boldsymbol{A}_p)$. In line with the notations for the Dirichlet Laplacian, we set

$$\boldsymbol{X}_\sigma^{s,p}(\Omega) := D(\boldsymbol{A}_p^{s/2}), \qquad \boldsymbol{X}_\sigma^{-s,p'}(\Omega) := \boldsymbol{X}_\sigma^{s,p}(\Omega)'$$

for $s \geq 0$ and $\|\boldsymbol{u}\|_{\boldsymbol{X}_\sigma^{s,p}(\Omega)} := \|\boldsymbol{A}_p^{s/2}\boldsymbol{u}\|_{\boldsymbol{L}_\sigma^p(\Omega)}$ for $\boldsymbol{u} \in \boldsymbol{X}_\sigma^{s,p}(\Omega)$. Thus, we have $\boldsymbol{X}_\sigma^{0,p}(\Omega) = \boldsymbol{L}_\sigma^p(\Omega)$, $\boldsymbol{X}_\sigma^{1,p}(\Omega) = \boldsymbol{W}_0^{1,p}(\Omega) \cap \boldsymbol{L}_\sigma^p(\Omega)$, $\boldsymbol{X}_\sigma^{2,p}(\Omega) = \boldsymbol{X}^{2,p}(\Omega) \cap \boldsymbol{L}_\sigma^p(\Omega)$ and $\boldsymbol{X}_\sigma^{3,p}(\Omega) = \boldsymbol{X}^{3,p}(\Omega) \cap \boldsymbol{L}_\sigma^p(\Omega)$. For the domains of the fractional powers of the Stokes operator, we refer to [39] in the case of smooth domains, and to [63] for three-dimensional Lipschitz domains. The analyticity of the Stokes semigroup in the $\boldsymbol{L}_\sigma^p(\Omega)$ spaces can be found in [38].

2.3. Lebesgue–Bochner Spaces. For time-dependent functions, we shall consider mainly the Lebesgue–Bochner spaces

$$W^{1,p}(I; X, Y) := \{u \in L^p(I; X) : \partial_t u \in L^p(I; Y)\}$$

for Banach spaces X and Y with $X \hookrightarrow Y$, where ∂_t is to be understood in the sense of vector-valued distributions. If $Y = X$ then we simply write $W^{1,p}(I;X)$ instead of $W^{1,p}(I;X,X)$. Also, $W^{1,p}(I;X,Y)$ is a Banach space when endowed with the graph norm

$$\|u\|_{W^{1,p}(I;X,Y)} := \|u\|_{L^p(I;X)} + \|\partial_t u\|_{L^p(I;Y)}.$$

The space of continuous functions on $\bar{I} = [0,T]$ into X with the supremum norm will be denoted by $C(\bar{I};X)$. Then $W^{1,p}(I;X,Y) \hookrightarrow W^{1,p}(I;Y) \hookrightarrow C(\bar{I};Y)$. We set

$$W_0^{1,p}(I;X) := \{u \in W^{1,p}(I;X) : u(0) = u(T) = 0\}$$

and $W^{-1,p'}(I;X') := W_0^{1,p}(I;X)'$ for $1 < p < \infty$.

2.4. The Linear and Nonlinear Parts. We now consider the decomposition of (1.1) as elucidated in the introduction. Suppose that the initial data and source functions can be written as follows:

$$(\phi_0, \theta_0, \boldsymbol{u}_0) = (\phi_{0L}, \theta_{0L}, \boldsymbol{u}_{0L}) + (\phi_{0N}, \theta_{0N}, \boldsymbol{u}_{0N}) \tag{2.1}$$

$$(\sigma, h, \boldsymbol{f}, \lambda) = (\sigma_L, h_L, \boldsymbol{f}_L, \lambda_L) + (\sigma_N, h_N, \boldsymbol{f}_N, \lambda_N). \tag{2.2}$$

The subscripts L and N stand for linear part and nonlinear part, respectively. Then, the components of the solution to (1.1) will be split according to

$$(\phi, \theta, \boldsymbol{u}, \mu, \mathfrak{p}) = (\phi_L, \theta_L, \boldsymbol{u}_L, \mu_L, \mathfrak{p}_L) + (\phi_N, \theta_N, \boldsymbol{u}_N, \mu_N, \mathfrak{p}_N). \tag{2.3}$$

In the above decomposition, on one hand, the first tuple $(\phi_L, \theta_L, \boldsymbol{u}_L, \mu_L, \mathfrak{p}_L)$ constitutes a weak or very weak solution of the linearized system

$$\left[\begin{array}{ll}
\partial_t\phi_L - m\Delta\mu_L = \sigma_L & \text{in } Q,\\
\mu_L = \tau\partial_t\phi_L - \epsilon\Delta\phi_L - \beta_1\phi_L + l_c\theta_L + \lambda_L & \text{in } Q,\\
\partial_t\theta_L - l_h\partial_t\phi_L - \kappa\Delta\theta_L = \alpha\boldsymbol{g}\cdot\boldsymbol{u}_L + h_L & \text{in } Q,\\
\partial_t\boldsymbol{u}_L - \nu\Delta\boldsymbol{u}_L + \nabla\mathfrak{p}_L = (\alpha_1\phi_L + \alpha_2\theta_L)\boldsymbol{g} + \boldsymbol{f}_L & \text{in } Q,\\
\operatorname{div}\boldsymbol{u}_L = 0 & \text{in } Q,\\
\phi_L = \Delta\phi_L = 0, \quad \theta_L = 0, \quad \boldsymbol{u}_L = \boldsymbol{0} & \text{on } \Sigma,\\
\phi_L(0) = \phi_{0L}, \quad \theta_L(0) = \theta_{0L}, \quad \boldsymbol{u}_L(0) = \boldsymbol{u}_{0L} & \text{in } \Omega.
\end{array}\right. \tag{2.4}$$

Notice that the linear system (2.4) is obtained by simply dropping the nonlinear terms in (1.1). On the other hand, the second tuple $(\phi_N, \theta_N, \boldsymbol{u}_N, \mu_N, \mathfrak{p}_N)$ satisfies the following nonlinear system with the frozen coefficients ϕ_L, μ_L, θ_L and $\boldsymbol{u}_L$:

$$\left[\begin{array}{ll}
\partial_t\phi_N + \operatorname{div}((\phi_L + \phi_N)(\boldsymbol{u}_L + \boldsymbol{u}_N)) - m\Delta\mu_N = \sigma_N & \text{in } Q,\\
\mu_N = \tau\partial_t\phi_N - \epsilon\Delta\phi_N + F(\phi_L + \phi_N) + l_c\theta_N + \beta_1\phi_L + \lambda_N & \text{in } Q,\\
\partial_t\theta_N - l_h\partial_t\phi_N + \operatorname{div}((\theta_L + \theta_N - l_h\phi_L - l_h\phi_N)(\boldsymbol{u}_L + \boldsymbol{u}_N)) - \kappa\Delta\theta_N & \\
\qquad = \alpha\boldsymbol{g}\cdot\boldsymbol{u}_N + h_N & \text{in } Q,\\
\partial_t\boldsymbol{u}_N + \operatorname{div}((\boldsymbol{u}_L + \boldsymbol{u}_N)\otimes(\boldsymbol{u}_L + \boldsymbol{u}_N)) - \nu\Delta\boldsymbol{u}_N + \nabla\mathfrak{p}_N & \\
\qquad = \mathcal{K}(\mu_L + \mu_N - l_c\theta_L - l_c\theta_N)\nabla(\phi_L + \phi_N) + \ell(\phi_N, \theta_N)\boldsymbol{g} + \boldsymbol{f}_N & \text{in } Q,\\
\operatorname{div}\boldsymbol{u}_N = 0 & \text{in } Q,\\
\phi_N = \Delta\phi_N = 0, \quad \theta_N = 0, \quad \boldsymbol{u}_N = \boldsymbol{0} & \text{on } \Sigma,\\
\phi_N(0) = \phi_{0N}, \quad \theta_N(0) = \theta_{0N}, \quad \boldsymbol{u}_N(0) = \boldsymbol{u}_{0N} & \text{in } \Omega.
\end{array}\right. \tag{2.5}$$

The precise functional analytic frameworks to (2.4) and (2.5) will be discussed in detail in the forthcoming sections.

3. Maximal Parabolic Regularity for the Linearized System

All throughout this section, we shall take $q, s, p, r \in (1, \infty)$ with $q \leq s$. We aim to present extensions of the MPR theorems for the Stokes, heat, and viscous biharmonic heat equations. We then combine these in order to prove the MPR for the linearized system (2.4). Generic positive constants will be denoted by c or with a subscript. In general, these constants depend in at least one of p, s, q, r, Ω, T and the parameters in the nonlinear system (1.1).

3.1. MPR for the Stokes Equation. We consider initial data for the Stokes equation in the following real interpolation spaces:

$$\begin{aligned}\boldsymbol{V}^1_{p,r}(\Omega) &:= (\boldsymbol{X}^{-1,p}_\sigma(\Omega), \boldsymbol{X}^{1,p}_\sigma(\Omega))_{1/r',r}\\ \boldsymbol{V}^0_{p,r}(\Omega) &:= (\boldsymbol{X}^{-2,p}_\sigma(\Omega), \boldsymbol{L}^{p}_\sigma(\Omega))_{1/r',r}.\end{aligned}$$

The superscript on the left signifies the order of weak differentiability with respect to the smaller function space in the interpolation. This is motivated from the fact that if r is large then the interpolated space is "closer" to the smaller function space. We have $\boldsymbol{V}^1_{p,r}(\Omega) \hookrightarrow \boldsymbol{V}^0_{p,r}(\Omega)$ since $\boldsymbol{X}^{-1,p}_\sigma(\Omega) \hookrightarrow \boldsymbol{X}^{-2,p}_\sigma(\Omega)$ and $\boldsymbol{X}^{1,p}_\sigma(\Omega) \hookrightarrow \boldsymbol{L}^{p}_\sigma(\Omega)$. The weak and very weak solution spaces we take into account are as follows:

$$\begin{aligned}\boldsymbol{\mathcal{V}}^1_{p,r}(Q) &:= W^{1,r}(I; \boldsymbol{X}^{1,p}_\sigma(\Omega), \boldsymbol{X}^{-1,p}_\sigma(\Omega))\\ \boldsymbol{\mathcal{V}}^0_{p,r}(Q) &:= W^{1,r}(I; \boldsymbol{L}^{p}_\sigma(\Omega), \boldsymbol{X}^{-2,p}_\sigma(\Omega)).\end{aligned}$$

In view of the previous embeddings, we have $\boldsymbol{\mathcal{V}}^1_{p,r}(Q) \hookrightarrow \boldsymbol{\mathcal{V}}^0_{p,r}(Q)$.

Define the continuous bilinear form $\boldsymbol{a}_p : \boldsymbol{W}^{1,p}_0(\Omega) \times \boldsymbol{W}^{1,p'}_0(\Omega) \to \mathbb{R}$ according to

$$\boldsymbol{a}_p(\boldsymbol{v}, \boldsymbol{\rho}) := \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{\rho} \,\mathrm{d}x = \sum_{j=1}^{2} \int_\Omega \nabla v_j \cdot \nabla \rho_j \,\mathrm{d}x \quad \forall\, (\boldsymbol{v}, \boldsymbol{\rho}) \in \boldsymbol{W}^{1,p}_0(\Omega) \times \boldsymbol{W}^{1,p'}_0(\Omega).$$

Definition 3.1. Consider a source function and initial data

$$\boldsymbol{f}_L \in L^r(I; \boldsymbol{X}^{-1,p}_\sigma(\Omega)), \quad \boldsymbol{u}_{0L} \in \boldsymbol{V}^1_{p,r}(\Omega). \tag{3.1}$$

We say that $\boldsymbol{u}_L \in \boldsymbol{\mathcal{V}}^1_{p,r}(Q)$ is a *weak solution* of the Stokes equation

$$\left[\begin{array}{l}\partial_t \boldsymbol{u}_L - \nu\Delta \boldsymbol{u}_L + \nabla \mathfrak{p}_L = \boldsymbol{f}_L \ \text{ in } Q,\\ \operatorname{div} \boldsymbol{u}_L = 0 \ \text{ in } Q, \ \boldsymbol{u}_L = \boldsymbol{0} \ \text{ on } \Sigma, \ \boldsymbol{u}_L(0) = \boldsymbol{u}_{0L} \ \text{ in } \Omega,\end{array}\right. \tag{3.2}$$

if $\boldsymbol{u}_L(0) = \boldsymbol{u}_{0L}$ in $\boldsymbol{V}^1_{p,r}(\Omega)$ and the following variational equation

$$\int_0^T \langle \partial_t \boldsymbol{u}_L, \boldsymbol{\rho}\rangle_{\boldsymbol{X}^{-1,p}_\sigma(\Omega), \boldsymbol{X}^{1,p'}_\sigma(\Omega)} \,\mathrm{d}t + \nu \int_0^T \boldsymbol{a}_p(\boldsymbol{u}_L, \boldsymbol{\rho}) \,\mathrm{d}t = \int_0^T \langle \boldsymbol{f}_L, \boldsymbol{\rho}\rangle_{\boldsymbol{X}^{-1,p}_\sigma(\Omega), \boldsymbol{X}^{1,p'}_\sigma(\Omega)} \,\mathrm{d}t$$

holds for every $\boldsymbol{\rho} \in L^{r'}(I; \boldsymbol{X}^{1,p'}_\sigma(\Omega))$. ◇

Thanks to the continuous embedding $\boldsymbol{\mathcal{V}}^1_{p,r}(Q) \hookrightarrow C(\bar{I}; \boldsymbol{V}^1_{p,r}(\Omega))$, see [7, Theorem III.4.10.2], the point-wise time evaluation $\boldsymbol{u}_L(0)$ is meaningful and lies in $\boldsymbol{V}^1_{p,r}(\Omega)$. As usual, the pressure has been eliminated in the weak formulation and it will be recovered by an application of de Rham's Theorem.

The following extension of the maximal parabolic L^r-L^p regularity theorem for the Stokes equation has been demonstrated in [18]. We also refer to [37, 40, 47] for related topics and relevant references. Here, we give an alternative demonstration for the existence of the pressure compared to those that were presented in [14, 18, 62]. In this direction, we follow the discussion provided in [27, Chapter 72] for the case of Hilbert spaces.

Theorem 3.2. *Let $p, r \in (1, \infty)$ and suppose that (3.1) is satisfied. Then the Stokes equation (3.2) has a unique weak solution $\boldsymbol{u}_L \in \boldsymbol{\mathcal{V}}^1_{p,r}(Q)$ and there exists a constant $c_1 > 0$ independent on $\boldsymbol{u}_L$, $\boldsymbol{f}_L$, and $\boldsymbol{u}_{0L}$ such that*

$$\|\boldsymbol{u}_L\|_{\boldsymbol{\mathcal{V}}^1_{p,r}(Q)} \le c_1\{\|\boldsymbol{f}_L\|_{L^r(I;\boldsymbol{X}^{-1,p}_\sigma(\Omega))} + \|\boldsymbol{u}_{0L}\|_{\boldsymbol{V}^1_{p,r}(\Omega)}\}. \tag{3.3}$$

In addition, if $\boldsymbol{f}_L \in L^r(I; \boldsymbol{W}^{-1,p}(\Omega))$, then there is a unique associated pressure $\mathfrak{p}_L \in W^{-1,r}(I; \widehat{L}^p(\Omega))$ in the sense that

$$\begin{aligned} &\langle \partial_t \boldsymbol{u}_L, \boldsymbol{\varrho}\rangle_{W^{-1,r}(I;\boldsymbol{W}^{-1,p}(\Omega)),W^{1,r'}_0(I;\boldsymbol{W}^{1,p'}_0(\Omega))} + \nu \int_0^T \boldsymbol{a}_p(\boldsymbol{u}_L, \boldsymbol{\varrho})\,\mathrm{d}t \\ &\quad - \langle \mathfrak{p}_L, \operatorname{div}\boldsymbol{\varrho}\rangle_{W^{-1,r}(I;\widehat{L}^p(\Omega)),W^{1,r'}_0(I;\widehat{L}^{p'}(\Omega))} = \int_0^T \langle \boldsymbol{f}_L, \boldsymbol{\varrho}\rangle_{\boldsymbol{W}^{-1,p}(\Omega),\boldsymbol{W}^{1,p'}_0(\Omega)}\,\mathrm{d}t \end{aligned} \tag{3.4}$$

for every $\boldsymbol{\varrho} \in W^{1,r'}_0(I; W^{1,p'}_0(\Omega))$ and there is a constant $c_2 > 0$ such that

$$\|\mathfrak{p}_L\|_{W^{-1,r}(I;\widehat{L}^p(\Omega))} \le c_2\{\|\boldsymbol{f}_L\|_{L^r(I;\boldsymbol{W}^{-1,p}(\Omega))} + \|\boldsymbol{u}_{0L}\|_{\boldsymbol{V}^1_{p,r}(\Omega)}\}. \tag{3.5}$$

Proof. The existence and uniqueness of the weak solution $\boldsymbol{u}_L \in \boldsymbol{\mathcal{V}}^1_{p,r}(Q)$ as well as the stability estimate (3.3) has been established in [18, Theorem 2.4] for $\boldsymbol{f}_L \in L^r(I; \boldsymbol{W}^{-1,p}(\Omega))$. Note that the proof of that theorem covers the case where $\boldsymbol{f}_L \in L^r(I; \boldsymbol{X}^{-1,p}_\sigma(\Omega))$. We point out that the definition of weak solutions to (3.2) in that paper is equivalent to the one prescribed by Definition 3.1 with space-time-dependent test functions. Let us provide an alternative proof for the existence and regularity of the associated pressure. The following argument will be utilized later in the associated pressure for the very weak formulation as well as for the nonlinear part.

Since $\boldsymbol{u}_L \in L^r(I; \boldsymbol{X}^{1,p}_\sigma(\Omega)) \hookrightarrow L^r(I; \boldsymbol{L}^p(\Omega)) \hookrightarrow L^r(I; \boldsymbol{W}^{-1,p}(\Omega))$, it follows that $\boldsymbol{u}_L$ has a distributional time derivative $\partial_t \boldsymbol{u}_L \in W^{-1,r}(I; \boldsymbol{W}^{-1,p}(\Omega))$, that is, the linear form given by

$$\begin{aligned} &\langle \partial_t \boldsymbol{u}_L, \boldsymbol{\varrho}\rangle_{W^{-1,r}(I;\boldsymbol{W}^{-1,p}(\Omega)),W^{1,r'}_0(I;\boldsymbol{W}^{1,p'}_0(\Omega))} \\ &:= -\int_0^T \langle \boldsymbol{u}_L, \partial_t \boldsymbol{\varrho}\rangle_{\boldsymbol{W}^{-1,p}(\Omega),\boldsymbol{W}^{1,p'}_0(\Omega)}\,\mathrm{d}t = -\int_0^T \langle \boldsymbol{u}_L, \partial_t \boldsymbol{\varrho}\rangle_{\boldsymbol{L}^p(\Omega),\boldsymbol{L}^{p'}(\Omega)}\,\mathrm{d}t \end{aligned}$$

for every $\boldsymbol{\varrho} \in W^{1,r'}_0(I; \boldsymbol{W}^{1,p'}_0(\Omega))$. On the other hand, using the density of $C^1(\bar{I}; \boldsymbol{X}^{1,p}_\sigma(\Omega))$ in $\boldsymbol{\mathcal{V}}^1_{p,r}(Q)$, see [57, Lemma 7.2] for instance, and then integrating by parts, we see that the weak derivative $\partial_t \boldsymbol{u}_L \in L^r(I;\boldsymbol{X}_\sigma^{-1,p}(\Omega))$ satisfies the equation

$$\begin{aligned}&\langle \partial_t \boldsymbol{u}_L, \boldsymbol{\rho}\rangle_{L^r(I;\boldsymbol{X}_\sigma^{-1,p}(\Omega)),L^{r'}(I;\boldsymbol{X}_\sigma^{1,p'}(\Omega))}\\ &= \int_0^T \langle \partial_t \boldsymbol{u}_L, \boldsymbol{\rho}\rangle_{\boldsymbol{X}_\sigma^{-1,p}(\Omega),\boldsymbol{X}_\sigma^{1,p'}(\Omega)}\,\mathrm{d}t = -\int_0^T \langle \boldsymbol{u}_L, \partial_t\boldsymbol{\rho}\rangle_{\boldsymbol{L}^p(\Omega),\boldsymbol{L}^{p'}(\Omega)}\,\mathrm{d}t\end{aligned}$$

for all $\boldsymbol{\rho} \in W_0^{1,r'}(I;\boldsymbol{X}_\sigma^{1,p'}(\Omega)) \hookrightarrow W_0^{1,r'}(I;\boldsymbol{W}_0^{1,p'}(\Omega))$.

The above equations imply that the weak and distributional time derivatives of $\boldsymbol{u}_L$ coincide in $W_0^{1,r'}(I;\boldsymbol{X}_\sigma^{1,p'}(\Omega))$, hence the use of the same notation for these derivatives. Moreover, it follows from the definition of the distributional derivative $\partial_t \boldsymbol{u}_L$ that

$$\|\partial_t \boldsymbol{u}_L\|_{W^{-1,r}(I;\boldsymbol{W}^{-1,p}(\Omega))} \le \|\boldsymbol{u}_L\|_{L^r(I;\boldsymbol{L}^p(\Omega))} \le c\|\boldsymbol{u}_L\|_{L^r(I;\boldsymbol{X}_\sigma^{1,p}(\Omega))} \tag{3.6}$$

where $c > 0$ is the constant associated with the continuous embedding $\boldsymbol{X}_\sigma^{1,p}(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$.

Let $\mathfrak{L} \in W^{-1,r}(I;\boldsymbol{W}^{-1,p}(\Omega))$ be the linear form defined by

$$\begin{aligned}&\langle \mathfrak{L}, \boldsymbol{\varrho}\rangle_{W^{-1,r}(I;\boldsymbol{W}^{-1,p}(\Omega)),W_0^{1,r'}(I;\boldsymbol{W}_0^{1,p'}(\Omega))}\\ &:= \int_0^T \langle \boldsymbol{f}_L, \boldsymbol{\varrho}\rangle_{\boldsymbol{W}^{-1,p}(\Omega),\boldsymbol{W}_0^{1,p'}(\Omega)}\,\mathrm{d}t - \nu\int_0^T \boldsymbol{a}_p(\boldsymbol{u}_L,\boldsymbol{\varrho})\,\mathrm{d}t\\ &\quad - \langle \partial_t\boldsymbol{u}_L, \boldsymbol{\varrho}\rangle_{W^{-1,r}(I;\boldsymbol{W}^{-1,p}(\Omega)),W_0^{1,r'}(I;\boldsymbol{W}_0^{1,p'}(\Omega))}\end{aligned}$$

for all $\boldsymbol{\varrho} \in W_0^{1,r'}(I;\boldsymbol{W}_0^{1,p'}(\Omega))$. According to the above discussion and the fact that $\boldsymbol{u}_L$ is a weak solution to (3.2), we see that $\mathfrak{L}$ annihilates $W_0^{1,r'}(I;\boldsymbol{X}_\sigma^{1,p'}(\Omega))$. Applying Proposition 7.1 in the Appendix with $k = 1$, we deduce the existence and uniqueness of an element $\mathfrak{p}_L \in W^{-1,r}(I;\widehat{L}^p(\Omega))$ such that $\mathfrak{L} = \nabla\mathfrak{p}_L$ in the sense of distributions, and for some $\tilde{c} > 0$ we have

$$\|\mathfrak{p}_L\|_{W^{-1,r}(I;\widehat{L}^p(\Omega))} \le \tilde{c}\|\mathfrak{L}\|_{W^{-1,r}(I;\boldsymbol{W}^{-1,p}(\Omega))}. \tag{3.7}$$

From the definition of the linear form $\mathfrak{L}$ and the distributional gradient, we see that (3.4) holds. In addition, one has

$$\begin{aligned}&\|\mathfrak{L}\|_{W^{-1,r}(I;\boldsymbol{W}^{-1,p}(\Omega))}\\ &\le \|\boldsymbol{f}_L\|_{L^r(I;\boldsymbol{W}^{-1,p}(\Omega))} + \nu\|\boldsymbol{u}_L\|_{L^r(I;\boldsymbol{X}_\sigma^{1,p}(\Omega))} + \|\partial_t\boldsymbol{u}_L\|_{W^{-1,r}(I;\boldsymbol{W}^{-1,p}(\Omega))}.\end{aligned} \tag{3.8}$$

The estimate (3.5) now follows from the inequalities (3.3), (3.6), (3.7), (3.8) and the continuous embedding $L^r(I;\boldsymbol{W}^{-1,p}(\Omega)) \hookrightarrow L^r(I;\boldsymbol{X}_\sigma^{-1,p}(\Omega))$. □

In general, the associated pressure may not exist when we merely have a source function $\boldsymbol{f}_L \in L^r(I;\boldsymbol{X}_\sigma^{-1,p}(\Omega))$. For instance, in the Hilbertian case $p = r = 2$, it was shown by Simon in [60] that $W^{-1,2}(I;\boldsymbol{W}^{-1,2}(\Omega))$ and $L^2(I;\boldsymbol{X}_\sigma^{-1,2}(\Omega))$ cannot be embedded in the same Hausdorff topological vector space, leading to the possible non-existence of the pressure.

Now, we turn to the definition of very weak solutions to the Stokes equation, see also [31].

Definition 3.3. Consider a source function and an initial data satisfying

$$\boldsymbol{f}_L \in L^r(I;\boldsymbol{X}_\sigma^{-2,p}(\Omega)), \quad \boldsymbol{u}_{0L} \in \boldsymbol{V}_{p,r}^0(\Omega). \tag{3.9}$$

A function $\boldsymbol{u}_L \in \boldsymbol{\mathcal{V}}_{p,r}^0(Q)$ will be called a *very weak solution* to (3.2) if $\boldsymbol{u}_L(0) = \boldsymbol{u}_{0L}$ in $\boldsymbol{V}_{p,r}^0(\Omega)$ and the following equation

$$\begin{aligned}\int_0^T \langle \partial_t \boldsymbol{u}_L, \boldsymbol{\rho}\rangle_{\boldsymbol{X}_\sigma^{-2,p}(\Omega),\boldsymbol{X}_\sigma^{2,p'}(\Omega)}\,\mathrm{d}t + \nu\int_0^T \langle \boldsymbol{u}_L, \boldsymbol{A}_{p'}\boldsymbol{\rho}\rangle_{\boldsymbol{L}_\sigma^p(\Omega),\boldsymbol{L}_\sigma^{p'}(\Omega)}\,\mathrm{d}t\\ = \int_0^T \langle \boldsymbol{f}_L, \boldsymbol{\rho}\rangle_{\boldsymbol{X}_\sigma^{-2,p}(\Omega),\boldsymbol{X}_\sigma^{2,p'}(\Omega)}\,\mathrm{d}t\end{aligned}$$

holds for every $\boldsymbol{\rho} \in L^{r'}(I;\boldsymbol{X}_\sigma^{2,p'}(\Omega))$. ◇

Time-evaluation is again valid in virtue of $\boldsymbol{\mathcal{V}}_{p,r}^0(Q) \hookrightarrow C(\bar{I};\boldsymbol{V}_{p,r}^0(\Omega))$. Due to the fact that $\boldsymbol{\mathcal{V}}_{p,r}^1(Q) \hookrightarrow \boldsymbol{\mathcal{V}}_{p,r}^0(Q)$, $L^r(I;\boldsymbol{X}_\sigma^{-1,p}(\Omega)) \hookrightarrow L^r(I;\boldsymbol{X}_\sigma^{-2,p}(\Omega))$, $\boldsymbol{V}_{p,r}^1(\Omega) \hookrightarrow \boldsymbol{V}_{p,r}^0(\Omega)$ and

$$\boldsymbol{a}_p(\boldsymbol{v},\boldsymbol{\varrho}) = \langle \boldsymbol{v}, \boldsymbol{A}_{p'}\boldsymbol{\varrho}\rangle_{\boldsymbol{L}_\sigma^p(\Omega),\boldsymbol{L}_\sigma^{p'}(\Omega)} \qquad \forall\,(\boldsymbol{v},\boldsymbol{\varrho}) \in \boldsymbol{X}_\sigma^{1,p}(\Omega)\times\boldsymbol{X}_\sigma^{2,p'}(\Omega), \tag{3.10}$$

a weak solution to the Stokes equation (3.2) in the sense of Definition 3.1 is necessarily a very weak solution in the sense of Definition 3.3. The equation (3.10) follows from Green's identity, divergence theorem, the definition of $\boldsymbol{P}_{p'}$ and that $\boldsymbol{v}$ is divergence-free in Ω.

Theorem 3.4. *Suppose that $p, r \in (1,\infty)$ and (3.9) are satisfied. Then (3.2) admits a unique very weak solution $\boldsymbol{u}_L \in \boldsymbol{\mathcal{V}}_{p,r}^0(Q)$ and we have*

$$\|\boldsymbol{u}_L\|_{\boldsymbol{\mathcal{V}}_{p,r}^0(Q)} \le c_1\{\|\boldsymbol{f}_L\|_{L^r(I;\boldsymbol{X}_\sigma^{-2,p}(\Omega))} + \|\boldsymbol{u}_{0L}\|_{\boldsymbol{V}_{p,r}^0(\Omega)}\} \tag{3.11}$$

for some $c_1 > 0$ independent on $\boldsymbol{u}_L$, $\boldsymbol{f}_L$, and $\boldsymbol{u}_{0L}$. In addition, if $\boldsymbol{f}_L \in L^r(I;\boldsymbol{X}^{-2,p}(\Omega))$, then we have a unique associated pressure $\mathfrak{p}_L \in W^{-1,r}(I;\widehat{W}^{-1,p}(\Omega))$ satisfying for every $\boldsymbol{\varrho} \in W_0^{1,r'}(I;\boldsymbol{W}_0^{2,p'}(\Omega))$ the variational equation

$$\begin{aligned}\langle \partial_t \boldsymbol{u}_L, \boldsymbol{\varrho}\rangle_{W^{-1,r}(I;\boldsymbol{W}^{-2,p}(\Omega)),W_0^{1,r'}(I;\boldsymbol{W}_0^{2,p'}(\Omega))} - \nu\int_0^T \langle \boldsymbol{u}_L, \Delta\boldsymbol{\varrho}\rangle_{\boldsymbol{L}^p(\Omega),\boldsymbol{L}^{p'}(\Omega)}\,\mathrm{d}t\\ - \langle \mathfrak{p}_L, \operatorname{div}\boldsymbol{\varrho}\rangle_{W^{-1,r}(I;\widehat{W}^{-1,p}(\Omega)),W_0^{1,r'}(I;\widehat{W}_0^{1,p'}(\Omega))} = \int_0^T \langle \boldsymbol{f}_L, \boldsymbol{\varrho}\rangle_{\boldsymbol{W}^{-2,p}(\Omega),\boldsymbol{W}_0^{2,p'}(\Omega)}\,\mathrm{d}t\end{aligned}$$

and for some constant $c_2 > 0$ it holds that

$$\|\mathfrak{p}_L\|_{W^{-1,r}(I;\widehat{W}^{-1,p}(\Omega))} \le c_2\{\|\boldsymbol{f}_L\|_{L^r(I;\boldsymbol{X}^{-2,p}(\Omega))} + \|\boldsymbol{u}_{0L}\|_{\boldsymbol{V}_{p,r}^0(\Omega)}\}. \tag{3.12}$$

Proof. The dual operator $\boldsymbol{A}_{p'}' : \boldsymbol{L}_\sigma^p(\Omega) \to \boldsymbol{X}_\sigma^{-2,p}(\Omega)$ of $\boldsymbol{A}_{p'} : \boldsymbol{X}_\sigma^{2,p'}(\Omega) \to \boldsymbol{L}_\sigma^{p'}(\Omega)$ is an isometric isomorphism and extends the Stokes operator $\boldsymbol{A}_p : \boldsymbol{X}_\sigma^{2,p}(\Omega) \to L_\sigma^p(\Omega)$. Indeed, given $\boldsymbol{u} \in \boldsymbol{X}_\sigma^{2,p}(\Omega)\cap\boldsymbol{X}_\sigma^{2,2}(\Omega)$ and $\boldsymbol{v} \in \boldsymbol{X}_\sigma^{2,p'}(\Omega)\cap\boldsymbol{X}_\sigma^{2,2}(\Omega)$ we have

$$\begin{aligned}\langle \boldsymbol{A}_{p'}'\boldsymbol{u}, \boldsymbol{v}\rangle_{\boldsymbol{X}_\sigma^{-2,p}(\Omega),\boldsymbol{X}_\sigma^{2,p'}(\Omega)} &= \langle \boldsymbol{u}, \boldsymbol{A}_{p'}\boldsymbol{v}\rangle_{\boldsymbol{L}_\sigma^p(\Omega),\boldsymbol{L}_\sigma^{p'}(\Omega)} = (\boldsymbol{u}, \boldsymbol{A}_2\boldsymbol{v})_{\boldsymbol{L}_\sigma^2(\Omega)}\\ &= (\boldsymbol{A}_2\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{L}_\sigma^2(\Omega)} = \langle \boldsymbol{A}_p\boldsymbol{u}, \boldsymbol{v}\rangle_{\boldsymbol{X}_\sigma^{-2,p}(\Omega),\boldsymbol{X}_\sigma^{2,p'}(\Omega)}\end{aligned}$$

since $\boldsymbol{A}_{\mathfrak{s}} = \boldsymbol{A}_2$ in $\boldsymbol{X}_\sigma^{2,\mathfrak{s}}(\Omega)\cap\boldsymbol{X}_\sigma^{2,2}(\Omega)$ for $\mathfrak{s} \in (1,\infty)$. Invoking the density of $\boldsymbol{X}_\sigma^{2,\mathfrak{s}}(\Omega)\cap\boldsymbol{X}_\sigma^{2,2}(\Omega)$ in $\boldsymbol{X}_\sigma^{2,\mathfrak{s}}(\Omega)$ yields $\boldsymbol{A}_{p'}'\boldsymbol{u} = \boldsymbol{A}_p\boldsymbol{u}$ in $\boldsymbol{L}_\sigma^p(\Omega)$ for every $\boldsymbol{u} \in \boldsymbol{X}_\sigma^{2,p}(\Omega)$. In particular, $(\boldsymbol{A}_{p'}')^{-1} = \boldsymbol{A}_p^{-1}$ as an isomorphism from $\boldsymbol{L}_\sigma^p(\Omega)$ onto $\boldsymbol{X}_\sigma^{2,p}(\Omega)$.

Define $\boldsymbol{g}_L := (\boldsymbol{A}'_{p'})^{-1}\boldsymbol{f}_L \in L^r(I;\boldsymbol{L}^p_\sigma(\Omega))$ and

$$\boldsymbol{v}_{0L} := (\boldsymbol{A}'_{p'})^{-1}\boldsymbol{u}_{0L} \in (\boldsymbol{A}'_{p'})^{-1}\boldsymbol{V}^0_{p,r}(\Omega) = (\boldsymbol{L}^p_\sigma(\Omega),\boldsymbol{X}^{2,p}_\sigma(\Omega))_{1/r',r} =: \boldsymbol{V}^2_{p,r}(\Omega),$$

where $\boldsymbol{f}_L$ and $\boldsymbol{u}_{0L}$ satisfy (3.9). From the classical maximal parabolic regularity for the Stokes equation, we obtain a unique weak solution $\boldsymbol{v}_L \in \boldsymbol{\mathcal{V}}^2_{p,r}(Q)$, where

$$\boldsymbol{\mathcal{V}}^2_{p,r}(Q) := W^{1,r}(I;\boldsymbol{X}^{2,p}_\sigma(\Omega),\boldsymbol{L}^p_\sigma(\Omega)),$$

to the evolution equation

$$\left[\begin{array}{ll} \partial_t\boldsymbol{v}_L + \nu\boldsymbol{A}_p\boldsymbol{v}_L = \boldsymbol{g}_L & \text{in } L^r(I;\boldsymbol{L}^p_\sigma(\Omega)),\\ \boldsymbol{v}_L(0) = \boldsymbol{v}_{0L} & \text{in } \boldsymbol{V}^2_{p,r}(\Omega), \end{array}\right. \tag{3.13}$$

and there is a constant $c>0$ independent on $\boldsymbol{v}_L$, $\boldsymbol{g}_L$, and $\boldsymbol{v}_{0L}$ such that

$$\|\boldsymbol{v}_L\|_{\boldsymbol{\mathcal{V}}^2_{p,r}(Q)} \le c\{\|\boldsymbol{g}_L\|_{L^r(I;\boldsymbol{L}^p_\sigma(\Omega))} + \|\boldsymbol{v}_{0L}\|_{\boldsymbol{V}^2_{p,r}(\Omega)}\}. \tag{3.14}$$

By applying $\boldsymbol{A}'_{p'}$ to (3.13), setting $\boldsymbol{u}_L = \boldsymbol{A}'_{p'}\boldsymbol{v}_L \in \boldsymbol{\mathcal{V}}^0_{p,r}(Q)$ and then using $\boldsymbol{A}'_{p'}\boldsymbol{A}_p\boldsymbol{v}_L = \boldsymbol{A}'_{p'}\boldsymbol{A}'_{p'}\boldsymbol{v}_L = \boldsymbol{A}'_{p'}\boldsymbol{u}_L$, we see that $\boldsymbol{u}_L$ satisfies

$$\left[\begin{array}{ll} \partial_t\boldsymbol{u}_L + \nu\boldsymbol{A}'_{p'}\boldsymbol{u}_L = \boldsymbol{f}_L & \text{in } L^r(I;\boldsymbol{X}^{-2,p}_\sigma(\Omega)),\\ \boldsymbol{u}_L(0) = \boldsymbol{u}_{0L} & \text{in } \boldsymbol{V}^0_{p,r}(\Omega). \end{array}\right.$$

Thus, $\boldsymbol{u}_L$ is a very weak solution to (3.2). The stability estimate (3.11) for this weak solution follows immediately from (3.14) and the definitions of $\boldsymbol{v}_L$, $\boldsymbol{g}_L$, and $\boldsymbol{v}_{0L}$. Furthermore, the uniqueness of this very weak solution is a consequence of the uniqueness of solutions to (3.13).

Finally, the existence and stability of the associated pressure can be established as in the proof of the preceding theorem. Indeed, consider the linear form $\mathfrak{L} \in W^{-1,r}(I;\boldsymbol{W}^{-2,p}(\Omega))$ given by

$$\begin{aligned} &\langle \mathfrak{L},\boldsymbol{\varrho}\rangle_{W^{-1,r}(I;\boldsymbol{W}^{-2,p}(\Omega)),W^{1,r'}_0(I;\boldsymbol{W}^{2,p'}_0(\Omega))}\\ &\quad := \int_0^T \langle \boldsymbol{f}_L,\boldsymbol{\varrho}\rangle_{\boldsymbol{W}^{-2,p}(\Omega),\boldsymbol{W}^{2,p'}_0(\Omega)}\,\mathrm{d}t - \int_0^T \nu\langle \boldsymbol{u}_L,\Delta\boldsymbol{\varrho}\rangle_{\boldsymbol{L}^p(\Omega),\boldsymbol{L}^{p'}(\Omega)}\,\mathrm{d}t\\ &\qquad - \langle \partial_t\boldsymbol{u}_L,\boldsymbol{\varrho}\rangle_{W^{-1,r}(I;\boldsymbol{W}^{-2,p}(\Omega)),W^{1,r'}_0(I;\boldsymbol{W}^{2,p'}_0(\Omega))} \end{aligned}$$

for all $\boldsymbol{\varrho} \in W^{1,r'}_0(I;\boldsymbol{W}^{2,p'}_0(\Omega))$. Note that the duality pairings on the right-hand side are well-defined because $\boldsymbol{u}_L \in L^r(I;\boldsymbol{L}^p(\Omega)) \hookrightarrow L^r(I;\boldsymbol{W}^{-2,p}(\Omega))$, so that $\partial_t\boldsymbol{u}_L \in W^{-1,r}(I;\boldsymbol{W}^{-2,p}(\Omega))$, and $\boldsymbol{f}_L \in L^r(I;\boldsymbol{X}^{-2,p}(\Omega)) \hookrightarrow L^r(I;\boldsymbol{W}^{-2,p}(\Omega))$. From the definition of the Leray–Helmholtz projector $\boldsymbol{P}_{p'}$, for every $\boldsymbol{\varrho} \in L^{r'}(I;\boldsymbol{X}^{2,p'}_\sigma(\Omega))$ there exists $\pi_{\boldsymbol{\varrho}} \in L^{r'}(I;\widehat{W}^{1,p'}(\Omega))$ such that

$$\begin{aligned} \int_0^T \langle \boldsymbol{u}_L,\Delta\boldsymbol{\varrho}\rangle_{\boldsymbol{L}^p(\Omega),\boldsymbol{L}^{p'}(\Omega)}\,\mathrm{d}t &= \int_0^T \{\langle \boldsymbol{u}_L,\boldsymbol{P}_{p'}\Delta\boldsymbol{\varrho}\rangle_{\boldsymbol{L}^p(\Omega),\boldsymbol{L}^{p'}(\Omega)} + \langle \boldsymbol{u}_L,\nabla\pi_{\boldsymbol{\varrho}}\rangle_{\boldsymbol{L}^p(\Omega),\boldsymbol{L}^{p'}(\Omega)}\}\,\mathrm{d}t\\ &= -\int_0^T \langle \boldsymbol{u}_L,\boldsymbol{A}_{p'}\boldsymbol{\varrho}\rangle_{\boldsymbol{L}^p_\sigma(\Omega),\boldsymbol{L}^{p'}_\sigma(\Omega)}\,\mathrm{d}t \end{aligned}$$

since $\operatorname{div}\boldsymbol{u}_L = 0$ in Q. This implies that $\mathfrak{L}$ vanishes on $\boldsymbol{W}^{1,r'}_0(I;\boldsymbol{W}^{2,p'}_0(\Omega)\cap\boldsymbol{L}^{p'}_\sigma(\Omega))$. Thus, we have a unique associated pressure $\mathfrak{p}_L \in W^{-1,r}(I;\widehat{W}^{-1,p}(\Omega))$ from Proposition 7.1 with $k=2$. The stability estimate (3.12) for $\mathfrak{p}_L$ follows from (3.11), the embedding $L^r(I;\boldsymbol{X}^{-2,p}(\Omega)) \hookrightarrow L^r(I;\boldsymbol{X}_\sigma^{-2,p}(\Omega))$ and the estimates similar to those with (3.7) and (3.8). □

Let $\boldsymbol{A}_p^{\mathrm{e}} \in \mathcal{L}(\boldsymbol{X}_\sigma^{1,p}(\Omega), \boldsymbol{X}_\sigma^{-1,p}(\Omega))$ be given by

$$\langle \boldsymbol{A}_p^{\mathrm{e}}\boldsymbol{v}, \boldsymbol{\varrho}\rangle_{\boldsymbol{X}_\sigma^{-1,p}(\Omega),\boldsymbol{X}_\sigma^{1,p'}(\Omega)} = \boldsymbol{a}_p(\boldsymbol{v},\boldsymbol{\varrho}) \qquad \forall\,(\boldsymbol{v},\boldsymbol{\varrho}) \in \boldsymbol{X}_\sigma^{1,p}(\Omega)\times\boldsymbol{X}_\sigma^{1,p'}(\Omega). \tag{3.15}$$

It has been shown in [18, Section 3] that $\boldsymbol{A}_p^{\mathrm{e}} = (\boldsymbol{A}_{p'}^{1/2})'\boldsymbol{A}_p^{1/2}$ and it is an isomorphism that extends the operator $\boldsymbol{A}_p : \boldsymbol{X}^{2,p}(\Omega) \to \boldsymbol{L}_\sigma^p(\Omega)$. The construction of weak solution to the Stokes equation in that paper was done by an application of the operator $\boldsymbol{A}_p^{\mathrm{e}}$, instead of $\boldsymbol{A}_{p'}'$ as in the above proof, to the evolution equation (3.13). We claim that $\boldsymbol{A}_p^{\mathrm{e}} = \boldsymbol{A}_{p'}'$ from $\boldsymbol{X}_\sigma^{1,p}(\Omega)$ to $\boldsymbol{X}_\sigma^{-1,p}(\Omega)$, so that $\boldsymbol{A}_{p'}'$ is also an extension of $\boldsymbol{A}_p^{\mathrm{e}}$. Indeed, from (3.10), (3.15), and the embedding $\boldsymbol{X}_\sigma^{-1,p}(\Omega) \hookrightarrow \boldsymbol{X}_\sigma^{-2,p}(\Omega)$, one has

$$\langle \boldsymbol{A}_p^{\mathrm{e}}\boldsymbol{v}, \boldsymbol{\varrho}\rangle_{\boldsymbol{X}_\sigma^{-2,p}(\Omega),\boldsymbol{X}_\sigma^{2,p'}(\Omega)} = \langle \boldsymbol{A}_{p'}'\boldsymbol{v}, \boldsymbol{\varrho}\rangle_{\boldsymbol{X}_\sigma^{-2,p}(\Omega),\boldsymbol{X}_\sigma^{2,p'}(\Omega)} \qquad \forall\,(\boldsymbol{v},\boldsymbol{\varrho}) \in \boldsymbol{X}_\sigma^{1,p}(\Omega)\times\boldsymbol{X}_\sigma^{2,p'}(\Omega).$$

Thus, for every $\boldsymbol{v} \in \boldsymbol{X}_\sigma^{1,p}(\Omega)$ we have $\boldsymbol{A}_p^{\mathrm{e}}\boldsymbol{v} = \boldsymbol{A}_{p'}'\boldsymbol{v}$ in $\boldsymbol{X}_\sigma^{-2,p}(\Omega)$, and hence in $\boldsymbol{X}_\sigma^{-1,p}(\Omega)$ since the left-hand side belongs to this space.

3.2. MPR for the Heat Equation. This short section deals with the analogous results to the heat equation. Initial data will be taken in the following interpolation spaces:

$$Z_{s,r}^1(\Omega) := (W^{-1,s}(\Omega), W_0^{1,s}(\Omega))_{1/r',r}$$
$$Z_{s,r}^0(\Omega) := (X^{-2,s}(\Omega), L^s(\Omega))_{1/r',r}.$$

The corresponding function spaces for the weak and very weak solutions will be

$$\mathcal{Z}_{s,r}^1(Q) := W^{1,r}(I; W_0^{1,s}(\Omega), W^{-1,s}(\Omega))$$
$$\mathcal{Z}_{s,r}^0(Q) := W^{1,r}(I; L^s(\Omega), X^{-2,s}(\Omega)).$$

It is clear that $Z_{s,r}^1(\Omega) \hookrightarrow Z_{s,r}^0(\Omega)$ and $\mathcal{Z}_{s,r}^1(Q) \hookrightarrow \mathcal{Z}_{s,r}^0(Q)$.

Let us introduce the continuous bilinear form $a_s : W_0^{1,s}(\Omega)\times W_0^{1,s'}(\Omega) \to \mathbb{R}$, which extends the Dirichlet Laplacian, defined by

$$a_s(\gamma,\varrho) := \int_\Omega \nabla\gamma\cdot\nabla\varrho\,\mathrm{d}x \quad \forall\,(\gamma,\varrho) \in W_0^{1,s}(\Omega)\times W_0^{1,s'}(\Omega).$$

Definition 3.5. Consider a source function and an initial data such that

$$h_L \in L^r(I; W^{-1,s}(\Omega)), \qquad \gamma_{0L} \in Z_{s,r}^1(\Omega). \tag{3.16}$$

A function $\gamma_L \in \mathcal{Z}_{s,r}^1(Q)$ is called a *weak solution* of the heat equation

$$\left[\begin{array}{l} \partial_t\gamma_L - \kappa\Delta\gamma_L = h_L \;\text{ in } Q, \\ \gamma_L = 0 \;\text{ on } \Sigma,\; \gamma_L(0) = \gamma_{0L} \;\text{ in } \Omega, \end{array}\right. \tag{3.17}$$

if $\gamma_L(0) = \gamma_{0L}$ in $Z_{s,r}^1(\Omega)$ and we have

$$\int_0^T \langle\partial_t\gamma_L,\varrho\rangle_{W^{-1,s}(\Omega),W_0^{1,s'}(\Omega)}\,\mathrm{d}t + \kappa\int_0^T a_s(\gamma_L,\varrho)\,\mathrm{d}t = \int_0^T \langle h_L,\rho\rangle_{W^{-1,s}(\Omega),W_0^{1,s'}(\Omega)}\,\mathrm{d}t$$

for every $\varrho \in L^{r'}(I; W_0^{1,s'}(\Omega))$. ◇

The initial condition is meaningful since $\mathcal{Z}^1_{s,r}(Q) \hookrightarrow C(\bar{I}; Z^1_{s,r}(\Omega))$ according to [9, Theorem III.4.10.2]. We have the following extended maximal regularity theorem for the heat equation.

Theorem 3.6. *Suppose that $s, r \in (1, \infty)$ and (3.16) are satisfied. The heat equation (3.17) admits a unique weak solution $\gamma_L \in \mathcal{Z}^1_{s,r}(Q)$ and there is a constant $c > 0$ independent of γ_L, h_L, and γ_{0L} such that*

$$\|\gamma_L\|_{\mathcal{Z}^1_{s,r}(Q)} \le c\{\|h_L\|_{L^r(I;W^{-1,s}(\Omega))} + \|\gamma_{0L}\|_{Z^1_{s,r}(\Omega)}\}.$$

Proof. The proof stated in Theorem 3.2 can be adapted to (3.17), and for this reason we omit the details. □

One may also introduce very weak solutions to the heat equation (3.17) similar to that of the Stokes equation, in such a way that weak solutions are also very weak solutions. We state without proof the corresponding result in the following theorem. Time-evaluation for very weak solutions is again well-defined due to $\mathcal{Z}^0_{s,r}(Q) \hookrightarrow C(\bar{I}; Z^0_{s,r}(\Omega))$.

Theorem 3.7. *Let $s, r \in (1, \infty)$ and*

$$h_L \in L^r(I; X^{-2,s}(\Omega)), \quad \gamma_{0L} \in Z^0_{s,r}(\Omega). \tag{3.18}$$

Then (3.17) has a unique very weak solution $\gamma_L \in \mathcal{Z}^0_{s,r}(Q)$ in the sense that $\gamma_L(0) = \gamma_{0L}$ in $Z^0_{s,r}(\Omega)$ and the following variational equation

$$\int_0^T \langle \partial_t \gamma_L, \varrho\rangle_{X^{-2,s}(\Omega),X^{2,s'}(\Omega)}\,\mathrm{d}t + \kappa \int_0^T \langle \gamma_L, A_{s'}\varrho\rangle_{L^s(\Omega),L^{s'}(\Omega)}\,\mathrm{d}t$$
$$= \int_0^T \langle h_L, \varrho\rangle_{X^{-2,s}(\Omega),X^{2,s'}(\Omega)}\,\mathrm{d}t$$

holds for every $\varrho \in L^{r'}(I; X^{2,s'}(\Omega))$. Moreover, there exists a constant $c > 0$ independent of γ_L, h_L, and γ_{0L} such that

$$\|\gamma_L\|_{\mathcal{Z}^0_{s,r}(Q)} \le c\{\|h_L\|_{L^r(I;X^{-2,s}(\Omega))} + \|\gamma_{0L}\|_{Z^0_{s,r}(\Omega)}\}.$$

In what follows, when the conditions (3.16) or (3.18) are referred in the context of the linear system (2.4), then γ_{0L} must be replaced by θ_{0L}.

3.3. MPR for the Viscous Biharmonic Heat Equation. We continue our discussion on the maximal parabolic regularity for the viscous biharmonic heat equation. The function spaces for the initial data in the weak and very weak formulations are given respectively by

$$Z^3_{q,r}(\Omega) := (W^{1,q}_0(\Omega), X^{3,q}(\Omega))_{1/r',r}$$
$$Z^2_{q,r}(\Omega) := (L^q(\Omega), X^{2,q}(\Omega))_{1/r',r}.$$

In the current situation, the weak and very weak solutions will be taken in

$$\mathcal{Z}^3_{q,r}(Q) := W^{1,r}(I; X^{3,q}(\Omega), W^{1,q}_0(\Omega))$$
$$\mathcal{Z}^2_{q,r}(Q) := W^{1,r}(I; X^{2,q}(\Omega), L^q(\Omega)).$$

Applying [9, Theorem III.4.10.2] once more, we deduce the continuity of the embeddings $\mathcal{Z}^3_{q,r}(Q) \hookrightarrow C(\bar{I}; Z^3_{q,r}(\Omega))$ and $\mathcal{Z}^2_{q,r}(Q) \hookrightarrow C(\bar{I}; Z^2_{q,r}(\Omega))$. It is easy to see that $Z^3_{q,r}(\Omega) \hookrightarrow Z^2_{q,r}(\Omega)$ and $\mathcal{Z}^3_{q,r}(Q) \hookrightarrow \mathcal{Z}^2_{q,r}(Q)$. Additional embedding properties are provided in the succeeding lemmas.

Lemma 3.8. *For any $q, r, s \in (1, \infty)$, we have the continuous embeddings $Z^3_{q,r}(\Omega) \hookrightarrow Z^1_{s,r}(\Omega)$ and $Z^2_{q,r}(\Omega) \hookrightarrow Z^0_{s,r}(\Omega)$. Similarly, $\mathcal{Z}^3_{q,r}(Q) \hookrightarrow \mathcal{Z}^1_{s,r}(Q)$ and $\mathcal{Z}^2_{q,r}(Q) \hookrightarrow \mathcal{Z}^0_{s,r}(Q)$.*

Proof. These follow immediately from the definition of real interpolation spaces and the continuity of $W^{1,q}_0(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow W^{-1,s}(\Omega)$, $X^{3,q}(\Omega) \hookrightarrow X^{2,2}(\Omega) \hookrightarrow W^{1,s}_0(\Omega)$, $L^q(\Omega) \hookrightarrow X^{-2,s}(\Omega)$ and $X^{2,q}(\Omega) \hookrightarrow X^{1,2}(\Omega) \hookrightarrow L^s(\Omega)$ by the Sobolev embedding theorem. □

Lemma 3.9. *If $q, r \in (1, \infty)$, then the continuous embeddings $Z^3_{q,r}(\Omega) \hookrightarrow W^{3-2/r-\delta,q}(\Omega) \cap W^{1,q}_0(\Omega)$ and $Z^2_{q,r}(\Omega) \hookrightarrow W^{2-2/r-\delta,q}(\Omega)$ hold for any $\delta > 0$.*

Proof. By definition, we have $Z^3_{q,r}(\Omega) \hookrightarrow W^{1,q}_0(\Omega)$. Since $X^{3,q}(\Omega) \hookrightarrow W^{3,q}(\Omega)$ and $W^{1,q}_0(\Omega) \hookrightarrow W^{1,q}(\Omega)$, by invoking [64, Theorem 4.3.1] we have

$$Z^3_{q,r}(\Omega) \hookrightarrow (W^{1,q}(\Omega), W^{3,q}(\Omega))_{1/r',r} = B^{3/r'+1/r}_{q,r}(\Omega) = B^{3-2/r}_{q,r}(\Omega)$$

where the right-hand side denotes a Besov space, see [64, Definition 4.2.1]. We note that $B^{3-2/r}_{q,r}(\Omega) \hookrightarrow W^{3-2/r-\delta,q}(\Omega)$ for any $\delta > 0$ by [64, Remark 2.3.3/4] and applying the extension property [64, Theorem 4.2.2]. This proves the first continuous embedding. The second one can be established with the same argument. □

Let us now consider the weak formulation for the viscous biharmonic heat equation.

Definition 3.10. Take a source function and an initial data that satisfy

$$\sigma_L \in L^r(I; W^{-1,q}(\Omega)), \quad \phi_{0L} \in Z^3_{q,r}(\Omega). \tag{3.19}$$

A function $\phi_L \in \mathcal{Z}^3_{q,r}(Q)$ is called a *weak solution* of the following viscous biharmonic heat equation

$$\left[\begin{array}{l} \partial_t(\phi_L - m\tau\Delta\phi_L) + m\epsilon\Delta^2\phi_L - \dfrac{\epsilon}{m\tau^2}\phi_L = \sigma_L \ \text{ in } Q, \\ \phi_L = \Delta\phi_L = 0 \ \text{ on } \Sigma, \ \phi_L(0) = \phi_{0L} \ \text{ in } \Omega, \end{array}\right. \tag{3.20}$$

if $\phi_L(0) = \phi_{0L}$ in $Z^3_{q,r}(\Omega)$ and the variational equation

$$\int_0^T \{\langle \partial_t\phi_L, \rho\rangle_{L^q(\Omega),L^{q'}(\Omega)} + m\tau a_q(\partial_t\phi_L, \rho)\}\,\mathrm{d}t + m\epsilon \int_0^T a_q(A_q\phi_L, \rho)\,\mathrm{d}t$$
$$- \frac{\epsilon}{m\tau^2}\int_0^T \langle \phi_L, \rho\rangle_{L^q(\Omega),L^{q'}(\Omega)}\,\mathrm{d}t = \int_0^T \langle \sigma_L, \rho\rangle_{W^{-1,q}(\Omega),W^{1,q'}_0(\Omega)}\,\mathrm{d}t$$

holds for every $\rho \in L^{r'}(I; W^{1,q'}_0(\Omega))$. ◇

Theorem 3.11. *Assume that $q,r \in (1,\infty)$ and (3.19) hold. Then the viscous biharmonic equation (3.20) possesses a unique weak solution $\phi_L \in \mathcal{Z}^3_{q,r}(Q)$ and there exists a constant $c > 0$ independent of ϕ_L, σ_L, and ϕ_{0L} for which*

$$\|\phi_L\|_{\mathcal{Z}^3_{q,r}(Q)} \leq c\{\|\sigma_L\|_{L^r(I;W^{-1,q}(\Omega))} + \|\phi_{0L}\|_{Z^3_{q,r}(\Omega)}\}.$$

Proof. We adapt the proof provided for the Stokes equation in [18] and utilize the maximal regularity for linear parabolic equations. Let us introduce the following isomorphism

$$B_q := I + m\tau A_q : X^{2,q}(\Omega) \to L^q(\Omega)$$

where A_q is the Dirichlet Laplacian on $L^q(\Omega)$. A simple algebraic calculation shows that in the space $\mathcal{L}(X^{2,q}(\Omega), L^q(\Omega))$ there holds

$$A_q^2 = \frac{1}{m^2\tau^2}(B_q^2 - 2B_q + I).$$

Step 1. *Extending the operator B_q.* Since $B_{q'} : X^{2,q'}(\Omega) \to L^{q'}(\Omega)$ and its square root $B_{q'}^{1/2} : W_0^{1,q'}(\Omega) \to L^{q'}(\Omega)$ are isomorphisms, it follows that the dual operator $(B_{q'}^{1/2})' : L^q(\Omega) \to W^{-1,q}(\Omega)$ is also an isomorphism. Consider the linear operator $B_q^{\mathrm{e}} : W_0^{1,q}(\Omega) \to W^{-1,q}(\Omega)$ defined by

$$\langle B_q^{\mathrm{e}}\phi, \rho\rangle_{W^{-1,q}(\Omega),W_0^{1,q'}(\Omega)} := \int_\Omega (\phi\rho + m\tau\nabla\phi\cdot\nabla\rho)\,\mathrm{d}x \quad \forall\,(\phi,\rho) \in W_0^{1,q}(\Omega) \times W_0^{1,q'}(\Omega).$$

This is an extension of the operator B_q since $B_q^{\mathrm{e}}\phi = B_q\phi$ for every $\phi \in X^{2,q}(\Omega)$. Moreover, it is an isomorphism because $B_q^{\mathrm{e}} = (B_{q'}^{1/2})'B_q^{1/2}$ in $W_0^{1,q}(\Omega)$, which can be established using a standard density argument, see [18, Section 3] for the details.

Step 2. *Existence of a weak solution.* By the maximal parabolic regularity for the heat equation, see for instance [23], given $\varsigma_L \in L^r(I;L^q(\Omega))$ and $\psi_{0L} \in Z^2_{q,r}(\Omega)$, the abstract differential equation

$$\left[\begin{array}{ll} \partial_t\psi_L + \dfrac{\epsilon}{m\tau^2}(B_q\psi_L - 2\psi_L) = \varsigma_L & \text{in } L^r(I;L^q(\Omega)), \\ \psi_L(0) = \psi_{0L} & \text{in } Z^2_{q,r}(\Omega), \end{array}\right. \tag{3.21}$$

admits a unique solution $\psi_L \in \mathcal{Z}^2_{q,r}(Q)$. In addition, there exists a constant $c > 0$ independent on ψ_L, ς_L, and ψ_{0L} such that

$$\|\psi_L\|_{\mathcal{Z}^2_{q,r}(Q)} \leq c(\|\varsigma_L\|_{L^r(I;L^q(\Omega))} + \|\psi_{0L}\|_{Z^2_{q,r}(\Omega)}). \tag{3.22}$$

Suppose that $\sigma_L \in L^r(I;W^{-1,q}(\Omega))$ and $\phi_{0L} \in Z^3_{q,r}(\Omega)$. Consider $\varsigma_L := B_q^{1/2}(B_q^{\mathrm{e}})^{-1}\sigma_L$ and $\psi_{0L} := B_q^{1/2}\phi_{0L}$. Since $B_q^{1/2}(B_q^{\mathrm{e}})^{-1} : W^{-1,q}(\Omega) \to L^q(\Omega)$ and $B_q^{1/2} : X^{3,q}(\Omega) \to X^{2,q}(\Omega)$ are isomorphisms, we obtain that $\varsigma_L \in L^r(I;L^q(\Omega))$ and $\psi_{0L} \in B_q^{1/2}(Z^3_{q,r}(\Omega)) = Z^2_{q,r}(\Omega)$, along with the estimates

$$\|\varsigma_L\|_{L^r(I;L^q(\Omega))} \leq c\|\sigma_L\|_{L^r(I;W^{-1,q}(\Omega))}, \quad \|\psi_{0L}\|_{Z^2_{q,r}(\Omega)} \leq c\|\phi_{0L}\|_{Z^3_{q,r}(\Omega)}. \tag{3.23}$$

Let ψ_L be the solution to (3.21) corresponding to these data. Applying $(B_q^{1/2}(B_q^{\mathrm{e}})^{-1})^{-1} = B_q^{\mathrm{e}}B_q^{-1/2}$ to the differential equation and $B_q^{-1/2}$ to the initial data in (3.21), and by setting $\phi_L := B_q^{-1/2}\psi_L \in \mathcal{Z}^3_{q,r}(Q)$, we have

$$\left[\begin{array}{ll} \partial_t B_q^{\mathrm{e}}\phi_L + \dfrac{\epsilon}{m\tau^2}(B_q^{\mathrm{e}}B_q\phi_L - 2B_q\phi_L) = \sigma_L & \text{in } L^r(I;W^{-1,q}(\Omega)), \\ \phi_L(0) = \phi_{0L} & \text{in } Z^3_{q,r}(\Omega). \end{array}\right. \tag{3.24}$$

Consider the isomorphism $A_q^{\mathrm{e}} := (A_{q'}^{1/2})'A_q^{1/2} : W_0^{1,q}(\Omega) \to W^{-1,q}(\Omega)$. This operator satisfies $A_q^{\mathrm{e}} = \frac{1}{m\tau}(B_q^{\mathrm{e}} - I)$ and

$$A_q^{\mathrm{e}}A_q = \frac{1}{m^2\tau^2}(B_q^{\mathrm{e}}B_q - 2B_q + I),$$

with the latter equation taken as equality of isomorphisms from $X^{3,q}(\Omega)$ onto $W^{-1,q}(\Omega)$. Hence, for each $\rho \in L^{r'}(I;W_0^{1,q'}(\Omega))$, there holds almost everywhere in I that

$$\langle B_q^{\mathrm{e}}B_q\phi_L - 2B_q\phi_L, \rho\rangle_{W^{-1,q}(\Omega)',W_0^{1,q'}(\Omega)} = m^2\tau^2 a_q(A_q\phi_L,\rho) - \langle \phi_L,\rho\rangle_{L^q(\Omega),L^{q'}(\Omega)}.$$

Substituting this in (3.24) and then using the definition of B_q^{e} in the time derivative, we see that ϕ_L is a weak solution to (3.20). Moreover, from (3.22), (3.23), and the definition of ϕ_L, we obtain the stability estimate stated by the theorem.

Step 3. *Uniqueness of the weak solution.* By linearity it suffices to show that we have a trivial solution corresponding to the equation with zero source term $\sigma_L = 0$ and initial data $\phi_{0L} = 0$. Indeed, suppose that $\phi_L \in \mathcal{Z}^3_{q,r}(Q)$ is a weak solution to such a system. Then, if we apply $B_q^{1/2}B_q^{-\mathrm{e}}$ to both sides of the differential equation in (3.24), we observe that $\psi_L := B_q^{1/2}\phi_L \in \mathcal{Z}^2_{q,r}(Q)$ is a solution to (3.21) with $\varsigma_L = 0$ and $\psi_{0L} = 0$. By uniqueness of solution to (3.21), it follows that $\psi_L = 0$, and consequently, we get $\phi_L = 0$. □

We also have the existence and uniqueness of very weak solutions to (3.20).

Theorem 3.12. *Let $q, r \in (1,\infty)$ and suppose that*

$$\sigma_L \in L^r(I;X^{-2,q}(\Omega)), \quad \phi_{0L} \in Z^2_{q,r}(\Omega). \tag{3.25}$$

Then (3.20) has a unique very weak solution $\phi_L \in \mathcal{Z}^2_{q,r}(Q)$ in the sense that $\phi_L(0) = \phi_{0L}$ in $Z^2_{q,r}(\Omega)$ and for every $\rho \in L^{r'}(I;X^{2,q'}(\Omega))$

$$\begin{aligned} &\int_0^T \{\langle \partial_t\phi_L,\rho\rangle_{L^q(\Omega),L^{q'}(\Omega)} + m\tau\langle A_{q'}'\partial_t\phi_L,\rho\rangle_{X^{-2,q}(\Omega),X^{2,q'}(\Omega)}\}\,\mathrm{d}t \\ &\quad + m\epsilon\int_0^T \langle A_q\phi_L, A_{q'}\rho\rangle_{L^q(\Omega),L^{q'}(\Omega)}\,\mathrm{d}t - \frac{\epsilon}{m\tau^2}\int_0^T \langle\phi_L,\rho\rangle_{L^q(\Omega),L^{q'}(\Omega)}\,\mathrm{d}t \\ &\quad = \int_0^T \langle\sigma_L,\rho\rangle_{X^{-2,q}(\Omega),X^{2,q'}(\Omega)}\,\mathrm{d}t. \end{aligned}$$

Furthermore, there is a constant $c > 0$ independent of ϕ_L, σ_L, and ϕ_{0L} for which

$$\|\phi_L\|_{\mathcal{Z}^2_{q,r}(Q)} \le c\{\|\sigma_L\|_{L^r(I;X^{-2,q}(\Omega))} + \|\phi_{0L}\|_{Z^2_{q,r}(\Omega)}\}.$$

Proof. Note that the dual operator $B'_{q'} : L^q(\Omega) \to X^{-2,q}(\Omega)$ of $B_{q'} : X^{2,q'}(\Omega) \to L^{q'}(\Omega)$ is an isomorphism and is an extension of $B_q : X^{2,q}(\Omega) \to L^q(\Omega)$. We then proceed as in the proof of the preceding theorem, but now by applying $B'_{q'}$ to the differential equation (3.24) with $\varsigma_L = (B'_{q'})^{-1}\sigma_L \in L^r(I; L^q(\Omega))$ and $\psi_{0L} = \phi_{0L} \in Z^2_{q,r}(\Omega)$. The required very weak solution would then be $\phi_L = \psi_L$. Note that in order to pass from the abstract differential equation to the variational equation in the very weak formulation, we utilize the fact that

$$A'_{q'}A_q = \frac{1}{m^2\tau^2}(B'_{q'}B_q - 2B_q + I)$$

as isomorphisms from $X^{2,q}(\Omega)$ onto $X^{-2,q}(\Omega)$. □

As in the case of the Stokes operator, we have $B'_{q'} = B^{\mathrm{e}}_q$ and $A'_{q'} = A^{\mathrm{e}}_q$ as isomorphisms from $W^{1,q}_0(\Omega)$ onto $W^{-1,q}(\Omega)$.

Remark 3.13. *Following the strategy in the succeeding subsection, one may drop the linear term $\frac{\epsilon}{m\tau^2}\phi_L$ in (3.20). However, since the above form of the biharmonic heat equation is sufficient to our analysis, we do not provide the details here.*

3.4. MPR for the Linearized System.

Having established maximal parabolic regularity theorems for each of the components in the linear system (2.4), we are now in position to establish the corresponding results for the coupled system. The main idea is to treat the coupling terms as *external* sources.

Definition 3.14. Suppose that the source functions and the initial data satisfy the conditions (3.1), (3.16), (3.19), and let

$$\lambda_L \in L^r(I; W^{1,q}_0(\Omega)). \tag{3.26}$$

A tuple $(\phi_L, \theta_L, \boldsymbol{u}_L, \mu_L) \in \mathcal{Z}^3_{q,r}(Q) \times \mathcal{Z}^1_{s,r}(Q) \times \boldsymbol{\mathcal{V}}^1_{p,r}(Q) \times L^r(I; W^{1,q}_0(\Omega))$ is said to be a *weak solution* to (2.4) provided that the initial condition $(\phi_L(0), \theta_L(0), \boldsymbol{u}_L(0)) = (\phi_{0L}, \theta_{0L}, \boldsymbol{u}_{0L})$ holds in $Z^3_{q,r}(\Omega) \times Z^1_{s,r}(\Omega) \times \boldsymbol{V}^1_{p,r}(\Omega)$, the variational equations

$$\text{(a)} \quad \int_0^T \{(\partial_t\phi_L, \rho)_{L^2(\Omega)} + ma_q(\mu_L, \rho)\}\,\mathrm{d}t = \int_0^T \langle \sigma_L, \rho\rangle_{W^{-1,q}(\Omega), W^{1,q'}_0(\Omega)}\,\mathrm{d}t$$

$$\text{(b)} \quad \int_0^T \{\langle \partial_t\theta_L, \varrho\rangle_{W^{-1,s}(\Omega), W^{1,s'}_0(\Omega)} - l_{\mathrm{h}}(\partial_t\phi_L, \varrho)_{L^2(\Omega)} + \kappa a_s(\theta_L, \varrho)\}\,\mathrm{d}t$$
$$= \int_0^T \{(\alpha\boldsymbol{g}\cdot\boldsymbol{u}_L, \varrho)_{L^2(\Omega)} + \langle h_L, \varrho\rangle_{W^{-1,s}(\Omega), W^{1,s'}_0(\Omega)}\}\,\mathrm{d}t$$

$$\text{(c)} \quad \int_0^T \{\langle \partial_t\boldsymbol{u}_L, \boldsymbol{\rho}\rangle_{\boldsymbol{X}^{-1,p}_\sigma(\Omega), \boldsymbol{X}^{1,p'}_\sigma(\Omega)} + \nu\boldsymbol{a}_p(\boldsymbol{u}_L, \boldsymbol{\rho})\}\,\mathrm{d}t$$
$$= \int_0^T \{((\alpha_1\phi_L + \alpha_2\theta_L)\boldsymbol{g}, \boldsymbol{\rho})_{\boldsymbol{L}^2(\Omega)} + \langle \boldsymbol{f}_L, \boldsymbol{\rho}\rangle_{\boldsymbol{X}^{-1,p}_\sigma(\Omega), \boldsymbol{X}^{1,p'}_\sigma(\Omega)}\}\,\mathrm{d}t$$

are satisfied for every $\rho \in L^{r'}(I; W^{1,q'}_0(\Omega))$, $\varrho \in L^{r'}(I; W^{1,s'}_0(\Omega))$, $\boldsymbol{\rho} \in L^{r'}(I; \boldsymbol{X}^{1,p'}_\sigma(\Omega))$, and we have

$$\mu_L = \tau\partial_t\phi_L - \epsilon\Delta\phi_L - \beta_1\phi_L + l_{\mathrm{c}}\theta_L + \lambda_L \quad \text{a.e. } Q. \tag{3.27}$$

If $\boldsymbol{f}_L \in L^r(I; W^{-1,p}(\Omega))$, then we call $\mathfrak{p}_L \in W^{-1,r}(I; \widehat{L}^p(\Omega))$ an *associated pressure* if

$$\begin{aligned}
&\langle \partial_t \boldsymbol{u}_L, \boldsymbol{\varrho} \rangle_{W^{-1,r}(I;\boldsymbol{W}^{-1,p}(\Omega)),\boldsymbol{W}_0^{1,r'}(I;\boldsymbol{W}_0^{1,p'}(\Omega))} \\
&\qquad + \nu \int_0^T \boldsymbol{a}_p(\boldsymbol{u}_L, \boldsymbol{\varrho})\,\mathrm{d}t - \langle \mathfrak{p}_L, \operatorname{div} \boldsymbol{\varrho} \rangle_{W^{-1,r}(I;\widehat{L}^p(\Omega)),W_0^{1,r'}(I;\widehat{L}^{p'}(\Omega))} \\
&\qquad = \int_0^T \{((\alpha_1 \phi_L + \alpha_2 \theta_L)\boldsymbol{g}, \boldsymbol{\varrho})_{\boldsymbol{L}^2(\Omega)} + \langle \boldsymbol{f}_L, \boldsymbol{\varrho} \rangle_{\boldsymbol{W}^{-1,p}(\Omega),\boldsymbol{W}_0^{1,p'}(\Omega)}\}\,\mathrm{d}t
\end{aligned}$$

is satisfied by every $\boldsymbol{\varrho} \in W_0^{1,r'}(I; \boldsymbol{W}_0^{1,p'}(\Omega))$. ◇

Observe that one can view (3.27) as an equality in the space $L^r(I; W_0^{1,q}(\Omega))$ when $q \le s$. On the left-hand side of the variational equation (a) and the right-hand sides of (b) and (c), where we have the appearances of the L^2 inner product, we used the fact that $W^{1,\mathfrak{s}}(\Omega) \hookrightarrow W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$ for every $\mathfrak{s} \ge 1$. In particular, we have in (c) the equation

$$\int_0^T ((\alpha_1 \phi_L + \alpha_2 \theta_L)\boldsymbol{g}, \boldsymbol{\rho})_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t = \int_0^T (\boldsymbol{P}_2\{(\alpha_1 \phi_L + \alpha_2 \theta_L)\boldsymbol{g}\}, \boldsymbol{\rho})_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t$$

where we recall that $\boldsymbol{P}_2$ is the Leray–Helmholtz projector from $\boldsymbol{L}^2(\Omega)$ onto $\boldsymbol{L}^2_\sigma(\Omega)$. Although the velocities coincide for these two formulations, the associated pressures will be different, see also Remark 3.17 below.

It will be advantageous to eliminate the linearized chemical potential μ_L in the system (2.4) and to introduce the new variable $\gamma_L := \theta_L - l_\mathrm{h}\phi_L$ along with the initial data $\gamma_{0L} := \theta_{0L} - l_\mathrm{h}\phi_{0L}$. In this direction, we have the following equivalent linear system

$$\left[\begin{array}{ll}
\partial_t(\phi_L - m\tau\Delta\phi_L) + m\{\epsilon\Delta^2\phi_L + (\beta_1 - l_\mathrm{c}l_\mathrm{h})\Delta\phi_L - l_\mathrm{c}\Delta\gamma_L - \Delta\lambda_L\} = \sigma_L & \text{in } Q, \\
\partial_t\gamma_L - \kappa\Delta\gamma_L - \kappa l_\mathrm{h}\Delta\phi_L = \alpha\boldsymbol{g}\cdot\boldsymbol{u}_L + h_L & \text{in } Q, \\
\partial_t\boldsymbol{u}_L - \nu\Delta\boldsymbol{u}_L + \nabla\mathfrak{p}_L = \{(\alpha_1 + \alpha_2 l_\mathrm{h})\phi_L + \alpha_2\gamma_L\}\boldsymbol{g} + \boldsymbol{f}_L & \text{in } Q, \\
\operatorname{div}\boldsymbol{u}_L = 0 & \text{in } Q, \\
\phi_L = \Delta\phi_L = 0, \quad \gamma_L = 0, \quad \boldsymbol{u}_L = \boldsymbol{0} & \text{on } \Sigma, \\
\phi_L(0) = \phi_{0L}, \quad \gamma_L(0) = \gamma_{0L}, \quad \boldsymbol{u}_L(0) = \boldsymbol{u}_{0L} & \text{in } \Omega.
\end{array}\right. \tag{3.28}$$

A similar definition of weak solutions to the equivalent linear system (3.28) can be formulated as in Definition 3.14, but we leave the details to the reader for the precise statements. In terms of the extended Dirichlet Laplacian and Stokes operator, the differential equations in (3.28) is equivalent to the following abstract evolution equations

$$\left[\begin{array}{l}
\partial_t(\phi_L + m\tau A_q\phi_L) + m\epsilon A_q^\mathrm{e}A_q\phi_L - m\{(\beta_1 - l_\mathrm{c}l_\mathrm{h})A_q\phi_L - l_\mathrm{c}A_q^\mathrm{e}\gamma_L\} = \sigma_L - mA_q^\mathrm{e}\lambda_L \\
\partial_t\gamma_L + \kappa A_s^\mathrm{e}\gamma_L + \kappa l_\mathrm{h}A_s^\mathrm{e}\phi_L = \alpha\boldsymbol{g}\cdot\boldsymbol{u}_L + h_L \\
\partial_t\boldsymbol{u}_L + \nu\boldsymbol{A}_p^\mathrm{e}\boldsymbol{u}_L = \{(\alpha_1 + \alpha_2 l_\mathrm{h})\phi_L + \alpha_2\gamma_L\}\boldsymbol{g} + \boldsymbol{f}_L
\end{array}\right.$$

The Laplace and Stokes operators in these equations have to be modified in the context of very weak solutions.

We wish is to establish maximal parabolic regularity theorems for (3.28). For this, we shall proceed by gradually decreasing the order of spatial differentiability. Let us start with the Hilbertian case, that is, $p = q = r = 2$. For the meantime we ignore the subscript L. The main tool is a classical theorem in [22] which we state for the convenience of the reader.

Theorem 3.15. *Let $r \in (1, \infty)$ and $A : D(A) \subset H \to H$ be a closed linear operator on a Hilbert space H such that $-A$ generates a strongly continuous analytic semigroup on H. For each $f \in L^r(I; H)$, the Cauchy problem*

$$\left[\begin{array}{ll} \partial_t z + Az = f & \text{in } L^r(I;H), \\ z(0) = 0 & \text{in } H, \end{array}\right.$$

admits a unique solution $z \in W^{1,r}(I; D(A), H)$. Moreover, there exists a constant $c > 0$ independent on z and f such that

$$\|z\|_{W^{1,r}(I;D(A),H)} \leq c\|f\|_{L^r(I;H)}. \tag{3.29}$$

The aim is to apply this theorem to (3.28) with homogeneous initial conditions. We introduce the Hilbert space $\boldsymbol{\mathcal{H}}_\omega := X^{2,2}(\Omega) \times L^2(\Omega) \times \boldsymbol{L}^2_\sigma(\Omega)$ equipped with the weighted inner product

$$((\phi, \gamma, \boldsymbol{u}), (\psi, \eta, \boldsymbol{v}))_{\boldsymbol{\mathcal{H}}_\omega} := \omega(\phi, \psi)_{X^{2,2}(\Omega)} + (\gamma, \eta)_{L^2(\Omega)} + (\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{L}^2_\sigma(\Omega)}$$

where $(\phi, \psi)_{X^{2,2}(\Omega)} := (B_2\phi, B_2\psi)_{L^2(\Omega)}$ and $\omega > 0$. Let us define the linear operator $\boldsymbol{\mathcal{A}} : D(\boldsymbol{\mathcal{A}}) \subset \boldsymbol{\mathcal{H}}_\omega \to \boldsymbol{\mathcal{H}}_\omega$ having the domain

$$D(\boldsymbol{\mathcal{A}}) = D(A_2^2) \times D(A_2) \times D(\boldsymbol{A}_2) = X^{4,2}(\Omega) \times X^{2,2}(\Omega) \times \boldsymbol{X}^{2,2}_\sigma(\Omega)$$

according to

$$\boldsymbol{\mathcal{A}}(\phi, \gamma, \boldsymbol{u}) = \begin{pmatrix} m\epsilon B_2^{-1} A_2^2 \phi - m(\beta_1 - l_{\mathrm{c}} l_{\mathrm{h}}) B_2^{-1} A_2 \phi + m l_{\mathrm{c}} B_2^{-1} A_2 \gamma \\ \kappa A_2 \gamma + \kappa l_{\mathrm{h}} A_2 \phi - \alpha \boldsymbol{g} \cdot \boldsymbol{u} \\ \nu \boldsymbol{A}_2 \boldsymbol{u} - \boldsymbol{P}_2\{((\alpha_1 + \alpha_2 l_{\mathrm{h}})\phi + \alpha_2\gamma)\boldsymbol{g}\} \end{pmatrix}. \tag{3.30}$$

Note that up to a constant factor, the principal term for the first component of $\boldsymbol{\mathcal{A}}$ is the Dirichlet Laplacian. Indeed, this component can be expressed solely in terms of A_2 using the identities

$$\begin{aligned} B_2^{-1} A_2 &= \frac{1}{m\tau}(I - (I + m\tau A_2)^{-1}) \\ B_2^{-1} A_2^2 &= \frac{1}{m^2\tau^2}(m\tau A_2 - I + (I + m\tau A_2)^{-1}). \end{aligned}$$

Here, $(I + m\tau A_2)^{-1}$ is a smoothing operator in the sense that it maps $X^{\mathfrak{s},2}(\Omega)$ onto $X^{\mathfrak{s}+2,2}(\Omega)$ for any $\mathfrak{s} \geq 0$. From these, we see that the map $\boldsymbol{\mathcal{A}}$ is well-defined.

It is standard to show that $-\boldsymbol{\mathcal{A}}$ generates an analytic C_0-semigroup on $\boldsymbol{\mathcal{H}}_\omega$ provided that $\omega > 0$ is small enough. Nevertheless, we present the proof in the Appendix for completeness, see Proposition 7.3. Translating this to the original linear system (2.4) with vanishing initial data leads to the following theorem. For the proof, we introduce the following strong solution space for the viscous biharmonic heat equation

$$\mathcal{Z}^4_{2,r}(Q) := W^{1,r}(I; X^{4,2}(\Omega), X^{2,2}(\Omega)).$$

Theorem 3.16. *Let $r \in (1,\infty)$. Suppose that $\sigma_L, h_L \in L^r(I;L^2(\Omega))$, $\lambda_L \in L^r(I;X^{2,2}(\Omega))$, $\boldsymbol{f}_L \in L^r(I;\boldsymbol{L}^2_\sigma(\Omega))$, $\phi_{0L} = 0$, $\theta_{0L} = 0$, and $\boldsymbol{u}_{0L} = \mathbf{0}$. Then the linear system (2.4) has a unique weak solution*

$$(\phi_L, \theta_L, \boldsymbol{u}_L, \mu_L) \in \mathcal{Z}^4_{2,r}(Q) \times \mathcal{Z}^2_{2,r}(Q) \times \boldsymbol{\mathcal{V}}^2_{2,r}(Q) \times L^r(I;X^{2,2}(\Omega)).$$

We have a unique associated pressure $\mathfrak{p}_L \in L^r(I;\widehat{W}^{1,2}(\Omega))$. Moreover, there is a constant $c > 0$ independent on the solution and the source functions such that

$$\begin{aligned}\|\phi_L\|_{\mathcal{Z}^4_{2,r}(Q)} + \|\theta_L\|_{\mathcal{Z}^2_{2,r}(Q)} + \|\boldsymbol{u}_L\|_{\boldsymbol{\mathcal{V}}^2_{2,r}(Q)} + \|\mu_L\|_{L^r(I;X^{2,2}(\Omega))} + \|\mathfrak{p}_L\|_{L^r(I;\widehat{W}^{1,2}(\Omega))} \\ \le c\{\|\sigma_L\|_{L^r(I;L^2(\Omega))} + \|h_L\|_{L^r(I;L^2(\Omega))} + \|\boldsymbol{f}_L\|_{L^r(I;\boldsymbol{L}^2_\sigma(\Omega))} + \|\lambda_L\|_{L^r(I;X^{2,2}(\Omega))}\}.\end{aligned} \tag{3.31}$$

Proof. From the assumptions on σ_L and λ_L, we have $B_2^{-1}(\sigma_L - A_2\lambda_L) \in L^r(I;X^{2,2}(\Omega))$ and

$$\|B_2^{-1}(\sigma_L - A_2\lambda_L)\|_{L^r(I;X^{2,2}(\Omega))} \le c\{\|\sigma_L\|_{L^r(I;L^2(\Omega))} + \|\lambda_L\|_{L^r(I;X^{2,2}(\Omega))}\}. \tag{3.32}$$

The analyticity of the C_0-semigroup generated by $-\boldsymbol{\mathcal{A}}$ (see Proposition 7.3) and Theorem 3.15 imply that the Cauchy problem

$$\partial_t(\phi_L, \gamma_L, \boldsymbol{u}_L) + \boldsymbol{\mathcal{A}}(\phi_L, \gamma_L, \boldsymbol{u}_L) = (B_2^{-1}(\sigma_L - A_2\lambda_L), h_L, \boldsymbol{f}_L)$$

with the homogeneous initial condition $(\phi_L(0), \gamma_L(0), \boldsymbol{u}_L(0)) = (0,0,\mathbf{0})$ admits a unique solution $(\phi_L, \gamma_L, \boldsymbol{u}_L) \in \mathcal{Z}^4_{2,r}(Q) \times \mathcal{Z}^2_{2,r}(Q) \times \boldsymbol{\mathcal{V}}^2_{2,r}(Q)$. Invoking the estimates (3.32) and (3.29), we deduce that

$$\begin{aligned}\|\phi_L\|_{\mathcal{Z}^4_{2,r}(Q)} + \|\gamma_L\|_{\mathcal{Z}^2_{2,r}(Q)} + \|\boldsymbol{u}_L\|_{\boldsymbol{\mathcal{V}}^2_{2,r}(Q)} \\ \le c\{\|\sigma_L\|_{L^r(I;L^2(\Omega))} + \|h_L\|_{L^r(I;L^2(\Omega))} + \|\boldsymbol{f}_L\|_{L^r(I;\boldsymbol{L}^2_\sigma(\Omega))} + \|\lambda_L\|_{L^r(I;X^{2,2}(\Omega))}\}.\end{aligned} \tag{3.33}$$

By applying B_2 to the first equation in the above Cauchy problem, we obtain that $(\phi_L, \gamma_L, \boldsymbol{u}_L)$ satisfies the evolution equation associated to the linear system (3.28) with zero initial conditions. Since $\mathcal{Z}^4_{2,r}(Q) \hookrightarrow \mathcal{Z}^2_{2,r}(Q)$, we have $\theta_L := \gamma_L + l_{\mathrm{h}}\phi_L \in \mathcal{Z}^2_{2,r}(Q)$, and by the triangle inequality

$$\|\theta_L\|_{\mathcal{Z}^2_{2,r}(Q)} \le c\{\|\gamma_L\|_{\mathcal{Z}^2_{2,r}(Q)} + \|\phi_L\|_{\mathcal{Z}^4_{2,r}(Q)}\}. \tag{3.34}$$

Also, μ_L defined by (3.27) is an element of $L^r(I;X^{2,2}(\Omega))$ and

$$\|\mu_L\|_{L^r(I;X^{2,2}(\Omega))} \le c\{\|\phi_L\|_{\mathcal{Z}^4_{2,r}(Q)} + \|\theta_L\|_{L^r(I;X^{2,2}(\Omega))} + \|\lambda_L\|_{L^r(I;X^{2,2}(\Omega))}\}. \tag{3.35}$$

Thus, $(\phi_L, \theta_L, \boldsymbol{u}_L, \mu_L) \in \mathcal{Z}^4_{2,r}(Q) \times \mathcal{Z}^2_{2,r}(Q) \times \boldsymbol{\mathcal{V}}^2_{2,r}(Q) \times L^r(I;X^{2,2}(\Omega))$ and it is the weak solution to (2.4). The existence of the associated pressure $\mathfrak{p}_L \in L^r(I;\widehat{W}^{1,2}(\Omega))$ is a consequence of de Rham's Theorem, see for instance [61, Section IV.1.4], and we have

$$\begin{aligned}\|\mathfrak{p}_L\|_{L^r(I;\widehat{W}^{1,2}(\Omega))} \le c\{\|\boldsymbol{u}_L\|_{\boldsymbol{\mathcal{V}}^2_{2,r}(Q)} + |\boldsymbol{g}|\|\phi_L\|_{L^r(I;L^2(\Omega))} \\ + |\boldsymbol{g}|\|\gamma_L\|_{L^r(I;L^2(\Omega))} + \|\boldsymbol{f}_L\|_{L^r(I;\boldsymbol{L}^2_\sigma(\Omega))}\}.\end{aligned} \tag{3.36}$$

Taking the sum of the estimates (3.33)–(3.36) leads to (3.31). □

Remark 3.17. *The preceding theorem is also valid when the condition* $\boldsymbol{f}_L \in L^2(I;\boldsymbol{L}^2_\sigma(\Omega))$ *is replaced by* $\boldsymbol{f}_L \in L^2(I;\boldsymbol{L}^2(\Omega))$. *Indeed, the weak solution* $\boldsymbol{u}_L$ *is the same for source functions* $\boldsymbol{f}_L \in L^2(I;\boldsymbol{L}^2(\Omega))$ *and its projection* $\boldsymbol{P}_2\boldsymbol{f}_L \in L^2(I;\boldsymbol{L}^2_\sigma(\Omega))$. *However, the pressures would be different, that is, if* $\mathfrak{p}_L$ *is the pressure corresponding to* $\boldsymbol{f}_L$, *then the pressure associated with* $\boldsymbol{P}_2\boldsymbol{f}_L$ *would be* $\mathfrak{p}_L - \pi_L$, *where* $\pi_L \in L^r(I;\widehat{W}^{1,2}(\Omega))$ *satisfies* $\boldsymbol{f}_L = \boldsymbol{P}_2\boldsymbol{f}_L + \nabla\pi_L$.

The remaining part of this section is concerned with the existence, uniqueness, and stability of weak and very weak solutions to the linear system (2.4). Let us start with weak solutions.

Theorem 3.18. *Let* $p,q,s,r \in (1,\infty)$ *where* $q \le s$. *Suppose that* (3.1), (3.16), (3.19) *and* (3.26) *hold. Then the linear system* (2.4) *admits a unique weak solution*

$$(\phi_L,\theta_L,\boldsymbol{u}_L,\mu_L) \in \mathcal{Z}^3_{q,r}(Q) \times \mathcal{Z}^1_{s,r}(Q) \times \boldsymbol{\mathcal{V}}^1_{p,r}(Q) \times L^r(I;W^{1,q}_0(\Omega)).$$

If $\boldsymbol{f}_L \in L^r(I;\boldsymbol{W}^{-1,p}(\Omega))$, *then there is a unique associated pressure* $\mathfrak{p}_L \in W^{-1,r}(I;\widehat{L}^p(\Omega))$. *In addition, there exists a constant* $c > 0$ *independent on the solution, source functions, and initial data for which*

$$\begin{aligned}
&\|\phi_L\|_{\mathcal{Z}^3_{q,r}(Q)} + \|\theta_L\|_{\mathcal{Z}^1_{s,r}(Q)} + \|\boldsymbol{u}_L\|_{\boldsymbol{\mathcal{V}}^1_{p,r}(Q)} + \|\mu_L\|_{L^r(I;W^{1,p}_0(\Omega))} + \|\mathfrak{p}_L\|_{W^{-1,r}(I;\widehat{L}^p(\Omega))} \\
&\quad \le c\{\|\phi_{0L}\|_{Z^3_{q,r}(\Omega)} + \|\theta_{0L}\|_{Z^1_{s,r}(\Omega)} + \|\boldsymbol{u}_{0L}\|_{\boldsymbol{V}^1_{p,r}(\Omega)} + \|\sigma_L\|_{L^r(I;W^{-1,q}(\Omega))} \\
&\qquad + \|\lambda_L\|_{L^r(I;W^{1,q}_0(\Omega))} + \|h_L\|_{L^r(I;W^{-1,s}(\Omega))} + \|\boldsymbol{f}_L\|_{L^r(I;\boldsymbol{W}^{-1,p}(\Omega))}\}.
\end{aligned} \tag{3.37}$$

Proof. Following the argument in Theorem 3.16, we first consider the equivalent linear system (3.28). From Lemma 3.8, note that $\gamma_{0L} = \theta_{0L} - l_{\mathrm{h}}\phi_{0L} \in Z^1_{s,r}(\Omega) + Z^3_{q,r}(\Omega) = Z^1_{s,r}(\Omega)$.

STEP 1. *Existence.* Consider the following linear system

$$\left[\begin{array}{ll}
\partial_t(\phi_1 - m\tau\Delta\phi_1) + m\epsilon\Delta^2\phi_1 - \dfrac{\epsilon}{m\tau^2}\phi_1 = \sigma_L + m\Delta\lambda_L + ml_{\mathrm{c}}\Delta\gamma_1 & \text{in } Q, \\
\partial_t\gamma_1 - \kappa\Delta\gamma_1 = h_L & \text{in } Q, \\
\partial_t\boldsymbol{u}_1 - \nu\Delta\boldsymbol{u}_1 + \nabla\mathfrak{p}_1 = \boldsymbol{f}_L & \text{in } Q, \\
\operatorname{div}\boldsymbol{u}_1 = 0 & \text{in } Q, \\
\phi_1 = \Delta\phi_1 = 0, \quad \gamma_1 = 0, \quad \boldsymbol{u}_1 = \boldsymbol{0} & \text{on } \Sigma, \\
\phi_1(0) = \phi_{0L}, \quad \gamma_1(0) = \gamma_{0L}, \quad \boldsymbol{u}_1(0) = \boldsymbol{u}_{0L} & \text{in } \Omega.
\end{array}\right. \tag{3.38}$$

Notice that the differential equations for $\boldsymbol{u}_1$ and γ_1 are independent to each other, while that of ϕ_1 depends only on γ_1. From Theorem 3.2 and Theorem 3.6, we infer that the Stokes and heat equations in this system have respective unique weak solutions $\boldsymbol{u}_1 \in \boldsymbol{\mathcal{V}}^1_{p,r}(Q)$ and $\gamma_1 \in \mathcal{Z}^1_{s,r}(Q)$, and these enjoy the following estimates:

$$\|\boldsymbol{u}_1\|_{\boldsymbol{\mathcal{V}}^1_{p,r}(Q)} \le c\{\|\boldsymbol{f}_L\|_{L^r(I;\boldsymbol{X}^{-1,p}_\sigma(\Omega))} + \|\boldsymbol{u}_{0L}\|_{\boldsymbol{V}^1_{p,r}(\Omega)}\} \tag{3.39}$$

$$\|\gamma_1\|_{\mathcal{Z}^1_{s,r}(Q)} \le c\{\|h_L\|_{L^r(I;W^{-1,s}(\Omega))} + \|\gamma_{0L}\|_{Z^1_{s,r}(\Omega)}\}. \tag{3.40}$$

In addition, if $\boldsymbol{f}_L \in L^r(I;\boldsymbol{W}^{-1,p}(\Omega))$, then we have a unique associated pressure $\mathfrak{p}_1 \in W^{-1,r}(I;\widehat{L}^p(\Omega))$ satisfying

$$\|\mathfrak{p}_1\|_{W^{-1,r}(I;\widehat{L}^p(\Omega))} \le c\{\|\boldsymbol{f}_L\|_{L^r(I;\boldsymbol{W}^{-1,p}(\Omega))} + \|\boldsymbol{u}_{0L}\|_{\boldsymbol{V}^1_{p,r}(\Omega)}\}. \tag{3.41}$$

From the condition $q \le s$, we obtain $W_0^{1,s}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$ and $W^{-1,s}(\Omega) \hookrightarrow W^{-1,q}(\Omega)$. Thus, $m\Delta\lambda_L + ml_c\Delta\gamma_1 \in L^r(I;W^{-1,q}(\Omega))$ and

$$\|m\Delta\lambda_L + ml_c\Delta\gamma_1\|_{L^r(I;W^{-1,q}(\Omega))} \le c\{\|\lambda_L\|_{L^r(I;W_0^{1,q}(\Omega))} + \|\gamma_1\|_{L^r(I;W_0^{1,s}(\Omega))}\}.$$

One can then use Theorem 3.11 to the viscous biharmonic heat equation in (3.38) to obtain a unique weak solution $\phi_1 \in \mathcal{Z}^3_{q,r}(Q)$ that satisfies the estimate

$$\begin{aligned}\|\phi_1\|_{\mathcal{Z}^3_{q,r}(Q)} \le c\{&\|\sigma_L\|_{L^r(I;W^{-1,q}(\Omega))} + \|\lambda_L\|_{L^r(I;W_0^{1,q}(\Omega))}\\ &+ \|\gamma_1\|_{L^r(I;W_0^{1,s}(\Omega))} + \|\psi_{0L}\|_{Z^3_{q,r}(\Omega)}\}.\end{aligned} \tag{3.42}$$

Let us define $\sigma_1 := -m(\beta_1 - l_c l_h)\Delta\phi_1 - \frac{\epsilon}{m\tau^2}\phi_1$, $\boldsymbol{f}_1 := ((\alpha_1+\alpha_2 l_h)\phi_1 + \alpha_2\gamma_1)\boldsymbol{g}$ and $h_1 := \kappa l_h\Delta\phi_1 + \alpha\boldsymbol{g}\cdot\boldsymbol{u}_1$, and consider the following linear system:

$$\left[\begin{array}{ll}\partial_t(\phi_2 - m\tau\Delta\phi_2) + m\{\epsilon\Delta^2\phi_2 + (\beta_1 - l_c l_h)\Delta\phi_2 - l_c\Delta\gamma_2\} = \sigma_1 & \text{in } Q,\\ \partial_t\gamma_2 - \kappa\Delta\gamma_2 - \kappa l_h\Delta\phi_2 = \alpha\boldsymbol{g}\cdot\boldsymbol{u}_2 + h_1 & \text{in } Q,\\ \partial_t\boldsymbol{u}_2 - \nu\Delta\boldsymbol{u}_2 + \nabla\mathfrak{p}_2 = \boldsymbol{P}_2\{((\alpha_1+\alpha_2 l_h)\phi_2 + \alpha_2\gamma_2)\boldsymbol{g}\} + \boldsymbol{f}_1 & \text{in } Q,\\ \operatorname{div}\boldsymbol{u}_2 = 0 & \text{in } Q,\\ \phi_2 = \Delta\phi_2 = 0, \quad \gamma_2 = 0, \quad \boldsymbol{u}_2 = \boldsymbol{0} & \text{on } \Sigma,\\ \phi_2(0) = 0, \quad \gamma_2(0) = 0, \quad \boldsymbol{u}_2(0) = \boldsymbol{0} & \text{in } \Omega.\end{array}\right. \tag{3.43}$$

In virtue of the Sobolev embedding theorem $W^{1,\mathfrak{s}}(\Omega) \hookrightarrow L^2(\Omega)$ for any $\mathfrak{s} \ge 1$, we deduce that $\sigma_1 \in L^r(I;W_0^{1,q}(\Omega)) \hookrightarrow L^r(I;L^2(\Omega))$, $h_1 \in L^r(I;W_0^{1,q}(\Omega)) + L^r(I;W_0^{1,p}(\Omega)) \hookrightarrow L^r(I;L^2(\Omega))$ and $\boldsymbol{f}_1 \in L^r(I;\boldsymbol{L}^2(\Omega))$. Furthermore, the following estimates hold

$$\|\sigma_1\|_{L^r(I;L^2(\Omega))} \le c\|\phi_1\|_{L^r(I;X^{3,q}(\Omega))} \tag{3.44}$$

$$\|h_1\|_{L^r(I;L^2(\Omega))} \le c\{\|\phi_1\|_{L^r(I;X^{3,q}(\Omega))} + \|\boldsymbol{u}_1\|_{L^r(I;\boldsymbol{X}^{1,p}_\sigma(\Omega))}\} \tag{3.45}$$

$$\|\boldsymbol{f}_1\|_{L^r(I;\boldsymbol{L}^2(\Omega))} \le c\{\|\phi_1\|_{L^r(I;W_0^{1,q}(\Omega))} + \|\gamma_1\|_{L^r(I;W_0^{1,s}(\Omega))}\}. \tag{3.46}$$

According to Theorem 3.16 and Remark 3.17, we have a unique weak solution $(\phi_2, \gamma_2, \boldsymbol{u}_2) \in \mathcal{Z}^4_{2,r}(Q) \times \mathcal{Z}^2_{2,r}(Q) \times \boldsymbol{\mathcal{V}}^2_{2,r}(Q)$ to the system (3.43), and moreover, the associated pressure satisfies $\mathfrak{p}_2 \in L^r(I;\widehat{W}^{1,2}(\Omega))$. Based on the estimates (3.31) and (3.44)–(3.46), we deduce

$$\begin{aligned}&\|\phi_2\|_{\mathcal{Z}^4_{2,r}(Q)} + \|\gamma_2\|_{\mathcal{Z}^2_{2,r}(Q)} + \|\boldsymbol{u}_2\|_{\boldsymbol{\mathcal{V}}^2_{2,r}(Q)} + \|\mathfrak{p}_2\|_{L^r(I;\widehat{W}^{1,2}(\Omega))}\\ &\quad\le c\{\|\phi_1\|_{L^r(I;X^{3,q}(\Omega))} + \|\boldsymbol{u}_1\|_{L^r(I;\boldsymbol{X}^{1,p}_\sigma(\Omega))} + \|\gamma_1\|_{L^r(I;W_0^{1,s}(\Omega))}\}.\end{aligned} \tag{3.47}$$

The Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^{\mathfrak{s}}(\Omega)$ for any $1 < \mathfrak{s} < \infty$ yields the following:

$$\mathcal{Z}^4_{2,r}(Q) \hookrightarrow \mathcal{Z}^3_{q,r}(Q), \qquad \mathcal{Z}^2_{2,r}(Q) \hookrightarrow \mathcal{Z}^1_{s,r}(Q), \qquad \boldsymbol{\mathcal{V}}^2_{2,r}(Q) \hookrightarrow \boldsymbol{\mathcal{V}}^1_{p,r}(Q). \tag{3.48}$$

As a consequence, the sum $(\psi_L, \gamma_L, \boldsymbol{u}_L) = (\psi_1, \gamma_1, \boldsymbol{u}_1) + (\psi_2, \gamma_2, \boldsymbol{u}_2)$ lies in $\mathcal{Z}^3_{q,r}(Q) \times \mathcal{Z}^1_{s,r}(Q) \times \boldsymbol{\mathcal{V}}^1_{p,r}(Q)$ and constitutes a weak solution to the equivalent linear system (3.28). With regard to the associated pressure, we have $\mathfrak{p}_L := \mathfrak{p}_1 + \mathfrak{p}_2 \in W^{-1,r}(I;\widehat{L}^p(\Omega))$ since $L^r(I;\widehat{W}^{1,2}(\Omega)) \hookrightarrow W^{-1,r}(I;\widehat{L}^p(\Omega))$. From (3.39)–(3.42), (3.47), and the triangle inequality, we obtain

$$\|\phi_L\|_{\mathcal{Z}^3_{q,r}(Q)} + \|\gamma_L\|_{\mathcal{Z}^2_{s,r}(Q)} + \|\boldsymbol{u}_L\|_{\boldsymbol{\mathcal{V}}^2_{p,r}(Q)} + \|\mathfrak{p}_L\|_{W^{-1,r}(I;\widehat{L}^p(\Omega))} \le \mathsf{R} \tag{3.49}$$

where R represents the right-hand side of (3.37).

Using the embedding $\mathcal{Z}^3_{q,r}(Q) \hookrightarrow \mathcal{Z}^1_{s,r}(Q)$ in the estimation of $\theta_L = \gamma_L + l_h\phi_L$ and $L^r(I; W^{1,s}_0(\Omega)) \hookrightarrow L^r(I; W^{1,q}_0(\Omega))$ for that of μ_L in (3.27), we obtain

$$\|\theta_L\|_{\mathcal{Z}^1_{s,r}(Q)} \le c\{\|\gamma_L\|_{\mathcal{Z}^1_{s,r}(Q)} + \|\phi_L\|_{\mathcal{Z}^3_{q,r}(Q)}\} \tag{3.50}$$

$$\|\mu_L\|_{L^r(I;W^{1,q}_0(\Omega))} \le c\{\|\phi_L\|_{\mathcal{Z}^3_{q,r}(Q)} + \|\theta_L\|_{L^r(I;W^{1,s}_0(\Omega))} + \|\lambda_L\|_{L^r(I;W^{1,q}_0(\Omega))}\}. \tag{3.51}$$

The inequalities (3.49)–(3.51) imply the estimate (3.37).

STEP 2. *Uniqueness.* Let us now establish the uniqueness of the weak solution to (3.28). By linearity it is enough to prove that the solution of (3.28) with zero source terms and initial data is trivial. Let $(\phi_L, \gamma_L, \boldsymbol{u}_L, \mathfrak{p}_L)$ be such a weak solution having the regularity as stated by the theorem. First, consider the heat equation

$$\left[\begin{array}{l} \partial_t\widetilde{\gamma}_L - \kappa\Delta\widetilde{\gamma}_L = \widetilde{h}_L := \kappa l_h\Delta\phi_L + \alpha\boldsymbol{g}\cdot\boldsymbol{u}_L \ \text{ in } Q, \\ \widetilde{\gamma}_L = 0 \ \text{ on } \Sigma, \ \widetilde{\gamma}_L(0) = 0 \ \text{ in } \Omega. \end{array}\right. \tag{3.52}$$

We have $\widetilde{h}_L \in L^r(I; W^{1,q}_0(\Omega)) + L^r(I; W^{1,p}_0(\Omega)) \hookrightarrow L^r(I; L^2(\Omega))$. The classical maximal parabolic regularity theory for the heat equation with homogeneous Dirichlet boundary condition imply that (3.52) has a weak solution $\widetilde{\gamma}_L \in \mathcal{Z}^2_{2,r}(Q)$. From the second embedding in (3.48), Theorem 3.6 and the uniqueness of solution to the heat equation in the class $\mathcal{Z}^1_{s,r}(Q)$, we have $\widetilde{\gamma}_L = \gamma_L$.

By adapting a similar argument to the Stokes part of the linear system, we have $\boldsymbol{u}_L \in \boldsymbol{\mathcal{V}}^2_{2,r}(Q)$ due to $((\alpha_1 + \alpha_2 l_h)\phi + \alpha_2\gamma)\boldsymbol{g} \in L^r(I; \boldsymbol{L}^2(\Omega))$ and the third embedding in (3.48). For the viscous biharmonic heat equation, we obtain that $\phi_L \in \mathcal{Z}^4_{2,r}(Q)$ from $ml_c\Delta\gamma_L - m(\beta_1 - l_c l_h)\Delta\phi_L - \frac{\epsilon}{m\tau^2}\phi_L \in L^r(I; L^2(\Omega))$ and the first embedding in (3.48). Here, we used the fact that $\gamma_L \in \mathcal{Z}^2_{2,r}(Q)$.

We conclude that $(\phi_L, \gamma_L, \boldsymbol{u}_L)$ must vanish according to the Theorem 3.16 since $\phi_L(0) = 0$, $\gamma_L(0) = 0$ and $\boldsymbol{u}_L(0) = \boldsymbol{0}$. Thus, we also have $\nabla\mathfrak{p}_L = \boldsymbol{0}$ almost everywhere in Q, so that $\mathfrak{p}_L$ is the zero element in $W^{-1,r}(I; \widehat{L}^p(\Omega))$. This completes the proof of uniqueness. □

Remark 3.19. *It can be seen from the first step in the proof of the preceding theorem that if $l_c = 0$, then we can drop the condition $q \le s$.*

The definition of very weak solutions to (2.4) can be formulated as in the previous subsections. We leave the details to the reader for this matter, see also the discussion in the succeeding section.

Theorem 3.20. *Suppose that $p, q, s, r \in (1, \infty)$ where $q \le s$. Let (3.9), (3.18), (3.25) and*

$$\lambda_L \in L^r(I; L^q(\Omega)) \tag{3.53}$$

be satisfied. Then the linear system (2.4) has a unique very weak solution

$$(\phi_L, \theta_L, \boldsymbol{u}_L, \mu_L) \in \mathcal{Z}^2_{q,r}(Q) \times \mathcal{Z}^0_{s,r}(Q) \times \boldsymbol{\mathcal{V}}^0_{p,r}(Q) \times L^r(I; L^q(\Omega)).$$

In addition, if $\boldsymbol{f}_L \in L^r(I; \boldsymbol{X}^{-2,p}(\Omega))$ holds, then there exists a unique associated pressure $\mathfrak{p}_L \in W^{-1,r}(I; \widehat{W}^{-1,p}(\Omega))$. Furthermore, for some constant $c > 0$ independent on the solution, source terms, and initial data, it holds that

$$\|\phi_L\|_{\mathcal{Z}^2_{q,r}(Q)} + \|\theta_L\|_{\mathcal{Z}^0_{s,r}(Q)} + \|\boldsymbol{u}_L\|_{\boldsymbol{\mathcal{V}}^0_{p,r}(Q)} + \|\mu_L\|_{L^r(I;L^p(\Omega))} + \|\mathfrak{p}_L\|_{W^{-1,r}(I;\widehat{W}^{-1,p}(\Omega))}$$

$$\begin{aligned}\le c\{\|\phi_{0L}\|_{Z^2_{q,r}(\Omega)} + \|\theta_{0L}\|_{Z^0_{s,r}(\Omega)} + \|\boldsymbol{u}_{0L}\|_{\boldsymbol{V}^0_{p,r}(\Omega)} + \|\sigma_L\|_{L^r(I;X^{-2,q}(\Omega))} \\ + \|\lambda_L\|_{L^r(I;L^q(\Omega))} + \|h_L\|_{L^r(I;X^{-2,s}(\Omega))} + \|\boldsymbol{f}_L\|_{L^r(I;\boldsymbol{X}^{-2,p}(\Omega))}\}.\end{aligned} \tag{3.54}$$

Proof. Let us pursue the strategy presented in Theorem 3.18 and follow the notations there. Under the given hypotheses, the Stokes equation in (3.38) possesses a very weak solution such that

$$\|\boldsymbol{u}_1\|_{\boldsymbol{\mathcal{V}}^0_{p,r}(Q)} \le c\{\|\boldsymbol{f}_L\|_{L^r(I;\boldsymbol{X}^{-2,p}_\sigma(\Omega))} + \|\boldsymbol{u}_{0L}\|_{\boldsymbol{V}^0_{p,r}(\Omega)}\} \tag{3.55}$$

$$\|\mathfrak{p}_1\|_{W^{-1,r}(I;\widehat{W}^{-1,p}(\Omega))} \le c\{\|\boldsymbol{f}_L\|_{L^r(I;\boldsymbol{X}^{-2,p}(\Omega))} + \|\boldsymbol{u}_{0L}\|_{\boldsymbol{V}^0_{p,r}(\Omega)}\} \tag{3.56}$$

thanks to Theorem 3.2, provided that $\boldsymbol{f}_L \in L^r(I;\boldsymbol{X}^{-2,p}(\Omega))$ in the case of (3.56). From Lemma 3.8 we see that $\gamma_{0L} \in Z^0_{s,r}(\Omega)$. Thus, Theorem 3.7 implies that the heat equation in (3.38) possesses a very weak solution with

$$\|\gamma_1\|_{\mathcal{Z}^0_{q,r}(Q)} \le c\{\|h_L\|_{L^r(I;X^{-2,s}(\Omega))} + \|\gamma_{0L}\|_{Z^0_{s,r}(\Omega)}\}. \tag{3.57}$$

We have $X^{-2,s}(\Omega) \hookrightarrow X^{-2,q}(\Omega)$ since $q \le s$. Hence, $m\Delta\lambda_L + ml_c\Delta\gamma_1 \in L^r(I;X^{-2,q}(\Omega))$ and we get

$$\|m\Delta\lambda_L + ml_c\Delta\gamma_1\|_{L^r(I;X^{-2,q}(\Omega))} \le c\{\|\lambda_L\|_{L^r(I;L^q(\Omega))} + \|\gamma_1\|_{L^r(I;L^s(\Omega))}\}.$$

By Theorem 3.12 and the previous estimate, we conclude that the biharmonic heat equation in (3.38) has a very weak solution such that

$$\begin{aligned}\|\phi_1\|_{\mathcal{Z}^2_{q,r}(Q)} \le c\{&\|\sigma_L\|_{L^r(I;X^{-2,q}(\Omega))} + \|\lambda_L\|_{L^r(I;L^q(\Omega))} \\ &+ \|\gamma_1\|_{L^r(I;L^s(\Omega))} + \|\psi_{0L}\|_{Z^2_{q,r}(\Omega)}\}.\end{aligned} \tag{3.58}$$

We utilize the above information in the other linear system (3.43). Observe that we have $\sigma_1 \in L^r(I;L^q(\Omega)) \hookrightarrow L^r(I;W^{-1,2}(\Omega))$, $\boldsymbol{f}_1 \in L^r(I;X^{2,q}(\Omega)) + L^r(I;L^s(\Omega)) \hookrightarrow L^r(I;\boldsymbol{W}^{-1,2}(\Omega))$ and $h_1 \in L^r(I;L^q(\Omega)) + L^r(I;L^p(\Omega)) \hookrightarrow L^r(I;W^{-1,2}(\Omega))$. Invoking Theorem 3.18 with $p = q = s = 2$, we see that (3.43) has a weak solution satisfying the estimate

$$\begin{aligned}&\|\phi_2\|_{\mathcal{Z}^3_{2,r}(Q)} + \|\gamma_2\|_{\mathcal{Z}^1_{2,r}(Q)} + \|\boldsymbol{u}_2\|_{\boldsymbol{\mathcal{V}}^1_{2,r}(Q)} + \|\mathfrak{p}_2\|_{W^{-1,r}(I;\widehat{L}^2(\Omega))} \\ &\quad\le c\{\|\phi_1\|_{L^r(I;X^{2,q}(\Omega))} + \|\boldsymbol{u}_1\|_{L^r(I;\boldsymbol{L}^p_\sigma(\Omega))} + \|\gamma_1\|_{L^r(I;L^s(\Omega))}\}.\end{aligned} \tag{3.59}$$

Due to the continuous embeddings $\mathcal{Z}^3_{2,r}(Q) \hookrightarrow \mathcal{Z}^2_{q,r}(Q)$, $\mathcal{Z}^1_{2,r}(Q) \hookrightarrow \mathcal{Z}^0_{s,r}(Q)$, $\boldsymbol{\mathcal{V}}^1_{2,r}(Q) \hookrightarrow \boldsymbol{\mathcal{V}}^0_{p,r}(Q)$ and $W^{-1,r}(I;\widehat{L}^2(\Omega)) \hookrightarrow W^{-1,r}(I;\widehat{W}^{-1,p}(\Omega))$, it follows that the sum $(\psi_L, \gamma_L, \boldsymbol{u}_L) = (\psi_1, \gamma_1, \boldsymbol{u}_1) + (\psi_2, \gamma_2, \boldsymbol{u}_2)$ belongs to $\mathcal{Z}^2_{q,r}(Q) \times \mathcal{Z}^0_{s,r}(Q) \times \boldsymbol{\mathcal{V}}^0_{p,r}(Q)$ and it is a very weak solution to the linear system (2.4) having the associated pressure $\mathfrak{p}_L = \mathfrak{p}_1 + \mathfrak{p}_2 \in W^{-1,r}(I;\widehat{W}^{-1,p}(\Omega))$. Moreover, (3.55)–(3.59) leads to

$$\|\phi_L\|_{\mathcal{Z}^2_{q,r}(Q)} + \|\gamma_L\|_{\mathcal{Z}^0_{s,r}(Q)} + \|\boldsymbol{u}_L\|_{\boldsymbol{\mathcal{V}}^0_{p,r}(Q)} + \|\mathfrak{p}_L\|_{W^{-1,r}(I;\widehat{W}^{-1,p}(\Omega))} \le \mathsf{R} \tag{3.60}$$

where R denotes the right-hand side of (3.54). The embeddings $\mathcal{Z}^2_{q,r}(Q) \hookrightarrow \mathcal{Z}^0_{s,r}(Q)$ and $L^r(I;L^s(\Omega)) \hookrightarrow L^r(I;L^q(\Omega))$ yield

$$\|\theta_L\|_{\mathcal{Z}^0_{s,r}(Q)} \le c\{\|\gamma_L\|_{\mathcal{Z}^0_{s,r}(Q)} + \|\phi_L\|_{\mathcal{Z}^2_{q,r}(Q)}\} \tag{3.61}$$

$$\|\mu_L\|_{L^r(I;L^q(\Omega))} \le c\{\|\phi_L\|_{\mathcal{Z}^2_{q,r}(Q)} + \|\theta_L\|_{L^r(I;L^s(\Omega))} + \|\lambda_L\|_{L^r(I;L^q(\Omega))}\}. \tag{3.62}$$

From (3.60)–(3.62), we deduce (3.54).

The uniqueness of the very weak solution can be established by following the same method as in the second step in the proof of Theorem 3.18. □

4. Well-Posedness of the Nonlinear System

In this section, we prove the existence, uniqueness, and stability of weak and very weak solutions to the nonlinear system (1.1). This section is divided into several parts, namely, nonlinear estimates, definition of weak and very weak solutions to (1.1), well-posedness of an auxiliary system that includes the nonlinear part (2.5), and finally that of (1.1). Application to sources with values in the duals of some Hölder spaces will be presented at the end.

4.1. Nonlinear Estimates. The aim here is to establish the continuity of the bilinear operators associated with the convection and surface tension terms. Let $\mathfrak{s}_0, \mathfrak{s}_1, \mathfrak{s}_2 \in [1, \infty]$ be such that $\frac{1}{\mathfrak{s}_0} + \frac{1}{\mathfrak{s}_1} + \frac{1}{\mathfrak{s}_2} \leq 1$. We define the trilinear forms $\boldsymbol{b} : \boldsymbol{L}^{\mathfrak{s}_0}(\Omega) \times \boldsymbol{L}^{\mathfrak{s}_1}(\Omega) \times \boldsymbol{W}^{1,\mathfrak{s}_2}(\Omega) \to \mathbb{R}$ and $b : \boldsymbol{L}^{\mathfrak{s}_0}(\Omega) \times L^{\mathfrak{s}_1}(\Omega) \times W^{1,\mathfrak{s}_2}(\Omega) \to \mathbb{R}$ according to

$$\boldsymbol{b}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) := -\int_\Omega (\boldsymbol{u} \otimes \boldsymbol{v}) : \nabla \boldsymbol{w} \,\mathrm{d}x$$
$$b(\boldsymbol{u}, \mu, \phi) := -\int_\Omega \boldsymbol{u} \cdot (\mu \nabla \phi) \,\mathrm{d}x$$

for $\boldsymbol{u} \in \boldsymbol{L}^{\mathfrak{s}_0}(\Omega)$, $\boldsymbol{v} \in \boldsymbol{L}^{\mathfrak{s}_1}(\Omega)$, $\boldsymbol{w} \in \boldsymbol{W}^{1,\mathfrak{s}_2}(\Omega)$, $\mu \in L^{\mathfrak{s}_1}(\Omega)$, and $\phi \in W^{1,\mathfrak{s}_2}(\Omega)$. One can easily check that $\boldsymbol{b}$ and b are well-defined and bounded due to the Hölder's inequality. Notice that the last argument for b and $\boldsymbol{b}$ contain the gradient. For time-dependent functions, we have the following lemma, which follows from the Hölder's inequality as well.

Lemma 4.1. *Let $\mathfrak{s}_1, \mathfrak{r}_1 \in [1, \infty]$ and $\mathfrak{s}_0, \mathfrak{s}_2, \mathfrak{r}_0, \mathfrak{r}_2 \in (1, \infty)$ with $\frac{1}{\mathfrak{s}_0} + \frac{1}{\mathfrak{s}_1} + \frac{1}{\mathfrak{s}_2} \leq 1$ and $\frac{1}{\mathfrak{r}_0} + \frac{1}{\mathfrak{r}_1} + \frac{1}{\mathfrak{r}_2} \leq 1$. Then the following bilinear operators*

$$\boldsymbol{B} : L^{\mathfrak{r}_0}(I; \boldsymbol{L}^{\mathfrak{s}_0}(\Omega)) \times L^{\mathfrak{r}_1}(I; \boldsymbol{L}^{\mathfrak{s}_1}(\Omega)) \to L^{\mathfrak{r}_2'}(I; \boldsymbol{W}^{-1,\mathfrak{s}_2'}(\Omega))$$
$$C : L^{\mathfrak{r}_0}(I; \boldsymbol{L}^{\mathfrak{s}_0}(\Omega)) \times L^{\mathfrak{r}_1}(I; L^{\mathfrak{s}_1}(\Omega)) \to L^{\mathfrak{r}_2'}(I; W^{-1,\mathfrak{s}_2'}(\Omega))$$
$$\boldsymbol{S} : L^{\mathfrak{r}_1}(I; \boldsymbol{L}^{\mathfrak{s}_1}(\Omega)) \times L^{\mathfrak{r}_2}(I; W^{1,\mathfrak{s}_2}(\Omega)) \to L^{\mathfrak{r}_0'}(I; \boldsymbol{L}^{\mathfrak{s}_0'}(\Omega))$$

defined respectively by

$$\langle \boldsymbol{B}(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w} \rangle_{L^{\mathfrak{r}_2'}(I; \boldsymbol{W}^{-1,\mathfrak{s}_2'}(\Omega)), L^{\mathfrak{r}_2}(I; \boldsymbol{W}_0^{1,\mathfrak{s}_2}(\Omega))} = \int_0^T \boldsymbol{b}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \,\mathrm{d}t$$
$$\langle C(\boldsymbol{u}, \mu), \phi \rangle_{L^{\mathfrak{r}_2'}(I; W^{-1,\mathfrak{s}_2'}(\Omega)), L^{\mathfrak{r}_2}(I; W_0^{1,\mathfrak{s}_2}(\Omega))} = \int_0^T b(\boldsymbol{u}, \mu, \phi) \,\mathrm{d}t$$
$$\langle \boldsymbol{S}(\mu, \phi), \boldsymbol{u} \rangle_{L^{\mathfrak{r}_0'}(I; \boldsymbol{L}^{\mathfrak{s}_0'}(\Omega)), L^{\mathfrak{r}_0}(I; \boldsymbol{L}^{\mathfrak{s}_0}(\Omega))} = \int_0^T b(\boldsymbol{u}, \mu, \phi) \,\mathrm{d}t$$

are continuous.

Based on the above definitions, we set div $(\boldsymbol{u}\otimes\boldsymbol{v}) = \boldsymbol{B}(\boldsymbol{u},\boldsymbol{v})$, div $(\mu\boldsymbol{u}) = C(\boldsymbol{u},\mu)$ and $\mu\nabla\phi = -\boldsymbol{S}(\mu,\phi)$. In the forthcoming analysis, in particular to the estimation of the time derivatives, we need the function spaces

$$\begin{aligned}\mathcal{X}^2_{\infty,2}(Q) &:= L^\infty(I;W^{1,2}_0(\Omega))\cap L^2(I;X^{2,2}(\Omega))\\ \mathcal{X}^1_{\infty,2}(Q) &:= L^\infty(I;L^2(\Omega))\cap L^2(I;W^{1,2}_0(\Omega))\\ \boldsymbol{\mathcal{U}}^1_{\infty,2}(Q) &:= L^\infty(I;\boldsymbol{L}^2_\sigma(\Omega))\cap L^2(I;\boldsymbol{X}^{1,2}_\sigma(\Omega)).\end{aligned}$$

The following lemma is concerned with some restrictions of the operators C, $\boldsymbol{B}$, and $\boldsymbol{S}$. These will later play important roles in the Faedo–Galerkin method.

Lemma 4.2. *Suppose that $q\in(\frac{4}{3},\infty)$ and $s,p,r\in[4,\infty)$. Then the following bilinear operators are continuous*

$$\begin{aligned}\boldsymbol{B} &: [\boldsymbol{\mathcal{V}}^0_{p,r}(Q)+\boldsymbol{\mathcal{U}}^1_{\infty,2}(Q)]\times[\boldsymbol{\mathcal{V}}^0_{p,r}(Q)+\boldsymbol{\mathcal{U}}^1_{\infty,2}(Q)]\to L^2(I;\boldsymbol{W}^{-1,2}(\Omega))\\ C &: [\boldsymbol{\mathcal{V}}^0_{p,r}(Q)+\boldsymbol{\mathcal{U}}^1_{\infty,2}(Q)]\times[\mathcal{Z}^0_{s,r}(Q)+\mathcal{X}^1_{\infty,2}(Q)]\to L^2(I;W^{-1,2}(\Omega))\\ \boldsymbol{S} &: [L^r(I;L^q(\Omega))+L^2(I;W^{1,2}_0(\Omega))]\times[\mathcal{Z}^2_{q,r}(Q)+\mathcal{X}^2_{\infty,2}(Q)]\to L^2(I;\boldsymbol{W}^{-1,2}(\Omega)).\end{aligned}$$

Proof. From the Gagliardo–Nirenberg and Hölder inequalities

$$\|\boldsymbol{u}\|_{L^4(I;\boldsymbol{L}^4(\Omega))}\le c\{\|\boldsymbol{u}\|_{L^\infty(I;\boldsymbol{L}^2_\sigma(\Omega))}\|\boldsymbol{u}\|_{L^2(I;\boldsymbol{X}^{1,2}_\sigma(\Omega))}\}$$

for each $\boldsymbol{u}\in\boldsymbol{\mathcal{U}}^1_{\infty,2}(Q)$. Thus, $\boldsymbol{\mathcal{U}}^1_{\infty,2}(Q)\hookrightarrow L^4(I;\boldsymbol{L}^4(\Omega))$. Also, $\boldsymbol{\mathcal{V}}^0_{p,r}(Q)\hookrightarrow L^4(I;\boldsymbol{L}^4(\Omega))$ since $p,r\ge 4$. The continuity of $\boldsymbol{B}$ with respect to the indicated function spaces follows immediately from $\boldsymbol{\mathcal{V}}^0_{p,r}(Q)+\boldsymbol{\mathcal{U}}^1_{\infty,2}(Q)\hookrightarrow L^4(I;\boldsymbol{L}^4(\Omega))$ and Lemma 4.1 with $\mathfrak{s}_0=\mathfrak{s}_1=\mathfrak{r}_0=\mathfrak{r}_1=4$ and $\mathfrak{s}_2=\mathfrak{r}_2=2$. In a similar way, we have the continuity of C with respect to the above function spaces since $\mathcal{Z}^0_{s,r}(Q)+\mathcal{X}^1_{\infty,2}(Q)\hookrightarrow L^4(I;L^4(\Omega))$.

Let us show the continuity of $\boldsymbol{S}$. First, we assume that $\frac{4}{3}<q<2$ and consider four scenarios. Here, we use Lemma 4.1 in each case.

First, we have $\mathcal{Z}^2_{q,r}(Q)\hookrightarrow L^r(I;W^{2,q}(\Omega))\hookrightarrow L^r(I;W^{1,2q/(2-q)}(\Omega))$ from the Sobolev embedding theorem. With $\mathfrak{s}_0=2q/(3q-4)$, $\mathfrak{s}_1=q$, $\mathfrak{s}_2=2q/(2-q)$, $\mathfrak{r}_0=2$, and $\mathfrak{r}_1=\mathfrak{r}_2=r$ in Lemma 4.1, we see that

$$\boldsymbol{S}: L^r(I;L^q(\Omega))\times\mathcal{Z}^2_{q,r}(Q)\to L^2(I;\boldsymbol{L}^{2q/(4-q)}(\Omega)) \tag{4.1}$$

is continuous since $\mathfrak{s}_0'=2q/(4-q)$.

Using the Gagliardo–Nirenberg and Hölder inequalities as before, we obtain the continuous embedding $\mathcal{X}^2_{\infty,2}(Q)\hookrightarrow L^4(I;W^{1,4}(\Omega))$. With the parameters $\mathfrak{s}_0=4q/(3q-4)$, $\mathfrak{s}_1=q$, $\mathfrak{s}_2=4$, $\mathfrak{r}_0=2$, $\mathfrak{r}_1=r$, and $\mathfrak{r}_2=4$ in Lemma 4.1, we have the continuity of

$$\boldsymbol{S}: L^r(I;L^q(\Omega))\times\mathcal{X}^2_{\infty,2}(Q)\to L^2(I;\boldsymbol{L}^{4q/(4+q)}(\Omega)) \tag{4.2}$$

because $\mathfrak{s}_0'=4q/(4+q)$.

From [9, Theorem III.4.10.2] and Lemma 3.9 with $0<\delta<\frac{1}{2}$, we have

$$\mathcal{Z}^2_{q,r}(Q)\hookrightarrow C(\bar{I};Z^2_{q,r}(\Omega))\hookrightarrow L^\infty(I;W^{2-2/r-\delta,q}(\Omega))\hookrightarrow L^\infty(I;W^{1,q}(\Omega)).$$

Also, $L^2(I;W^{1,2}_0(\Omega))\hookrightarrow L^2(I;L^{2q/(q-1)}(\Omega))$ by the Sobolev embedding theorem. Hence, using the parameters $\mathfrak{s}_0=\mathfrak{s}_1=2q/(q-1)$, $\mathfrak{s}_2=q$, $\mathfrak{r}_0=\mathfrak{r}_1=2$, and

$\mathfrak{r}_2 = \infty$ in Lemma 4.1, we deduce the continuity of

$$\boldsymbol{S} : L^2(I; W_0^{1,2}(\Omega)) \times \mathcal{Z}^2_{q,r}(Q) \to L^2(I; \boldsymbol{L}^{2q/(q+1)}(\Omega)) \tag{4.3}$$

since $\mathfrak{s}_0' = 2q/(q+1)$.

Finally, notice that $L^2(I; W_0^{1,2}(\Omega)) \hookrightarrow L^2(I; L^4(\Omega))$ and $\mathcal{X}^2_{\infty,2}(Q) \hookrightarrow L^\infty(I; W^{1,2}(\Omega))$. Using $\mathfrak{s}_0 = 4$, $\mathfrak{s}_1 = 4$, $\mathfrak{s}_2 = 2$, $\mathfrak{r}_0 = \mathfrak{r}_1 = 2$, and $\mathfrak{r}_2 = \infty$ in Lemma 4.1, we obtain the continuity of

$$\boldsymbol{S} : L^2(I; W_0^{1,2}(\Omega)) \times \mathcal{X}^2_{\infty,2}(Q) \to L^2(I; \boldsymbol{L}^{4/3}(\Omega)). \tag{4.4}$$

Invoking the continuity of the $\boldsymbol{S}$ provided by (4.1)–(4.4), along with the definition of the norm for the sum of Banach spaces and $L^2(I; \boldsymbol{L}^{\mathfrak{s}}(\Omega)) \hookrightarrow L^2(I; \boldsymbol{W}^{-1,2}(\Omega))$ for any $\mathfrak{s} \in (1,\infty)$, we obtain the continuity of the bilinear operator $\boldsymbol{S}$ under the function spaces stated by the lemma.

Now for the case $q \geq 2$, we just need to use the continuous embeddings $L^r(I; L^q(\Omega)) \hookrightarrow L^r(I; L^{q^*}(\Omega))$ and $\mathcal{Z}^2_{q,r}(Q) \hookrightarrow \mathcal{Z}^2_{q^*,r}(Q)$ for a $q^* \in (\frac{4}{3}, 2)$, and apply the above result. □

Corollary 4.3. *Let $q \in (1,\infty)$, $s, p \in [\frac{4}{3}, \infty)$, and $r \in [4,\infty)$. Then the following bilinear operators are continuous*

$$\begin{aligned}
\boldsymbol{B} &: [\boldsymbol{\mathcal{V}}^1_{p,r}(Q) + \boldsymbol{\mathcal{U}}^1_{\infty,2}(Q)] \times [\boldsymbol{\mathcal{V}}^1_{p,r}(Q) + \boldsymbol{\mathcal{U}}^1_{\infty,2}(Q)] \to L^2(I; \boldsymbol{W}^{-1,2}(\Omega))\\
C &: [\boldsymbol{\mathcal{V}}^1_{p,r}(Q) + \boldsymbol{\mathcal{U}}^1_{\infty,2}(Q)] \times [\mathcal{Z}^1_{s,r}(Q) + \mathcal{X}^1_{\infty,2}(Q)] \to L^2(I; W^{-1,2}(\Omega))\\
\boldsymbol{S} &: [L^r(I; W_0^{1,q}(\Omega)) + L^2(I; W_0^{1,2}(\Omega))] \times [\mathcal{Z}^3_{q,r}(Q) + \mathcal{X}^2_{\infty,2}(Q)] \to L^2(I; \boldsymbol{W}^{-1,2}(\Omega)).
\end{aligned}$$

Proof. The assumptions on p, s, and q give us, due to the Sobolev embedding theorem, the following continuous embeddings:

$$\boldsymbol{\mathcal{V}}^1_{p,r}(Q) \hookrightarrow \boldsymbol{\mathcal{V}}^0_{4,r}(Q), \qquad \mathcal{Z}^1_{s,r}(Q) \hookrightarrow \mathcal{Z}^0_{4,r}(Q), \tag{4.5}$$

$$\mathcal{Z}^3_{q,r}(Q) \hookrightarrow \mathcal{Z}^2_{2,r}(Q), \qquad W_0^{1,q}(\Omega) \hookrightarrow L^2(\Omega). \tag{4.6}$$

We obtain the corollary by simply applying Lemma 4.2 with $q = 2$ and $s = p = 4$. □

The next lemma will imply the continuity of the cubic function F that appears in the equation for the chemical potential.

Lemma 4.4. *Let $q \in (\frac{4}{3}, \infty)$, $r \in [4,\infty)$, and k be a positive integer. Then $\phi_1 \cdots \phi_k \in L^2(I; W_0^{1,2}(\Omega))$ for every $\phi_1, \dots, \phi_k \in \mathcal{Z}^2_{q,r}(Q) + \mathcal{X}^2_{\infty,2}(Q)$. Moreover, there exists a constant $c_k > 0$ independent on $\phi_1, \dots, \phi_k$ such that*

$$\|\phi_1 \cdots \phi_k\|_{L^2(I; W_0^{1,2}(\Omega))} \leq c_k \|\phi_1\|_{\mathcal{Z}^2_{q,r}(Q) + \mathcal{X}^2_{\infty,2}(Q)} \cdots \|\phi_k\|_{\mathcal{Z}^2_{q,r}(Q) + \mathcal{X}^2_{\infty,2}(Q)}.$$

Proof. Since $\mathcal{Z}^2_{q,r}(Q) + \mathcal{X}^2_{\infty,2}(Q) \hookrightarrow L^2(I; W_0^{1,2}(\Omega))$, we may assume that $k > 1$. Taking $0 < \delta < \frac{3q-4}{2q}$ and using $r \geq 4$, we have $(2 - \frac{2}{r} - \delta)q \geq (\frac{3}{2} - \delta)q > 2$, and thus

$$\mathcal{Z}^2_{q,r}(Q) \hookrightarrow L^\infty(I; W^{2-2/r-\delta,q}(\Omega)) \hookrightarrow L^\infty(I; L^\infty(\Omega))$$

by Lemma 3.9 and the Sobolev embedding. Likewise, $\mathcal{X}^2_{\infty,2}(Q) \hookrightarrow L^\infty(I; L^{\mathfrak{s}}(\Omega))$ for any $\mathfrak{s} \in [1,\infty)$. Hence, $\mathcal{Z}^2_{q,r}(Q) + \mathcal{X}^2_{\infty,2}(Q) \hookrightarrow L^{2k}(I; L^{2k}(\Omega))$ and by the Hölder's inequality

$$\begin{aligned}\|\phi_1\cdots\phi_k\|_{L^2(I;L^2(\Omega))} &\le \|\phi_1\|_{L^{2k}(I;L^{2k}(\Omega))}\cdots\|\phi_k\|_{L^{2k}(I;L^{2k}(\Omega))}\\ &\le c_k\|\phi_1\|_{\mathcal{Z}^2_{q,r}(Q)+\mathcal{X}^2_{\infty,2}(Q)}\cdots\|\phi_k\|_{\mathcal{Z}^2_{q,r}(Q)+\mathcal{X}^2_{\infty,2}(Q)}.\end{aligned}$$

We compute the gradient by the product rule, so that

$$\nabla(\phi_1\cdots\phi_k)=\sum_{j=1}^{k}\phi_1\cdots\phi_{j-1}\phi_{j+1}\cdots\phi_k\nabla\phi_j.$$

Note that $\mathcal{Z}^2_{q,r}(Q)+\mathcal{X}^2_{\infty,2}(Q)\hookrightarrow L^2(I;W^{1,4}(\Omega))\cap L^\infty(I;L^{4(k-1)}(\Omega))$. Thus, by applying Hölder's inequality to the latter equation, one has

$$\begin{aligned}\|\nabla(\phi_1\cdots\phi_k)\|_{L^2(I;\boldsymbol{L}^2(\Omega))} &\le \sum_{j=1}^{k}\|\nabla\phi_j\|_{L^2(I;\boldsymbol{L}^4(\Omega))}\prod_{l\neq j}\|\phi_l\|_{L^\infty(I;L^{4(k-1)}(\Omega))}\\ &\le c_k\|\phi_1\|_{\mathcal{Z}^2_{q,r}(Q)+\mathcal{X}^2_{\infty,2}(Q)}\cdots\|\phi_k\|_{\mathcal{Z}^2_{q,r}(Q)+\mathcal{X}^2_{\infty,2}(Q)}.\end{aligned}$$

Combining this with the earlier estimate establishes the lemma. □

The first statement of the succeeding corollary follows immediately from Lemma 4.4, while the second is a result of the first and the embedding $\mathcal{Z}^3_{q,r}(Q)\hookrightarrow\mathcal{Z}^2_{2,r}(Q)$ for $q\in(1,\infty)$.

Corollary 4.5. *Let $q\in(\frac{4}{3},\infty)$ and $r\in[4,\infty)$. Then the map $F:\mathcal{Z}^2_{q,r}(Q)+\mathcal{X}^2_{\infty,2}(Q)\to L^2(I;W^{1,2}_0(\Omega))$ defined by $F(\phi)=\beta_0\phi^3-\beta_1\phi$ is continuous and there is a constant $c>0$ such that for every $\phi\in\mathcal{Z}^2_{q,r}(Q)+\mathcal{X}^2_{\infty,2}(Q)$, we have*

$$\|F(\phi)\|_{L^2(I;W^{1,2}_0(\Omega))}\le c\{\|\phi\|^3_{\mathcal{Z}^2_{q,r}(Q)+\mathcal{X}^2_{\infty,2}(Q)}+\|\phi\|_{\mathcal{Z}^2_{q,r}(Q)+\mathcal{X}^2_{\infty,2}(Q)}\}.$$

If $q\in(1,\infty)$ and $r\in[4,\infty)$, then $F:\mathcal{Z}^3_{q,r}(Q)+\mathcal{X}^2_{\infty,2}(Q)\to L^2(I;W^{1,2}_0(\Omega))$ is continuous and there exists a constant $c>0$ such that for every $\phi\in\mathcal{Z}^3_{q,r}(Q)+\mathcal{X}^2_{\infty,2}(Q)$, it holds that

$$\|F(\phi)\|_{L^2(I;W^{1,2}_0(\Omega))}\le c\{\|\phi\|^3_{\mathcal{Z}^3_{q,r}(Q)+\mathcal{X}^2_{\infty,2}(Q)}+\|\phi\|_{\mathcal{Z}^3_{q,r}(Q)+\mathcal{X}^2_{\infty,2}(Q)}\}.$$

4.2. Definitions of Weak and Very Weak Solutions to the Nonlinear System. Let us begin with the definition of weak solutions to the nonlinear system (1.1). We consider the following assumption:

$$1<q<2,\quad \frac{4}{3}\le s,p<2,\quad 4\le r<\infty,\quad q\le s. \tag{4.7}$$

For the regularity of the source functions, we shall take the following:

$$\left[\begin{array}{l}\sigma\in L^r(I;W^{-1,q}(\Omega))+L^2(I;W^{-1,2}(\Omega)),\\ h\in L^r(I;W^{-1,s}(\Omega))+L^2(I;W^{-1,2}(\Omega)),\\ \boldsymbol{f}\in L^r(I;\boldsymbol{W}^{-1,p}(\Omega))+L^2(I;\boldsymbol{W}^{-1,2}(\Omega)),\\ \lambda\in L^r(I;W^{1,q}_0(\Omega))+L^2(I;W^{1,2}_0(\Omega)).\end{array}\right. \tag{4.8}$$

As for the initial data, we consider

$$\phi_0\in Z^3_{q,r}(\Omega)+X^{2,2}(\Omega),\quad \theta_0\in Z^1_{s,r}(\Omega)+L^2(\Omega),\quad \boldsymbol{u}_0\in\boldsymbol{V}^1_{p,r}(\Omega)+\boldsymbol{L}^2_\sigma(\Omega). \tag{4.9}$$

These conditions include the situation where a Hilbert space framework can be used, by simply taking the first components in the above sums to be zero.

Definition 4.6. Suppose that (4.7) is satisfied. A tuple $(\phi, \theta, \boldsymbol{u}, \mu)$ having components $\phi \in \mathcal{Z}^3_{q,r}(Q) + \mathcal{Z}^3_{2,2}(Q)$, $\theta \in \mathcal{Z}^1_{s,r}(Q) + \mathcal{Z}^1_{2,2}(Q)$, $\boldsymbol{u} \in \boldsymbol{\mathcal{V}}^1_{p,r}(Q) + \boldsymbol{\mathcal{V}}^1_{2,2}(Q)$ and $\mu \in L^r(I; W^{1,q}_0(\Omega)) + L^2(I; W^{1,2}_0(\Omega))$ is called a *weak solution* to (1.1) if the initial condition $(\phi(0), \theta(0), \boldsymbol{u}(0)) = (\phi_0, \theta_0, \boldsymbol{u}_0)$ holds in $[Z^3_{q,r}(\Omega) + X^{2,2}(\Omega)] \times [Z^1_{s,r}(\Omega) + L^2(\Omega)] \times [\boldsymbol{V}^1_{p,r}(\Omega) + \boldsymbol{L}^2_\sigma(\Omega)]$, the following variational equations

$$\text{(a)} \quad \int_0^T \{(\partial_t\phi, \rho)_{L^2(\Omega)} + b(\boldsymbol{u}, \phi, \rho) + ma_q(\mu, \rho)\}\,\mathrm{d}t = \int_0^T \langle \sigma, \rho\rangle_{W^{-1,q}(\Omega), W^{1,q'}_0(\Omega)}\,\mathrm{d}t$$

$$\begin{aligned}\text{(b)} \quad &\int_0^T \{\langle \partial_t\theta, \varrho\rangle_{W^{-1,s}(\Omega), W^{1,s'}_0(\Omega)} - l_{\mathrm{h}}(\partial_t\phi, \varrho)_{L^2(\Omega)}\}\,\mathrm{d}t \\ &+ \int_0^T \{b(\boldsymbol{u}, \theta - l_{\mathrm{h}}\phi, \varrho) + \kappa a_s(\theta, \varrho) - (\alpha\boldsymbol{g}\cdot\boldsymbol{u}, \varrho)_{L^2(\Omega)}\}\,\mathrm{d}t \\ &= \int_0^T \langle h, \varrho\rangle_{W^{-1,s}(\Omega), W^{1,s'}_0(\Omega)}\,\mathrm{d}t\end{aligned}$$

$$\begin{aligned}\text{(c)} \quad &\int_0^T \{\langle \partial_t\boldsymbol{u}, \boldsymbol{\rho}\rangle_{\boldsymbol{X}^{-1,p}_\sigma(\Omega), \boldsymbol{X}^{1,p'}_\sigma(\Omega)} + \boldsymbol{b}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\rho}) + \nu\boldsymbol{a}_p(\boldsymbol{u}, \boldsymbol{\rho})\}\,\mathrm{d}t \\ &= \int_0^T \{(\ell(\phi,\theta)\boldsymbol{g}, \boldsymbol{\rho})_{\boldsymbol{L}^2(\Omega)} - \mathcal{K}b(\boldsymbol{\rho}, \mu - l_{\mathrm{c}}\theta, \phi) + \langle \boldsymbol{f}, \boldsymbol{\rho}\rangle_{\boldsymbol{X}^{-1,p}_\sigma(\Omega), \boldsymbol{X}^{1,p'}_\sigma(\Omega)}\}\,\mathrm{d}t\end{aligned}$$

hold for every test functions $\rho \in L^{r'}(I; W^{1,q'}_0(\Omega)) \cap L^2(I; W^{1,2}_0(\Omega))$, $\varrho \in L^{r'}(I; W^{1,s'}_0(\Omega)) \cap L^2(I; W^{1,2}_0(\Omega))$, $\boldsymbol{\rho} \in L^{r'}(I; \boldsymbol{X}^{1,p'}_\sigma(\Omega)) \cap L^2(I; \boldsymbol{X}^{1,2}_\sigma(\Omega))$, and

$$\mu = \tau\partial_t\phi - \epsilon\Delta\phi + F(\phi) + l_{\mathrm{c}}\theta + \lambda \quad \text{a.e. } Q. \tag{4.10}$$

A function $\mathfrak{p} \in \mathcal{P}^1_{p,r}(Q) := W^{-1,r}(I; \widehat{L}^p(\Omega)) + W^{-1,2}(I; \widehat{L}^2(\Omega))$ is called an *associated pressure* if the fourth equation in (1.1) is satisfied in $W^{-1,r}(I; W^{-1,p}(\Omega)) + W^{-1,2}(I; W^{-1,2}(\Omega))$, that is, we have

$$\begin{aligned}&\langle \partial_t\boldsymbol{u}, \boldsymbol{\varrho}\rangle_{\mathcal{Y}^1_{p,r}(Q)', \mathcal{Y}^1_{p,r}(Q)} + \int_0^T \{\boldsymbol{b}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\varrho}) + \nu\boldsymbol{a}_p(\boldsymbol{u}, \boldsymbol{\varrho})\}\,\mathrm{d}t - \langle \mathfrak{p}, \operatorname{div}\boldsymbol{\varrho}\rangle_{\mathcal{P}^1_{p,r}(Q), \mathcal{P}^1_{p,r}(Q)'} \\ &= \int_0^T \{(\ell(\phi,\theta)\boldsymbol{g}, \boldsymbol{\varrho})_{\boldsymbol{L}^2(\Omega)} - \mathcal{K}b(\boldsymbol{\varrho}, \phi, \mu - l_{\mathrm{c}}\theta) + \langle \boldsymbol{f}, \boldsymbol{\varrho}\rangle_{\boldsymbol{W}^{-1,p}(\Omega), \boldsymbol{W}^{1,p'}_0(\Omega)}\}\,\mathrm{d}t\end{aligned}$$

for every $\boldsymbol{\varrho} \in \mathcal{Y}^1_{p,r}(Q) := W^{1,r'}_0(I; W^{1,p'}_0(\Omega)) \cap W^{1,2}_0(I; W^{1,2}_0(\Omega))$. ◇

Let us give some comments in the above definition. The duality pairings in the above variational equations are meaningful. We consider the right-hand side of (a) as an illustration. From the duality properties between the sum and intersection of reflexive Banach spaces, we obtain

$$[L^{r'}(I; W^{1,q'}_0(\Omega)) \cap L^2(I; W^{1,2}_0(\Omega))]' = L^r(I; W^{-1,q}(\Omega)) + L^2(I; W^{-1,2}(\Omega)).$$

Given $\rho \in L^{r'}(I; W^{1,q'}_0(\Omega)) \cap L^2(I; W^{1,2}_0(\Omega))$, we have $\rho(t) \in W^{1,q'}_0(\Omega)$ for almost every $t \in I$. Also, $\sigma(t) \in W^{-1,q}(\Omega) + W^{-1,2}(\Omega) = W^{-1,q}(\Omega)$ for almost every $t \in I$ since $q < 2$. Thus,

$$\langle \sigma, \rho\rangle_{L^r(I;W^{-1,q}(\Omega)) + L^2(I;W^{-1,2}(\Omega)), L^{r'}(I;W^{1,q'}_0(\Omega)) \cap L^2(I;W^{1,2}_0(\Omega))}$$

$$= \int_0^T \langle \sigma, \rho \rangle_{W^{-1,q}(\Omega), W_0^{1,q'}(\Omega)} \, dt.$$

The other bilinear terms in (a)–(c) can be dealt with a similar reasoning.

Next, we justify that the trilinear terms in (a)–(c) are also well-defined. First, note that

$$b(\boldsymbol{u}, \phi, \rho) = \langle C(\boldsymbol{u}, \phi), \rho \rangle_{W^{-1,2}(\Omega), W_0^{1,2}(\Omega)}$$

is integrable over I thanks to Corollary 4.3 and the embeddings $\boldsymbol{\mathcal{V}}^1_{2,2}(Q) \hookrightarrow \boldsymbol{\mathcal{U}}^1_{\infty,2}(Q)$, $\mathcal{Z}^3_{q,r}(Q) \hookrightarrow \mathcal{Z}^1_{s,r}(Q)$, and $\mathcal{Z}^3_{2,2}(Q) \hookrightarrow \mathcal{X}^2_{\infty,2}(Q)$. Second, by using the same argument along with $\mathcal{Z}^2_{2,2}(Q) \hookrightarrow \mathcal{X}^1_{\infty,2}(Q)$, the following convection terms

$$b(\boldsymbol{u}, \theta - l_{\mathrm{h}}\phi, \varrho) = \langle C(\boldsymbol{u}, \theta - l_{\mathrm{h}}\phi), \varrho \rangle_{W^{-1,2}(\Omega), W_0^{1,2}(\Omega)}$$
$$\boldsymbol{b}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\rho}) = \langle \boldsymbol{B}(\boldsymbol{u}, \boldsymbol{u}), \boldsymbol{\rho} \rangle_{\boldsymbol{W}^{-1,2}(\Omega), \boldsymbol{W}_0^{1,2}(\Omega)}$$

are integrable over I. Finally, the trilinear term

$$b(\boldsymbol{\rho}, \mu - l_{\mathrm{c}}\theta, \phi) = \langle \boldsymbol{S}(\phi, \mu - l_{\mathrm{c}}\theta), \boldsymbol{\rho} \rangle_{\boldsymbol{W}^{-1,2}(\Omega), \boldsymbol{W}_0^{1,2}(\Omega)}$$

associated with the surface tension is also integrable over I thanks to the continuity of the embedding $\mathcal{Z}^1_{s,r}(Q) + \mathcal{Z}^1_{2,2}(Q) \hookrightarrow L^r(I; W_0^{1,q}(\Omega)) + L^2(I; W_0^{1,2}(\Omega))$ since $q \le s$. Furthermore, the equation for the chemical potential given by (4.10) can be taken as equality in the function space $L^r(I; W_0^{1,q}(\Omega)) + L^2(I; W_0^{1,2}(\Omega))$ by Corollary 4.5.

We now turn to the definition of very weak solutions to (1.1). In this case, we take source functions such that

$$\left[\begin{array}{l} \sigma \in L^r(I; X^{-2,q}(\Omega)) + L^2(I; W^{-1,2}(\Omega)), \\ h \in L^r(I; X^{-2,s}(\Omega)) + L^2(I; W^{-1,2}(\Omega)), \\ \boldsymbol{f} \in L^r(I; \boldsymbol{X}^{-2,p}(\Omega)) + L^2(I; \boldsymbol{W}^{-1,2}(\Omega)), \\ \lambda \in L^r(I; L^q(\Omega)) + L^2(I; W_0^{1,2}(\Omega)), \end{array}\right. \tag{4.11}$$

and initial data for which

$$\phi_0 \in Z^2_{q,r}(\Omega) + X^{2,2}(\Omega), \quad \theta_0 \in Z^0_{s,r}(\Omega) + L^2(\Omega), \quad \boldsymbol{u}_0 \in \boldsymbol{V}^0_{p,r}(\Omega) + \boldsymbol{L}^2_\sigma(\Omega). \tag{4.12}$$

Compared to that of the weak solutions, observe that the main differences are the first components in the sums. These are in fact the function spaces corresponding to the linearized system. The definition of very weak solutions to the nonlinear system (1.1) can be adapted as in the previous section. Nevertheless, we present the precise formulation here for the sake of clarity and completeness, in particular, to those nonlinear terms corresponding to convection and surface tension. In this direction, the following assumption will be considered

$$\frac{4}{3} < q < \infty, \quad 4 \le s, p, r < \infty, \quad q \le s. \tag{4.13}$$

Definition 4.7. Let (4.13) be satisfied. We say that $(\phi, \theta, \boldsymbol{u}, \mu)$ having the components $\phi \in \mathcal{Z}^2_{q,r}(Q) + \mathcal{Z}^3_{2,2}(Q)$, $\theta \in \mathcal{Z}^0_{s,r}(Q) + \mathcal{Z}^1_{2,2}(Q)$, $\boldsymbol{u} \in \boldsymbol{\mathcal{V}}^0_{p,r}(Q) + \boldsymbol{\mathcal{V}}^1_{2,2}(Q)$ and $\mu \in L^r(I; L^q(\Omega)) + L^2(I; W_0^{1,2}(\Omega))$ a *very weak solution* to (1.1) if $(\phi(0), \theta(0), \boldsymbol{u}(0)) = (\phi_0, \theta_0, \boldsymbol{u}_0)$ holds in $[Z^2_{q,r}(\Omega) + X^{2,2}(\Omega)] \times [Z^0_{s,r}(\Omega) + L^2(\Omega)] \times [\boldsymbol{V}^0_{p,r}(\Omega) + \boldsymbol{L}^2_\sigma(\Omega)]$, the following variational equations

$$\text{(a)} \quad \int_0^T \{\langle \partial_t \phi, \rho\rangle_{L^q(\Omega),L^{q'}(\Omega)} + b(\boldsymbol{u},\phi,\rho) - m\langle \mu, \Delta\rho\rangle_{L^q(\Omega),L^{q'}(\Omega)}\}\,\mathrm{d}t$$
$$= \int_0^T \langle \sigma, \rho\rangle_{X^{-2,q}(\Omega),X^{2,q'}(\Omega)}\,\mathrm{d}t$$

$$\text{(b)} \quad \int_0^T \{\langle \partial_t \theta, \varrho\rangle_{X^{-2,s}(\Omega),X^{2,s'}(\Omega)} - l_{\mathrm{h}}\langle \partial_t\phi, \varrho\rangle_{L^q(\Omega),L^{q'}(\Omega)}\}\,\mathrm{d}t$$
$$+ \int_0^T \{b(\boldsymbol{u},\theta - l_{\mathrm{h}}\phi,\varrho) - \kappa\langle \theta, \Delta\varrho\rangle_{L^s(\Omega),L^{s'}(\Omega)} - \langle \alpha\boldsymbol{g}\cdot\boldsymbol{u}, \varrho\rangle_{L^p(\Omega),L^{p'}(\Omega)}\}\,\mathrm{d}t$$
$$= \int_0^T \langle h, \varrho\rangle_{X^{-2,s}(\Omega),X^{2,s'}(\Omega)}\}\,\mathrm{d}t$$

$$\text{(c)} \quad \int_0^T \{\langle \partial_t \boldsymbol{u}, \boldsymbol{\rho}\rangle_{\boldsymbol{X}_\sigma^{-2,p}(\Omega),\boldsymbol{X}_\sigma^{2,p'}(\Omega)} + \boldsymbol{b}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\rho}) - \nu\langle \boldsymbol{u}, \boldsymbol{P}_{p'}\Delta\boldsymbol{\rho}\rangle_{\boldsymbol{L}^p_\sigma(\Omega),\boldsymbol{L}^{p'}_\sigma(\Omega)}\}\,\mathrm{d}t$$
$$= \int_0^T \{\langle \ell(\phi,\theta)\boldsymbol{g}, \boldsymbol{\rho}\rangle_{\boldsymbol{L}^q(\Omega),\boldsymbol{L}^{q'}(\Omega)} - \mathcal{K}b(\boldsymbol{\rho},\mu - l_{\mathrm{c}}\theta,\phi)\}\,\mathrm{d}t$$
$$+ \int_0^T \langle \boldsymbol{f}, \boldsymbol{\rho}\rangle_{\boldsymbol{X}_\sigma^{-2,p}(\Omega),\boldsymbol{X}_\sigma^{2,p'}(\Omega)}\,\mathrm{d}t$$

are satisfied by any test functions $\rho \in L^{r'}(I;X^{2,q'}(\Omega)) \cap L^2(I;W_0^{1,2}(\Omega))$, $\varrho \in L^{r'}(I;X^{2,s'}(\Omega)) \cap L^2(I;W_0^{1,2}(\Omega))$, $\boldsymbol{\rho} \in L^{r'}(I;\boldsymbol{X}_\sigma^{2,p'}(\Omega)) \cap L^2(I;\boldsymbol{X}_\sigma^{1,2}(\Omega))$, and (4.10) holds. Also, $\mathfrak{p} \in \mathcal{P}^0_{p,r}(Q) := W^{-1,r}(I;\widehat{W}^{-1,p}(\Omega)) + W^{-1,2}(I;\widehat{L}^2(\Omega))$ is called an *associated pressure* if the fourth equation in (1.1) is satisfied in $W^{-1,r}(I;W^{-2,p}(\Omega)) + W^{-1,2}(I;W^{-1,2}(\Omega))$, that is, the equation

$$\langle \partial_t \boldsymbol{u}, \boldsymbol{\varrho}\rangle_{\mathcal{Y}^0_{p,r}(Q)',\mathcal{Y}^0_{p,r}(Q)} + \int_0^T \{\boldsymbol{b}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\varrho}) - \nu\langle \boldsymbol{u}, \Delta\boldsymbol{\varrho}\rangle_{\boldsymbol{L}^p(\Omega),\boldsymbol{L}^{p'}(\Omega)}\}\,\mathrm{d}t$$
$$- \langle \mathfrak{p}, \operatorname{div}\boldsymbol{\varrho}\rangle_{\mathcal{P}^0_{p,r}(Q),\mathcal{P}^0_{p,r}(Q)'} = \int_0^T \{\langle \ell(\phi,\theta)\boldsymbol{g}, \boldsymbol{\varrho}\rangle_{\boldsymbol{L}^q(\Omega),\boldsymbol{L}^{q'}(\Omega)} - \mathcal{K}b(\boldsymbol{\varrho},\phi,\mu - l_{\mathrm{c}}\theta)\}\,\mathrm{d}t$$
$$+ \int_0^T \langle \boldsymbol{f}, \boldsymbol{\varrho}\rangle_{\boldsymbol{W}^{-2,p}(\Omega),\boldsymbol{W}_0^{2,p'}(\Omega)}\,\mathrm{d}t$$

holds for every $\boldsymbol{\varrho} \in \mathcal{Y}^0_{p,r}(Q) := W_0^{1,r'}(I;W_0^{2,p'}(\Omega)) \cap W_0^{1,2}(I;W_0^{1,2}(\Omega))$. ◇

By arguing as in the case of the weak solutions, each term in the above variational equations is well-defined. In particular, due to Lemma 4.2 and Corollary 4.5, the nonlinear terms are meaningful.

Remark 4.8. *A weak solution in the sense of Definition 4.6 is also a very weak solution in the sense of Definition 4.7. This follows immediately from the continuous embeddings (4.5) and (4.6). For the associated pressure, we use the fact that* $\widehat{L}^p(\Omega) \hookrightarrow \widehat{W}^{-1,p}(\Omega) \hookrightarrow \widehat{W}^{-1,4}(\Omega)$ *for* $p \geq 4$.

4.3. WELL-POSEDNESS OF AN AUXILIARY PDE SYSTEM. To accommodate the analysis both for the existence and uniqueness of weak solutions to the nonlinear part (2.5), we consider the following auxiliary PDE system:

$$\left[\begin{array}{ll}\partial_t\phi_N + \chi\mathrm{div}\,(\phi_N\boldsymbol{u}_N) + \mathrm{div}\,(\phi_N\widetilde{\boldsymbol{u}}) + \mathrm{div}\,(\widetilde{\phi}\boldsymbol{u}_N) - m\Delta\mu_N = \widetilde{\sigma} & \text{in } Q,\\ \mu_N = \tau\partial_t\phi_N - \epsilon\Delta\phi_N + \chi(F(\widetilde{\phi}+\phi_N) - F(\widetilde{\phi})) + G(\widetilde{\phi})\phi_N + l_{\mathrm{c}}\theta_N + \widetilde{\lambda} & \text{in } Q,\\ \partial_t\theta_N - l_{\mathrm{h}}\partial_t\phi_N + \chi\mathrm{div}\,((\theta_N - l_{\mathrm{h}}\phi_N)\boldsymbol{u}_N) + \mathrm{div}\,((\theta_N - l_{\mathrm{h}}\phi_N)\widetilde{\boldsymbol{u}}) & \\ \qquad + \mathrm{div}\,((\widetilde{\theta} - l_{\mathrm{h}}\widetilde{\phi})\boldsymbol{u}_N) - \kappa\Delta\theta_N = \alpha\boldsymbol{g}\cdot\boldsymbol{u}_N + \widetilde{h} & \text{in } Q,\\ \partial_t\boldsymbol{u}_N + \chi\mathrm{div}\,(\boldsymbol{u}_N\otimes\boldsymbol{u}_N) + \mathrm{div}\,(\boldsymbol{u}_N\otimes\widetilde{\boldsymbol{u}}) + \mathrm{div}\,(\widetilde{\boldsymbol{u}}\otimes\boldsymbol{u}_N) & \\ \qquad - \nu\Delta\boldsymbol{u}_N + \nabla\mathfrak{p}_N = \chi\mathcal{K}(\mu_N - l_{\mathrm{c}}\theta_N)\nabla\phi_N + \mathcal{K}(\widetilde{\mu} - l_{\mathrm{c}}\widetilde{\theta})\nabla\phi_N & \\ \qquad + \mathcal{K}(\mu_N - l_{\mathrm{c}}\theta_N)\nabla\widetilde{\phi} + \ell(\phi_N,\theta_N)\boldsymbol{g} + \widetilde{\boldsymbol{f}} & \text{in } Q,\\ \mathrm{div}\,\boldsymbol{u}_N = 0 & \text{in } Q,\\ \phi_N = \Delta\phi_N = 0, \quad \theta_N = 0, \quad \boldsymbol{u}_N = \boldsymbol{0} & \text{on } \Sigma,\\ \phi_N(0) = \phi_{0N}, \quad \theta_N(0) = \theta_{0N}, \quad \boldsymbol{u}_N(0) = \boldsymbol{u}_{0N} & \text{in } \Omega,\end{array}\right. \tag{4.14}$$

where $\chi \geq 0$ and $G(\widetilde{\phi})$ is a polynomial of degree at most 2, that is, $G(\widetilde{\phi}) = \chi_1\widetilde{\phi}^2 + \chi_2\widetilde{\phi} + \chi_3$ with $\chi_1, \chi_2, \chi_3 \in \mathbb{R}$. This auxiliary system also appears later in the proof of smoothness of the operator that maps the source functions and the initial data to the weak or very weak solution. In (4.14), $(\phi_N, \theta_N, \boldsymbol{u}_N, \mu_N, \mathfrak{p}_N)$ is the unknown vector function, while the components of $(\widetilde{\phi}, \widetilde{\theta}, \widetilde{\boldsymbol{u}}, \widetilde{\mu})$ are called the *frozen coefficients*. These coefficients will correspond to the solution of the linear part (2.4).

We now proceed with a classical spectral Faedo–Galerkin method for the well-posedness of (4.14). To this end, let $\{\boldsymbol{w}_j\}_{j=1}^\infty \subset \boldsymbol{X}_\sigma^{2,2}(\Omega)$ and $\{\rho_j\}_{j=1}^\infty \subset X^{2,2}(\Omega)$ be orthonormal bases for $\boldsymbol{L}_\sigma^2(\Omega)$ and $L^2(\Omega)$ that consist of eigenfunctions of the Stokes operator $\boldsymbol{A}_2$ and the Dirichlet Laplacian A_2, respectively. The existence of such bases is guaranteed from the fact that $\boldsymbol{A}_2 : \boldsymbol{X}_\sigma^{2,2}(\Omega) \to \boldsymbol{L}_\sigma^2(\Omega)$ and $A_2 : X^{2,2}(\Omega) \to L^2(\Omega)$ are positive operators, respectively, having compact resolvents.

Denote by $\boldsymbol{W}_k$ and R_k the linear spans of $\{\boldsymbol{w}_j\}_{j=1}^k$ and $\{\rho_j\}_{j=1}^k$, respectively. Define the orthogonal projections $\boldsymbol{\Pi}_k : \boldsymbol{L}_\sigma^2(\Omega) \to \boldsymbol{W}_k$ and $P_k : L^2(\Omega) \to R_k$ by

$$\boldsymbol{\Pi}_k\boldsymbol{w} = \sum_{j=1}^k(\boldsymbol{w}, \boldsymbol{w}_j)_{\boldsymbol{L}_\sigma^2(\Omega)}\boldsymbol{u}_j, \qquad P_k\varphi = \sum_{j=1}^k(\varphi, \rho_j)_{L^2(\Omega)}\rho_j.$$

Note that $\boldsymbol{\Pi}_k \in \mathcal{L}(\boldsymbol{W}_k, \boldsymbol{X}_\sigma^{1,2}(\Omega))$ and $P_k \in \mathcal{L}(R_k, W_0^{1,2}(\Omega))$, and hence for the duals we have $\boldsymbol{\Pi}_k' \in \mathcal{L}(\boldsymbol{X}_\sigma^{1,2}(\Omega)', \boldsymbol{W}_k)$ and $P_k' \in \mathcal{L}(W^{-1,2}(\Omega), R_k)$. Here, we have identified the duals of the finite-dimensional spaces $\boldsymbol{W}_k$ and R_k with themselves.

Theorem 4.9. *Let* (4.13) *be satisfied and suppose that we have source functions* $\widetilde{\sigma}, \widetilde{h} \in L^2(I; W^{-1,2}(\Omega))$, $\widetilde{\boldsymbol{f}} \in L^2(I; \boldsymbol{W}^{-1,2}(\Omega))$, $\widetilde{\lambda} \in L^2(I; W_0^{1,2}(\Omega))$, *and initial data* $\phi_{0N} \in X^{2,2}(\Omega)$, $\theta_{0N} \in L^2(\Omega)$, $\boldsymbol{u}_{0N} \in \boldsymbol{L}_\sigma^2(\Omega)$ *in* (4.14). *Moreover, suppose that the frozen coefficients satisfy* $\widetilde{\phi} \in \mathcal{Z}_{q,r}^2(Q) + \mathcal{X}_{\infty,2}^2(Q)$, $\widetilde{\theta} \in \mathcal{Z}_{s,r}^0(Q) + \mathcal{X}_{\infty,2}^1(Q)$, $\widetilde{\boldsymbol{u}} \in \boldsymbol{\mathcal{V}}_{p,r}^0(Q) + \boldsymbol{\mathcal{U}}_{\infty,2}^1(Q)$ *and* $\widetilde{\mu} \in L^r(I; L^q(\Omega)) + L^2(I; W_0^{1,2}(\Omega))$. *Then* (4.14) *has a unique weak solution*

$$(\phi_N, \theta_N, \boldsymbol{u}_N, \mu_N) \in \mathcal{Z}_{2,2}^3(Q) \times \mathcal{Z}_{2,2}^1(Q) \times \boldsymbol{\mathcal{V}}_{2,2}^1(Q) \times L^2(I; W_0^{1,2}(\Omega)). \tag{4.15}$$

Furthermore, there is a unique associated pressure $\mathfrak{p}_N \in W^{-1,2}(I;\widehat{L}^2(\Omega))$ *and a continuous and monotone increasing function* $\mathscr{C} : [0,\infty) \to [0,\infty)$*, depending continuously on the norms of the frozen coefficients but not on the source functions, initial data, and weak solution, for which* $\mathscr{C}(0) = 0$ *and*

$$\begin{aligned}\|\phi_N\|_{\mathcal{Z}^3_{2,2}(Q)} &+ \|\theta_N\|_{\mathcal{Z}^1_{2,2}(Q)} + \|\boldsymbol{u}_N\|_{\boldsymbol{\mathcal{V}}^1_{2,2}(Q)} + \|\mu_N\|_{L^2(I;W^{1,2}_0(\Omega))} + \|\mathfrak{p}_N\|_{W^{-1,2}(I;\widehat{L}^2(\Omega))}\\ &\le \mathscr{C}(\|\phi_{0N}\|_{X^{2,2}(\Omega)} + \|\theta_{0N}\|_{L^2(\Omega)} + \|\boldsymbol{u}_{0N}\|_{\boldsymbol{L}^2_\sigma(\Omega)} + \|\widetilde{\sigma}\|_{L^2(I;W^{-1,2}(\Omega))}\\ &+ \|\widetilde{\lambda}\|_{L^2(I;W^{1,2}_0(\Omega))} + \|\widetilde{h}\|_{L^2(I;W^{-1,2}(\Omega))} + \|\widetilde{\boldsymbol{f}}\|_{L^2(I;\boldsymbol{W}^{-1,2}(\Omega))} + |\alpha_0\boldsymbol{g}|).\end{aligned} \tag{4.16}$$

Proof. We follow the demonstration given in [54] and divide the proof into 4 steps: discretization, a priori estimates, passage to limit, and uniqueness. The derivation of the a priori estimates is more involved due to the limited regularity of the frozen coefficients. In the proof, $\mathscr{C} : [0,\infty) \to [0,\infty)$ will denote a generic monotone increasing and continuous function such that $\mathscr{C}(0) = 0$.

STEP 1. *Discretization.* Given a positive integer k, consider the projected initial data $\phi_{0Nk} = P_k\phi_{0N} \in R_k$, $\theta_{0Nk} = P_k\theta_{0N} \in R_k$, $\boldsymbol{u}_{0Nk} = \boldsymbol{\Pi}_k\boldsymbol{u}_{0N} \in \boldsymbol{W}_k$, and the ansatz

$$\phi_k(t) = \sum_{j=1}^k \alpha_{kj}(t)\rho_j, \qquad \theta_k(t) = \sum_{j=1}^k \gamma_{kj}(t)\rho_j, \qquad \boldsymbol{u}_k(t) = \sum_{j=1}^k \beta_{kj}(t)\boldsymbol{w}_j,$$

where $\alpha_{kj}, \beta_{kj}, \gamma_{kj} \in W^{1,2}(I)$ for $j = 1,\dots,k$, to the following finite-dimensional approximation of (4.14):

$$\left[\begin{aligned}&\partial_t\phi_k + P_k'[\chi C(\boldsymbol{u}_k,\phi_k) + C(\widetilde{\boldsymbol{u}},\phi_k) + C(\boldsymbol{u}_k,\widetilde{\phi})] + mA_2\mu_k = P_k'\widetilde{\sigma}\\ &\mu_k = \tau\partial_t\phi_k + \epsilon A_2\phi_k + P_k[\chi(F(\widetilde{\phi}+\phi_k) - F(\widetilde{\phi})) + G(\widetilde{\phi})\phi_k + \widetilde{\lambda}] + l_c\theta_k\\ &\partial_t\theta_k - l_h\partial_t\phi_k + P_k'[\chi C(\boldsymbol{u}_k,\theta_k - l_h\phi_k) + C(\widetilde{\boldsymbol{u}},\theta_k - l_h\phi_k) + C(\boldsymbol{u}_k,\widetilde{\theta} - l_h\widetilde{\phi})]\\ &\qquad + \kappa A_2\theta_k = P_k'[\alpha\boldsymbol{g}\cdot\boldsymbol{u}_k + \widetilde{h}]\\ &\partial_t\boldsymbol{u}_k + \boldsymbol{\Pi}_k'[\chi\boldsymbol{B}(\boldsymbol{u}_k,\boldsymbol{u}_k) + \boldsymbol{B}(\widetilde{\boldsymbol{u}},\boldsymbol{u}_k) + \boldsymbol{B}(\boldsymbol{u}_k,\widetilde{\boldsymbol{u}})] + \nu\boldsymbol{A}_2\boldsymbol{u} = \boldsymbol{\Pi}_k'[\ell(\phi_k,\theta_k)\boldsymbol{g}]\\ &\qquad + \boldsymbol{\Pi}_k'[\widetilde{\boldsymbol{f}} - \chi\mathcal{K}\boldsymbol{S}(\mu_k - l_c\theta_k,\phi_k) - \mathcal{K}\boldsymbol{S}(\widetilde{\mu} - l_c\widetilde{\theta},\phi_k) - \mathcal{K}\boldsymbol{S}(\mu_k - l_c\theta_k,\widetilde{\phi})]\\ &\phi_k(0) = \phi_{0Nk}, \quad \theta_k(0) \in \theta_{0Nk}, \quad \boldsymbol{u}_k(0) = \boldsymbol{u}_{0Nk}.\end{aligned}\right. \tag{4.17}$$

The first three equations are to be understood in the function space $L^2(I;R_k)$, while the fourth equation in $L^2(I;\boldsymbol{W}_k)$. From the Cauchy–Lipschitz Theorem, this system has a unique maximal solution with components ϕ_k, μ_k, $\theta_k \in W^{1,2}(I_k;R_k)$ and $\boldsymbol{u}_k \in W^{1,2}(I_k;\boldsymbol{W}_k)$ for some time interval $I_k = (0,t_k)$ with $0 < t_k \le T$. The a priori estimates that we will derive along with a standard continuation argument will show that $I_k = I$ for each k.

STEP 2. *A priori estimates.* This is the bulk of the proof. For clarity we derive these estimates in several steps, with the corresponding result in bullets.

• $L^\infty(I;W^{1,2}_0(\Omega)) \cap L^2(I;X^{2,2}(\Omega))$*-estimate for* ϕ_k. Taking $\phi_k(t)$ as a test function in the first equation in (4.17), using $b(\chi\boldsymbol{u}_k + \widetilde{\boldsymbol{u}},\phi_k,\phi_k) = 0$ and integrating by parts for the term involving μ_k, we get

$$\frac{1}{2}\frac{d}{dt}\|\phi_k\|^2_{L^2(\Omega)} + b(\boldsymbol{u}_k,\phi_k,\widetilde{\phi}) = m(\mu_k,\Delta\phi_k)_{L^2(\Omega)} + \langle\widetilde{\sigma},\phi_k\rangle_{W^{-1,2}(\Omega),W^{1,2}_0(\Omega)}. \tag{4.18}$$

The trilinear term can be estimated by Hölder's inequality and $W_0^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$ as follows:

$$\begin{aligned} |b(\boldsymbol{u}_k, \phi_k, \widetilde{\phi})| &\le \|\boldsymbol{u}_k\|_{\boldsymbol{L}^2(\Omega)} \|\phi_k\|_{L^4(\Omega)} \|\nabla\widetilde{\phi}\|_{L^4(\Omega)} \\ &\le c\{\|\boldsymbol{u}_k\|^2_{\boldsymbol{L}^2_\sigma(\Omega)} + \|\widetilde{\phi}\|^2_{W_0^{1,4}(\Omega)} \|\phi_k\|^2_{W_0^{1,2}(\Omega)}\}. \end{aligned} \tag{4.19}$$

For the first term on right-hand side in (4.18), we use the equation for μ_k given by the second equation in (4.17) and integrate by parts to obtain

$$\begin{aligned} m(\mu_k, \Delta\phi_k)_{L^2(\Omega)} = &-\frac{m\tau}{2}\frac{d}{dt}\|\nabla\phi_k\|^2_{\boldsymbol{L}^2(\Omega)} - m\epsilon\|\Delta\phi_k\|^2_{L^2(\Omega)} + m(l_c\theta_k + \widetilde{\lambda}, \Delta\phi_k)_{L^2(\Omega)} \\ &+ m(\chi(F(\widetilde{\phi} + \phi_k) - F(\widetilde{\phi})) + G(\widetilde{\phi})\phi_k, \Delta\phi_k)_{L^2(\Omega)}. \end{aligned} \tag{4.20}$$

For the second term on the right-hand side of (4.18), we have

$$|\langle\widetilde{\sigma}, \phi_k\rangle_{W^{-1,2}(\Omega), W_0^{1,2}(\Omega)}| \le c\{\|\widetilde{\sigma}\|^2_{W^{-1,2}(\Omega)} + \|\phi_k\|^2_{W_0^{1,2}(\Omega)}\}. \tag{4.21}$$

Applying Young's inequality to the third term on the right-hand side of (4.20) yields

$$|m(l_c\theta_k + \widetilde{\lambda}, \Delta\phi_k)_{L^2(\Omega)}| \le \delta\|\Delta\phi_k\|^2_{L^2(\Omega)} + c_\delta\{\|\theta_k\|^2_{L^2(\Omega)} + \|\widetilde{\lambda}\|^2_{L^2(\Omega)}\}. \tag{4.22}$$

In what follows, δ will denote a positive constant, taken to be sufficiently small.

Expanding the cubic term in $F(\widetilde{\phi} + \phi_k)$ and rearranging the terms yield

$$\begin{aligned} &\chi(F(\widetilde{\phi} + \phi_k) - F(\widetilde{\phi})) + G(\widetilde{\phi})\phi_k \\ &\quad = \chi\beta_0\phi_k^3 + 3\chi\beta_0\widetilde{\phi}\phi_k^2 + [G(\widetilde{\phi}) + 3\chi\beta_0\widetilde{\phi}^2 - \chi\beta_1]\phi_k. \end{aligned}$$

Using Green's identity and the Hölder and Young inequalities, one has

$$m(\chi\beta_0\phi_k^3, \Delta\phi_k)_{L^2(\Omega)} = -3m\chi\beta_0\|\phi_k\nabla\phi_k\|^2_{\boldsymbol{L}^2(\Omega)} \tag{4.23}$$

$$\begin{aligned} &|m([G(\widetilde{\phi}) + 3\chi\beta_0\widetilde{\phi}^2 - \chi\beta_1]\phi_k, \Delta\phi_k)_{L^2(\Omega)}| \\ &\quad \le \delta\|\Delta\phi_k\|^2_{L^2(\Omega)} + c_\delta\{\|\widetilde{\phi}\|^4_{L^8(\Omega)} + 1\}\|\phi_k\|^2_{W_0^{1,2}(\Omega)}. \end{aligned} \tag{4.24}$$

In the last inequality we used the assumption that G is a quadratic form. Integrating by parts, using Hölder's inequality and invoking the embeddings $W_0^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$ and $W_0^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$, we have

$$\begin{aligned} &|m(3\chi\beta_0\widetilde{\phi}\phi_k^2, \Delta\phi_k)_{L^2(\Omega)}| \\ &\le m\chi\beta_0\{6|(\phi_k\nabla\phi_k, \widetilde{\phi}\nabla\phi_k)_{\boldsymbol{L}^2(\Omega)}| + 3|(\phi_k\nabla\phi_k, \phi_k\nabla\widetilde{\phi})_{\boldsymbol{L}^2(\Omega)}|\} \\ &\le m\chi\beta_0\{6\|\phi_k\nabla\phi_k\|_{\boldsymbol{L}^2(\Omega)}\|\widetilde{\phi}\|_{L^\infty(\Omega)}\|\nabla\phi_k\|_{\boldsymbol{L}^2(\Omega)} + 3\|\phi_k\nabla\phi_k\|_{\boldsymbol{L}^2(\Omega)}\|\phi_k\|_{L^4(\Omega)}\|\nabla\widetilde{\phi}\|_{\boldsymbol{L}^4(\Omega)}\} \\ &\le m\chi\beta_0\|\phi_k\nabla\phi_k\|^2_{\boldsymbol{L}^2(\Omega)} + c\|\widetilde{\phi}\|^2_{W_0^{1,4}(\Omega)}\|\phi_k\|^2_{W_0^{1,2}(\Omega)}. \end{aligned} \tag{4.25}$$

Define $J_1 := \|\widetilde{\phi}\|^2_{W_0^{1,4}(\Omega)} + \|\widetilde{\phi}\|^4_{L^8(\Omega)} + 1$. Note that $J_1 \in L^1(I)$ since $\mathcal{Z}^2_{q,r}(Q) + \mathcal{X}^2_{\infty,2}(Q) \hookrightarrow L^\infty(I; L^{\mathfrak{s}}(\Omega)) \cap L^4(I; W_0^{1,4}(\Omega))$ for any $1 < \mathfrak{s} < \infty$. In particular, we have

$$\|J_1\|_{L^1(I)} \le \mathscr{C}(\|\widetilde{\phi}\|_{\mathcal{Z}^2_{q,r}(Q)+\mathcal{X}^2_{\infty,2}(Q)} + 1). \tag{4.26}$$

Here, $\mathscr{C}$ is a continuous function as described from the beginning of the proof. Now, by plugging the estimates (4.19)–(4.25) in (4.18), it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\{\|\phi_k\|_{L^2(\Omega)}^2 + m\tau\|\nabla\phi_k\|_{\boldsymbol{L}^2(\Omega)}^2\} + (m\epsilon - 2\delta)\|\Delta\phi_k\|_{L^2(\Omega)}^2 \\ + 2m\chi\beta_0\|\phi_k\nabla\phi_k\|_{\boldsymbol{L}^2(\Omega)}^2 \le c_\delta\{\|\widetilde{\sigma}\|_{W^{-1,2}(\Omega)}^2 + \|\widetilde{\lambda}\|_{L^2(\Omega)}^2\} \\ + c_\delta J_1\{\|\phi_k\|_{W_0^{1,2}(\Omega)}^2 + \|\theta_k\|_{L^2(\Omega)}^2 + \|\boldsymbol{u}_k\|_{\boldsymbol{L}_\sigma^2(\Omega)}^2\}.\end{aligned} \tag{4.27}$$

• *$L^2(I;L^2(\Omega))$-estimate for $\partial_t\phi_k$ and $L^2(I;W_0^{1,2}(\Omega))$-estimate for μ_k.* Applying the test function $\mu_k(t)$ on the first equation in the approximate system (4.17) and using the antisymmetry of b with respect to its second and third arguments, we get

$$\begin{aligned}(\partial_t\phi_k, \mu_k)_{L^2(\Omega)} - \chi b(\boldsymbol{u}_k, \mu_k, \phi_k) + b(\widetilde{\boldsymbol{u}}, \phi_k, \mu_k) + b(\boldsymbol{u}_k, \widetilde{\phi}, \mu_k) \\ + m\|\nabla\mu_k\|_{\boldsymbol{L}^2(\Omega)}^2 = \langle\widetilde{\sigma}, \mu_k\rangle_{W^{-1,2}(\Omega), W_0^{1,2}(\Omega)}.\end{aligned} \tag{4.28}$$

From the Poincaré inequality, we can estimate the right-hand side by

$$|\langle\widetilde{\sigma}, \mu_k\rangle_{W^{-1,2}(\Omega), W_0^{1,2}(\Omega)}| \le \delta\|\nabla\mu_k\|_{\boldsymbol{L}^2(\Omega)}^2 + c_\delta\|\widetilde{\sigma}\|_{W^{-1,2}(\Omega)}^2. \tag{4.29}$$

For the second trilinear term in (4.28) we use $W_0^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$ and for the third we apply $W_0^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$, so that

$$\begin{aligned}|b(\widetilde{\boldsymbol{u}}, \phi_k, \mu_k)| &\le \|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^4(\Omega)}\|\phi_k\|_{L^4(\Omega)}\|\nabla\mu_k\|_{\boldsymbol{L}^2(\Omega)} \\ &\le \delta\|\nabla\mu_k\|_{\boldsymbol{L}^2(\Omega)}^2 + c_\delta\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^4(\Omega)}^2\|\phi_k\|_{W_0^{1,2}(\Omega)}^2\end{aligned} \tag{4.30}$$

$$\begin{aligned}|b(\boldsymbol{u}_k, \widetilde{\phi}, \mu_k)| &\le \|\boldsymbol{u}_k\|_{\boldsymbol{L}^2(\Omega)}\|\widetilde{\phi}\|_{L^\infty(\Omega)}\|\nabla\mu_k\|_{\boldsymbol{L}^2(\Omega)} \\ &\le \delta\|\nabla\mu_k\|_{\boldsymbol{L}^2(\Omega)}^2 + c_\delta\|\widetilde{\phi}\|_{W_0^{1,4}(\Omega)}^2\|\boldsymbol{u}_k\|_{\boldsymbol{L}_\sigma^2(\Omega)}^2.\end{aligned} \tag{4.31}$$

Now, by taking the L^2-inner product of μ_k and $\partial_t\phi_k$, one has the equation

$$\begin{aligned}(\mu_k, \partial_t\phi_k)_{L^2(\Omega)} = \tau\|\partial_t\phi_k\|_{L^2(\Omega)}^2 + \frac{\epsilon}{2}\frac{d}{dt}\|\nabla\phi_k\|_{\boldsymbol{L}^2(\Omega)}^2 \\ + (\chi(F(\widetilde{\phi} + \phi_k) - F(\widetilde{\phi})) + G(\widetilde{\phi})\phi_k, \partial_t\phi_k)_{L^2(\Omega)} + (l_c\theta_k + \widetilde{\lambda}, \partial_t\phi_k)_{L^2(\Omega)}.\end{aligned} \tag{4.32}$$

By Young's inequality the last term on the right-hand side of (4.32) can be estimated by

$$|(l_c\theta_k + \widetilde{\lambda}, \partial_t\phi_k)_{L^2(\Omega)}| \le \delta\|\partial_t\phi_k\|_{L^2(\Omega)}^2 + c_\delta\{\|\theta_k\|_{L^2(\Omega)}^2 + \|\widetilde{\lambda}\|_{L^2(\Omega)}^2\}. \tag{4.33}$$

On the other hand, for the term involving F and G in (4.32), we can adapt the methods presented in the previous step to deduce the following bound from below

$$\begin{aligned}|(\chi(F(\widetilde{\phi} + \phi_k) - F(\widetilde{\phi})) + G(\widetilde{\phi})\phi_k, \partial_t\phi_k)_{L^2(\Omega)}| \\ \ge \frac{\chi\beta_0}{4}\frac{d}{dt}\|\phi_k\|_{L^4(\Omega)}^4 - \delta\|\partial_t\phi_k\|_{L^2(\Omega)}^2 - c_\delta J_1\{\chi\beta_0\|\phi_k\|_{L^4(\Omega)}^4 + \|\phi_k\|_{W_0^{1,2}(\Omega)}^2\}.\end{aligned} \tag{4.34}$$

Thus, upon substitution of (4.29)–(4.34) in (4.28), we obtain

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\{2\epsilon\|\nabla\phi_k\|_{\boldsymbol{L}^2(\Omega)}^2 + \chi\beta_0\|\phi_k\|_{L^4(\Omega)}^4\} + (\tau - 2\delta)\|\partial_t\phi_k\|_{L^2(\Omega)}^2 \\ + (m - 3\delta)\|\nabla\mu_k\|_{\boldsymbol{L}^2(\Omega)}^2 \le c_\delta\{\|\widetilde{\sigma}\|_{W^{-1,2}(\Omega)}^2 + \|\widetilde{\lambda}\|_{L^2(\Omega)}^2\} + \chi b(\boldsymbol{u}_k, \mu_k, \phi_k)\end{aligned}$$

$$+ c_\delta J_2\{\chi\beta_0\|\phi_k\|_{L^4(\Omega)}^4 + \|\phi_k\|_{W_0^{1,2}(\Omega)}^2 + \|\theta_k\|_{L^2(\Omega)}^2 + \|\boldsymbol{u}_k\|_{\boldsymbol{L}_\sigma^2(\Omega)}^2\} \tag{4.35}$$

where $J_2 := J_1 + \|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^4(\Omega)}^2$. Note that $J_2 \in L^1(I)$ since $\boldsymbol{\mathcal{V}}_{p,r}^0(Q) + \boldsymbol{\mathcal{U}}_{\infty,2}^1(Q) \hookrightarrow L^4(I;\boldsymbol{L}^4(\Omega))$, and in particular,

$$\|J_2\|_{L^1(I)} \le \|J_1\|_{L^1(I)} + \mathscr{C}(\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{\mathcal{V}}_{p,r}^0(Q)+\boldsymbol{\mathcal{U}}_{\infty,2}^1(Q)}). \tag{4.36}$$

• $L^\infty(I;L^2(\Omega))\cap L^2(I;W_0^{1,2}(\Omega))$*-estimate for* θ_k. Choosing $\theta_k(t)$ as a test function in the third equation of the approximate system (4.17) and using $b(\chi\boldsymbol{u}_k+\widetilde{\boldsymbol{u}},\theta_k,\theta_k) = 0$ give us

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta_k\|_{L^2(\Omega)}^2 &+ \chi l_{\rm h} b(\boldsymbol{u}_k,\theta_k,\phi_k) - l_{\rm h} b(\widetilde{\boldsymbol{u}},\phi_k,\theta_k) + b(\boldsymbol{u}_k,\widetilde{\theta} - l_{\rm h}\widetilde{\phi},\theta_k)\\ &+ \kappa\|\nabla\theta_k\|_{\boldsymbol{L}^2(\Omega)}^2 = (\alpha\boldsymbol{g}\cdot\boldsymbol{u}_k + l_{\rm h}\partial_t\phi_k,\theta_k)_{L^2(\Omega)} + \langle\widetilde{h},\theta_k\rangle_{W^{-1,2}(\Omega),W_0^{1,2}(\Omega)}.\end{aligned} \tag{4.37}$$

By the Cauchy–Schwarz inequality, we can estimate the terms on right-hand side by

$$|\langle\widetilde{h},\theta_k\rangle_{W^{-1,2}(\Omega),W_0^{1,2}(\Omega)}| \le \delta\|\nabla\theta_k\|_{\boldsymbol{L}^2(\Omega)}^2 + c_\delta\|\widetilde{h}\|_{W^{-1,2}(\Omega)}^2 \tag{4.38}$$

$$|(\alpha\boldsymbol{g}\cdot\boldsymbol{u}_k + l_{\rm h}\partial_t\phi_k,\theta_k)_{L^2(\Omega)}| \le \delta\|\partial_t\phi_k\|_{L^2(\Omega)}^2 + c_\delta\{\|\theta_k\|_{L^2(\Omega)}^2 + \|\boldsymbol{u}_k\|_{\boldsymbol{L}_\sigma^2(\Omega)}^2\}. \tag{4.39}$$

For the last two trilinear terms in (4.37), we estimate as follows:

$$\begin{aligned}|l_{\rm h} b(\widetilde{\boldsymbol{u}},\phi_k,\theta_k)| &\le l_{\rm h}\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^4(\Omega)}\|\phi_k\|_{L^4(\Omega)}\|\nabla\theta_k\|_{\boldsymbol{L}^2(\Omega)}\\ &\le \delta\|\nabla\theta_k\|_{\boldsymbol{L}^2(\Omega)}^2 + c_\delta\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^4(\Omega)}^2\|\phi_k\|_{W_0^{1,2}(\Omega)}^2\end{aligned} \tag{4.40}$$

$$\begin{aligned}|b(\boldsymbol{u}_k,\widetilde{\theta} - l_{\rm h}\widetilde{\phi},\theta_k)| &\le c\|\boldsymbol{u}_k\|_{\boldsymbol{L}^2(\Omega)}^{1/2}\|\nabla\boldsymbol{u}_k\|_{\boldsymbol{L}^2(\Omega)^2}^{1/2}\|\widetilde{\theta} - l_{\rm h}\widetilde{\phi}\|_{L^4(\Omega)}\|\nabla\theta_k\|_{\boldsymbol{L}^2(\Omega)}\\ &\le \delta\|\nabla\theta_k\|_{\boldsymbol{L}^2(\Omega)}^2 + \delta\|\nabla\boldsymbol{u}_k\|_{\boldsymbol{L}^2(\Omega)^2}^2 + c_\delta\{\|\widetilde{\theta}\|_{L^4(\Omega)}^4 + \|\widetilde{\phi}\|_{L^4(\Omega)}^4\}\|\boldsymbol{u}_k\|_{\boldsymbol{L}^2(\Omega)}^2\end{aligned} \tag{4.41}$$

where we used the Gagliardo–Nirenberg inequality in the second trilinear term.

Let $J_3 := \|\widetilde{\phi}\|_{L^4(\Omega)}^4 + \|\widetilde{\theta}\|_{L^4(\Omega)}^4 + \|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^4(\Omega)}^2 + 1$. Then, we have $J_3 \in L^1(I)$ and

$$\|J_3\|_{L^1(I)} \le \mathscr{C}(\|\widetilde{\phi}\|_{\mathcal{Z}_{q,r}^2(Q)+\mathcal{X}_{\infty,2}^2(Q)} + \|\widetilde{\theta}\|_{\mathcal{Z}_{s,r}^0(Q)+\mathcal{X}_{\infty,2}^1(Q)} + \|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{\mathcal{V}}_{p,r}^0(Q)+\boldsymbol{\mathcal{U}}_{\infty,2}^1(Q)} + 1). \tag{4.42}$$

Thus, plugging the inequalities (4.38)–(4.41) into (4.37), we obtain the estimate

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta_k\|_{L^2(\Omega)}^2 &+ (\kappa - 2\delta)\|\nabla\theta_k\|_{\boldsymbol{L}^2(\Omega)}^2 - \delta\|\partial_t\phi_k\|_{L^2(\Omega)}^2 - \delta\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)^2}^2\\ &\le c_\delta\|\widetilde{h}\|_{W^{-1,2}(\Omega)}^2 + c_\delta J_3\{\|\phi_k\|_{W_0^{1,2}(\Omega)}^2 + \|\theta_k\|_{L^2(\Omega)}^2 + \|\boldsymbol{u}_k\|_{\boldsymbol{L}_\sigma^2(\Omega)}^2\} - \chi l_{\rm h} b(\boldsymbol{u}_k,\theta_k,\phi_k).\end{aligned} \tag{4.43}$$

• $L^\infty(I;\boldsymbol{L}_\sigma^2(\Omega))\cap L^2(I;\boldsymbol{X}_\sigma^{1,2}(\Omega))$*-estimate for* $\boldsymbol{u}_k$. Testing the fourth equation in the system (4.17) by $\boldsymbol{u}_k(t)$ leads to the following equation

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\|\boldsymbol{u}_k\|_{\boldsymbol{L}_\sigma^2(\Omega)}^2 + \boldsymbol{b}(\boldsymbol{u}_k,\widetilde{\boldsymbol{u}},\boldsymbol{u}_k) + \nu\|\nabla\boldsymbol{u}_k\|_{\boldsymbol{L}^2(\Omega)^2}^2\\ &= (\ell(\phi_k,\theta_k)\boldsymbol{g},\boldsymbol{u}_k)_{\boldsymbol{L}^2(\Omega)} + \langle\widetilde{\boldsymbol{f}},\boldsymbol{u}_k\rangle_{\boldsymbol{W}^{-1,2}(\Omega),\boldsymbol{W}_0^{1,2}(\Omega)}\\ &- \chi\mathcal{K} b(\boldsymbol{u}_k,\mu_k - l_{\rm c}\theta_k,\phi_k) - \mathcal{K} b(\boldsymbol{u}_k,\widetilde{\mu} - l_{\rm c}\widetilde{\theta},\phi_k) + \mathcal{K} b(\boldsymbol{u}_k,\widetilde{\phi},\mu_k - l_{\rm c}\theta_k)\end{aligned} \tag{4.44}$$

where we used $\boldsymbol{b}(\chi\boldsymbol{u}_k+\widetilde{\boldsymbol{u}},\boldsymbol{u}_k,\boldsymbol{u}_k)=0$. By the Cauchy–Schwartz inequality

$$|\langle\widetilde{\boldsymbol{f}},\boldsymbol{u}_k\rangle_{\boldsymbol{W}^{-1,2}(\Omega),\boldsymbol{W}_0^{1,2}(\Omega)}|\le\delta\|\nabla\boldsymbol{u}_k\|^2_{\boldsymbol{L}^2(\Omega)^2}+c_\delta\|\widetilde{\boldsymbol{f}}\|^2_{\boldsymbol{W}^{-1,2}(\Omega)} \tag{4.45}$$

$$|(\ell(\phi_k,\theta_k)\boldsymbol{g},\boldsymbol{u}_k)_{\boldsymbol{L}^2(\Omega)}|\le c\{|\alpha_0\boldsymbol{g}|^2+\|\phi_k\|^2_{L^2(\Omega)}+\|\theta_k\|^2_{L^2(\Omega)}+\|\boldsymbol{u}_k\|^2_{\boldsymbol{L}^2_\sigma(\Omega)}\}. \tag{4.46}$$

Here, we recall that $\ell(\phi_k,\theta_k)\boldsymbol{g}=(\alpha_0+\alpha_1\phi_k+\alpha_2\theta_k)\boldsymbol{g}$. The trilinear term on the left-hand side of (4.44) is bounded from above by

$$\begin{aligned}|\boldsymbol{b}(\boldsymbol{u}_k,\widetilde{\boldsymbol{u}},\boldsymbol{u}_k)|&\le c\|\boldsymbol{u}_k\|^{1/2}_{\boldsymbol{L}^2(\Omega)}\|\nabla\boldsymbol{u}_k\|^{3/2}_{\boldsymbol{L}^2(\Omega)^2}\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^4(\Omega)}\\&\le\delta\|\nabla\boldsymbol{u}_k\|^2_{\boldsymbol{L}^2(\Omega)^2}+c_\delta\|\widetilde{\boldsymbol{u}}\|^4_{\boldsymbol{L}^4(\Omega)}\|\boldsymbol{u}_k\|^2_{\boldsymbol{L}^2_\sigma(\Omega)}.\end{aligned} \tag{4.47}$$

The estimation of the second trilinear term in (4.44) is more delicate. In this direction, consider an arbitrary representation $\widetilde{\mu}=\widetilde{\mu}_L+\widetilde{\mu}_N$, where $\widetilde{\mu}_L\in L^r(I;L^q(\Omega))$ and $\widetilde{\mu}_N\in L^2(I;W_0^{1,2}(\Omega))$. We write $b(\boldsymbol{u}_k,\widetilde{\mu}-l_c\widetilde{\theta},\phi_k)=b(\boldsymbol{u}_k,\widetilde{\mu}_L,\phi_k)+b(\boldsymbol{u}_k,\widetilde{\mu}_N-l_c\widetilde{\theta},\phi_k)$ and estimate the terms on the right-hand side. Using $\boldsymbol{W}_0^{1,2}(\Omega)\hookrightarrow\boldsymbol{L}^{4q/(3q-4)}(\Omega)$ and the Hölder and Gagliardo–Nirenberg inequalities, we have

$$\begin{aligned}|\mathcal{K}b(\boldsymbol{u}_k,\widetilde{\mu}_L,\phi_k)|&\le c\|\boldsymbol{u}_k\|_{\boldsymbol{L}^{4q/(3q-4)}(\Omega)}\|\widetilde{\mu}_L\|_{L^q(\Omega)}\|\nabla\phi_k\|_{\boldsymbol{L}^4(\Omega)}\\&\le c\|\nabla\boldsymbol{u}_k\|_{\boldsymbol{L}^2(\Omega)^2}\|\widetilde{\mu}_L\|_{L^q(\Omega)}\|\nabla\phi_k\|^{1/2}_{\boldsymbol{L}^2(\Omega)}\|\Delta\phi_k\|^{1/2}_{L^2(\Omega)}\\&\le\delta\|\nabla\boldsymbol{u}_k\|^2_{\boldsymbol{L}^2(\Omega)^2}+\delta\|\Delta\phi_k\|^2_{L^2(\Omega)}+c_\delta\|\widetilde{\mu}_L\|^4_{L^q(\Omega)}\|\phi_k\|^2_{W_0^{1,2}(\Omega)}\end{aligned} \tag{4.48}$$

$$\begin{aligned}|\mathcal{K}b(\boldsymbol{u}_k,\widetilde{\mu}_N-l_c\widetilde{\theta},\phi_k)|&\le c\|\boldsymbol{u}_k\|_{\boldsymbol{L}^4(\Omega)}\|\widetilde{\mu}_N-l_c\widetilde{\theta}\|_{L^4(\Omega)}\|\nabla\phi_k\|_{\boldsymbol{L}^2(\Omega)}\\&\le c\|\boldsymbol{u}_k\|^{1/2}_{\boldsymbol{L}^2(\Omega)}\|\nabla\boldsymbol{u}_k\|^{1/2}_{\boldsymbol{L}^2(\Omega)^2}\|\widetilde{\mu}_N-l_c\widetilde{\theta}\|_{L^4(\Omega)}\|\phi_k\|^{1/2}_{W_0^{1,2}(\Omega)}\|\Delta\phi_k\|^{1/2}_{L^2(\Omega)}\\&\le\delta\|\nabla\boldsymbol{u}_k\|^2_{\boldsymbol{L}^2(\Omega)^2}+\delta\|\Delta\phi_k\|^2_{L^2(\Omega)}\\&\quad+c_\delta\{\|\widetilde{\mu}_N\|^2_{L^4(\Omega)}+\|\widetilde{\theta}\|^2_{L^4(\Omega)}\}\{\|\phi_k\|^2_{W_0^{1,2}(\Omega)}+\|\boldsymbol{u}_k\|^2_{\boldsymbol{L}^2_\sigma(\Omega)}\}.\end{aligned} \tag{4.49}$$

Finally, the remaining trilinear term in (4.44) satisfies

$$\begin{aligned}|\mathcal{K}b(\boldsymbol{u}_k,\widetilde{\phi},\mu_k-l_c\theta_k)|&\le c\|\boldsymbol{u}_k\|_{\boldsymbol{L}^2(\Omega)}\|\widetilde{\phi}\|_{L^\infty(\Omega)}\|\nabla(\mu_k-l_c\theta_k)\|_{\boldsymbol{L}^2(\Omega)}\\&\le\delta\|\nabla\theta_k\|^2_{\boldsymbol{L}^2(\Omega)}+\delta\|\nabla\mu_k\|^2_{\boldsymbol{L}^2(\Omega)}+c_\delta\|\widetilde{\phi}\|^2_{W_0^{1,4}(\Omega)}\|\boldsymbol{u}_k\|^2_{\boldsymbol{L}^2_\sigma(\Omega)}.\end{aligned} \tag{4.50}$$

Utilizing the estimates (4.45)–(4.50) in (4.44) and by setting $J_4:=\|\widetilde{\boldsymbol{u}}\|^4_{\boldsymbol{L}^4(\Omega)}+\|\widetilde{\mu}_L\|^4_{L^q(\Omega)}+\|\widetilde{\mu}_N\|^2_{L^4(\Omega)}+\|\widetilde{\theta}\|^2_{L^4(\Omega)}+\|\widetilde{\phi}\|^2_{W_0^{1,4}(\Omega)}+1$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_k\|^2_{\boldsymbol{L}^2_\sigma(\Omega)}+(\nu-4\delta)\|\nabla\boldsymbol{u}_k\|^2_{\boldsymbol{L}^2(\Omega)^2}-2\delta\|\Delta\phi_k\|^2_{L^2(\Omega)}\\&\quad-\delta\|\nabla\theta_k\|^2_{\boldsymbol{L}^2(\Omega)}-\delta\|\nabla\mu_k\|^2_{\boldsymbol{L}^2(\Omega)}\le c_\delta\{|\alpha_0\boldsymbol{g}|^2+\|\widetilde{\boldsymbol{f}}\|^2_{\boldsymbol{W}^{-1,2}(\Omega)}\}\\&\quad+c_\delta J_4\{\|\phi_k\|^2_{W_0^{1,2}(\Omega)}+\|\theta_k\|^2_{L^2(\Omega)}+\|\boldsymbol{u}_k\|^2_{\boldsymbol{L}^2_\sigma(\Omega)}\}-\chi\mathcal{K}b(\boldsymbol{u}_k,\mu_k-l_c\theta_k,\phi_k).\end{aligned} \tag{4.51}$$

Furthermore, $J_4\in L^1(I)$ and it holds that

$$\begin{aligned}\|J_4\|_{L^1(I)}&\le\mathscr{C}(\|\widetilde{\phi}\|_{\mathcal{Z}^2_{q,r}(Q)+\mathcal{X}^2_{\infty,2}(Q)}+\|\widetilde{\theta}\|_{\mathcal{Z}^0_{s,r}(Q)+\mathcal{X}^1_{\infty,2}(Q)}\\&\quad+\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{\mathcal{V}}^0_{p,r}(Q)+\boldsymbol{\mathcal{U}}^1_{\infty,2}(Q)}+\|\widetilde{\mu}_L\|_{L^r(I;L^q(\Omega))}+\|\widetilde{\mu}_N\|_{L^2(I;W_0^{1,2}(\Omega))}+1).\end{aligned} \tag{4.52}$$

We now combine the above a priori estimates. Multiply (4.43) by $\mathcal{K}l_{\mathrm{c}}/l_{\mathrm{h}}$ and (4.35) by $\mathcal{K}$, and then take the sum of the resulting inequalities with (4.27) and (4.51). After that we choose $\delta > 0$ small enough to obtain the differential inequality

$$\frac{1}{2}\frac{d}{dt}\mathsf{E}_k + c_1\mathsf{D}_k \le c_2(\mathsf{F} + \mathsf{J}\mathsf{E}_k) \quad \text{in } I_k. \tag{4.53}$$

for some constants $c_1, c_2 > 0$, where $\mathsf{J} := J_1 + J_2 + J_3 + J_4$,

$$\mathsf{E}_k := \frac{\mathcal{K}\chi\beta_0}{2}\|\phi_k\|_{L^4(\Omega)}^4 + \|\phi_k\|_{L^2(\Omega)}^2 + (\mathcal{K}\epsilon + m\tau)\|\nabla\phi_k\|_{\boldsymbol{L}^2(\Omega)}^2 + \frac{\mathcal{K}l_{\mathrm{c}}}{l_{\mathrm{h}}}\|\theta_k\|_{L^2(\Omega)}^2 + \|\boldsymbol{u}_k\|_{\boldsymbol{L}^2_\sigma(\Omega)}^2$$

$$\mathsf{D}_k := \chi\|\phi_k\nabla\phi_k\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\Delta\phi_k\|_{L^2(\Omega)}^2 + \|\nabla\mu_k\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla\theta_k\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla\boldsymbol{u}_k\|_{\boldsymbol{L}^2(\Omega)^2}^2$$

$$\mathsf{F} := \|\widetilde{\sigma}\|_{W^{-1,2}(\Omega)}^2 + \|\widetilde{\lambda}\|_{L^2(\Omega)}^2 + \|\widetilde{h}\|_{W^{-1,2}(\Omega)}^2 + \|\widetilde{\boldsymbol{f}}\|_{\boldsymbol{W}^{-1,2}(\Omega)}^2 + |\alpha_0\boldsymbol{g}|^2.$$

From (4.26), (4.36), (4.42) and (4.52) we have $\mathsf{J} \in L^1(I)$. Also, $\mathsf{F} \in L^1(I)$ based on the assumptions on the source functions. Using Gronwall's Lemma to (4.53), we obtain that $\mathsf{E}_k \in L^\infty(I_k)$ and

$$\|\mathsf{E}_k\|_{L^\infty(I_k)} \le (\mathsf{E}_k(0) + 2c_2\|\mathsf{F}\|_{L^1(I)})e^{2c_2\|\mathsf{J}\|_{L^1(I)}} \tag{4.54}$$

and as a result, by integrating (4.53) over I_k, one has $\mathsf{D}_k \in L^1(I_k)$ and

$$2c_1\|\mathsf{D}_k\|_{L^1(I_k)} \le (\mathsf{E}_k(0) + 2c_2\|\mathsf{F}\|_{L^1(I)} + 2c_2\|\mathsf{J}\|_{L^1(I)}\|\mathsf{E}_k\|_{L^\infty(I_k)}). \tag{4.55}$$

From the definition of the approximate initial data and the uniform boundedness of the projection operators $P_k \in \mathcal{L}(L^2(\Omega)) \cap \mathcal{L}(X^{2,2}(\Omega))$ and $\boldsymbol{\Pi}_k \in \mathcal{L}(\boldsymbol{L}^2_\sigma(\Omega))$, for each k one has

$$\mathsf{E}_k(0) \le \mathscr{C}(\|\phi_{0N}\|_{X^{2,2}(\Omega)} + \|\theta_{0N}\|_{L^2(\Omega)} + \|\boldsymbol{u}_{0N}\|_{\boldsymbol{L}^2_\sigma(\Omega)}). \tag{4.56}$$

Let us denote the right-hand side of the inequality (4.16) by R. With abuse of notation, we shall write R in place of $\mathscr{C}(\mathsf{R})$, that is, the function $\mathscr{C}$ in the definition of R has to be modified at each step. With this convention, we get $\mathsf{E}_k(0) \le \mathsf{R}$ from (4.56) and $\|\mathsf{F}\|_{L^1(I)} \le \mathsf{R}$. Plugging these in (4.54) and (4.55), we have

$$\|\mathsf{E}_k\|_{L^\infty(I_k)} + \|\mathsf{D}_k\|_{L^1(I_k)} \le \mathsf{R}.$$

Here, we took the infimum over all representations of $\widetilde{\mu}$ in $L^r(I; L^q(\Omega)) + L^2(I; W^{1,2}_0(\Omega))$ to pass from the estimate involving $\widetilde{\mu}_L$ and $\widetilde{\mu}_N$ to that of $\widetilde{\mu}$ that appears in $\|\mathsf{J}\|_{L^1(I)}$. Based on the definitions of E_k and D_k, we get the priori estimate

$$\|\phi_k\|_{\mathcal{X}^2_{\infty,2}(I_k\times\Omega)} + \|\theta_k\|_{\mathcal{X}^1_{\infty,2}(I_k\times\Omega)} + \|\boldsymbol{u}_k\|_{\boldsymbol{\mathcal{U}}^1_{\infty,2}(I_k\times\Omega)} + \|\mu_k\|_{L^2(I_k;W^{1,2}_0(\Omega))} \le \mathsf{R}. \tag{4.57}$$

• *$L^2(I; W^{1,2}_0(\Omega))$-estimates for $\Delta\phi_k$ and $\partial_t\phi_k$.* Testing the first equation in the approximate system (4.17) by $-\Delta\phi_k(t)$ leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\phi_k\|_{\boldsymbol{L}^2(\Omega)}^2 - b(\chi\boldsymbol{u}_k + \widetilde{\boldsymbol{u}}, \phi_k, \Delta\phi_k) - b(\boldsymbol{u}_k, \widetilde{\phi}, \Delta\phi_k)&\\ + m(\Delta\mu_k, \Delta\phi_k)_{L^2(\Omega)} = -\langle\widetilde{\sigma}, \Delta\phi_k\rangle_{W^{-1,2}(\Omega),W^{1,2}_0(\Omega)}.&\end{aligned} \tag{4.58}$$

For this equation, we have the following estimates:

$$|b(\boldsymbol{u}_k, \widetilde{\phi}, \Delta\phi_k)| \le \delta\|\nabla\Delta\phi_k\|_{\boldsymbol{L}^2(\Omega)}^2 + c_\delta\|\widetilde{\phi}\|_{W^{1,4}_0(\Omega)}^2\|\boldsymbol{u}_k\|_{\boldsymbol{L}^2(\Omega)}^2 \tag{4.59}$$

$$|b(\chi \boldsymbol{u}_k + \widetilde{\boldsymbol{u}}, \phi_k, \Delta\phi_k)| \le \delta\|\nabla\Delta\phi_k\|^2_{\boldsymbol{L}^2(\Omega)} + c_\delta\{\|\boldsymbol{u}_k\|^2_{\boldsymbol{L}^4(\Omega)} + \|\widetilde{\boldsymbol{u}}\|^2_{\boldsymbol{L}^4(\Omega)}\}\|\phi_k\|^2_{W_0^{1,2}(\Omega)} \tag{4.60}$$

$$|\langle\widetilde{\sigma}, \Delta\phi_k\rangle_{W^{-1,2}(\Omega), W_0^{1,2}(\Omega)}| \le \delta\|\nabla\Delta\phi_k\|^2_{\boldsymbol{L}^2(\Omega)} + c_\delta\|\widetilde{\sigma}\|^2_{W^{-1,2}(\Omega)}. \tag{4.61}$$

From the equation for μ_k in (4.17), we obtain

$$\begin{aligned} m(\Delta\mu_k, \Delta\phi_k)_{L^2(\Omega)} &= \frac{m\tau}{2}\frac{d}{dt}\|\Delta\phi_k\|^2_{L^2(\Omega)} - m(\nabla(l_c\theta_k + \widetilde{\lambda}), \nabla\Delta\varphi_k)_{\boldsymbol{L}^2(\Omega)} \\ &+ m\epsilon\|\nabla\Delta\phi_k\|^2_{\boldsymbol{L}^2(\Omega)} - m(\chi\nabla(F(\widetilde{\phi} + \phi_k) - F(\widetilde{\phi})) + \nabla(G(\widetilde{\phi})\phi_k), \nabla\Delta\phi_k)_{\boldsymbol{L}^2(\Omega)}. \end{aligned} \tag{4.62}$$

By Young's inequality, one can estimate the second and fourth terms on the right-hand side of (4.62) according to

$$\begin{aligned} &|m(\nabla(l_c\theta_k + \widetilde{\lambda}), \nabla\Delta\varphi_k)_{\boldsymbol{L}^2(\Omega)}| \\ &\qquad \le \delta\|\nabla\Delta\phi_k\|^2_{\boldsymbol{L}^2(\Omega)} + c_\delta\{\|\nabla\theta_k\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\widetilde{\lambda}\|^2_{\boldsymbol{L}^2(\Omega)}\} \end{aligned} \tag{4.63}$$

$$\begin{aligned} &|m(\chi(\nabla F(\widetilde{\phi} + \phi_k) - \nabla F(\widetilde{\phi})) + \nabla(G(\widetilde{\phi})\phi_k), \nabla\Delta\phi_k)_{\boldsymbol{L}^2(\Omega)}| \\ &\qquad \le \delta\|\nabla\Delta\phi_k\|^2_{\boldsymbol{L}^2(\Omega)} + c_\delta\{\|\nabla F(\widetilde{\phi} + \phi_k) - \nabla F(\widetilde{\phi})\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla(G(\widetilde{\phi})\phi_k)\|^2_{\boldsymbol{L}^2(\Omega)}\}. \end{aligned} \tag{4.64}$$

Let us set $J_5 := \|\nabla F(\widetilde{\phi} + \phi_k) - \nabla F(\widetilde{\phi})\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla(G(\widetilde{\phi})\phi_k)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\theta_k\|^2_{\boldsymbol{L}^2(\Omega)}$ and $J_6 := \|\widetilde{\phi}\|^2_{W_0^{1,4}(\Omega)} + \|\boldsymbol{u}_k\|^2_{\boldsymbol{L}^4(\Omega)} + \|\widetilde{\boldsymbol{u}}\|^2_{\boldsymbol{L}^4(\Omega)} + 1$. Note that $J_6 \in L^1(I)$, and by invoking Lemma 4.4 for the first two terms in J_5, we obtain $J_5 \in L^1(I)$. In fact, we have

$$\|J_5\|_{L^1(I_k)} \le \mathscr{C}(\|\phi_k\|_{\mathcal{X}^2_{\infty,2}(I_k\times\Omega)} + \|\theta_k\|_{\mathcal{X}^1_{\infty,2}(I_k\times\Omega)}) \tag{4.65}$$

$$\|J_6\|_{L^1(I_k)} \le \mathscr{C}(\|\widetilde{\phi}\|_{\mathcal{Z}^2_{q,r}(Q)+\mathcal{X}^2_{\infty,2}(Q)} + \|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{\mathcal{V}}^0_{p,r}(Q)+\boldsymbol{\mathcal{U}}^1_{\infty,2}(Q)} + \|\boldsymbol{u}_k\|_{\boldsymbol{\mathcal{U}}^1_{\infty,2}(I_k\times\Omega)} + 1). \tag{4.66}$$

Here, we used $\mathcal{X}^1_{\infty,2}(I_k\times\Omega) \hookrightarrow L^2(I_k; W_0^{1,2}(\Omega))$ and $\boldsymbol{\mathcal{U}}^1_{\infty,2}(I_k\times\Omega) \hookrightarrow L^2(I; \boldsymbol{L}^4(\Omega))$. Applying the estimates (4.59)–(4.64) in (4.58), we obtain after integrating by parts

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\{\|\nabla\phi_k\|^2_{\boldsymbol{L}^2(\Omega)} + m\tau\|\Delta\phi_k\|^2_{L^2(\Omega)}\} + (m\epsilon - 5\delta)\|\nabla\Delta\phi_k\|^2_{\boldsymbol{L}^2(\Omega)} \\ &\qquad \le c_\delta\{J_5 + \|\widetilde{\sigma}\|^2_{W^{-1,2}(\Omega)} + \|\widetilde{\lambda}\|^2_{W_0^{1,2}(\Omega)}\} + c_\delta J_6\{\|\boldsymbol{u}_k\|^2_{\boldsymbol{L}^2(\Omega)} + \|\phi_k\|^2_{W_0^{1,2}(\Omega)}\}. \end{aligned} \tag{4.67}$$

Taking the gradient of the equation for μ_k leads to the following estimate:

$$\|\nabla\partial_t\phi_k\|^2_{\boldsymbol{L}^2(\Omega)} \le c_\delta\{J_5 + \|\nabla\mu_k\|^2_{\boldsymbol{L}^2(\Omega)} + \|\widetilde{\lambda}\|^2_{W_0^{1,2}(\Omega)}\}. \tag{4.68}$$

Applying Gronwall's Lemma in (4.67), taking the sum of the resulting estimate with (4.68), and then using (4.65), (4.66) and (4.57), one would get

$$\|\nabla\Delta\phi_k\|_{L^2(I_k;\boldsymbol{L}^2(\Omega))} + \|\nabla\partial_t\phi_k\|_{L^2(I_k;\boldsymbol{L}^2(\Omega))} \le \mathsf{R}. \tag{4.69}$$

Thanks to the estimates for the trilinear terms in Lemma 4.2, one can also bound the norm of the time derivatives $\partial_t\theta_k$ in $L^2(I_k; W^{-1,2}(\Omega))$ and $\partial_t\boldsymbol{u}_k$ in $L^2(I_k; \boldsymbol{X}_\sigma^{-1,2}(\Omega))$ according to

$$\|\partial_t\theta_k\|_{L^2(I_k;W^{-1,2}(\Omega))} + \|\partial_t\boldsymbol{u}_k\|_{L^2(I_k;\boldsymbol{X}_\sigma^{-1,2}(\Omega))} \le \mathsf{R}. \tag{4.70}$$

Inequalities (4.57), (4.69) and (4.70) lead to the a priori estimate

$$\|\phi_k\|_{\mathcal{Z}^3_{2,2}(I_k\times\Omega)} + \|\theta_k\|_{\mathcal{Z}^1_{2,2}(I_k\times\Omega)} + \|\boldsymbol{u}_k\|_{\boldsymbol{\mathcal{V}}^1_{2,2}(I_k\times\Omega)} + \|\mu_k\|_{L^2(I_k;W^{1,2}_0(\Omega))} \le \mathsf{R}. \quad (4.71)$$

This uniform bound implies that the approximate system (4.17) has a unique solution over the whole interval I and we can replace the time interval I_k in (4.71) by I.

Step 3. *Passage to limit.* From the uniform a priori bound (4.71) with I_k replaced by I, we deduce the existence of $(\phi_N, \theta_N, \boldsymbol{u}_N, \mu_N)$ satisfying (4.15) and such that for appropriate subsequences (using the same index k for simplicity), the following weak and weak* convergence hold:

$$\begin{aligned}
&\phi_k \overset{*}{\rightharpoonup} \phi_N \text{ in } L^\infty(I;X^{2,2}(\Omega)), && \boldsymbol{u}_k \overset{*}{\rightharpoonup} \boldsymbol{u}_N \text{ in } L^\infty(I;\boldsymbol{L}^2_\sigma(\Omega)),\\
&\theta_k \overset{*}{\rightharpoonup} \theta_N \text{ in } L^\infty(I;L^2(\Omega)), && \phi_k \rightharpoonup \phi_N \text{ in } L^2(I;X^{3,2}(\Omega)),\\
&\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}_N \text{ in } L^2(I;\boldsymbol{X}^{1,2}_\sigma(\Omega)), && \theta_k \rightharpoonup \theta_N \text{ in } L^2(I;W^{1,2}_0(\Omega)),\\
&\partial_t\phi_k \rightharpoonup \partial_t\phi_N \text{ in } L^2(I;W^{1,2}_0(\Omega)), && \partial_t\boldsymbol{u}_k \rightharpoonup \partial_t\boldsymbol{u}_N \text{ in } L^2(I;\boldsymbol{X}^{-1,2}_\sigma(\Omega)),\\
&\partial_t\theta_k \rightharpoonup \partial_t\theta_N \text{ in } L^2(I;W^{-1,2}(\Omega)), && \mu_k \rightharpoonup \mu_N \text{ in } L^2(I;W^{1,2}_0(\Omega)).
\end{aligned}$$

In addition to these, we have the strong convergence $\phi_k \to \phi_N$ in $L^2(I;X^{2,2}(\Omega))$, $\theta_k \to \theta_N$ in $L^2(I;L^2(\Omega))$, and $\boldsymbol{u}_k \to \boldsymbol{u}_N$ in $L^2(I;\boldsymbol{L}^2_\sigma(\Omega))$ by the Aubin–Lions–Simon Lemma [59]. The a priori estimate (4.16) follows by taking the limit inferior to (4.71) and using the lower semicontinuity of the norms with respect to the underlying weak topologies.

It is now standard to pass to the limit in the variational formulation of the approximate system and obtain a weak solution to (4.14). We outline this process for the sake of the reader. The only crucial parts are the passage to the limit for the nonlinear terms.

For each $\boldsymbol{\rho} \in L^\infty(I;\boldsymbol{W}^{1,2}_0(\Omega))$, we have $\phi_k\boldsymbol{\rho} \to \phi_N\boldsymbol{\rho}$ in $L^2(I;\boldsymbol{L}^2(\Omega))$ due to the estimate

$$\|\phi_k\boldsymbol{\rho} - \phi_N\boldsymbol{\rho}\|_{L^2(I;\boldsymbol{L}^2(\Omega))} \le c\|\phi_k - \phi_N\|_{L^2(I;X^{2,2}(\Omega))}\|\boldsymbol{\rho}\|_{L^\infty(I;\boldsymbol{W}^{1,2}_0(\Omega))}.$$

Together with $\nabla\mu_k \rightharpoonup \nabla\mu_N$ in $L^2(I;\boldsymbol{L}^2(\Omega))$, one obtains

$$\begin{aligned}
&\langle \boldsymbol{S}(\mu_k,\phi_k) - \boldsymbol{S}(\mu_N,\phi_N), \boldsymbol{\rho}\rangle_{L^2(I;\boldsymbol{W}^{-1,2}(\Omega)),L^2(\boldsymbol{W}^{1,2}_0(\Omega))}\\
&\qquad = \int_0^T \{(\nabla\mu_N, \phi_N\boldsymbol{\rho})_{\boldsymbol{L}^2(\Omega)} - (\nabla\mu_k, \phi_k\boldsymbol{\rho})_{\boldsymbol{L}^2(\Omega)}\}\,\mathrm{d}t \to 0.
\end{aligned}$$

Using the density of $L^\infty(I;\boldsymbol{W}^{1,2}_0(\Omega))$ in $L^2(I;\boldsymbol{W}^{1,2}_0(\Omega))$ and applying the boundedness of the sequence $\{\boldsymbol{S}(\mu_k,\phi_k)\}_{k=1}^\infty$ in $L^2(I;\boldsymbol{W}^{-1,2}(\Omega))$, we deduce that

$$\boldsymbol{S}(\mu_k,\phi_k) \rightharpoonup \boldsymbol{S}(\mu_N,\phi_N) \text{ in } L^2(I;\boldsymbol{W}^{-1,2}(\Omega)).$$

Similarly, $\boldsymbol{S}(\theta_k,\phi_k) \rightharpoonup \boldsymbol{S}(\theta_N,\phi_N)$ in $L^2(I;\boldsymbol{W}^{-1,2}(\Omega))$. We can adapt the same idea to prove that $\boldsymbol{B}(\boldsymbol{u}_k,\boldsymbol{u}_k) \rightharpoonup \boldsymbol{B}(\boldsymbol{u}_N,\boldsymbol{u}_N)$ in $L^2(I;\boldsymbol{W}^{-1,2}(\Omega))$, $C(\boldsymbol{u}_k,\theta_k) \to C(\boldsymbol{u}_N,\theta_N)$ in $L^2(I;W^{-1,2}(\Omega))$, and $C(\boldsymbol{u}_k,\phi_k) \to C(\boldsymbol{u}_N,\phi_N)$ in $L^2(I;W^{-1,2}(\Omega))$. Let us note that a weak convergence in $L^2(I;\boldsymbol{W}^{-1,2}(\Omega))$ implies a weak convergence in $L^2(I;\boldsymbol{X}^{-1,2}_\sigma(\Omega))$ since the former space is continuously embedded to the latter space.

Finally, passing to the weak limit in $L^2(I; L^2(\Omega))$ to the second equation of the approximate system (4.17), we obtain the second equation in (4.14) since $\chi(F(\widetilde{\phi}+\phi_k) - F(\widetilde{\phi})) + G(\widetilde{\phi})\phi_k \to \chi(F(\widetilde{\phi}+\phi_N) - F(\widetilde{\phi})) + G(\widetilde{\phi})\phi_N$ in $L^2(I; L^2(\Omega))$.

From the above discussion, together with the weak convergence for the linear terms, we conclude that $(\phi_N, \theta_N, \boldsymbol{u}_N, \mu_N)$ is a weak solution to (4.14). As usual, the existence of a unique pressure $\mathfrak{p}_N \in W^{-1,2}(I; \widehat{L}^2(\Omega))$ as well as the required stability estimate follows from de Rham's Theorem, see Proposition 7.1 with $k = 1$ and $p = 2$. The details are similar to that in the proof of Theorem 3.2.

STEP 4. *Uniqueness.* Let $(\phi_N^j, \theta_N^j, \boldsymbol{u}_N^j, \mu_N^j, \mathfrak{p}_N^j)$ for $j = 1, 2$ be two weak solutions of (4.14), and denote their difference by

$$(\phi_N, \theta_N, \boldsymbol{u}_N, \mu_N, \mathfrak{p}_N) = (\phi_N^1, \theta_N^1, \boldsymbol{u}_N^1, \mu_N^1, \mathfrak{p}_N^1) - (\phi_N^2, \theta_N^2, \boldsymbol{u}_N^2, \mu_N^2, \mathfrak{p}_N^2).$$

Then, this difference is a weak solution to the following nonlinear system:

$$\left[\begin{array}{ll}
\partial_t\phi_N + \mathrm{div}\,((\chi\phi_N^1 + \widetilde{\phi})\boldsymbol{u}_N) + \mathrm{div}\,(\phi_N(\chi\boldsymbol{u}_N^2 + \widetilde{\boldsymbol{u}})) - m\Delta\mu_N = 0 & \text{in } Q,\\
\mu_N = \tau\partial_t\phi_N - \epsilon\Delta\phi_N + [\chi G_0(\widetilde{\phi}, \phi_N^1, \phi_N^2) + G(\widetilde{\phi})]\phi_N + l_{\mathrm{c}}\theta_N & \text{in } Q,\\
\partial_t\theta_N - l_{\mathrm{h}}\partial_t\phi_N + \mathrm{div}\,((\chi\theta_N^1 - \chi l_{\mathrm{h}}\phi_N^1 + \widetilde{\theta} - l_{\mathrm{h}}\widetilde{\phi})\boldsymbol{u}_N) & \\
\qquad + \mathrm{div}\,((\theta_N - l_{\mathrm{h}}\phi_N)(\chi\boldsymbol{u}_N^2 + \widetilde{\boldsymbol{u}})) - \kappa\Delta\theta_N = \alpha\boldsymbol{g}\cdot\boldsymbol{u}_N & \text{in } Q,\\
\partial_t\boldsymbol{u}_N + \mathrm{div}\,(\boldsymbol{u}_N \otimes (\chi\boldsymbol{u}_N^1 + \widetilde{\boldsymbol{u}})) + \mathrm{div}\,((\chi\boldsymbol{u}_N^2 + \widetilde{\boldsymbol{u}}) \otimes \boldsymbol{u}_N) & \\
\qquad - \nu\Delta\boldsymbol{u}_N + \nabla\mathfrak{p}_N = \mathcal{K}(\mu_N - l_{\mathrm{c}}\theta_N)\nabla(\chi\phi_N^1 + \widetilde{\phi}) & \\
\qquad + \mathcal{K}(\chi\mu_N^2 - \chi l_{\mathrm{c}}\theta_N^2 + \widetilde{\mu} - l_{\mathrm{c}}\widetilde{\theta})\nabla\phi_N + (\alpha_1\phi_N + \alpha_2\theta_N)\boldsymbol{g} & \text{in } Q,\\
\mathrm{div}\,\boldsymbol{u}_N = 0 & \text{in } Q,\\
\phi_N = \Delta\phi_N = 0, \quad \theta_N = 0, \quad \boldsymbol{u}_N = \boldsymbol{0} & \text{on } \Sigma,\\
\phi_N(0) = 0, \quad \theta_N(0) = 0, \quad \boldsymbol{u}_N(0) = \boldsymbol{0} & \text{in } \Omega,
\end{array}\right. \tag{4.72}$$

where $G_0(\widetilde{\phi}, \phi_N^1, \phi_N^2)$ is a quadratic function in three variables.

Notice that (4.72) has the same form as that of (4.14) with $\chi = 0$ and $\alpha_0 = 0$ in the latter equation. The main difference though is that there are more frozen coefficients in the convection and surface tension terms, namely, the components of the two weak solutions. Nonetheless, these components belong to the function spaces that are required by the theorem for the frozen coefficients. Therefore, we can follow the derivation of the a priori estimates provided in STEP 2 and obtain (4.16) for the solution of (4.72). Since we have vanishing source functions and initial data as well as the absence of the term $\alpha_0\boldsymbol{g}$, we deduce that $(\phi_N, \theta_N, \boldsymbol{u}_N, \mu_N, \mathfrak{p}_N)$ must be the trivial solution to (4.72). Hence, the weak solution to (4.14) is unique. The proof of Theorem 4.9 is now complete. □

Remark 4.10. *Adapting the process in the uniqueness proof, it follows that the map*

$$((\widetilde{\sigma}, \widetilde{h}, \widetilde{\boldsymbol{f}}, \widetilde{\lambda}), (\phi_{0N}, \theta_{0N}, \boldsymbol{u}_{0N})) \mapsto (\phi_N, \theta_N, \boldsymbol{u}_N, \mu_N, \mathfrak{p}_N)$$

is locally Lipschitz continuous with respect to the function spaces for the sources, initial data, and weak solution as stated by Theorem 4.9. In fact, this map is of class C^∞, and this will be shown in the next section.

4.4. Existence and Uniqueness of Weak and Very Weak Solutions. We now combine the results for the linear and nonlinear parts to establish the well-posedness of (1.1). In contrast with the discussion with the linear system, we shall start on the very weak solutions, see Definition 4.7.

Theorem 4.11. *Suppose that* (4.13) *holds, the source functions satisfy* (4.11)*, and the initial data satisfy* (4.12)*. Then the nonlinear system* (1.1) *admits a unique very weak solution*

$$\begin{aligned}(\phi,\theta,\boldsymbol{u},\mu) \in [\mathscr{Z}^2_{q,r}(Q)+\mathscr{Z}^3_{2,2}(Q)] \times [\mathscr{Z}^0_{s,r}(Q)+\mathscr{Z}^1_{2,2}(Q)] \\ \times [\boldsymbol{\mathcal{V}}^0_{p,r}(Q)+\boldsymbol{\mathcal{V}}^1_{2,2}(Q)] \times [L^r(I;L^q(\Omega))+L^2(I;W^{1,2}_0(\Omega))]\end{aligned} \tag{4.73}$$

with an associated pressure $\mathfrak{p} \in W^{-1,r}(I;\widehat{W}^{-1,p}(\Omega)) + W^{-1,2}(I;\widehat{L}^2(\Omega))$ *in the sense of Definition* 4.7*. Furthermore, the solution depends continuously on the initial data and source functions, that is, there is a continuous and monotone increasing function* $\mathscr{C}:[0,\infty)\to[0,\infty)$ *such that* $\mathscr{C}(0)=0$ *and*

$$\begin{aligned}&\|\phi\|_{\mathscr{Z}^2_{q,r}(Q)+\mathscr{Z}^3_{2,2}(Q)} + \|\theta\|_{\mathscr{Z}^0_{s,r}(Q)+\mathscr{Z}^1_{2,2}(Q)} + \|\boldsymbol{u}\|_{\boldsymbol{\mathcal{V}}^0_{p,r}(Q)+\boldsymbol{\mathcal{V}}^1_{2,2}(Q)} \\ &\quad + \|\mu\|_{L^r(I;L^q(\Omega))+L^2(I;W^{1,2}_0(\Omega))} + \|\mathfrak{p}\|_{W^{-1,r}(I;\widehat{W}^{-1,p}(\Omega))+W^{-1,2}(I;\widehat{L}^2(\Omega))} \\ &\quad \le \mathscr{C}(\|\phi_0\|_{Z^2_{q,r}(\Omega)+X^{2,2}(\Omega)} + \|\theta_0\|_{Z^0_{s,r}(\Omega)+L^2(\Omega)} + \|\boldsymbol{u}_0\|_{\boldsymbol{V}^0_{p,r}(\Omega)+\boldsymbol{L}^2_\sigma(\Omega)} \\ &\quad + |\alpha_0\boldsymbol{g}| + \|\sigma\|_{L^r(I;X^{-2,q}(\Omega))+L^2(I;W^{-1,2}(\Omega))} + \|\lambda\|_{L^r(I;L^q(\Omega))+L^2(I;W^{1,2}_0(\Omega))} \\ &\quad + \|h\|_{L^r(I;X^{-2,s}(\Omega))+L^2(I;W^{-1,2}(\Omega))} + \|\boldsymbol{f}\|_{L^r(I;\boldsymbol{X}^{-2,p}(\Omega))+L^2(I;\boldsymbol{W}^{-1,2}(\Omega))}).\end{aligned} \tag{4.74}$$

Finally, the map $((\sigma,h,\boldsymbol{f},\lambda),(\phi_0,\theta_0,\boldsymbol{u}_0)) \mapsto (\phi,\theta,\boldsymbol{u},\mu,\mathfrak{p})$ *is locally Lipschitz continuous with respect to the above function spaces for the sources, initial data, and very weak solutions.*

Proof. Let us express the source functions and initial data according to (2.2) and (2.1), respectively. First, recall from Theorem 3.20 that the linear system (2.4) admits a very weak solution. Second, by taking $\chi=1$, $G\equiv 0$, $\widetilde{\phi}=\phi_L \in \mathscr{Z}^2_{q,r}(Q)$, $\widetilde{\theta}=\theta_L \in Z^0_{s,r}(Q)$, $\widetilde{\boldsymbol{u}}=\boldsymbol{u}_L \in \boldsymbol{\mathcal{V}}_{p,r}(Q)$, $\widetilde{\mu}=\mu_L \in L^r(I;W^{1,q}_0(\Omega))$,

$$\widetilde{\sigma} = \sigma_N - \operatorname{div}(\phi_L\boldsymbol{u}_L) \in L^2(I;W^{-1,2}(\Omega)) \tag{4.75}$$

$$\widetilde{h} = h_N - \operatorname{div}((\theta_L - l_{\mathrm{h}}\phi_L)\boldsymbol{u}_L) \in L^2(I;\boldsymbol{W}^{-1,2}(\Omega)) \tag{4.76}$$

$$\widetilde{\boldsymbol{f}} = \boldsymbol{f}_N - \operatorname{div}(\boldsymbol{u}_L\otimes\boldsymbol{u}_L) \in L^2(I;W^{-1,2}(\Omega)) \tag{4.77}$$

$$\widetilde{\lambda} = \lambda_N + \beta_1\phi_L + F(\phi_L) \in L^2(I;W^{1,2}_0(\Omega)) \tag{4.78}$$

in (4.14), we obtain from Theorem 4.9 that (2.5) admits a weak solution. Note that (4.75)–(4.77) follows from Lemma 4.2 while (4.78) is a consequence of Corollary 4.5. Then, the sum (2.3) constitutes a very weak solution to (1.1). Indeed, we obtain the variational equations in Definition 4.7 by simply taking the sum of the variational equations from the very weak formulation of (2.4) and the weak formulation of (2.5). Here, one has to take the intersection of the space of test functions for each system, which is precisely the one being prescribed by Definition 4.7. Furthermore, we have an associated pressure $\mathfrak{p}=\mathfrak{p}_L+\mathfrak{p}_N \in W^{-1,r}(I;\widehat{W}^{-1,p}(\Omega)) + W^{-1,2}(I;\widehat{L}^2(\Omega))$

The constructed very weak solution, along with the associated pressure, satisfies the stability estimate stated by theorem due to Theorem 3.20 and Theorem 4.9, and after taking the infima over all possible sum representations for the source functions and initial data.

Let us prove that the very weak solution is unique. As usual, let $(\phi^k, \theta^k, \boldsymbol{u}^k, \mu^k, \mathfrak{p}^k)$ for $k = 1, 2$ be very weak solutions to (1.1). Then their difference

$$(\phi, \theta, \boldsymbol{u}, \mu, \mathfrak{p}) = (\phi^1, \theta^1, \boldsymbol{u}^1, \mu^1, \mathfrak{p}^1) - (\phi^2, \theta^2, \boldsymbol{u}^2, \mu^2, \mathfrak{p}^2)$$

is a very weak solution to the following system:

$$\left[\begin{array}{ll} \partial_t\phi - m\Delta\mu = -\operatorname{div}(\phi\boldsymbol{u}^1) - \operatorname{div}(\phi^2\boldsymbol{u}) & \text{in } Q, \\ \mu = \tau\partial_t\phi - \epsilon\Delta\phi + l_{\mathrm{c}}\theta + G_1(\phi^1,\phi^2)\phi & \text{in } Q, \\ \partial_t\theta - l_{\mathrm{h}}\partial_t\phi - \kappa\Delta\theta - \alpha\boldsymbol{g}\cdot\boldsymbol{u} = -\operatorname{div}((\theta - l_{\mathrm{h}}\phi)\boldsymbol{u}^1) - \operatorname{div}((\theta^2 - l_{\mathrm{h}}\phi^2)\boldsymbol{u}) & \text{in } Q, \\ \partial_t\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla\mathfrak{p} - (\alpha_1\phi + \alpha_2\theta)\boldsymbol{g} = -\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}^1) - \operatorname{div}(\boldsymbol{u}^2\otimes\boldsymbol{u}) & \\ \qquad + \mathcal{K}(\mu^1 - l_{\mathrm{c}}\theta^1)\nabla\phi + \mathcal{K}(\mu - l_{\mathrm{c}}\theta)\nabla\phi^2 & \text{in } Q, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } Q, \\ \phi = \Delta\phi = 0, \quad \theta = 0, \quad \boldsymbol{u} = \boldsymbol{0} & \text{on } \Sigma, \\ \phi(0) = 0, \quad \theta(0) = 0, \quad \boldsymbol{u}(0) = \boldsymbol{0} & \text{in } \Omega, \end{array}\right. \tag{4.79}$$

for some quadratic function $G_1(\phi^1, \phi^2)$.

According to the following list of continuous embeddings: $X^{3,2}(\Omega) \hookrightarrow X^{2,q}(\Omega)$, $W_0^{1,2}(\Omega) \hookrightarrow L^s(\Omega) \hookrightarrow L^q(\Omega)$, $W^{-1,2}(\Omega) \hookrightarrow X^{-2,s}(\Omega)$, $\boldsymbol{X}_\sigma^{1,2}(\Omega) \hookrightarrow \boldsymbol{L}_\sigma^p(\Omega)$, $\boldsymbol{W}^{-1,2}(\Omega) \hookrightarrow \boldsymbol{X}_\sigma^{-2,p}(\Omega)$, and $\widehat{L}^2(\Omega) \hookrightarrow \widehat{W}^{-1,p}(\Omega)$, we deduce that

$$\begin{aligned} &\mathcal{Z}_{2,2}^3(Q) \times \mathcal{Z}_{2,2}^1(Q) \times \boldsymbol{\mathcal{V}}_{2,2}^1(Q) \times L^2(I; W_0^{1,2}(\Omega)) \times W^{-1,2}(I; \widehat{L}^2(\Omega)) \\ &\hookrightarrow \mathcal{Z}_{q,2}^2(Q) \times \mathcal{Z}_{s,2}^0(Q) \times \boldsymbol{\mathcal{V}}_{p,2}^0(Q) \times L^2(I; L^q(\Omega)) \times W^{-1,2}(I; \widehat{W}^{-1,p}(\Omega)). \end{aligned} \tag{4.80}$$

From this embedding, together with $r \geq 4$, we get

$$(\phi, \theta, \boldsymbol{u}, \mu, \mathfrak{p}) \in \mathcal{Z}_{q,2}^2(Q) \times \mathcal{Z}_{s,2}^0(Q) \times \boldsymbol{\mathcal{V}}_{p,2}^0(Q) \times L^2(I; L^q(\Omega)) \times W^{-1,2}(I; \widehat{W}^{-1,p}(\Omega)).$$

Observe that $\operatorname{div}(\phi\boldsymbol{u}^1)$, $\operatorname{div}(\phi^2\boldsymbol{u})$, $\operatorname{div}((\theta - l_{\mathrm{h}}\phi)\boldsymbol{u}^1)$, $\operatorname{div}((\theta^2 - l_{\mathrm{h}}\phi^2)\boldsymbol{u}) \in L^2(I; W^{-1,2}(\Omega))$ and $\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}^1)$, $\operatorname{div}(\boldsymbol{u}^2\otimes\boldsymbol{u})$, $(\mu^1 - l_{\mathrm{c}}\theta^1)\nabla\phi$, $(\mu - l_{\mathrm{c}}\theta)\nabla\phi^2 \in L^2(I; \boldsymbol{W}^{-1,2}(\Omega))$ from Lemma 4.2. Moreover, $G_1(\phi^1, \phi^2)\phi \in L^2(I; W_0^{1,2}(\Omega))$ according to Lemma 4.4. From these, we obtain from Theorem 3.18 a weak solution

$$(\widetilde{\phi}, \widetilde{\theta}, \widetilde{\boldsymbol{u}}, \widetilde{\mu}, \widetilde{\mathfrak{p}}) \in \mathcal{Z}_{2,2}^3(Q) \times \mathcal{Z}_{2,2}^1(Q) \times \boldsymbol{\mathcal{V}}_{2,2}^1(Q) \times L^2(I; W_0^{1,2}(\Omega)) \times W^{-1,2}(I; \widehat{L}^2(\Omega))$$

to the system (4.79) for the difference. In view of the uniqueness of very weak solutions in Theorem 3.20 and the embedding (4.80), we have $(\widetilde{\phi}, \widetilde{\theta}, \widetilde{\boldsymbol{u}}, \widetilde{\mu}, \widetilde{\mathfrak{p}}) = (\phi, \theta, \boldsymbol{u}, \mu, \mathfrak{p})$.

Since (4.79) is in the form of (4.14) and only differs on the frozen coefficients, if we adapt the proof of Theorem 4.9, then we obtain that $(\phi, \theta, \boldsymbol{u}, \mu, \mathfrak{p})$ must be the trivial solution. Therefore, the very weak solution to (1.1) is unique. The local Lipschitz continuity of the solution operator is a direct consequence of the same property for the solution operators of the linear part (2.4) and the nonlinear part (2.5). □

The case of weak solutions can be easily shown.

Theorem 4.12. *Let (4.7), (4.8) and (4.9) be satisfied. Then the system (1.1) has unique weak solution*

$$(\phi, \theta, \boldsymbol{u}, \mu) \in [\mathcal{Z}^3_{q,r}(Q) + \mathcal{Z}^3_{2,2}(Q)] \times [\mathcal{Z}^1_{s,r}(Q) + \mathcal{Z}^1_{2,2}(Q)] \\ \times [\boldsymbol{\mathcal{V}}^1_{p,r}(Q) + \boldsymbol{\mathcal{V}}^1_{2,2}(Q)] \times [L^r(I; W^{1,q}_0(\Omega)) + L^2(I; W^{1,2}_0(\Omega))] \quad (4.81)$$

with an associated pressure $\mathfrak{p} \in W^{-1,r}(I; \widehat{L}^p(\Omega)) + W^{-1,2}(I; \widehat{L}^2(\Omega))$ in the sense of Definition 4.6. Moreover, as in Theorem 4.11, the solution depends continuously with respect to the source functions and the initial data, and the map $((\sigma, h, \boldsymbol{f}, \lambda), (\phi_0, \theta_0, \boldsymbol{u}_0)) \mapsto (\phi, \theta, \boldsymbol{u}, \mu, \mathfrak{p})$ is locally Lipschitz continuous.

Proof. Follow the proof of the preceding theorem, but now using Theorem 3.18 in place of Theorem 3.20. Moreover, the uniqueness of the weak solution follows from the fact that any weak solution is also a very weak solution, see Remark 4.8, and that the very weak solutions are unique according to Theorem 4.11. □

Corollary 4.13. *The conclusions of Theorem 4.12 are also valid in the case where $q, s, p, r \geq 2$ and $s \geq q$.*

Proof. The assumptions on q, s, p and r imply that $\mathcal{Z}^3_{q,r}(Q) \hookrightarrow \mathcal{Z}^3_{2,2}(Q) \hookrightarrow \mathcal{X}^2_{\infty,2}(Q)$, $\mathcal{Z}^1_{s,r}(Q) \hookrightarrow \mathcal{Z}^1_{2,2}(Q) \hookrightarrow \mathcal{X}^1_{\infty,2}(Q)$, $\boldsymbol{\mathcal{V}}^1_{p,r}(Q) \hookrightarrow \boldsymbol{\mathcal{V}}^1_{2,2}(Q) \hookrightarrow \boldsymbol{\mathcal{U}}^2_{\infty,2}(Q)$ and $L^r(I; W^{1,q}_0(\Omega)) \hookrightarrow L^2(I; W^{1,2}_0(\Omega))$. These mean that the components of the weak solution for the linearized system (2.4) satisfy the regularity requirements for the frozen coefficients in Theorem 4.9. We can then proceed as before to obtain the conclusions of Theorem 4.12. □

4.5. Sources with Values in Duals of Hölder Spaces. Let $C_0(\bar{\Omega})$ be the Banach space of all continuous functions on the closure of Ω that vanish on Γ equipped with the supremum norm, $C^{k,\mathfrak{a}}(\bar{\Omega})$ be the Hölder space, where k is a nonnegative integer and $\mathfrak{a} \in (0,1)$, and set $C^{k,\mathfrak{a}}_0(\bar{\Omega}) := C^{k,\mathfrak{a}}(\bar{\Omega}) \cap C_0(\bar{\Omega})$ and $\boldsymbol{C}^{k,\mathfrak{a}}_0(\bar{\Omega}) := C^{k,\mathfrak{a}}_0(\bar{\Omega}) \times C^{k,\mathfrak{a}}_0(\bar{\Omega})$.

For $1 < r < \infty$, let $L^r_{\mathrm{w}}(I; C^{k,\mathfrak{a}}_0(\bar{\Omega})')$ be the Banach space of equivalence classes of $C^{k,\mathfrak{a}}_0(\bar{\Omega})$-weakly measurable functions from I into $C^{k,\mathfrak{a}}_0(\bar{\Omega})'$ equipped with the norm

$$\|\sigma\|_{L^r_{\mathrm{w}}(I; C^{k,\mathfrak{a}}_0(\bar{\Omega})')} := \left(\inf_{\rho \approx \sigma} \inf_{\varphi \geq \|\rho\|_{C^{k,\mathfrak{a}}_0(\bar{\Omega})'}} \int_0^T \varphi(t)^r \, \mathrm{d}t \right)^{1/r}$$

where the inner infimum is taken over all Lebesgue measurable functions $\varphi : I \to \mathbb{R}$ and $\rho \approx \sigma$ in the outer infimum means that for each $\phi \in C^{k,\mathfrak{a}}_0(\bar{\Omega})$ there exists $I_\phi \subset I$ with Lebesgue measure zero and $\langle \rho, \phi \rangle_{C^{k,\mathfrak{a}}_0(\bar{\Omega})', C^{k,\mathfrak{a}}_0(\bar{\Omega})} = \langle \sigma, \phi \rangle_{C^{k,\mathfrak{a}}_0(\bar{\Omega})', C^{k,\mathfrak{a}}_0(\bar{\Omega})}$ in $I \setminus I_\phi$. Then we have $L^r_{\mathrm{w}}(I; C^{k,\mathfrak{a}}_0(\bar{\Omega})') = L^{r'}(I; C^{k,\mathfrak{a}}_0(\bar{\Omega}))'$. In the same manner, $L^r_{\mathrm{w}}(I; \boldsymbol{C}^{k,\mathfrak{a}}_0(\bar{\Omega})') = L^{r'}(I; \boldsymbol{C}^{k,\mathfrak{a}}_0(\bar{\Omega}))'$. We refer the reader to [32, Section 12.9] or [65, Chapter 7] for the details and the proof of the duality identification.

Consider the framework of Theorem 4.12, but now we have the source functions

$$\sigma \in L^r_{\mathrm{w}}(I; C^{0,2/q-1}_0(\bar{\Omega})'), \quad h \in L^r_{\mathrm{w}}(I; C^{0,2/s-1}_0(\bar{\Omega})'), \quad \boldsymbol{f} \in L^r_{\mathrm{w}}(I; \boldsymbol{C}^{0,2/p-1}_0(\bar{\Omega})') \quad (4.82)$$

where the parameters q, s, p and r satisfy (4.7). By the Sobolev embedding theorem, see [28, Section 5.6.3] for instance, we have the continuous embedding $W_0^{1,q'}(\Omega) \hookrightarrow C_0^{0,1-2/q'}(\bar{\Omega}) = C_0^{0,2/q-1}(\bar{\Omega})$. Thus, one has $L^r_{\mathrm{w}}(I; C_0^{0,2/q-1}(\bar{\Omega})') \hookrightarrow L^r_{\mathrm{w}}(I; W^{-1,q}(\Omega)) = L^r(I; W^{-1,q}(\Omega))$ by duality and the equality is due to the fact that $W_0^{1,q'}(\Omega)$ is reflexive [32, Example 12.9.6]. In a similar way, $L^r_{\mathrm{w}}(I; C_0^{0,2/s-1}(\bar{\Omega})') \hookrightarrow L^r(I; W^{-1,s}(\Omega))$ and $L^r_{\mathrm{w}}(I; \boldsymbol{C}_0^{0,2/p-1}(\bar{\Omega})') \hookrightarrow L^r_{\mathrm{w}}(I; \boldsymbol{W}^{-1,p}(\Omega))$. With these, we deduce from Theorem 4.12 that (1.1) with source functions in (4.82) admits a unique weak solution satisfying (4.81).

Now, let us suppose that the assumptions of Theorem 4.11 hold, with q, s, p and r obeying (4.13), and in addition, we have $q > 2$ and source functions that satisfy

$$\sigma \in L^r_{\mathrm{w}}(I; C_0^{0,2/q}(\bar{\Omega})'), \quad h \in L^r_{\mathrm{w}}(I; C_0^{0,2/s}(\bar{\Omega})'), \quad \boldsymbol{f} \in L^r_{\mathrm{w}}(I; \boldsymbol{C}_0^{0,2/p}(\bar{\Omega})'). \tag{4.83}$$

Using the Sobolev embedding theorem once more, we have $X^{2,q'}(\Omega) \hookrightarrow C_0^{0,2-2/q'}(\bar{\Omega}) = C_0^{0,2/q}(\bar{\Omega})$. Hence, we get $L^r_{\mathrm{w}}(I; C_0^{0,2/q}(\bar{\Omega})') \hookrightarrow L^r(I; X^{-2,q}(\Omega))$. Likewise, we also have $L^r_{\mathrm{w}}(I; C_0^{0,2/s}(\bar{\Omega})') \hookrightarrow L^r(I; X^{-2,s}(\Omega))$ and $L^r_{\mathrm{w}}(I; \boldsymbol{C}_0^{0,2/p}(\bar{\Omega})') \hookrightarrow L^r(I; \boldsymbol{W}^{-2,p}(\Omega))$. Based on these embeddings, Theorem 4.11 implies that (1.1) with the source functions (4.83) has a unique very weak solution with the regularity (4.73). If we have $\frac{4}{3} < q < 2$, then the same conclusion holds when $\sigma \in L^r_{\mathrm{w}}(I; C_0^{1,2/q-1}(\bar{\Omega})')$ since $X^{2,q'}(\Omega) \hookrightarrow C_0^{1,1-2/q'}(\bar{\Omega}) = C_0^{1,2/q-1}(\bar{\Omega})$.

As $M(\Omega) := C_0(\bar{\Omega})' \hookrightarrow C_0^{0,\mathfrak{a}}(\bar{\Omega})'$ for any $\mathfrak{a} \in (0,1)$, the previous statements are also valid for source functions

$$\sigma \in L^r_{\mathrm{w}}(I; M(\Omega)), \quad h \in L^r_{\mathrm{w}}(I; M(\Omega)), \quad \boldsymbol{f} \in L^r_{\mathrm{w}}(I; \boldsymbol{M}(\Omega))$$

where as usual $\boldsymbol{M}(\Omega) = M(\Omega) \times M(\Omega)$. Recall that $M(\Omega)$ can be topologically identified with the Banach space of real and regular Borel measures in Ω equipped with the total variation norm. An optimal control problem for the nonlinear system (1.1) with controls taking values in the space of regular Borel measures will be considered in future work.

5. Differentiability of the Solution Operator

In this section, we prove that the operator mapping the source functions and initial data to the very weak or weak solution is of class C^∞. For convenience, we denote the space of source functions associated with the very weak and weak solutions by

$$\begin{aligned}
&\boldsymbol{\mathcal{F}}^{\mathrm{vw},r}_{q,s,p}(Q) \\
&:= [L^r(I; X^{-2,q}(\Omega)) + L^2(I; W^{-1,2}(\Omega))] \times [L^r(I; X^{-2,s}(\Omega)) + L^2(I; W^{-1,2}(\Omega))] \\
&\times [L^r(I; \boldsymbol{X}^{-2,p}(\Omega)) + L^2(I; \boldsymbol{W}^{-1,2}(\Omega))] \times [L^r(I; L^q(\Omega)) + L^2(I; W_0^{1,2}(\Omega))] \\
&\boldsymbol{\mathcal{F}}^{\mathrm{w},r}_{q,s,p}(Q) \\
&:= [L^r(I; W^{-1,q}(\Omega)) + L^2(I; W^{-1,2}(\Omega))] \times [L^r(I; W^{-1,s}(\Omega)) + L^2(I; W^{-1,2}(\Omega))] \\
&\times [L^r(I; \boldsymbol{W}^{-1,p}(\Omega)) + L^2(I; \boldsymbol{W}^{-1,2}(\Omega))] \times [L^r(I; W_0^{1,q}(\Omega)) + L^2(I; W_0^{1,2}(\Omega))].
\end{aligned}$$

For the function spaces of initial data, we introduce the notation

$$\boldsymbol{U}^{\mathrm{vw},r}_{q,s,p}(\Omega) := [Z^2_{q,r}(\Omega) + X^{2,2}(\Omega)] \times [Z^0_{s,r}(\Omega) + L^2(\Omega)] \times [\boldsymbol{V}^0_{p,r}(\Omega) + \boldsymbol{L}^2_\sigma(\Omega)]$$

$$\boldsymbol{U}^{\mathrm{w},r}_{q,s,p}(\Omega) := [Z^3_{q,r}(\Omega) + X^{2,2}(\Omega)] \times [Z^1_{s,r}(\Omega) + L^2(\Omega)] \times [\boldsymbol{V}^1_{p,r}(\Omega) + \boldsymbol{L}^2_\sigma(\Omega)].$$

The corresponding very weak and weak solution spaces are then denoted by

$$\begin{aligned}\boldsymbol{\mathcal{U}}^{\mathrm{vw},r}_{q,s,p}(Q) &:= [\mathcal{Z}^2_{q,r}(Q) + \mathcal{Z}^3_{2,2}(Q)] \times [\mathcal{Z}^0_{s,r}(Q) + \mathcal{Z}^1_{2,2}(Q)] \\ &\quad \times [\boldsymbol{\mathcal{V}}^0_{p,r}(Q) + \boldsymbol{\mathcal{V}}^1_{2,2}(Q)] \times [L^r(I; L^q(\Omega)) + L^2(I; W^{1,2}_0(\Omega))] \\ \boldsymbol{\mathcal{U}}^{\mathrm{w},r}_{q,s,p}(Q) &:= [\mathcal{Z}^3_{q,r}(Q) + \mathcal{Z}^3_{2,2}(Q)] \times [\mathcal{Z}^1_{s,r}(Q) + \mathcal{Z}^1_{2,2}(Q)] \\ &\quad \times [\boldsymbol{\mathcal{V}}^1_{p,r}(Q) + \boldsymbol{\mathcal{V}}^1_{2,2}(Q)] \times [L^r(I; W^{1,q}_0(\Omega)) + L^2(I; W^{1,2}_0(\Omega))].\end{aligned}$$

Let us start with the case of very weak solutions and later state the corresponding result for weak solutions. Consider the so-called *solution operator*

$$\mathfrak{S} : \boldsymbol{\mathcal{F}}^{\mathrm{vw},r}_{q,s,p}(Q) \times \boldsymbol{U}^{\mathrm{vw},r}_{q,s,p}(\Omega) \to \boldsymbol{\mathcal{U}}^{\mathrm{vw},r}_{q,s,p}(Q)$$

defined as follows: $\mathfrak{S}((\sigma, h, \boldsymbol{f}, \lambda), (\phi_0, \theta_0, \boldsymbol{u}_0)) = (\phi, \theta, \boldsymbol{u}, \mu)$ if and only if $(\phi, \theta, \boldsymbol{u}, \mu) \in \boldsymbol{\mathcal{U}}^{\mathrm{vw},r}_{q,s,p}(Q)$ is the very weak solution of (1.1) in the sense of Definition 4.7, having the source terms $(\sigma, h, \boldsymbol{f}, \lambda) \in \boldsymbol{\mathcal{F}}^{\mathrm{vw},r}_{q,s,p}(Q)$ and initial data $(\phi_0, \theta_0, \boldsymbol{u}_0) \in \boldsymbol{U}^{\mathrm{vw},r}_{q,s,p}(\Omega)$.

Theorem 5.1. *Under the condition* (4.13)*, we have*

$$\mathfrak{S} \in C^\infty(\boldsymbol{\mathcal{F}}^{\mathrm{vw},r}_{q,s,p}(Q) \times \boldsymbol{U}^{\mathrm{vw},r}_{q,s,p}(\Omega), \boldsymbol{\mathcal{U}}^{\mathrm{vw},r}_{q,s,p}(Q)).$$

Proof. We shall proceed with the implicit function theorem. Let $\boldsymbol{\mathcal{G}}^{\mathrm{vw},r}_{q,s,p}(Q)$ be as that of $\boldsymbol{\mathcal{F}}^{\mathrm{vw},r}_{q,s,p}(Q)$ but with the third function space replaced by $L^r(I; \boldsymbol{X}^{-2,p}_\sigma(\Omega)) + L^2(I; \boldsymbol{X}^{-1,2}_\sigma(\Omega))$. Also, let $\widetilde{\mathfrak{S}} : \boldsymbol{\mathcal{G}}^{\mathrm{vw},r}_{q,s,p}(Q) \to \boldsymbol{\mathcal{U}}^{\mathrm{vw},r}_{q,s,p}(Q)$ be the associated solution operator.

Consider the linear operators

$$\begin{aligned}&A : L^r(I; L^s(\Omega)) + L^2(I; X^{1,2}(\Omega)) \to L^r(I; X^{-2,s}(\Omega)) + L^2(I; W^{-1,2}(\Omega)) \\ &\qquad A\theta = A'_{s'}\theta_L + A_2\theta_N, \quad \theta = \theta_L + \theta_N,\ \theta_L \in L^r(I; L^s(\Omega)),\ \theta_N \in L^2(I; X^{1,2}(\Omega)) \\ &\boldsymbol{A} : L^r(I; \boldsymbol{L}^p_\sigma(\Omega)) + L^2(I; \boldsymbol{X}^{1,2}_\sigma(\Omega)) \to L^r(I; \boldsymbol{X}^{-2,p}_\sigma(\Omega)) + L^2(I; \boldsymbol{X}^{-1,2}_\sigma(\Omega)) \\ &\qquad \boldsymbol{A}\boldsymbol{u} = \boldsymbol{A}'_{p'}\boldsymbol{u}_L + \boldsymbol{A}_2\boldsymbol{u}_N, \quad \boldsymbol{u} = \boldsymbol{u}_L + \boldsymbol{u}_N,\ \boldsymbol{u}_L \in L^r(I; \boldsymbol{L}^p_\sigma(\Omega)),\ \boldsymbol{u}_N \in L^2(I; \boldsymbol{X}^{1,2}_\sigma(\Omega)).\end{aligned}$$

With abuse of notation, we shall also define the linear operator

$$\begin{aligned}&A : L^r(I; X^{2,q}(\Omega)) + L^2(I; X^{3,2}(\Omega)) \to L^r(I; L^q(\Omega)) + L^2(I; X^{1,2}(\Omega)) \\ &\qquad A\phi = A_q\phi_L + A_2\phi_N, \qquad \phi = \phi_L + \phi_N,\ \phi_L \in L^r(I; X^{2,q}(\Omega)),\ \phi_N \in L^2(I; X^{3,2}(\Omega)).\end{aligned}$$

These operators are well-defined, that is, they are independent with respect to the representation of the arguments as sums, and moreover, they are continuous.

We introduce the nonlinear operator

$$\mathfrak{N} : \boldsymbol{\mathcal{U}}^{\mathrm{vw},r}_{q,s,p}(Q) \times \boldsymbol{\mathcal{G}}^{\mathrm{vw},r}_{q,s,p}(Q) \times \boldsymbol{U}^{\mathrm{vw},r}_{q,s,p}(\Omega) \to \boldsymbol{\mathcal{G}}^{\mathrm{vw},r}_{q,s,p}(Q) \times \boldsymbol{U}^{\mathrm{vw},r}_{q,s,p}(\Omega)$$

with components $\mathfrak{N}(\mathfrak{u}, \mathfrak{f}, \boldsymbol{\mathfrak{u}}_0) = (\mathfrak{N}_1(\mathfrak{u}, \mathfrak{f}), \mathfrak{N}_0(\mathfrak{u}, \mathfrak{u}_0))$, where

$$\mathfrak{N}_0(\mathfrak{u}, \mathfrak{u}_0) := \begin{pmatrix} \phi(0) - \phi_0 \\ \theta(0) - \theta_0 \\ \boldsymbol{u}(0) - \boldsymbol{u}_0 \end{pmatrix}$$

$$\mathfrak{N}_1(\mathfrak{u},\mathfrak{f}) := \begin{pmatrix} \partial_t\phi + mA\mu + C(\boldsymbol{u},\phi) - \sigma \\ \partial_t(\theta - l_{\rm h}\phi) + \kappa A\theta - \alpha\boldsymbol{g}\cdot\boldsymbol{u} + C(\boldsymbol{u},\theta - l_{\rm h}\phi) - h \\ \partial_t\boldsymbol{u} + \nu\boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}(\boldsymbol{u},\boldsymbol{u}) - \ell(\phi,\theta)\boldsymbol{g} + \mathcal{K}\boldsymbol{S}(\mu - l_{\rm c}\theta,\phi) - \boldsymbol{f} \\ \mu - \tau\partial_t\phi - \epsilon A\phi - l_{\rm c}\theta - F(\phi) - \lambda \end{pmatrix}$$

with $\mathfrak{u} = (\phi,\theta,\boldsymbol{u},\mu)$, $\mathfrak{u}_0 = (\phi_0,\theta_0,\boldsymbol{u}_0)$ and $\mathfrak{f} = (\sigma,h,\boldsymbol{f},\lambda)$. The linear terms are clearly of class C^∞. Also, the bilinear terms and the function F are of class C^∞ thanks to Lemma 4.2 and Lemma 4.4. Therefore, $\mathfrak{N}$ is a C^∞ mapping.

Given $\mathfrak{f} = (\sigma,h,\boldsymbol{f},\lambda) \in \boldsymbol{\mathcal{G}}^{\rm vw,\it r}_{q,s,p}(Q)$ and $\mathfrak{u}_0 = (\phi_0,\theta_0,\boldsymbol{u}_0) \in \boldsymbol{U}^{\rm vw,\it r}_{q,s,p}(\Omega)$, we see from the definition of $\mathfrak{N}$ that the very weak solution $\mathfrak{u} = (\phi,\theta,\boldsymbol{u},\mu) = \widetilde{\mathfrak{S}}(\mathfrak{f},\mathfrak{u}_0)$ to (1.1) satisfies $\mathfrak{N}(\widetilde{\mathfrak{S}}(\mathfrak{f},\mathfrak{u}_0),\mathfrak{f},\mathfrak{u}_0) = 0$. Taking the derivative with respect to the very weak solution, we have

$$\partial_{\mathfrak{u}}\mathfrak{N}(\widetilde{\mathfrak{S}}(\mathfrak{f},\mathfrak{u}_0),\mathfrak{f},\mathfrak{u}_0) = \mathfrak{A}(\mathfrak{u})$$

where $\mathfrak{A} : \boldsymbol{\mathcal{U}}^{\rm vw,\it r}_{q,s,p}(Q) \to \mathcal{L}(\boldsymbol{\mathcal{U}}^{\rm vw,\it r}_{q,s,p}(Q), \boldsymbol{\mathcal{G}}^{\rm vw,\it r}_{q,s,p}(Q) \times \boldsymbol{U}^{\rm vw,\it r}_{q,s,p}(\Omega))$ is the operator-valued mapping given by

$$[\mathfrak{A}(\mathfrak{u})]\mathfrak{y} = ((\widetilde{\sigma},\widetilde{h},\widetilde{\boldsymbol{f}},\widetilde{\lambda}),(\psi(0),\zeta(0),\boldsymbol{w}(0)))$$

with $\mathfrak{y} = (\psi,\zeta,\boldsymbol{w},\xi) \in \boldsymbol{\mathcal{U}}^{\rm vw,\it r}_{q,s,p}(Q)$, and the first four components are:

$$\begin{aligned}
\widetilde{\sigma} &:= \partial_t\psi + mA\xi + C(\boldsymbol{w},\phi) + C(\boldsymbol{u},\psi) \\
\widetilde{h} &:= \partial_t(\zeta - l_{\rm h}\psi) + \kappa A\zeta - \alpha\boldsymbol{g}\cdot\boldsymbol{w} + C(\boldsymbol{w},\theta - l_{\rm h}\phi) + C(\boldsymbol{u},\zeta - l_{\rm h}\psi) \\
\widetilde{\boldsymbol{f}} &:= \partial_t\boldsymbol{w} + \nu\boldsymbol{A}\boldsymbol{w} + \boldsymbol{B}(\boldsymbol{w},\boldsymbol{u}) + \boldsymbol{B}(\boldsymbol{u},\boldsymbol{w}) - (\alpha_1\psi + \alpha_2\zeta)\boldsymbol{g} \\
&\qquad + \mathcal{K}\boldsymbol{S}(\xi - l_{\rm c}\zeta,\phi) + \mathcal{K}\boldsymbol{S}(\mu - l_{\rm c}\theta,\psi) \\
\widetilde{\lambda} &:= \xi - \tau\partial_t\psi - \epsilon A\psi - l_{\rm c}\zeta - F'(\phi)\psi.
\end{aligned}$$

Following the proof of Theorem 4.11, it can be shown that $\mathfrak{A}(\mathfrak{u})$ is an isomorphism from $\boldsymbol{\mathcal{U}}^{\rm vw,\it r}_{q,s,p}(Q)$ onto $\boldsymbol{\mathcal{G}}^{\rm vw,\it r}_{q,s,p}(Q) \times \boldsymbol{U}^{\rm vw,\it r}_{q,s,p}(\Omega)$ for every $\mathfrak{u} \in \boldsymbol{\mathcal{U}}^{\rm vw,\it r}_{q,s,p}(Q)$. Hence, by the implicit function theorem [68, Section 4.7], we deduce that $\widetilde{\mathfrak{S}} \in C^\infty(\boldsymbol{\mathcal{G}}^{\rm vw,\it r}_{q,s,p}(Q) \times \boldsymbol{U}^{\rm vw,\it r}_{q,s,p}(\Omega), \boldsymbol{\mathcal{U}}^{\rm vw,\it r}_{q,s,p}(Q))$. Now, the result follows from $\mathfrak{S} = \widetilde{\mathfrak{S}} \circ \mathfrak{I}$, where $\mathfrak{I}$ is the canonical injection from $\boldsymbol{\mathcal{F}}^{\rm vw,\it r}_{q,s,p}(Q) \times \boldsymbol{U}^{\rm vw,\it r}_{q,s,p}(\Omega)$ into $\boldsymbol{\mathcal{G}}^{\rm vw,\it r}_{q,s,p}(Q) \times \boldsymbol{U}^{\rm vw,\it r}_{q,s,p}(\Omega)$ that is obviously of class C^∞, and the chain rule. □

Let us present the action of the first two derivatives of $\mathfrak{S}$. These play important roles in the area of optimal control, for instance, to the first and second order necessary and sufficient conditions for local optimality. The action of the first and second derivatives

$$\begin{aligned}
\mathrm{D}\mathfrak{S} &: \boldsymbol{\mathcal{F}}^{\rm vw,\it r}_{q,s,p}(Q) \times \boldsymbol{U}^{\rm vw,\it r}_{q,s,p}(\Omega) \to \mathcal{L}(\boldsymbol{\mathcal{F}}^{\rm vw,\it r}_{q,s,p}(Q) \times \boldsymbol{U}^{\rm vw,\it r}_{q,s,p}(\Omega), \boldsymbol{\mathcal{U}}^{\rm vw,\it r}_{q,s,p}(Q)) \\
\mathrm{D}^2\mathfrak{S} &: \boldsymbol{\mathcal{F}}^{\rm vw,\it r}_{q,s,p}(Q) \times \boldsymbol{U}^{\rm vw,\it r}_{q,s,p}(\Omega) \to \mathcal{L}([\boldsymbol{\mathcal{F}}^{\rm vw,\it r}_{q,s,p}(Q) \times \boldsymbol{U}^{\rm vw,\it r}_{q,s,p}(\Omega)]^2, \boldsymbol{\mathcal{U}}^{\rm vw,\it r}_{q,s,p}(Q))
\end{aligned}$$

are given as follows: For every $\mathfrak{f},\mathfrak{f}_1,\mathfrak{f}_2 \in \boldsymbol{\mathcal{F}}^{\rm vw,\it r}_{q,s,p}(Q)$ and $\mathfrak{u}_0,\mathfrak{u}_{01},\mathfrak{u}_{02} \in \boldsymbol{U}^{\rm vw,\it r}_{q,s,p}(\Omega)$, we have

$$\begin{aligned}
&\mathrm{D}\mathfrak{S}(\mathfrak{f},\mathfrak{u}_0)(\mathfrak{f}_1,\mathfrak{u}_{01}) = \mathfrak{A}(\mathfrak{u})^{-1}(\mathfrak{f}_1,\mathfrak{u}_{01}) \\
&\mathrm{D}^2\mathfrak{S}(\mathfrak{f},\mathfrak{u}_0)((\mathfrak{f}_1,\mathfrak{u}_{01}),(\mathfrak{f}_2,\mathfrak{u}_{02})) = -\mathfrak{A}(\mathfrak{u})^{-1}((\widetilde{\sigma},\widetilde{h},\widetilde{\boldsymbol{f}},\widetilde{\lambda}),(0,0,\boldsymbol{0})) \\
&\qquad \widetilde{\sigma} := C(\boldsymbol{w}_1,\psi_2) + C(\boldsymbol{w}_2,\psi_1)
\end{aligned}$$

$$\begin{aligned}\widetilde{h} &:= C(\boldsymbol{w}_1, \zeta_2 - l_{\mathrm{h}}\psi_2) + C(\boldsymbol{w}_2, \zeta_1 - l_{\mathrm{h}}\psi_1)\\ \widetilde{\boldsymbol{f}} &:= \boldsymbol{B}(\boldsymbol{w}_1, \boldsymbol{w}_2) + \boldsymbol{B}(\boldsymbol{w}_2, \boldsymbol{w}_1) + \mathcal{K}\boldsymbol{S}(\xi_1 - l_{\mathrm{c}}\zeta_1, \psi_2) + \mathcal{K}\boldsymbol{S}(\xi_2 - l_{\mathrm{c}}\zeta_2, \psi_1)\\ \widetilde{\lambda} &:= -F''(\phi)\psi_1\psi_2\end{aligned}$$

where $\mathfrak{u} = (\phi, \theta, \boldsymbol{u}, \mu) = \mathfrak{S}(\mathfrak{f}, \mathfrak{u}_0)$ and $(\psi_k, \zeta_k, \boldsymbol{w}_k, \xi_k) = \mathrm{D}\mathfrak{S}(\mathfrak{f}, \mathfrak{u}_0)(\mathfrak{f}_k, \mathfrak{u}_{0k})$ for $k = 1, 2$. It is possible to write down the corresponding linear PDE systems for the actions of these derivatives by simply applying the operator $\mathfrak{A}(\mathfrak{u})$ to these equations, see for instance [54, Section 4]. In particular, the PDE system for the second order derivative has homogeneous initial data.

We close this section by stating without proof the corresponding result for the case of weak solutions. We still denote the associated solution operator by $\mathfrak{S}$.

Theorem 5.2. *With respect to the assumption (4.7), we have*

$$\mathfrak{S} \in C^\infty(\boldsymbol{\mathcal{F}}^{\mathrm{w},r}_{q,s,p}(Q) \times \boldsymbol{U}^{\mathrm{w},r}_{q,s,p}(\Omega), \boldsymbol{\mathcal{U}}^{\mathrm{w},r}_{q,s,p}(Q)).$$

6. Higher Time Integrability

In this section, we consider higher integrability conditions with respect to time on the source functions. Here, we follow the ideas of the papers [17, 18], see also Remark 6.5 below. The crucial part here is on how to deal with the coupling terms in the nonlinear system.

First, let us state the following theorem in [9] for the sake of the reader.

Theorem 6.1. [9, Theorem 3] *Let X_1 and X_0 be Banach spaces such that X_1 is dense in X_0. If $1 \le \mathfrak{r} < \infty$, $0 < \mathfrak{s} < 1/\mathfrak{r}$, and $0 < \mathfrak{t} < 1 - \mathfrak{s}$, then*

$$W^{1,\mathfrak{r}}(I; X_1, X_0) \hookrightarrow L^{\mathfrak{r}/(1-\mathfrak{r}\mathfrak{s})}(I; (X_0, X_1)_{\mathfrak{t},1}).$$

We shall start with a simplified version of the auxiliary PDE system (4.14).

Theorem 6.2. *Assume that $4 \le q, s, p < \infty$, $q \le s$ and $8 \le r < \infty$. Let $\chi = 1$ and $G \equiv 0$ in (4.14) and consider source functions $\widetilde{\sigma}, \widetilde{h} \in L^{r/2}(I; W^{-1,2}(\Omega))$, $\widetilde{\boldsymbol{f}} \in L^{r/2}(I; \boldsymbol{W}^{-1,2}(\Omega))$, $\widetilde{\lambda} \in L^{r/2}(I; W^{1,2}_0(\Omega))$ and initial data $\phi_{0N} \in Z^3_{2,r/2}(\Omega)$, $\theta_{0N} \in Z^1_{2,r/2}(\Omega)$, $\boldsymbol{u}_{0N} \in \boldsymbol{V}^1_{2,r/2}(\Omega)$. Moreover, suppose that the frozen coefficients satisfy $\widetilde{\phi} \in \mathcal{Z}^2_{q,r}(Q)$, $\widetilde{\theta} \in \mathcal{Z}^0_{s,r}(Q)$, $\widetilde{\boldsymbol{u}} \in \boldsymbol{\mathcal{V}}^0_{p,r}(Q)$ and $\widetilde{\mu} \in L^r(I; L^q(\Omega))$. Then (4.14) admits a unique weak solution*

$$(\phi_N, \theta_N, \boldsymbol{u}_N, \mu_N) \in \mathcal{Z}^3_{2,r/2}(Q) \times \mathcal{Z}^1_{2,r/2}(Q) \times \boldsymbol{\mathcal{V}}^1_{2,r/2}(Q) \times L^{r/2}(I; W^{1,2}_0(\Omega)) \tag{6.1}$$

with an associated pressure $\mathfrak{p}_N \in W^{-1,r/2}(I; \widehat{L}^2(\Omega))$.

Proof. As we have done in the linear case, let us introduce $\gamma_N := \theta_N - l_{\mathrm{h}}\phi_N$, $\gamma_{0N} := \theta_{0N} - l_{\mathrm{h}}\phi_{0N}$, and $\widetilde{\gamma} := \widetilde{\theta} - l_{\mathrm{h}}\widetilde{\phi}$. With these, (4.14) with $\chi = 1$ and $G \equiv 0$ is equivalent to

$$\left[\begin{array}{ll}\partial_t\phi_N + \mathrm{div}\,(\phi_N \boldsymbol{u}_N) + \mathrm{div}\,(\phi_N \widetilde{\boldsymbol{u}}) + \mathrm{div}\,(\widetilde{\phi}\boldsymbol{u}_N) - m\Delta\mu_N = \widetilde{\sigma} & \text{in } Q,\\ \mu_N = \tau\partial_t\phi_N - \epsilon\Delta\phi_N + F(\widetilde{\phi}+\phi_N) - F(\widetilde{\phi}) + l_{\mathrm{c}}l_{\mathrm{h}}\phi_N + l_{\mathrm{c}}\gamma_N + \widetilde{\lambda} & \text{in } Q,\\ \partial_t\gamma_N + \mathrm{div}\,(\gamma_N\boldsymbol{u}_N) + \mathrm{div}\,(\gamma_N\widetilde{\boldsymbol{u}}) + \mathrm{div}\,(\widetilde{\gamma}\boldsymbol{u}_N) - \kappa\Delta\gamma_N - \kappa l_{\mathrm{h}}\Delta\phi_N & \\ \qquad = \alpha\boldsymbol{g}\cdot\boldsymbol{u}_N + \widetilde{h} & \text{in } Q,\\ \partial_t\boldsymbol{u}_N + \mathrm{div}\,(\boldsymbol{u}_N\otimes\boldsymbol{u}_N) + \mathrm{div}\,(\boldsymbol{u}_N\otimes\widetilde{\boldsymbol{u}}) + \mathrm{div}\,(\widetilde{\boldsymbol{u}}\otimes\boldsymbol{u}_N) - \nu\Delta\boldsymbol{u}_N + \nabla\mathfrak{p}_N & \\ \qquad = \mathcal{K}(\mu_N - l_{\mathrm{c}}\gamma_N - l_{\mathrm{c}}l_{\mathrm{h}}\phi_N)\nabla\phi_N + \mathcal{K}(\widetilde{\mu} - l_{\mathrm{c}}\widetilde{\theta} - l_{\mathrm{c}}l_{\mathrm{h}}\widetilde{\phi})\nabla\phi_N & \\ \qquad + \mathcal{K}(\mu_N - l_{\mathrm{c}}\gamma_N - l_{\mathrm{c}}l_{\mathrm{h}}\phi_N)\nabla\widetilde{\phi} + (\alpha_0 + (\alpha_1+\alpha_2 l_{\mathrm{h}})\phi_N + \alpha_2\gamma_N)\boldsymbol{g} + \widetilde{\boldsymbol{f}} & \text{in } Q,\\ \mathrm{div}\,\boldsymbol{u}_N = 0 & \text{in } Q,\\ \phi_N = \Delta\phi_N = 0, \quad \gamma_N = 0, \quad \boldsymbol{u}_N = \boldsymbol{0} & \text{on } \Sigma,\\ \phi_N(0) = \phi_{0N}, \quad \gamma_N(0) = \gamma_{0N}, \quad \boldsymbol{u}_N(0) = \boldsymbol{u}_{0N} & \text{in } \Omega.\end{array}\right. \tag{6.2}$$

We shall proceed with a fixed point argument as in [18].

Step 1. *Local Existence.* Consider $\widetilde{\phi}_N \in L^r(I; W^{2,4}(\Omega))$, $\widetilde{\gamma}_N \in L^r(I; L^4(\Omega))$, and $\widetilde{\boldsymbol{u}}_N \in L^r(I; \boldsymbol{L}^4(\Omega))$. First, let us take the following heat equation

$$\left[\begin{array}{l}\partial_t\gamma_N - \kappa\Delta\gamma_N = \widetilde{h}_N \ \text{ in } Q,\\ \gamma_N = 0 \ \text{ on } \Sigma, \ \gamma_N(0) = \gamma_{0N} \ \text{ in } \Omega,\end{array}\right. \tag{6.3}$$

with the source function

$$\widetilde{h}_N := \widetilde{h} + \alpha\boldsymbol{g}\cdot\widetilde{\boldsymbol{u}}_N + \kappa l_{\mathrm{h}}\Delta\widetilde{\phi}_N - \mathrm{div}\,(\widetilde{\gamma}_N\widetilde{\boldsymbol{u}}_N) - \mathrm{div}\,(\widetilde{\gamma}_N\widetilde{\boldsymbol{u}}) - \mathrm{div}\,(\widetilde{\gamma}\widetilde{\boldsymbol{u}}_N).$$

Using Hölder's inequality, it is not difficult to see that $\widetilde{h}_N \in L^{r/2}(I; W^{-1,2}(\Omega))$ and

$$\begin{aligned}\|\widetilde{h}_N\|_{L^{r/2}(I;W^{-1,2}(\Omega))} &\le c\{\|\widetilde{h}\|_{L^{r/2}(I;W^{-1,2}(\Omega))} + \|\widetilde{\boldsymbol{u}}_N\|_{L^{r/2}(I;\boldsymbol{L}^2(\Omega))} + \|\widetilde{\phi}_N\|_{L^{r/2}(I;W^{1,2}(\Omega))}\\ &\quad + \|\widetilde{\gamma}_N\|^2_{L^r(I;L^4(\Omega))} + \|\widetilde{\boldsymbol{u}}_N\|^2_{L^r(I;\boldsymbol{L}^4(\Omega))} + \|\widetilde{\gamma}\|^2_{L^r(I;L^4(\Omega))} + \|\widetilde{\boldsymbol{u}}\|^2_{L^r(I;\boldsymbol{L}^4(\Omega))}\}.\end{aligned} \tag{6.4}$$

Thus, according to the maximal parabolic regularity for the heat equation in Theorem 3.6, (6.3) possesses a weak solution $\gamma_N \in \mathcal{Z}^1_{2,r/2}(Q)$, and we have

$$\|\gamma_N\|_{\mathcal{Z}^1_{2,r/2}(Q)} \le c\{\|\widetilde{h}_N\|_{L^{r/2}(I;W^{-1,2}(\Omega))} + \|\gamma_{0N}\|_{Z^1_{2,r/2}(\Omega)}\}. \tag{6.5}$$

Applying the properties of real interpolation spaces, one has

$$(W^{-1,2}(\Omega), W^{1,2}_0(\Omega))_{\frac{3}{4},1} \hookrightarrow (W^{-1,2}(\Omega), W^{1,2}_0(\Omega))_{\frac{3}{4},2} = W^{\frac{1}{2},2}(\Omega) \hookrightarrow L^4(\Omega).$$

Hence, we obtain from Theorem 6.1 with $\mathfrak{r} = \frac{r}{2}$, $\mathfrak{s} = \frac{1}{r}$ and $\mathfrak{t} = \frac{3}{4}$ the compact embedding $\mathcal{Z}^1_{2,r/2}(Q) \overset{c}{\hookrightarrow} L^r(I; L^4(\Omega))$. On the other hand, applying [64, Theorem 4.3.1 and Theorem 4.6.1(d)] we get

$$Z^1_{2,4}(\Omega) \hookrightarrow (W^{-1,2}(\Omega), W^{1,2}(\Omega))_{\frac{3}{4},4} = B^{1/2}_{2,4}(\Omega) \hookrightarrow L^4(\Omega).$$

Since $r \ge 8$, this leads us to the embeddings $\mathcal{Z}^1_{2,r/2}(Q) \hookrightarrow \mathcal{Z}^1_{2,4}(Q) \hookrightarrow C(\bar{I}; Z^1_{2,4}(\Omega)) \hookrightarrow C(\bar{I}; L^4(\Omega))$, and as a result it holds that

$$\|\gamma_N\|_{L^r((0,t);L^4(\Omega))} \le t^{1/r}\|\gamma_N\|_{C([0,t];L^4(\Omega))} \le ct^{1/r}\|\gamma_N\|_{\mathcal{Z}^1_{2,r/2}(Q)}. \tag{6.6}$$

Next, we turn our attention to the biharmonic heat equation

$$\left[\begin{array}{l}\partial_t(\phi_N - m\tau\Delta\phi_N) + m\epsilon\Delta^2\phi_N - \dfrac{\epsilon}{m\tau^2}\phi_N = m\Delta\widetilde{\lambda}_N + \widetilde{\sigma}_N \quad \text{in } Q, \\ \phi_N = \Delta\phi_N = 0 \quad \text{on } \Sigma, \;\; \phi_N(0) = \phi_{0N} \quad \text{in } \Omega,\end{array}\right. \tag{6.7}$$

where the right-hand sides are given by

$$\begin{aligned}\widetilde{\lambda}_N &:= \widetilde{\lambda} + F(\widetilde{\phi} + \widetilde{\phi}_N) - F(\widetilde{\phi}) + l_c\gamma_N + l_c l_h\widetilde{\phi}_N \\ \widetilde{\sigma}_N &:= \widetilde{\sigma} - \frac{\epsilon}{m\tau^2}\widetilde{\phi}_N - \operatorname{div}(\widetilde{\phi}_N\widetilde{\boldsymbol{u}}_N) - \operatorname{div}(\widetilde{\phi}_N\widetilde{\boldsymbol{u}}) - \operatorname{div}(\widetilde{\phi}\widetilde{\boldsymbol{u}}_N).\end{aligned}$$

Here, γ_N is the solution to (6.3). With Hölder's inequality, we can estimate the second and third terms in the definition of $\widetilde{\lambda}_N$ according to

$$\begin{aligned}&\|F(\widetilde{\phi} + \widetilde{\phi}_N) - F(\widetilde{\phi})\|_{L^{r/2}(I;W_0^{1,2}(\Omega))} \\ &\quad \leq c(\|\widetilde{\phi}\|^2_{L^\infty(I;L^\infty(\Omega))} + \|\widetilde{\phi}_N\|^2_{L^\infty(I;L^\infty(\Omega))})(\|\widetilde{\phi}\|_{L^{r/2}(I;W^{1,2}(\Omega))} + \|\widetilde{\phi}_N\|_{L^{r/2}(I;W^{1,2}(\Omega))}) \\ &\quad \leq \mathscr{C}(\|\widetilde{\phi}\|_{\mathcal{Z}^2_{q,r}(Q)} + \|\widetilde{\phi}_N\|_{Z^3_{2,2}(Q)})\end{aligned}$$

where $\mathscr{C} : [0,\infty) \to [0,\infty)$ is a cubic polynomial with $\mathscr{C}(0) = 0$. Thus, we have $\widetilde{\lambda}_N \in L^{r/2}(I;W_0^{1,2}(\Omega))$, $\widetilde{\sigma}_N \in L^{r/2}(I;W^{-1,2}(\Omega))$ and these satisfy

$$\begin{aligned}&\|\Delta\widetilde{\lambda}_N\|_{L^{r/2}(I;W^{-1,2}(\Omega))} \leq \|\widetilde{\lambda}_N\|_{L^{r/2}(I;W_0^{1,2}(\Omega))} \\ &\quad \leq c\{\|\widetilde{\lambda}\|_{L^{r/2}(I;W_0^{1,2}(\Omega))} + \mathscr{C}(\|\widetilde{\phi}\|_{\mathcal{Z}^2_{q,r}(Q)} + \|\widetilde{\phi}_N\|_{Z^3_{2,2}(Q)}) + \|\gamma_N\|_{L^{r/2}(I;W_0^{1,2}(\Omega))}\}\end{aligned} \tag{6.8}$$

$$\begin{aligned}&\|\widetilde{\sigma}_N\|_{L^{r/2}(I;W^{-1,2}(\Omega))} \leq c\{\|\widetilde{\sigma}\|_{L^{r/2}(I;W^{-1,2}(\Omega))} + \|\widetilde{\phi}_N\|_{L^r(I;L^2(\Omega))} + \|\widetilde{\phi}_N\|^2_{L^r(I;L^4(\Omega))} \\ &\quad + \|\widetilde{\boldsymbol{u}}_N\|^2_{L^r(I;\boldsymbol{L}^4(\Omega))} + \|\widetilde{\phi}\|^2_{L^r(I;L^4(\Omega))} + \|\widetilde{\boldsymbol{u}}\|^2_{L^r(I;\boldsymbol{L}^4(\Omega))}\}.\end{aligned} \tag{6.9}$$

The maximal parabolic regularity for the biharmonic heat equation provided in Theorem 3.11 is applicable, and hence, (6.7) admits a weak solution $\phi_N \in \mathcal{Z}^3_{2,r/2}(Q)$ and

$$\|\phi_N\|_{\mathcal{Z}^3_{2,r/2}(Q)} \leq c\{\|\widetilde{\lambda}_N\|_{L^{r/2}(I;W_0^{1,2}(\Omega))} + \|\widetilde{\sigma}_N\|_{L^{r/2}(I;W^{-1,2}(\Omega))} + \|\phi_{0N}\|_{Z^3_{2,r/2}(\Omega)}\}. \tag{6.10}$$

Using Theorem 6.1 with $\mathfrak{r} = \frac{r}{2}$, $\mathfrak{s} = \frac{1}{r}$ and $\mathfrak{t} = \frac{3}{4}$, we obtain the compact embedding $\mathcal{Z}^3_{2,r/2}(Q) \hookrightarrow\hookrightarrow L^r(I;W^{2,4}(\Omega))$ thanks to

$$\begin{aligned}(W_0^{1,2}(\Omega), X^{3,2}(\Omega))_{\frac{3}{4},1} &= (W_0^{1,2}(\Omega), W^{3,2}(\Omega) \cap W_0^{1,2}(\Omega))_{\frac{3}{4},2} \\ &\hookrightarrow W^{\frac{5}{2},2}(\Omega) \cap W_0^{1,2}(\Omega) \hookrightarrow W^{2,4}(\Omega).\end{aligned}$$

Invoking [64, Theorem 4.3.1 and Theorem 4.6.1(d)] leads to the continuous embeddings

$$Z^3_{2,4}(\Omega) \hookrightarrow (W^{1,2}(\Omega), W^{3,2}(\Omega))_{\frac{3}{4},4} = B^{5/2}_{2,4}(\Omega) \hookrightarrow W^{2,4}(\Omega).$$

This implies that $\mathcal{Z}^3_{2,r/2}(Q) \hookrightarrow \mathcal{Z}^3_{2,4}(Q) \hookrightarrow C(\bar{I}; Z^3_{2,4}(\Omega)) \hookrightarrow C(\bar{I}; W^{2,4}(\Omega))$, and thus we have the estimate

$$\|\phi_N\|_{L^r((0,t);W^{2,4}(\Omega))} \leq t^{1/r}\|\phi_N\|_{C([0,t];W^{2,4}(\Omega))} \leq ct^{1/r}\|\phi_N\|_{\mathcal{Z}^3_{2,r/2}(Q)}. \tag{6.11}$$

Finally, we will deal with the following Stokes equation

$$\left[\begin{array}{l}\partial_t \boldsymbol{u}_N - \nu\Delta \boldsymbol{u}_N + \nabla \mathfrak{p}_N = \widetilde{\boldsymbol{f}}_N \ \text{ in } Q,\\ \operatorname{div} \boldsymbol{u}_N = 0 \ \text{ in } \Omega,\ \boldsymbol{u}_N = \boldsymbol{0} \ \text{ on } \Sigma,\ \boldsymbol{u}_N(0) = \boldsymbol{u}_{0N} \ \text{ in } \Omega,\end{array}\right. \tag{6.12}$$

where the source function $\boldsymbol{f}_N$ is given by

$$\begin{aligned}\widetilde{\boldsymbol{f}}_N &:= \widetilde{\boldsymbol{f}} + (\alpha_0 + (\alpha_1 + \alpha_2 l_{\rm h})\phi_N + \alpha_2\gamma_N)\boldsymbol{g} - \operatorname{div}(\widetilde{\boldsymbol{u}}_N \otimes \widetilde{\boldsymbol{u}}_N)\\ &\quad - \operatorname{div}(\widetilde{\boldsymbol{u}}_N \otimes \widetilde{\boldsymbol{u}}) - \operatorname{div}(\widetilde{\boldsymbol{u}} \otimes \widetilde{\boldsymbol{u}}_N) + \mathcal{K}(\widetilde{\mu}_N - l_{\rm c}\gamma_N - l_{\rm c}l_{\rm h}\phi_N)\nabla\phi_N\\ &\quad + \mathcal{K}(\widetilde{\mu} - l_{\rm c}\widetilde{\gamma} - l_{\rm c}l_{\rm h}\widetilde{\phi})\nabla\phi_N + \mathcal{K}(\widetilde{\mu}_N - l_{\rm c}\gamma_N - l_{\rm c}l_{\rm h}\phi_N)\nabla\widetilde{\phi}\\ \widetilde{\mu}_N &:= \widetilde{\lambda}_N + \tau\partial_t\phi_N - \epsilon\Delta\phi_N.\end{aligned}$$

Here, γ_N and ϕ_N are the weak solutions to (6.3) and (6.7), respectively.

Applying Hölder's inequality, we see that the $\widetilde{\boldsymbol{f}}_N$ and $\widetilde{\mu}_N$ obey the following estimates:

$$\begin{aligned}&\|\widetilde{\boldsymbol{f}}_N\|_{L^{r/2}(I;\boldsymbol{W}^{-1,2}(\Omega))}\\ &\quad \le c\{|\alpha_0\boldsymbol{g}| + \|\widetilde{\boldsymbol{f}}\|_{L^{r/2}(I;\boldsymbol{W}^{-1,2}(\Omega))} + \|\phi_N\|_{L^{r/2}(I;L^2(\Omega))} + \|\gamma_N\|_{L^{r/2}(I;L^2(\Omega))}\\ &\quad + \|\widetilde{\boldsymbol{u}}_N\|^2_{L^r(I;\boldsymbol{L}^4(\Omega))} + \|\widetilde{\boldsymbol{u}}\|^2_{L^r(I;\boldsymbol{L}^4(\Omega))} + \|\gamma_N\|^2_{L^{r/2}(I;L^4(\Omega))} + \|\phi_N\|^2_{L^{r/2}(I;L^4(\Omega))}\\ &\quad + \|\nabla\phi_N\|^2_{L^\infty(I;\boldsymbol{L}^4(\Omega))} + \|\nabla\widetilde{\phi}\|^2_{L^\infty(I;\boldsymbol{L}^4(\Omega))} + \|\widetilde{\mu}\|^2_{L^{r/2}(I;L^q(\Omega))} + \|\widetilde{\gamma}\|^2_{L^{r/2}(I;L^4(\Omega))}\\ &\quad + \|\widetilde{\phi}\|^2_{L^{r/2}(I;L^4(\Omega))} + \|\widetilde{\mu}_N\|_{L^{r/2}(I;L^4(\Omega))}(\|\nabla\phi_N\|_{L^\infty(I;L^4(\Omega))} + \|\nabla\widetilde{\phi}\|_{L^\infty(I;\boldsymbol{L}^4(\Omega))})\}\end{aligned} \tag{6.13}$$

$$\|\widetilde{\mu}_N\|_{L^{r/2}(I;W_0^{1,2}(\Omega))} \le c\{\|\widetilde{\lambda}_N\|_{L^{r/2}(I;W_0^{1,2}(\Omega))} + \|\phi_N\|_{\mathcal{Z}^3_{2,r/2}(Q)}\}. \tag{6.14}$$

We use the fact that $q \ge 4$ in the estimate involving the term $\widetilde{\mu}$. Invoking the maximal parabolic regularity for the Stokes equation stated by Theorem 3.2, (6.12) has a weak solution $\boldsymbol{u}_N \in \boldsymbol{\mathcal{V}}^1_{2,r/2}(Q)$ such that

$$\|\boldsymbol{u}_N\|_{\boldsymbol{\mathcal{V}}^1_{2,r/2}(Q)} \le c\{\|\widetilde{\boldsymbol{f}}_N\|_{L^{r/2}(I;\boldsymbol{W}^{-1,2}(\Omega))} + \|\boldsymbol{u}_{0N}\|_{\boldsymbol{V}^1_{2,r/2}(\Omega)}\}. \tag{6.15}$$

On one hand, owing to the interpolation theory for complemented subspaces in [64, Section 1.17.1] and the fact that $\boldsymbol{A}_2$ is an isomorphism from $\boldsymbol{X}^{1,2}_\sigma(\Omega)$ onto $\boldsymbol{X}^{-1,2}_\sigma(\Omega)$ and from $\boldsymbol{X}^{3,2}_\sigma(\Omega)$ onto $\boldsymbol{X}^{1,2}_\sigma(\Omega)$, we have

$$\begin{aligned}(\boldsymbol{X}^{-1,2}_\sigma(\Omega), \boldsymbol{X}^{1,2}_\sigma(\Omega))_{\frac{3}{4},1} &= \boldsymbol{A}_2(\boldsymbol{X}^{1,2}_\sigma(\Omega), \boldsymbol{X}^{3,2}_\sigma(\Omega))_{\frac{3}{4},1}\\ &= \boldsymbol{A}_2((\boldsymbol{X}^{1,2}(\Omega), \boldsymbol{X}^{3,2}(\Omega))_{\frac{3}{4},1} \cap \boldsymbol{L}^2_\sigma(\Omega))\\ &\hookrightarrow \boldsymbol{A}_2((\boldsymbol{X}^{1,2}(\Omega), \boldsymbol{X}^{3,2}(\Omega))_{\frac{3}{4},2} \cap \boldsymbol{L}^2_\sigma(\Omega))\\ &= \boldsymbol{A}_2(\boldsymbol{W}^{\frac{5}{2},2}(\Omega) \cap \boldsymbol{W}^{1,2}_0(\Omega) \cap \boldsymbol{L}^2_\sigma(\Omega)) \hookrightarrow \boldsymbol{W}^{\frac{1}{2},2}(\Omega) \hookrightarrow \boldsymbol{L}^4(\Omega).\end{aligned}$$

These embeddings and Theorem 6.1 with $\mathfrak{r} = \frac{r}{2}$, $\mathfrak{s} = \frac{1}{r}$ and $\mathfrak{t} = \frac{3}{4}$ give us the compact embedding $\boldsymbol{\mathcal{V}}^1_{2,r/2}(Q) \rightrightarrows L^r(I;\boldsymbol{L}^4(\Omega))$. On the other hand, by invoking [64, Theorem 4.3.1 and Theorem 4.6.1(d)] once again, we deduce that

$$\begin{aligned}\boldsymbol{V}^1_{2,4}(\Omega) &= \boldsymbol{A}_2((\boldsymbol{X}^{1,2}(\Omega), \boldsymbol{X}^{3,2}(\Omega))_{\frac{3}{4},4} \cap \boldsymbol{L}^2_\sigma(\Omega))\\ &\hookrightarrow \boldsymbol{A}_2((\boldsymbol{W}^{1,2}(\Omega), \boldsymbol{W}^{3,2}(\Omega))_{\frac{3}{4},4} \cap \boldsymbol{W}^{1,2}_0(\Omega) \cap \boldsymbol{L}^2_\sigma(\Omega))\end{aligned}$$

$$
\begin{aligned}
&= \boldsymbol{A}_2(\boldsymbol{B}^{5/2}_{2,4}(\Omega) \cap \boldsymbol{W}^{1,2}_0(\Omega) \cap \boldsymbol{L}^2_\sigma(\Omega)) \\
&\hookrightarrow \boldsymbol{A}_2(\boldsymbol{W}^{2,4}(\Omega) \cap \boldsymbol{W}^{1,2}_0(\Omega) \cap \boldsymbol{L}^2_\sigma(\Omega)) \hookrightarrow \boldsymbol{L}^4(\Omega). \qquad (6.16)
\end{aligned}
$$

Thus, the following continuous embeddings hold

$$
\boldsymbol{\mathcal{V}}^1_{2,r/2}(Q) \hookrightarrow \boldsymbol{\mathcal{V}}^1_{2,4}(Q) \hookrightarrow C(\bar{I}; \boldsymbol{V}^1_{2,4}(\Omega)) \hookrightarrow C(\bar{I}; \boldsymbol{L}^4(\Omega)),
$$

and moreover, we have the estimate

$$
\|\boldsymbol{u}_N\|_{L^r((0,t);\boldsymbol{L}^4(\Omega))} \le t^{1/r} \|\boldsymbol{u}_N\|_{C([0,t];\boldsymbol{L}^4(\Omega))} \le ct^{1/r} \|\boldsymbol{u}_N\|_{\boldsymbol{\mathcal{V}}^1_{2,r/2}(Q)}. \qquad (6.17)
$$

Given $R > 0$, let $B_{R,t}$ denote the closed ball with radius R in the space $L^r((0,t); W^{2,4}(\Omega)) \times L^r((0,t); L^4(\Omega)) \times L^r((0,t); \boldsymbol{L}^4(\Omega))$. Based on the above discussion, we have

$$
\begin{aligned}
&\mathcal{Z}^3_{2,r/2}((0,t)\times\Omega) \times \mathcal{Z}^1_{2,r/2}((0,t)\times\Omega) \times \boldsymbol{\mathcal{V}}^1_{2,r/2}((0,t)\times\Omega) \\
&\hookrightarrow\hookrightarrow L^r((0,t); W^{2,4}(\Omega)) \times L^r((0,t); L^4(\Omega)) \times L^r((0,t); \boldsymbol{L}^4(\Omega)). \qquad (6.18)
\end{aligned}
$$

It can be deduced from the inequalities (6.4)–(6.6), (6.8)–(6.11) and (6.13)–(6.17) that $(\widetilde{\phi}_N, \widetilde{\gamma}_N, \widetilde{\boldsymbol{u}}_N) \mapsto (\phi_N, \gamma_N, \boldsymbol{u}_N)$ maps $B_{R,t}$ into itself for sufficiently small t and this map is compact. By the Schauder Fixed Point Theorem, we obtain the existence of a fixed point, and this corresponds to a local solution of (6.2) belonging to the function space on the left-hand side of (6.18), and in turn, we obtain a local solution to (4.14) with $\chi = 1$ and $G \equiv 0$. Note that this solution coincides with that in Theorem 4.9.

Step 2. *Existence over the interval I.* Let $(0,t^*)$ be the maximal interval of existence. There are two alternatives, namely, $t^* = T$ or $t^* < T$ and

$$
\lim_{t\uparrow t^*} \{\|\phi_N\|_{\mathcal{Z}^3_{2,r/2}((0,t)\times\Omega)} + \|\gamma_N\|_{\mathcal{Z}^1_{2,r/2}((0,t)\times\Omega)} + \|\boldsymbol{u}_N\|_{\boldsymbol{\mathcal{V}}^1_{2,r/2}((0,t)\times\Omega)}\} = \infty. \qquad (6.19)
$$

We shall show that the second alternative is not possible, that is, blow-up does not occur. For the meantime, let us temporarily assume that

$$
(\gamma_N, \boldsymbol{u}_N) \in L^{r/2}((0,t); L^4(\Omega)) \times L^{r/2}((0,t); \boldsymbol{L}^4(\Omega)) \qquad \forall\, t \in (0,t^*). \qquad (6.20)
$$

In order to simplify the succeeding estimates, we introduce the following notations:

$$
\begin{aligned}
\mathsf{N} &:= \|\phi_N\|_{\mathcal{Z}^3_{2,2}(Q)} + \|\gamma_N\|_{\mathcal{Z}^1_{2,2}(Q)} + \|\boldsymbol{u}_N\|_{\boldsymbol{\mathcal{V}}^1_{2,2}(Q)} \\
\mathsf{F} &:= \|\widetilde{\phi}\|_{\mathcal{Z}^2_{q,r}(Q)} + \|\widetilde{\gamma}\|_{\mathcal{Z}^0_{s,r}(Q)} + \|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{\mathcal{V}}^0_{p,r}(Q)} + \|\widetilde{\mu}\|_{L^r(I;L^q(\Omega))} \\
\mathsf{B} &:= \|\widetilde{\sigma}\|_{L^{r/2}(I;W^{-1,2}(\Omega))} + \|\widetilde{h}\|_{L^{r/2}(I;W^{-1,2}(\Omega))} + \|\widetilde{\boldsymbol{f}}\|_{L^{r/2}(I;\boldsymbol{W}^{-1,2}(\Omega))} \\
&\quad + \|\widetilde{\lambda}\|_{L^{r/2}(I;W^{1,2}_0(\Omega))} + |\alpha_0 \boldsymbol{g}| \\
\mathsf{D}(t) &:= \|\phi_N(t)\|_{Z^3_{2,r/2}(\Omega)} + \|\gamma_N(t)\|_{Z^1_{2,r/2}(\Omega)} + \|\boldsymbol{u}_N(t)\|_{\boldsymbol{V}^1_{2,r/2}(\Omega)}.
\end{aligned}
$$

Given $\delta > 0$, (6.20) and the absolute continuity of the Lebesgue integral imply the existence of $\eta_\delta > 0$ such that

$$
\|\gamma_N\|_{L^{r/2}((t_0,t^*);L^4(\Omega))} + \|\boldsymbol{u}_N\|_{L^{r/2}((t_0,t^*);\boldsymbol{L}^4(\Omega))} < \delta \qquad (6.21)
$$

whenever $0 < t^* - t_0 < \eta_\delta$. Let $(\sigma_N, h_N, \boldsymbol{f}_N, \lambda_N)$ be as that with $(\widetilde{\sigma}_N, \widetilde{h}_N, \widetilde{\boldsymbol{f}}_N, \widetilde{\lambda}_N)$ in Step 1 but with $(\widetilde{\phi}_N, \widetilde{\gamma}_N, \widetilde{\boldsymbol{u}}_N, \widetilde{\mu}_N)$ replaced by $(\phi_N, \gamma_N, \boldsymbol{u}_N, \mu_N)$. In what follows, the estimates in the previous step involving the time interval $(0,t)$ will be replaced by (t_0,t).

By Hölder's inequality, $\mathscr{Z}^1_{2,r/2}((t_0,t)\times\Omega)\hookrightarrow C([t_0,t];L^4(\Omega))$ and (6.21), we have

$$\begin{aligned}\|\gamma_N\|^2_{L^r((t_0,t);L^4(\Omega))} &\le \|\gamma_N\|_{L^{r/2}((t_0,t^*);L^4(\Omega))}\|\gamma_N\|_{C([t_0,t];L^4(\Omega))}\\ &\le \delta\|\gamma_N\|_{\mathscr{Z}^1_{2,r/2}((t_0,t)\times\Omega)}. \end{aligned}\tag{6.22}$$

Similarly, using $\boldsymbol{\mathcal{V}}^1_{2,r/2}((t_0,t)\times\Omega)\hookrightarrow C([t_0,t];\boldsymbol{L}^4(\Omega))$, we obtain

$$\|\boldsymbol{u}_N\|^2_{L^r((t_0,t);\boldsymbol{L}^4(\Omega))}\le\delta\|\boldsymbol{u}_N\|_{\boldsymbol{\mathcal{V}}^1_{2,r/2}((t_0,t)\times\Omega)}.\tag{6.23}$$

From the embeddings $\boldsymbol{\mathcal{V}}^1_{2,2}(Q)\hookrightarrow L^{r/2}((t_0,t);\boldsymbol{L}^2(\Omega))$, $\mathscr{Z}^3_{2,2}(Q)\hookrightarrow L^{r/2}((t_0,t);W^{1,2}(\Omega))$, $\mathscr{Z}^2_{q,r}(Q)\hookrightarrow L^r((t_0,t);L^4(\Omega))$ and $\boldsymbol{\mathcal{V}}^0_{p,r}(Q)\hookrightarrow L^r((t_0,t);\boldsymbol{L}^4(\Omega))$, we have the following

$$\begin{aligned}\|\widetilde{\gamma}\|_{L^r((t_0,t);L^4(\Omega))}+\|\widetilde{\boldsymbol{u}}\|_{L^r((t_0,t);\boldsymbol{L}^4(\Omega))}&\le c\mathsf{F}\\ \|\boldsymbol{u}_N\|_{L^{r/2}((t_0,t);\boldsymbol{L}^2(\Omega))}+\|\phi_N\|_{L^{r/2}((t_0,t);W^{1,2}(\Omega))}&\le c\mathsf{N}.\end{aligned}$$

Substituting these, along with (6.22) and (6.23), in (6.4) with I replaced by (t_0,t), we get

$$\|h_N\|_{L^{r/2}((t_0,t);W^{-1,2}(\Omega))}\le c\{\mathsf{N}+\mathsf{B}+\mathsf{F}^2+\delta\|\gamma_N\|_{\mathscr{Z}^1_{2,r/2}((t_0,t)\times\Omega)}+\delta\|\boldsymbol{u}_N\|_{\boldsymbol{\mathcal{V}}^1_{2,r/2}((t_0,t)\times\Omega)}\}.$$

Here, $c>0$ is a constant that is independent on δ. Plugging this in (6.5) yields

$$(1-c\delta)\|\gamma_N\|_{\mathscr{Z}^1_{2,r/2}((t_0,t)\times\Omega)}-c\delta\|\boldsymbol{u}_N\|_{\boldsymbol{\mathcal{V}}^1_{2,r/2}((t_0,t)\times\Omega)}\le c\{\mathsf{N}+\mathsf{B}+\mathsf{F}^2+\mathsf{D}(t_0)\}.\tag{6.24}$$

For the source term in the viscous biharmonic equation, using $\mathscr{Z}^3_{2,2}(Q)\hookrightarrow C(\bar{I};W^{2,2}(\Omega))\hookrightarrow L^r((t_0,t);W^{1,2}(\Omega))$ and $\mathscr{Z}^1_{2,r/2}((t_0,t)\times\Omega)\hookrightarrow L^{r/2}((t_0,t);W^{1,2}_0(\Omega))$, we obtain from (6.8), (6.9) and (6.23) the following estimates:

$$\begin{aligned}\|\lambda_N\|_{L^{r/2}((t_0,t);W^{1,2}_0(\Omega))}&\le c\{\mathscr{C}(\mathsf{F}+\mathsf{N})+\mathsf{B}+\|\gamma_N\|_{\mathscr{Z}^1_{2,r/2}((t_0,t)\times\Omega)}\}\\ \|\sigma_N\|_{L^{r/2}((t_0,t);W^{-1,2}(\Omega))}&\le c\{\mathsf{N}^2+\mathsf{N}+\mathsf{B}+\mathsf{F}^2+c\delta\|\boldsymbol{u}_N\|_{\boldsymbol{\mathcal{V}}^1_{2,r/2}((t_0,t)\times\Omega)}\}.\end{aligned}$$

Recall that $\mathscr{C}$ is a cubic polynomial. Utilizing these in (6.10), multiplying (6.14) by $\frac{1}{2c}$, taking the sum of the resulting inequalities, and then rearranging the terms, one has

$$\begin{aligned}&\frac{1}{2}\|\phi_N\|_{\mathscr{Z}^3_{2,r/2}((t_0,t)\times\Omega)}+\frac{1}{2c}\|\mu_N\|_{L^{r/2}((t_0,t);W^{1,2}_0(\Omega))}-c\|\gamma_N\|_{\mathscr{Z}^1_{2,r/2}((t_0,t)\times\Omega)}\\ &\quad-c\delta\|\boldsymbol{u}_N\|_{\boldsymbol{\mathcal{V}}^1_{2,r/2}((t_0,t)\times\Omega)}\le c\{\mathscr{C}(\mathsf{F}+\mathsf{N})+\mathsf{N}^2+\mathsf{N}+\mathsf{B}+\mathsf{F}^2+\mathsf{D}(t_0)\}.\end{aligned}\tag{6.25}$$

Now, let us consider the source term $\boldsymbol{f}_N$ in the Stokes equation. It is not difficult to deduce from the definitions of F and N the inequalities

$$\begin{aligned}&\|\widetilde{\boldsymbol{u}}\|_{L^r((t_0,t);\boldsymbol{L}^4(\Omega))}+\|\widetilde{\gamma}\|_{L^{r/2}((t_0,t);L^4(\Omega))}+\|\widetilde{\mu}\|_{L^{r/2}((t_0,t);L^q(\Omega))}\\ &\qquad+\|\widetilde{\phi}\|_{L^{r/2}((t_0,t);L^4(\Omega))}+\|\widetilde{\phi}\|_{C([t_0,t];\boldsymbol{W}^{1,4}(\Omega))}\le c\mathsf{F}\\ &\|\phi_N\|_{L^{r/2}((t_0,t);L^2(\Omega))}+\|\gamma_N\|_{L^{r/2}((t_0,t);L^2(\Omega))}+\|\phi_N\|_{L^{r/2}((t_0,t);L^4(\Omega))}\\ &\qquad+\|\phi_N\|_{C([t_0,t];\boldsymbol{W}^{1,4}(\Omega))}\le c\mathsf{N}.\end{aligned}$$

These inequalities, together with (6.22) and (6.23), when applied in (6.13) lead to

$$\|\boldsymbol{f}_N\|_{L^{r/2}((t_0,t);\boldsymbol{W}^{-1,2}(\Omega))}\le c\{\mathsf{N}^2+\mathsf{N}+\mathsf{B}+\mathsf{F}^2+\delta\|\gamma_N\|_{\mathscr{Z}^1_{2,r/2}((t_0,t)\times\Omega)}$$

$$+ \delta\|\boldsymbol{u}_N\|_{\boldsymbol{\mathcal{V}}^1_{2,r/2}((t_0,t)\times\Omega)} + (\mathsf{N}+\mathsf{F})\|\mu_N\|_{L^{r/2}((t_0,t);W^{1,2}_0(\Omega))}\}.$$

Thus, we obtain from (6.15) the remaining estimate that we need

$$\begin{aligned}(1-c\delta)\|\boldsymbol{u}_N\|_{\boldsymbol{\mathcal{V}}^1_{2,r/2}((t_0,t)\times\Omega)} - c(\mathsf{N}+\mathsf{F})\|\mu_N\|_{L^{r/2}((t_0,t);W^{1,2}_0(\Omega))}\\ - c\delta\|\gamma_N\|_{\mathcal{Z}^1_{2,r/2}((t_0,t)\times\Omega)} \le c\{\mathsf{N}^2+\mathsf{N}+\mathsf{B}+\mathsf{F}^2+\mathsf{D}(t_0)\}.\end{aligned} \tag{6.26}$$

We combine the above estimates with suitable weights. Multiplying both sides of (6.25) by ξ and (6.26) by ξ^2, and then taking the sum of the resulting inequalities to (6.24), we get

$$\begin{aligned}(1-c(\delta+\xi+\delta\xi^2))\|\gamma_N\|_{\mathcal{Z}^1_{2,r/2}((t_0,t)\times\Omega)} + \frac{\xi}{2}\|\phi_N\|_{\mathcal{Z}^3_{2,r/2}((t_0,t)\times\Omega)}\\ + \xi\left(\frac{1}{2c} - c\xi(\mathsf{N}+\mathsf{F})\right)\|\mu_N\|_{L^{r/2}((t_0,t);W^{1,2}_0(\Omega))}\\ + (\xi^2(1-c\delta) - c\delta(1+\xi))\|\boldsymbol{u}_N\|_{\boldsymbol{\mathcal{V}}^1_{2,r/2}((t_0,t)\times\Omega)}\\ \le c_\xi\{\mathscr{C}(\mathsf{F}+\mathsf{N}) + \mathsf{N}^2+\mathsf{N}+\mathsf{B}+\mathsf{F}^2+\mathsf{D}(t_0)\}.\end{aligned} \tag{6.27}$$

Note that we can choose $\xi > 0$ and $\delta > 0$ small enough so that the coefficients on the left-hand side are positive. Indeed, one may take

$$0 < \xi < \min\left\{\frac{1}{2c^2(\mathsf{N}+\mathsf{F})}, \frac{1}{2c}\right\}, \qquad 0 < \delta < \min\left\{\frac{\xi^2}{c(1+\xi+\xi^2)}, \frac{1}{2c(1+\xi^2)}\right\}. \tag{6.28}$$

After choosing δ, we take t_0 close enough to t^* such that $0 < t^* - t_0 < \eta_\delta$. Observe that the right-hand side of (6.27) is independent of t.

We finish the proof by a simple bootstrap argument. From Theorem 4.9, it is easy to see that (6.20) is satisfied when $r = 8$. Indeed, from the Gagliardo–Nirenberg inequality we have $(\gamma_N, \boldsymbol{u}_N) \in \mathcal{Z}^1_{2,2}(Q) \times \boldsymbol{\mathcal{V}}^1_{2,2}(Q) \hookrightarrow L^4(I; L^4(\Omega)) \times L^4(I; \boldsymbol{L}^4(\Omega))$. Thus, (6.27) with (6.28) implies that the blow-up (6.19) with $r = 8$ is not possible. As a consequence, this proves that we have a solution over the whole interval I satisfying (6.1) in the case where $r = 8$. Moreover, from Step 1 we know that

$$\mathcal{Z}^1_{2,4}(Q) \times \boldsymbol{\mathcal{V}}^1_{2,4}(Q) \hookrightarrow C(\bar{I}; L^4(\Omega)) \times C(\bar{I}; \boldsymbol{L}^4(\Omega)).$$

Now, suppose that $r > 8$. Applying the previous case $r = 8$ and the above embedding, we infer that (6.20) holds when $r > 8$. Again, the uniform a priori bound (6.27) with (6.28) implies that the blow-up scenario (6.19) will not occur, thereby proving that the weak solution constructed from the previous step exists in I. Finally, the regularity of the associated pressure $\mathfrak{p}_N \in W^{-1,r/2}(I; \widehat{L}^2(\Omega))$ follows from Theorem 3.2 with $p = 2$ and r replaced by $r/2$. This completes the proof of the theorem. □

Let us now state the main result of this section. In the following, the source functions are taken such that

$$\left[\begin{array}{l} \sigma \in L^r(I; X^{-2,q}(\Omega)) + L^{r/2}(I; W^{-1,2}(\Omega)), \\ h \in L^r(I; X^{-2,s}(\Omega)) + L^{r/2}(I; W^{-1,2}(\Omega)), \\ \boldsymbol{f} \in L^r(I; \boldsymbol{X}^{-2,p}(\Omega)) + L^{r/2}(I; \boldsymbol{W}^{-1,2}(\Omega)), \\ \lambda \in L^r(I; L^q(\Omega)) + L^{r/2}(I; W_0^{1,2}(\Omega)) \end{array}\right. \tag{6.29}$$

and the initial data satisfy

$$\phi_0 \in Z^2_{q,r}(\Omega) + Z^3_{2,r/2}(\Omega), \quad \theta_0 \in Z^0_{s,r}(\Omega) + Z^1_{2,r/2}(\Omega), \quad \boldsymbol{u}_0 \in \boldsymbol{V}^0_{p,r}(\Omega) + \boldsymbol{V}^1_{2,r/2}(\Omega). \tag{6.30}$$

Theorem 6.3. *Suppose that $4 \le q, s, p < \infty$, $q \le s$, $8 \le r < \infty$, (6.29) and (6.30) hold. Then the nonlinear system (1.1) possesses a unique very weak solution*

$$\begin{aligned}(\phi, \theta, \boldsymbol{u}, \mu) \in {} & [\mathcal{Z}^2_{q,r}(Q) + \mathcal{Z}^3_{2,r/2}(Q)] \times [\mathcal{Z}^0_{s,r}(Q) + \mathcal{Z}^1_{2,r/2}(Q)] \\ & \times [\boldsymbol{\mathcal{V}}^0_{p,r}(Q) + \boldsymbol{\mathcal{V}}^1_{2,r/2}(Q)] \times [L^r(I; L^q(\Omega)) + L^{r/2}(I; W_0^{1,2}(\Omega))]\end{aligned}$$

with an associated pressure $\mathfrak{p} \in W^{-1,r}(I; \widehat{W}^{-1,p}(\Omega)) + W^{-1,r/2}(I; \widehat{L}^2(\Omega))$.

Proof. One can follow the proof provided in Theorem 4.11 and apply the result of Theorem 6.2 with $(\widetilde{\phi}, \widetilde{\theta}, \widetilde{\boldsymbol{u}}, \widetilde{\mu}) = (\phi_L, \theta_L, \boldsymbol{u}_L, \mu_L)$. Note that the functions defined in (4.75)–(4.78) satisfy $\widetilde{\sigma}, \widetilde{h} \in L^{r/2}(I; W^{-1,2}(\Omega))$, $\widetilde{\lambda} \in L^{r/2}(I; W_0^{1,2}(\Omega))$ and $\widetilde{\boldsymbol{f}} \in L^{r/2}(I; \boldsymbol{W}^{-1,2}(\Omega))$. □

Consider a set of initial data for which

$$\phi_0 \in Z^3_{q,r}(\Omega) + Z^3_{2,r/2}(\Omega), \quad \theta_0 \in Z^1_{s,r}(\Omega) + Z^1_{2,r/2}(\Omega), \quad \boldsymbol{u}_0 \in \boldsymbol{V}^1_{p,r}(\Omega) + \boldsymbol{V}^1_{2,r/2}(\Omega) \tag{6.31}$$

and let the source functions satisfy

$$\left[\begin{array}{l} \sigma \in L^r(I; W^{-1,q}(\Omega)) + L^{r/2}(I; W^{-1,2}(\Omega)), \\ h \in L^r(I; W^{-1,s}(\Omega)) + L^{r/2}(I; W^{-1,2}(\Omega)), \\ \boldsymbol{f} \in L^r(I; \boldsymbol{W}^{-1,p}(\Omega)) + L^{r/2}(I; \boldsymbol{W}^{-1,2}(\Omega)), \\ \lambda \in L^r(I; W_0^{1,q}(\Omega)) + L^{r/2}(I; W_0^{1,2}(\Omega)). \end{array}\right. \tag{6.32}$$

Under these conditions, the analogue of Theorem 6.3 in the context of weak solutions is given in the following theorem.

Theorem 6.4. *Let $\frac{4}{3} \le q, s, p < 2$, $q \le s$, $8 \le r < \infty$, (6.31) and (6.32) be satisfied. Then the nonlinear system (1.1) has a unique weak solution*

$$\begin{aligned}(\phi, \theta, \boldsymbol{u}, \mu) \in {} & [\mathcal{Z}^3_{q,r}(Q) + \mathcal{Z}^3_{2,r/2}(Q)] \times [\mathcal{Z}^1_{s,r}(Q) + \mathcal{Z}^1_{2,r/2}(Q)] \\ & \times [\boldsymbol{\mathcal{V}}^1_{p,r}(Q) + \boldsymbol{\mathcal{V}}^1_{2,r/2}(Q)] \times [L^r(I; W_0^{1,q}(\Omega)) + L^{r/2}(I; W_0^{1,2}(\Omega))]\end{aligned}$$

with an associated pressure $\mathfrak{p} \in W^{-1,r}(I; \widehat{L}^p(\Omega)) + W^{-1,r/2}(I; \widehat{L}^2(\Omega))$.

Proof. Adapt the proof in Theorem 4.12, apply the continuous embeddings $\mathcal{Z}^3_{q,r}(Q) \hookrightarrow \mathcal{Z}^2_{4,r}(Q)$, $\mathcal{Z}^1_{s,r}(Q) \hookrightarrow \mathcal{Z}^0_{4,r}(Q)$, $\boldsymbol{\mathcal{V}}^1_{p,r}(Q) \hookrightarrow \boldsymbol{\mathcal{V}}^0_{4,r}(Q)$ and $L^r(I; W_0^{1,q}(\Omega)) \hookrightarrow L^r(I; L^4(\Omega))$ for the frozen coefficients, and then utilize

Theorem 6.3. □

Remark 6.5. *We would like to point out that the proof for the embedding*

$$\boldsymbol{\mathcal{V}}^1_{2,4}(Q) = W^{1,4}(I; \boldsymbol{X}^{1,2}_\sigma(\Omega), \boldsymbol{X}^{-1,2}_\sigma(\Omega)) \hookrightarrow C(\bar{I}; \boldsymbol{L}^4(\Omega))$$

provided in [18, Theorem 2.9] *was not entirely correct. The mistake was due to the use of the invalid embedding* $\boldsymbol{W}^{-1,2}(\Omega) \hookrightarrow \boldsymbol{W}^{-1,4}(\Omega)$*. Nevertheless, we have resolved this issue thanks to* (6.16)*.*

7. Appendix

7.1. A Space-Time Version of de Rham's Theorem. We prove a space-time version of the classical de Rham's theorem. The following proposition is an extension of the one stated in [27, Lemma 72.8], in particular, the case where $p = r = 2$ and $k = 1$.

Proposition 7.1. *Let* $p, r \in (1, \infty)$ *and* k *be a positive integer. Then* $\mathfrak{L} \in W^{-1,r}(I; \boldsymbol{W}^{-k,p}(\Omega))$ *satisfies*

$$\langle \mathfrak{L}, \boldsymbol{\varrho} \rangle_{W^{-1,r}(I;\boldsymbol{W}^{-k,p}(\Omega)), W^{1,r'}_0(I;\boldsymbol{W}^{k,p'}_0(\Omega))} = 0 \qquad \forall\, \boldsymbol{\varrho} \in W^{1,r'}_0(I; \boldsymbol{W}^{k,p'}_0(\Omega) \cap \boldsymbol{L}^{p'}_\sigma(\Omega))$$

if and only if there exists a unique $\mathfrak{p} \in W^{-1,r}(I; \widehat{W}^{1-k,p}(\Omega))$ *such that* $\mathfrak{L} = \nabla \mathfrak{p}$ *in the distributional sense, that is,*

$$\langle \mathfrak{L}, \boldsymbol{\rho} \rangle_{W^{-1,r}(I;\boldsymbol{W}^{-k,p}(\Omega)), W^{1,r'}_0(I;\boldsymbol{W}^{k,p'}_0(\Omega))} = -\langle \mathfrak{p}, \operatorname{div} \boldsymbol{\rho} \rangle_{W^{-1,r}(I;\widehat{W}^{1-k,p}(\Omega)), W^{1,r'}_0(I;\widehat{W}^{k-1,p'}_0(\Omega))}$$

for every $\boldsymbol{\rho} \in W^{1,r'}_0(I; \boldsymbol{W}^{k,p'}_0(\Omega))$*. In this case, there exists a constant* $c > 0$ *such that*

$$\|\mathfrak{p}\|_{W^{-1,r}(I;\widehat{W}^{1-k,p}(\Omega))} \le c \|\mathfrak{L}\|_{W^{-1,r}(I;\boldsymbol{W}^{-k,p}(\Omega))}.$$

Proof. We proceed by a duality argument. First, let us note that the linear operator

$$\operatorname{div} : L^{r'}(I; \boldsymbol{W}^{k,p'}_0(\Omega)) \to L^{r'}(I; \widehat{W}^{k-1,p'}_0(\Omega))$$

is bounded and surjective, see for instance Lemma II.2.1.1 and Lemma II.2.3.1 in [61] for the time-independent case. We claim that the restriction

$$\widetilde{\operatorname{div}} := \operatorname{div} : W^{1,r'}_0(I; \boldsymbol{W}^{k,p'}_0(\Omega)) \to W^{1,r'}_0(I; \widehat{W}^{k-1,p'}_0(\Omega)) \tag{7.1}$$

is also bounded and surjective. It is clear that (7.1) is well-defined, linear and bounded.

Let $g \in W^{1,r'}_0(I; \widehat{W}^{k-1,p'}_0(\Omega)) \hookrightarrow C(\bar{I}; \widehat{W}^{k-1,p'}_0(\Omega)) \hookrightarrow L^{r'}(I; \widehat{W}^{k-1,p'}_0(\Omega))$. Then there is a $\boldsymbol{v} \in L^{r'}(I; \boldsymbol{W}^{k,p'}_0(\Omega))$ such that $\operatorname{div} \boldsymbol{v} = \partial_t g$ almost everywhere in Q. For each $t \in [0, T]$, let us define

$$\boldsymbol{w}(t) := \frac{T-t}{T} \int_0^T \boldsymbol{v}(s)\, \mathrm{d}s - \int_t^T \boldsymbol{v}(s)\, \mathrm{d}s.$$

It is easy to see that $\boldsymbol{w} \in W^{1,r'}_0(I; \boldsymbol{W}^{k,p'}_0(\Omega))$, and for all $t \in [0, T]$ we have

$$\widetilde{\operatorname{div}}\, \boldsymbol{w}(t) = \frac{T-t}{T} \int_0^T \partial_t g(s)\, \mathrm{d}s - \int_t^T \partial_t g(s)\, \mathrm{d}s = g(t)$$

since $g(0) = g(T) = 0$ in $\widehat{W}_0^{k-1,p'}(\Omega)$. We point out that the insertion of the divergence operator inside the integral is valid since div is linear and continuous, see for instance [48, Chap. III, Theorem 3.7.12]. This shows that the map (7.1) is surjective.

It follows from the closed range theorem [67, page 205] that the dual operator $-\nabla = \widetilde{\mathrm{div}}' : W^{-1,r}(I; \widehat{W}^{1-k,p}(\Omega)) \to W^{-1,r}(I; \boldsymbol{W}^{-k,p}(\Omega))$ has a trivial kernel and a range $\mathrm{Ran}(-\nabla)$ that is closed with respect to the topology of $W^{-1,r}(I; \boldsymbol{W}^{-k,p}(\Omega))$. As a consequence, the inverse $(-\nabla)^{-1}$ is a well-defined, linear and bounded operator from $\mathrm{Ran}(-\nabla)$ onto $W^{-1,r}(I; \widehat{W}^{1-k,p}(\Omega))$. Thus, if $\mathfrak{L} \in W^{-1,r}(I; \boldsymbol{W}^{-k,p}(\Omega))$ vanishes on $W_0^{1,r'}(I; \boldsymbol{W}_0^{k,p'}(\Omega) \cap \boldsymbol{L}_\sigma^{p'}(\Omega)) = \mathrm{Ker}(\widetilde{\mathrm{div}})$, then $\mathfrak{L} \in \mathrm{Ker}(\widetilde{\mathrm{div}})^\perp = \mathrm{Ran}(-\nabla)$. Therefore, we may take $\mathfrak{p} = -(-\nabla)^{-1}\mathfrak{L} = \nabla^{-1}\mathfrak{L} \in W^{-1,r}(I; \widehat{W}^{1-k,p}(\Omega))$ with norm

$$\|\mathfrak{p}\|_{W^{-1,r}(I;\widehat{W}^{1-k,p}(\Omega))} \le \|(-\nabla)^{-1}\|_{\mathcal{L}(\mathrm{Ran}(-\nabla), W^{-1,r}(I;\widehat{W}^{1-k,p}(\Omega)))} \|\mathfrak{L}\|_{W^{-1,r}(I;\boldsymbol{W}^{-k,p}(\Omega))}.$$

The converse of the first statement in the proposition is trivial. □

7.2. Analyticity of the Semigroup for the Linearized System.

In the following, we prove that the linear operator $-\boldsymbol{\mathcal{A}}$ generates a strongly continuous analytic semigroup on $\boldsymbol{\mathcal{H}}_\omega$, where $\boldsymbol{\mathcal{A}}$ is defined by (3.30). The sesqui-linear form associated with $\boldsymbol{\mathcal{A}}$ is given by

$$(\boldsymbol{\mathcal{A}}(\phi, \gamma, \boldsymbol{u}), (\psi, \eta, \boldsymbol{v}))_{\boldsymbol{\mathcal{H}}_\omega} = \mathfrak{a}((\phi, \gamma, \boldsymbol{u}), (\psi, \eta, \boldsymbol{v}))$$

where $\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2$ and

$$\mathfrak{a}_1((\phi, \gamma, \boldsymbol{u}), (\psi, \eta, \boldsymbol{v})) := \int_\Omega \{\omega m^2 \tau \epsilon \nabla\Delta\phi \cdot \nabla\Delta\overline{\psi} + \kappa\nabla\gamma \cdot \nabla\overline{\eta} + \nu\nabla\boldsymbol{u} : \nabla\overline{\boldsymbol{v}}\}\,\mathrm{d}x$$

$$\begin{aligned}\mathfrak{a}_2((\phi, \gamma, \boldsymbol{u}), (\psi, \eta, \boldsymbol{v})) :=& -\int_\Omega \omega m[(\beta_1 - l_{\mathrm{c}} l_{\mathrm{h}})\Delta\phi - l_{\mathrm{c}}\Delta\gamma](m\tau\Delta\overline{\psi} - \overline{\psi})\,\mathrm{d}x \\ &- \int_\Omega \{(\kappa l_{\mathrm{h}}\Delta\phi)\overline{\eta} + (\alpha\boldsymbol{g}\cdot\boldsymbol{u})\overline{\eta} + [(\alpha_1 + \alpha_2 l_{\mathrm{h}})\phi + \alpha_2\gamma]\boldsymbol{g}\cdot\overline{\boldsymbol{v}}\}\,\mathrm{d}x.\end{aligned}$$

Given $\beta \in (0, \pi)$, we denote the sector $\Sigma_\beta := \{\zeta \in \mathbb{C} \setminus \{0\} : |\arg\zeta| < \pi - \beta\}$. First, we prove the following elementary inequality.

Lemma 7.2. *For each $\beta \in (0, \pi)$, there exists $\tau_\beta > 0$ such that for every $a, b \ge 0$ and $\zeta \in \Sigma_\beta$ there holds $|a\zeta + b| \ge \tau_\beta(a|\zeta| + b)$.*

Proof. Suppose $a, b > 0$. Setting $z = a\zeta/b$, it suffices to show that $|z + 1| \ge \tau_\beta(|z| + 1)$ for every $z \in \Sigma_\beta$. Write z in its polar form $z = re^{i\vartheta}$ where $|\vartheta| < \pi - \beta$ and $r > 0$. Let $\delta_\beta := \cos(\pi - \beta) > -1$. Then

$$\frac{|z+1|^2}{(|z|+1)^2} = \frac{r^2 + 2r\cos\vartheta + 1}{(r+1)^2} \ge 1 - \frac{2(1-\delta_\beta)r}{(r+1)^2} \ge 1 - \frac{1-\delta_\beta}{2} =: c_\beta$$

for every $r > 0$, where $c_\beta > 0$. We may then take $\tau_\beta = \min\{1, c_\beta^{1/2}\}$, and this clearly covers the case when $a = 0$ or $b = 0$. □

Proposition 7.3. *For small enough $\omega > 0$, the linear operator $-\boldsymbol{\mathcal{A}} : D(\boldsymbol{\mathcal{A}}) \subset \boldsymbol{\mathcal{H}}_\omega \to \boldsymbol{\mathcal{H}}_\omega$ generates an analytic C_0-semigroup on $\boldsymbol{\mathcal{H}}_\omega$.*

Proof. It is clear that $\boldsymbol{\mathcal{A}}$ is a closed and densely defined linear operator. Let $\delta > 0$ be a constant to be chosen later. Applying integration by parts and Young's inequality, it is not hard to see that for each $(\phi, \gamma, \boldsymbol{u}) \in X^{3,2}(\Omega) \times W_0^{1,2}(\Omega) \times \boldsymbol{X}_\sigma^{1,2}(\Omega)$, we have

$$\mathfrak{a}_1((\phi,\gamma,\boldsymbol{u}),(\phi,\gamma,\boldsymbol{u})) \geq c\{\omega\|\phi\|^2_{X^{3,2}(\Omega)} + \|\gamma\|^2_{W_0^{1,2}(\Omega)} + \|\boldsymbol{u}\|^2_{\boldsymbol{X}_\sigma^{1,2}(\Omega)}\}$$
$$|\mathfrak{a}_2((\phi,\gamma,\boldsymbol{u}),(\phi,\gamma,\boldsymbol{u}))| \leq c_\omega\|(\phi,\gamma,\boldsymbol{u})\|^2_{\boldsymbol{\mathcal{H}}_\omega} + \omega\{\delta\|\phi\|^2_{X^{3,2}(\Omega)} + c_\delta\|\gamma\|^2_{W_0^{1,2}(\Omega)}\}$$

where $c = \min\{m^2\tau\epsilon, \kappa, \nu\} > 0$ and $c_\omega, c_\delta > 0$ are independent of $(\phi, \gamma, \boldsymbol{u})$. Let $\beta \in (0, \pi)$ be fixed. If $\varpi \geq 0$ and $\zeta \in \Sigma_\beta$, then by invoking the estimate in the previous lemma, we obtain

$$\begin{aligned}
&|(\zeta+\varpi)\|(\phi,\gamma,\boldsymbol{u})\|^2_{\boldsymbol{\mathcal{H}}_\omega} + \mathfrak{a}((\phi,\gamma,\boldsymbol{u}),(\phi,\gamma,\boldsymbol{u}))| \\
&\quad\geq |(\zeta+\varpi)\|(\phi,\gamma,\boldsymbol{u})\|^2_{\boldsymbol{\mathcal{H}}_\omega} + \mathfrak{a}_1((\phi,\gamma,\boldsymbol{u}),(\phi,\gamma,\boldsymbol{u}))| - |\mathfrak{a}_2((\phi,\gamma,\boldsymbol{u}),(\phi,\gamma,\boldsymbol{u}))| \\
&\quad\geq \tau_\beta\{(|\zeta|+\varpi)\|(\phi,\gamma,\boldsymbol{u})\|^2_{\boldsymbol{\mathcal{H}}_\omega} + \mathfrak{a}_1((\phi,\gamma,\boldsymbol{u}),(\phi,\gamma,\boldsymbol{u}))\} - |\mathfrak{a}_2((\phi,\gamma,\boldsymbol{u}),(\phi,\gamma,\boldsymbol{u}))| \\
&\quad\geq \{\tau_\beta(|\zeta|+\varpi) - c_\omega\}\|(\phi,\gamma,\boldsymbol{u})\|^2_{\boldsymbol{\mathcal{H}}_\omega} + c_{\omega,\delta,\beta}\|(\phi,\gamma,\boldsymbol{u})\|^2_{X^{3,2}(\Omega)\times W_0^{1,2}(\Omega)\times \boldsymbol{X}_\sigma^{1,2}(\Omega)}
\end{aligned}$$

where $c_{\omega,\delta,\beta} = \min\{\omega(c\tau_\beta - \delta), c\tau_\beta - \omega c_\delta\}$. Taking $0 < \delta < c\tau_\beta$, $0 < \omega < c\tau_\beta/c_\delta$, and $\varpi \geq c_\omega/\tau_\beta > 0$, we have $c_{\omega,\delta,\beta} > 0$ and

$$\begin{aligned}
&|(\zeta+\varpi)\|(\phi,\gamma,\boldsymbol{u})\|^2_{\boldsymbol{\mathcal{H}}_\omega} + \mathfrak{a}((\phi,\gamma,\boldsymbol{u}),(\phi,\gamma,\boldsymbol{u}))| \\
&\qquad\geq \tau_\beta|\zeta|\|(\phi,\gamma,\boldsymbol{u})\|^2_{\boldsymbol{\mathcal{H}}_\omega} + c_{\omega,\delta,\beta}\|(\phi,\gamma,\boldsymbol{u})\|^2_{X^{3,2}(\Omega)\times W_0^{1,2}(\Omega)\times \boldsymbol{X}_\sigma^{1,2}(\Omega)}.
\end{aligned} \tag{7.2}$$

Thus, the sesqui-linear form $(\zeta+\varpi)(\cdot,\cdot)_{\boldsymbol{\mathcal{H}}_\omega} + \mathfrak{a}$ is bounded and coercive on $X^{3,2}(\Omega) \times W_0^{1,2}(\Omega) \times \boldsymbol{X}_\sigma^{1,2}(\Omega)$.

For each $(\sigma, h, \boldsymbol{f}) \in X^{2,2}(\Omega) \times L^2(\Omega) \times \boldsymbol{L}^2_\sigma(\Omega)$ the following variational equation for all $(\psi, \eta, \boldsymbol{v}) \in X^{3,2}(\Omega) \times W_0^{1,2}(\Omega) \times \boldsymbol{X}_\sigma^{1,2}(\Omega)$

$$(\zeta+\varpi)((\phi,\gamma,\boldsymbol{u}),(\psi,\eta,\boldsymbol{v}))_{\boldsymbol{\mathcal{H}}_\omega} + \mathfrak{a}((\phi,\gamma,\boldsymbol{u}),(\psi,\eta,\boldsymbol{v})) = ((\sigma,h,\boldsymbol{f}),(\psi,\eta,\boldsymbol{v}))_{\boldsymbol{\mathcal{H}}_\omega} \tag{7.3}$$

admits a unique solution $(\phi, \gamma, \boldsymbol{u}) \in X^{3,2}(\Omega) \times W_0^{1,2}(\Omega) \times \boldsymbol{X}_\sigma^{1,2}(\Omega)$ in virtue of the Lax–Milgram Lemma. Moreover, it follows from the definition of $\boldsymbol{\mathcal{A}}$ that $(\phi, \gamma, \boldsymbol{u})$ is a weak solution to the following system of boundary value problems:

$$\left[\begin{array}{ll}
(\zeta+\varpi)(\phi - m\tau\Delta\phi) + m\epsilon\Delta^2\phi + m(\beta_1 - l_{\mathrm{c}}l_{\mathrm{h}})\Delta\phi - ml_{\mathrm{c}}\Delta\gamma = \sigma - m\tau\Delta\sigma & \text{in } \Omega, \\
(\zeta+\varpi)\gamma - \kappa\Delta\gamma - \kappa l_{\mathrm{h}}\Delta\phi - \alpha\boldsymbol{g}\cdot\boldsymbol{u} + \nabla\mathfrak{p} = h & \text{in } \Omega, \\
(\zeta+\varpi)\boldsymbol{u} - \nu\Delta\boldsymbol{u} - ((\alpha_1 + \alpha_2 l_{\mathrm{h}})\phi + \alpha_2\gamma)\boldsymbol{g} = \boldsymbol{f} & \text{in } \Omega, \\
\operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \\
\phi = \Delta\phi = 0, \quad \gamma = 0, \quad \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma.
\end{array}\right.$$

By classical elliptic regularity theory for the Poisson and stationary Stokes equations, we have $\gamma \in X^{2,2}(\Omega)$ and $\boldsymbol{u} \in \boldsymbol{X}_\sigma^{2,2}(\Omega)$. Thus, we also have $\phi \in X^{4,2}(\Omega)$ for the solution of the above bi-Laplace equation since $\sigma - m\tau\Delta\sigma + ml_{\mathrm{c}}\Delta\gamma \in L^2(\Omega)$. Consequently, it holds that $(\phi, \gamma, \boldsymbol{u}) \in D(\boldsymbol{\mathcal{A}})$.

The variational equation (7.3) is equivalent to $[\zeta\boldsymbol{I} + (\varpi\boldsymbol{I} + \boldsymbol{\mathcal{A}})](\phi, \gamma, \boldsymbol{u}) = (\sigma, h, \boldsymbol{f})$, and moreover, from (7.2) and the Cauchy–Schwarz inequality, one has

$$\tau_\beta|\zeta|\|(\phi,\gamma,\boldsymbol{u})\|_{\boldsymbol{\mathcal{H}}_\omega} \leq \|(\sigma,h,\boldsymbol{f})\|_{\boldsymbol{\mathcal{H}}_\omega}. \tag{7.4}$$

Hence, the sector Σ_β lies in the resolvent set of $-(\varpi \boldsymbol{I}+\boldsymbol{\mathcal{A}})$, and for every $\zeta \in \Sigma_\beta$ the resolvent estimate $\|[\zeta \boldsymbol{I} + (\varpi \boldsymbol{I} + \boldsymbol{\mathcal{A}})]^{-1}\|_{\mathcal{L}(\boldsymbol{\mathcal{H}}_\omega)} \leq \tau_\beta^{-1}/|\zeta|$ holds due to (7.4). These show that $-(\varpi \boldsymbol{I}+\boldsymbol{\mathcal{A}})$ is sectorial, and hence it generates an analytic C_0-semigroup on $\boldsymbol{\mathcal{H}}_\omega$ by [26, Theorem 4.6]. Thanks to the bounded perturbation theorem in [53, Chapter 3, Corollary 2.2], we conclude that $-\boldsymbol{\mathcal{A}} = -(\varpi \boldsymbol{I} + \boldsymbol{\mathcal{A}}) + \varpi \boldsymbol{I}$ is also a generator of an analytic C_0-semigroup on $\boldsymbol{\mathcal{H}}_\omega$. The proposition is now established. □

References

[1] H. Abels. Existence of weak solutions for a diffuse interface model for viscous, incompressible fluids with general densities. *Comm. Math. Phys.*, 289:45–73, 2009. doi:10.1007/s00220-009-0806-4.

[2] H. Abels. On a diffuse interface model for two-phase flows of viscous, incompressible fluids with matched densities. *Arch. Ration. Mech. Anal.*, 194:463–506, 2009. doi:10.1007/s00205-008-0160-2.

[3] H. Abels, D. Depner, and H. Garcke. Existence of weak solutions for a diffuse interface model for two-phase flows of incompressible fluids with different densities. *J. Math. Fluid Mech.*, 15:453–480, 2013. doi:10.1007/s00021-012-0118-x.

[4] H. Abels, D. Depner, and H. Garcke. On an incompressible Navier–Stokes/Cahn–Hilliard system with degenerate mobility. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 30:1175–1190, 2013. doi:10.1016/j.anihpc.2013.01.002.

[5] H. Abels and M. Wilke. Convergence to equilibrium for the Cahn–Hilliard equation with a logarithmic free energy. *Nonlinear Anal.*, 67:3176–3193, 2007. doi:10.1016/j.na.2006.10.002.

[6] R. A. Adams. *Sobolev Spaces.* Academic Press, New York, 1975.

[7] H. Amann. *Linear and Quasilinear Parabolic Problems. Vol. 1.* Birkhäuser, Boston, 1995.

[8] H. Amann. On the strong solvability of the Navier–Stokes equations. *J. Math. Fluid Mech.*, 2:16–98, 2000. doi:10.1007/s000210050018.

[9] H. Amann. Linear parabolic problems involving measures. *Rev. R. Acad. Cien. Serie A. Mat.*, 95:85–119, 2001. doi:10.5167/uzh-21986.

[10] H. Amann. Nonhomogeneous Navier–Stokes equations with integrable low-regularity data. In V.A. Solonnikov M.Sh. Birman, S. Hildebrandt and N.N. Uraltseva, editors, *Non-linear Problems in Mathematical Physics and Related Problems II*, pages 1–26. Kluwer Academic/Plenum Publ., New York, 2002. doi:10.5167/uzh-21915.

[11] H. Amann. Navier–Stokes equations with nonhomogeneous Dirichlet data. *J. Nonlinear Math. Phys.*, 10 (Suppl. 1):1–11, 2003. doi:10.2991/jnmp.2003.10.s1.1.

[12] J. Bergh and J. Löfström. *Interpolation Spaces: An Introduction*. Springer-Verlag, Berlin, 1976.

[13] J. Boussinesq. *Théorie analytique de la chaleur: Mise en harmonie avec la thermodynamique et avec la théorie mécanique de la lumière*, volume 2. Gauthier-Villars, Paris, 1903.

[14] F. Boyer and P. Fabrie. *Mathematical Tools for the Study of the Incompressible Navier–Stokes Equation and Related Models*. Springer, New York, 2013.

[15] G. Caginalp. An analysis of a phase-field model of a free boundary. *Arch. Ration. Mech. Anal.*, 92:205–245, 1986. doi:10.1007/BF00254827.

[16] J. W. Cahn and J. E. Hilliard. Free energy in nonuniform system. I. Interfacial free energy. *J. Chem. Phys.*, 28:258–267, 1958. doi:10.1063/1.1744102.

[17] E. Casas and K. Kunisch. Optimal control of the two-dimensional evolutionary Navier–Stokes equations with measure valued controls. *SIAM J. Control Optim.*, 59:2223–2246, 2021. doi:10.1137/20M1351400.

[18] E. Casas and K. Kunisch. Well-posedness of evolutionary Navier–Stokes equations with forces of low regularity on two dimensional domains. *ESAIM Control Optim. Calc. Var.*, 27:61, 2021. doi:10.1051/cocv/2021058.

[19] R. Chella and J. Viñals. Mixing of a two-phase fluid by a cavity flow. *Phys. Rev. E*, 53:3832–3840, 1996. doi:10.1103/PhysRevE.53.3832.

[20] L. Cherfils and M. Petcu. On the viscous Cahn–Hilliard–Navier–Stokes equations with dynamic boundary conditions. *Commun. Pure Appl. Math.*, 15:1419–1449, 2016. doi:10.3934/cpaa.2016.15.1419.

[21] P. Colli, G. Gilardi, and J. Sprekels. A boundary control problem for the viscous Cahn–Hilliard equation with dynamic boundary conditions. *Appl. Math. Optim.*, 73:195–225, 2016. doi:10.1007/s00245-015-9299-z.

[22] L. de Simon. Un'applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineari astratte del primo ordine. *Rend. Sem. Mat. Univ. Padova*, 34:205–223, 1964. Available from: http://www.numdam.org/article/RSMUP_1964__34__205_0.pdf.

[23] R. Denk, M. Hieber, and J. Prüss. $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Am. Math. Soc. 166*, 166:viii+114, 2003. doi:10.1090/memo/0788.

[24] M. Doi. Dynamics of domains and textures. In *Theoretical Challenges in the Dynamics of Complex Fluids*, pages 293–314, 1997. doi:10.1007/978-94-011-5480-2_18.

[25] C. M. Elliott and A. M. Stuart. Viscous Cahn–Hilliard equation II. Analysis. *J. Differ. Equation.*, 128:387–414, 1996. doi:10.1006/jdeq.1996.0101.

[26] K. J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations*. Springer-Verlag, New York, 2000.

[27] A. Ern and J.-L. Guermond. *Finite Elements III: First-Order and Time-Dependent PDEs*. Springer, Cham, 2021.

[28] L. C. Evans. *Partial Differential Equations*. American Mathematical Society, Providence, 2nd edition, 2010.

[29] R. Farwig, H. Kozono, and H. Sohr. An L^q-approach to Stokes and Navier–Stokes equations in general domains. *Acta Math.*, 195:21–53, 2005. Available from: https://www.jstor.org/stable/24902762.

[30] R. Farwig, H. Kozono, and H. Sohr. The Stokes operator in general unbounded domains. *Hokkaido Math. J.*, 38:111–136, 2009. doi:10.14492/hokmj/1248787007.

[31] R. Farwig and P.F. Riechwald. Very weak solutions to the Navier–Stokes system in general unbounded domains. *J. Evol. Equ.*, 15:253–279, 2015. doi:10.1007/s00028-014-0258-y.

[32] H.O. Fattorini. *Infinite Dimensional Optimization and Control Theory*. Cambridge University Press, Cambridge, 1999.

[33] S. Frigeri, C. G. Gal, M. Grasselli, and J. Sprekels. Two-dimensional nonlocal Cahn–Hilliard–Navier–Stokes systems with variable viscosity, degenerate mobility and singular potential. *Nonlinearity*, 32:678–727, 2019. doi:10.1088/1361-6544/aaedd0.

[34] D. Fujiwara and H. Morimoto. An L_r-theorem of the Helmholtz decomposition of vector fields. *J. Fac. Science Univ. Tokyo Sec.*, 124:685–700, 1977.

[35] C. G. Gal and M. Grasselli. Asymptotic behavior of a Cahn–Hilliard–Navier–Stokes system in 2D. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 27:401–436, 2010. doi:10.1016/j.anihpc.2009.11.013.

[36] C. G. Gal, M. Grasselli, and H. Wu. Global weak solutions to a diffuse interface model for incompressible two-phase flows with moving contact lines and different densities. *Arch. Ration. Mech. Anal.*, 234:1–56, 2019. doi:10.1007/s00205-019-01383-8.

[37] M. Geissert, M. Hess, M. Hieber, C. Schwarz, and K. Stavrakidis. Maximal L^p-L^q-estimates for the Stokes equation: a short proof of Solonnikov's theorem. *J. Math. Fluid Mech.*, 12:47–60, 2010. doi:10.1007/s00021-008-0275-0.

[38] Y. Giga. Analyticity of the semigroup generated by the Stokes operator in L_r spaces. *Math. Zeitschrift*, 178:297–328, 1981. doi:10.1007/BF01214869.

[39] Y. Giga. Domains of fractional powers of the Stokes operator in L^r spaces. *Arch. Rational Mech. Anal.*, 89:251–265, 1985. doi:10.1007/BF00276874.

[40] Y. Giga and H. Sohr. Abstract L^p estimates for the Cauchy problem with applications to the Navier–Stokes equations in exterior domains. *J. Funct. Anal.*, 102:72–94, 1991. doi:10.1016/0022-1236(91)90136-S.

[41] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, Berlin, 1977.

[42] A. Giorgini, A. Miranville, and R. Temam. Uniqueness and regularity for the Navier–Stokes–Cahn–Hilliard system. *SIAM J. Math. Anal.*, 51:2535–2574, 2019. doi:10.1137/18M1223459.

[43] A. Giorgini and R. Temam. Weak and strong solutions to the nonhomogeneous incompressible Navier–Stokes–Cahn–Hilliard system. *J. Math. Pures Appl.*, 144:194–249, 2020. doi:10.1016/j.matpur.2020.08.009.

[44] M. Grinfeld and A. Novick-Cohen. The viscous Cahn–Hilliard equation: Morse decomposition and structure of the global attractor. *Trans. Amer. Math. Soc.*, 351:2375–2406, 1999. Available from: https://www.jstor.org/stable/117876.

[45] M. E. Gurtin. Generalized Ginzburg–Landau and Cahn–Hilliard equations based on microforce balance. *Physica D*, 92:178–192, 1996. doi:10.1016/0167-2789(95)00173-5.

[46] M. E. Gurtin, D. Polignone, and J. Viñals. Two-phase binary fluids and immiscible fluids described by an order parameter. *Math. Models Methods Appl. Sci.*, 6:815–831, 1996. doi:10.1142/S0218202596000341.

[47] M. Hieber and J. Saal. The Stokes equation in the L^p-setting: Well-posedness and regularity properties. In Y. Giga and A. Novotný, editors, *Handbook of Mathematical Analysis in Mechanics of Viscous Fluids*, pages 117–206. Springer, Cham, 2018. doi:10.1007/978-3-319-13344-7_3.

[48] E. Hille and R. S. Phillips. *Functional Analysis and Semi-groups*. American Mathematical Society, Providence, 1957.

[49] H. Kim. Existence and regularity of very weak solutions of the stationary Navier–Stokes equations. *Arch. Rational Mech. Anal.*, 193:117–152, 2009. doi:10.1007/s00205-008-0168-7.

[50] H. Kim. The existence and uniqueness of very weak solutions of the stationary Boussinesq system. *Nonlinear Analysis*, 75:317–330, 2012. doi:10.1016/j.na.2011.08.035.

[51] A. Lunardi. *Analytic Semigroups and Optimal Regularity in Parabolic Problems*. Birkhäuser, Basel, 1995.

[52] A. Oberbeck. Ueber die Wärmeleitung der Flüssigkeiten bei Berücksichtigung der Strömungen infolge von Temperaturdifferenzen. *Ann. Phys.*, 243:271–292, 1879. doi:10.1002/andp.18792430606.

[53] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag, New York, 1983.

[54] G. Peralta. Distributed optimal control of the 2D Cahn–Hilliard–Oberbeck–Boussinesq system for nonisothermal viscous two-phase flows. *Appl. Math. Optim.*, 2021. doi:10.1007/s00245-021-09759-7.

[55] J. Prüss, R. Racke, and S. Zheng. Maximal regularity and asymptotic behavior of solutions for the Cahn–Hilliard equation with dynamic boundary conditions. *Ann. di Mat. Pura ed Appl.*, 185:627–648, 2006. doi:10.1007/s10231-005-0175-3.

[56] J. Prüss and M. Wilke. Maximal L_p-regularity and long-time behaviour of the non-isothermal Cahn–Hilliard equation with dynamic boundary conditions. In E. Koelink, J. van Neerven, B. de Pagter, G. Sweers, A. Luger, and H. Woracek,

editors, *Partial Differential Equations and Functional Analysis. Operator Theory: Advances and Applications*, volume 168, pages 209–236. Birkhäuser, Basel, 2006. doi:10.1007/3-7643-7601-5_13.

[57] T. Roubíček. *Nonlinear Partial Differential Equations with Applications*. Birkhäuser, Basel, 2nd edition, 2013.

[58] D. Serre. Équations de Navier–Stokes stationnaries avec donnes peu reguliéres. *Ann. Sci. Norm. Sup. Pisa*, 10:543–559, 1983. Available from: http://www.numdam.org/item/ASNSP_1983_4_10_4_543_0/.

[59] J. Simon. Compact sets in $L^p(0,T;B)$. *Ann. Mat. Pur. Appel.*, 146:65–96, 1987. doi:10.1007/BF01762360.

[60] J. Simon. On the existence of the pressure for solutions of the variational Navier–Stokes equations. *J. Math. Fluid Mech.*, 1:225–234, 1999. doi:10.1007/s000210050010.

[61] H. Sohr. *The Navier–Stokes Equations: An Elementary Functional Analytic Approach*. Birkhäuser, Berlin, 2001.

[62] R. Temam. *Navier–Stokes Equations, Theory and Numerical Analysis*. AMS Chelsea Publishing, Providence, RI, 2001.

[63] P. Tolksdorf. *On the L^p-theory of the Navier–Stokes Equations on Lipschitz Domains*. PhD thesis, Darmstadt University, 2017. Available from: https://tuprints.ulb.tu-darmstadt.de/5960/.

[64] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators*. North-Holland, Berlin, 1978.

[65] A. Ionescu Tulcea and C. Ionescu Tulcea. *Topics in the Theory of Lifting*. Springer, Heidelberg, 1969.

[66] V. Vergara. Maximal regularity and global well-posedness for a phase field system with memory. *J. Integral Equ. Appl.*, 19:93–115, 2007. doi:10.1216/jiea/1181075424.

[67] K. Yosida. *Functional Analysis*. Springer-Verlag, Heidelberg, sixth edition, 1980.

[68] E. Zeidler. *Nonlinear Functional Analysis and its Applications*, volume I. Springer-Verlag, New York, 1986.