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Energy Method for Exponential Stability of Coupled One-Dimensional Hyperbolic PDE-ODE Systems

ENERGY METHOD FOR EXPONENTIAL STABILITY OF COUPLED ONE-DIMENSIONAL HYPERBOLIC PDE-ODE SYSTEMS

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Abstract.
We consider a hyperbolic system of partial differential equations on a bounded interval coupled with ordinary differential equations on both ends. The evolution is governed by linear balance laws, which we treat with semigroup and time-space methods. Our goal is to establish the exponential stability in the natural state space by utilizing the stability with respect to the first-order energy of the system. Derivation of a priori estimates plays a crucial role in obtaining energy and dissipation functionals. The theory is then applied to specific physical models.

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1. Introduction

When a rigid body on the peripheries of a medium interact with waves in the interior, the dynamics can be described by coupling hyperbolic partial differential equations (PDEs) with ordinary differential equations (ODEs) on the boundaries. In applications, these hybrid systems model physical phenomena such as blood flow [6, 8, 11, 23, 24], valveless pumping [4, 18, 22], fluid-particle interaction, and traffic flow [3], to name a few.

Consider a distributed parameter system that is described by the following one-dimensional hyperbolic system of n linear balance laws with dynamic boundary conditions:

$$\begin{cases} u_t(t,x) + \Lambda u_x(t,x) + Lu(t,x) = 0, & t>0,\ 0<x<\ell, \\ \begin{bmatrix} u^+(t,0) \\ u^-(t,\ell) \end{bmatrix} = K \begin{bmatrix} u^+(t,\ell) \\ u^-(t,0) \end{bmatrix} + Qh(t), & t>0, \\ h'(t) + Hh(t) + G \begin{bmatrix} u^+(t,\ell) \\ u^-(t,0) \end{bmatrix} = 0, & t>0, \\ u(0,x) = u_0(x), & 0<x<\ell, \\ h(0) = h_0, \end{cases} \tag{1.1}$$

where the independent variables t and x denote time and space, respectively. The unknown state variables are $u : (0,\infty) \times (0,\ell) \to \mathbb{R}^n$ and $h : (0,\infty) \to \mathbb{R}^m$ with corresponding initial data u_0 and h_0. The constant matrix $\Lambda \in \mathbb{R}^{n\times n}$ is diagonal with entries

$$\lambda_n \leq \cdots \leq \lambda_{p+1} < 0 < \lambda_1 \leq \cdots \leq \lambda_p$$

for some $1 \leq p \leq n$. Consequently, we can decompose $u = (u^+, u^-)^T$ with the components $u^-(t,x) \in \mathbb{R}^p$ and $u^+(t,x) \in \mathbb{R}^{n-p}$ that propagate in the negative and positive directions, respectively. Here, the superscript T denotes the transposition of a vector or a matrix. The constant matrices $L \in \mathbb{R}^{n\times n}$ and $H \in \mathbb{R}^{m\times m}$ act as damping mechanisms for (1.1) provided that they are positive semi-definite. Without the source term L and damping on the boundaries, the state variables in (1.1) are conserved.

On the other hand, the constant matrices $Q \in \mathbb{R}^{n\times m}$ and $G \in \mathbb{R}^{m\times n}$ can be viewed as feedback interconnections between u and h at the boundaries, see [21] for instance. In the absence of the PDE-ODE coupling on the boundary, that is, when Q and G are both zero, the second equation in (1.1) reduces to the static boundary condition

$$\begin{bmatrix} u^+(t,0) \\ u^-(t,\ell) \end{bmatrix} = K \begin{bmatrix} u^+(t,\ell) \\ u^-(t,0) \end{bmatrix}, \quad t > 0,$$

where $K \in \mathbb{R}^{n\times n}$ is constant. In this setting, the incoming characteristics are determined by the outgoing characteristics, regarded as reflections of waves on the boundaries [25].

Existence and uniqueness of the solutions of systems with static boundary conditions are presented in [7, 9, 12]. Well-posedness in Hilbert spaces is proved in [25, Theorem 3.1], as well as in [1] by semigroup methods.

In this study, the well-posedness of (1.1) will be established using the classical Lumer-Phillips Theorem. As usual, the first step is to reformulate the system as an initial value problem on a suitable state space. The infinitesimal generator of the semigroup associated with the resulting system is then shown to satisfy properties required by the classical theorem. In [1, Appendix A], this is done by applying the theorem to a closed operator and its adjoint. In contrast to their work, we shall consider a more straightforward approach where the existence of a weak solution for a two-point boundary value problem is directly established thanks to the diagonal form of the system.

We transform (1.1) so that u is in one direction of the input, and in this case, to the right. To this end, we introduce v defined by $v(t,x) := u^-(t,\ell - x)$. Setting $\tilde{u} := (u^+, v)^T$, system (1.1) is equivalent to

$$\begin{cases} \tilde{u}_t(t,x) + \tilde{\Lambda}\tilde{u}_x(t,x) + L\tilde{u}(t,x) = 0, & t>0,\ 0<x<\ell, \\ h'(t) + Hh(t) + G\tilde{u}(t,\ell) = 0, & t > 0, \\ \tilde{u}(t,0) = K\tilde{u}(t,\ell) + Qh(t), & t>0, \\ \tilde{u}(0,x) = \tilde{u}_0(x), & 0<x<\ell, \\ h(0) = h_0, \end{cases} \tag{1.2}$$

where $\tilde{u}_0(x) = (u_0^+(x),\ u_0^-(\ell - x))^T$ and $\tilde{\Lambda}(x) = \mathrm{diag}(\lambda_1, \dots, \lambda_p, -\lambda_{p+1}, \dots, -\lambda_n)$. Observe that $\tilde{\Lambda}$ has positive diagonal entries. For simplicity, we remove the tildes in the following discussions.

Under suitable conditions, we shall construct a stability theory for systems of the form (1.2), and apply it to specific physical models. In certain instances, the exponential stability with respect to the state space energy can be directly derived using Lyapunov, energy, and spectral methods. The energy method is a popular strategy in showing the stability of systems defined in the entire space. However, employing the energy method to some physical systems on bounded domains necessitates the inclusion of first-order derivatives of the solution, and this requires additional regularity and compatibility conditions on the data. With this observation, we shall first establish stability with respect to the first-order energy by developing appropriate energy and dissipation functionals of the system. Then, with additional assumptions on the semigroup, the stability will be lifted back to the state space energy. Systems of the form (1.2) may have nontrivial equilibrium states, and in such situations, we shall decompose the state space in terms of the equilibrium states.

We organize this paper as follows. In the next section, we establish the well-posedness of the system (1.2). The main results are presented in Section 3. Finally, we apply the results to specific physical examples in Section 4: (1) a low loss electrical line, (2) linearized two-tank model and (3) wave equation with oscillatory boundary conditions.

Notations. Given $\ell > 0$ and a Banach space Z, the set of all Lebesgue square-integrable functions from $(0,\ell)$ to Z is denoted by $L^2(0,\ell;Z)$. The set of all k-times continuously differentiable functions from $[0,\ell]$ to Z will be written as $C^k([0,\ell];Z)$. Throughout the paper, we set $X := L^2(0,\ell;\mathbb{R}^n)\times\mathbb{R}^m$, and its subspace $Y := H^1(0,\ell;\mathbb{R}^n)\times\mathbb{R}^m$. We use the notation $|M|$ for the operator norm of a constant matrix M.

2. WELL-POSEDNESS OF THE HYPERBOLIC PDE-ODE SYSTEM

We establish the well-posedness of (1.2) using semigroups of bounded linear operators. The first step is to recast the system as a differential equation in an infinite-dimensional state space. Given $\mu \in \mathbb{R}$, we denote by $L^2_\mu(0,\ell)$ the space of L^2-functions $u : (0,\ell) \to \mathbb{R}^n$ equipped with the weighted norm

$$\|u\|_{L^2_\mu} := \left(\int_0^\ell e^{\mu(x-\ell)} u(x)^2 \,\mathrm{d}x\right)^{1/2}.$$

Note that $L^2_\mu(0,\ell)$ coincides topologically with the usual Lebesgue space $L^2(0,\ell)$.

Let X be endowed with the inner product

$$\langle (u,h),(\varphi,\psi)\rangle_X := \langle u,\varphi\rangle_{L^2_\mu} + \nu h^T\psi,$$

where $\nu > 0$. Consider the linear operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ defined by

$$\mathcal{A}\begin{bmatrix} u \\ h\end{bmatrix} := \begin{bmatrix} -\Lambda u_x - Lu \\ -Hh - Gu(\ell)\end{bmatrix}, \tag{2.1}$$

with domain $D(\mathcal{A}) := \{(u,h) \in Y : u(0) = Ku(\ell) + Qh\}$. Defining $z := (u,h)$, we write system (1.2) in an abstract form on X:

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}t}(t) = \mathcal{A}z(t), \quad t > 0, \\ z(0) = z_0, \end{cases} \tag{2.2}$$

where $z_0 = (u_0,h_0)$. The *first-order energy* of a solution z at time t of (2.2) is defined by

$$\|z(t)\|_{D(\mathcal{A})} := (\|z(t)\|_X^2 + \|\mathcal{A}z(t)\|_X^2)^{\frac{1}{2}}.$$

We now state our well-posedness result.

Theorem 2.1. *The linear operator $\mathcal{A}$ in (2.1) generates a strongly continuous semigroup, also called $\mathcal{C}_0$-semigroup, of bounded linear operators in X. In particular, for every $z_0 \in D(\mathcal{A})$, system (2.2) has a unique classical solution $z \in C([0,T]; D(\mathcal{A})) \cap C^1([0,T]; X)$.*

Proof. According to the bounded perturbation theorem for operator semigroups [10, p. 158], it is enough to prove the theorem when L and H are both zero matrices. We proceed in two steps.

Step 1. Quasi-Dissipativity. Let $\mu > 0$ and $\nu > 0$ be constants that will be chosen below. Denote by λ_s and λ_g the smallest and largest positive diagonal entries of Λ, respectively, so that $\lambda_s \le |\Lambda| \le \lambda_g$. For $z := (u, h) \in D(\mathcal{A})$, we have

$$\langle \mathcal{A}z, z\rangle_X = -\langle \Lambda u_x, u\rangle_{L^2_\mu} - \nu h^T G u(\ell).$$

Applying integration by parts yields

$$-\langle \Lambda u_x, u\rangle_{L^2_\mu} = \frac{1}{2}e^{-\mu\ell}u(0)^T\Lambda u(0) - \frac{1}{2}u(\ell)^T\Lambda u(\ell) + \frac{\mu}{2}\langle u, \Lambda u\rangle_{L^2_\mu}.$$

With the boundary condition $u(0) = Ku(\ell) + Qh$, we can estimate this from above:

$$-\langle \Lambda u_x, u\rangle_{L^2_\mu} \le \frac{1}{2}\mu\lambda_g\|u\|^2_{L^2_\mu} + \frac{1}{2}(e^{-\mu\ell}(P_1 + 2P_2) - P_3),$$

where $P_1 = h^TQ^T\Lambda Qh$, $P_2 = h^TQ^T\Lambda Ku(\ell)$, and $P_3 = u(\ell)^T\left(\Lambda - e^{-\mu\ell}K^T\Lambda K\right)u(\ell)$. The Cauchy-Schwarz inequality allows us to estimate further: $P_1 \le \lambda_g|Q|^2|h|^2$, $2P_2 \le \lambda_g(|Q|^2|h|^2 + |K|^2|u(\ell)|^2)$, and $P_3 \ge \left(\lambda_s - \lambda_g|K|^2e^{-\mu\ell}\right)|u(\ell)|^2$. Gathering these computations, we obtain

$$-\langle \Lambda u_x, u\rangle_{L^2_\mu} \le \frac{1}{2}\mu\lambda_g\|u\|^2_{L^2_\mu} + \lambda_g|Q|^2|h|^2e^{-\mu\ell} - \frac{1}{2}\left(\lambda_s - 2\lambda_g|K|^2e^{-\mu\ell}\right)|u(\ell)|^2. \quad (2.3)$$

Using Cauchy-Schwarz inequality once more,

$$|\nu h^TGu(\ell)| \le \frac{\nu}{2}\left(|h|^2 + |G|^2|u(\ell)|^2\right). \quad (2.4)$$

Now, inequalities (2.3) and (2.4) imply

$$\langle \mathcal{A}z, z\rangle_X \le c_1\|u\|^2_{L^2_\mu} + c_2|h|^2 - c_3|u(\ell)|^2,$$

where $c_1 = \frac{1}{2}\mu\lambda_g$, $c_2 = \lambda_g|Q|^2e^{-\mu\ell} + \frac{\nu}{2}$ and $c_3 = \frac{1}{2}(\lambda_s - 2\lambda_g|K|^2e^{-\mu\ell} - \nu|G|^2)$. We can then choose $\mu > 0$ to be large enough, and $\nu > 0$ to be small enough such that $c_3 > 0$. Consequently,

$$\langle \mathcal{A}z, z\rangle_X \le c\|z\|_X, \quad \text{with } c := \max\{c_1, c_2\} > 0, \quad (2.5)$$

allowing us to conclude that $\mathcal{A}$ is quasi-dissipative.

Step 2. Range Condition. We prove that $R(\lambda I - \mathcal{A}) = X$ for a constant $\lambda > c$, where c is defined in (2.5). This is equivalent to solving the two-point boundary value problem: Given $(f, g) \in X$, find $(u, h) \in Y$ such that

$$\begin{cases} \Lambda\dfrac{\mathrm{d}u}{\mathrm{d}x} + \lambda u = f, \quad 0 < x < \ell, \\ Gu(\ell) + \lambda h = g, \\ u(0) = Ku(\ell) + Qh. \end{cases} \quad (2.6)$$

From the variation of parameters formula, the solution of the first equation in (2.6) is given by

$$u(x) = e^{-\lambda x\Lambda^{-1}}u(0) + \int_0^x e^{-\lambda(y-x)\Lambda^{-1}}\Lambda^{-1}f(y)\,\mathrm{d}y. \tag{2.7}$$

Using (2.7) and the second equation in (2.6), we obtain

$$h = \frac{1}{\lambda}\left(g - Ge^{-\lambda\ell\Lambda^{-1}}u(0) - \int_0^\ell Ge^{-\lambda(y-\ell)\Lambda^{-1}}\Lambda^{-1}f(y)\,\mathrm{d}y\right). \tag{2.8}$$

By the continuity of the operator norm, there exists $\lambda > c$ such that $|(K - \lambda^{-1}QG)e^{-\lambda\ell\Lambda^{-1}}| < 1$. This implies that the matrix $S_\lambda := I - (K - \lambda^{-1}QG)e^{-\lambda\ell\Lambda^{-1}}$ is invertible. Taking $x = \ell$ in (2.6), and using the third equation in (2.6) as well as (2.8), it holds that

$$u(0) = S_\lambda^{-1}\left(\frac{1}{\lambda}Qg + \int_0^\ell (K - \lambda^{-1}QG)e^{-\lambda(y-\ell)\Lambda^{-1}}\Lambda^{-1}f(y)\,\mathrm{d}y\right). \tag{2.9}$$

Substituting (2.9) in (2.7) and (2.8) then yields a pair $(u, h) \in Y$ that is a solution of (2.6).

Steps 1 and 2 allow us to conclude from the Lumer-Phillips Theorem that $\mathcal{A}$ generates a $\mathcal{C}_0$-semigroup on X, which we denote by $(e^{t\mathcal{A}})_{t\geq 0}$. The well-posedness of (2.2) immediately follows. □

3. Stability for a One-Dimensional Hyperbolic PDE-ODE System

We equip the domain $D(\mathcal{A})$ with the graph norm

$$\|(u,h)\|^2_{D(\mathcal{A})} := \|\mathcal{A}(u,h)\|^2_X + \|(u,h)\|^2_X. \tag{3.1}$$

Note that $D(\mathcal{A}) \subset X$ is a Hilbert space with respect to the inner product induced by (3.1). We also consider a weighted graph norm on $D(\mathcal{A})$ defined by

$$\|(u,h)\|^2_{D(\mathcal{A}),\varepsilon,\delta} := \varepsilon\|\Lambda u_x + Lu\|^2_{L^2(0,\ell;\mathbb{R}^n)} + \varepsilon\delta|Hh + Gu(\ell)|^2 + \|u\|^2_{L^2(0,\ell;\mathbb{R}^n)} + |h|^2, \tag{3.2}$$

where $\varepsilon, \delta > 0$. If $\varepsilon = \delta = 1$, then (3.2) coincides with the graph norm (3.1), and for sufficiently small $\varepsilon, \delta > 0$, they are equivalent. In the following proposition, we prove that the weighted norm (3.2) is equivalent to the norm on Y.

Proposition 3.1. *There exist constants $\varepsilon > 0$ and $\delta > 0$ such that the weighted graph norm (3.2) is equivalent to the usual norm on Y, that is, for some $c_2 \geq c_1 > 0$,*

$$c_1\|(u,h)\|^2_Y \leq \|(u,h)\|^2_{D(\mathcal{A}),\varepsilon,\delta} \leq c_2\|(u,h)\|^2_Y, \quad \forall (u,h) \in D(\mathcal{A}).$$

Proof. The estimate $\|(u,h)\|^2_{D(\mathcal{A}),\varepsilon,\delta} \leq c_2\|(u,h)\|^2_Y$ follows immediately from the Cauchy-Schwarz inequality and the trace theorem: there exists a constant $c > 0$ such that

$$|u(0)| + |u(\ell)| \leq c\|u\|_Y, \quad \forall u \in Y,$$

see for instance [5, Theorem 8.2].

For the reverse inequality, we use a consequence of Young's inequality: there exist constants $\gamma, c_\gamma > 0$ such that for every $a, b \in \mathbb{R}^n$, it holds that

$$|a+b|^2 \geq (1-\gamma)|a|^2 - (c_\gamma - 1)|b|^2.$$

This allows for the computation

$$\|\Lambda u_x + Lu\|_{L^2}^2 \geq \lambda_s^2(1-\gamma)\|u_x\|_{L^2}^2 - |L|^2(c_\gamma - 1)\|u\|_{L^2}^2.$$

Again by the trace theorem, there exists a constant $c > 0$ such that

$$-|u(\ell)|^2 \geq -c(\|u_x\|_{L^2}^2 + \|u\|_{L^2}^2).$$

Setting $C_1 := \varepsilon((1-\gamma)\lambda_s^2 - 2c\delta|G|^2)$, $C_2 := 1 - \varepsilon(|L|^2(c_\gamma - 1) + 2c\delta|G|^2)$, and $C_3 := 1 - 2\varepsilon\delta|H|^2$, we estimate from below:

$$\|(u,h)\|_{D(A),\varepsilon,\delta}^2 \geq C_1\|u_x\|_{L^2}^2 + C_2\|u\|_{L^2}^2 + C_3|h|^2.$$

We choose $\delta, \gamma, \varepsilon > 0$ in succession as follows:

$$\delta < \frac{\lambda_s^2}{2c|G|^2}, \quad \gamma < 1 - \frac{2c\delta}{\lambda_s^2}|G|^2, \quad \varepsilon < \min\left\{\frac{1}{2\delta|H|^2}, \frac{1}{|L|^2(c_\gamma - 1) + 2c\delta|G|^2}\right\},$$

so that $C_1, C_2, C_3 > 0$. Finally, by taking $c_1 := \min\{C_1, C_2, C_3\}$, we arrive at the desired estimate. □

In the following, we denote $X_0 := \ker \mathcal{A}$ and its orthogonal complement by $X_0^\perp := (\ker \mathcal{A})^\perp$. The part of $\mathcal{A}$ in $X_0^\perp$ is defined as $\mathcal{A}_0 : D(\mathcal{A}_0) \to X_0^\perp$, together with its domain $D(\mathcal{A}_0) := D(\mathcal{A}) \cap X_0^\perp$. According to [26, Proposition 2.4.4], if the closed subspace $X_0^\perp$ of X is invariant under $(e^{t\mathcal{A}})_{t\geq 0}$, then $\mathcal{A}_0$ generates a $\mathcal{C}_0$-semigroup $(e^{t\mathcal{A}_0})_{t\geq 0}$ on $X_0^\perp$. Furthermore, the restriction of the semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ on $X_0^\perp$ is $(e^{t\mathcal{A}_0})_{t\geq 0}$, that is,

$$(e^{t\mathcal{A}})|_{X_0^\perp} = (e^{t\mathcal{A}_0}), \qquad \forall t \geq 0. \tag{3.3}$$

The next lemma gives a necessary condition for the semigroup invariance of $X_0^\perp$ under $(e^{t\mathcal{A}})_{t\geq 0}$. It involves the adjoint operator of $\mathcal{A}$, which we denote by $\mathcal{A}^*$.

Lemma 3.2. *The subspace $X_0^\perp$ is invariant under $(e^{t\mathcal{A}})_{t\geq 0}$ whenever $X_0 \subset \ker \mathcal{A}^*$.*

Proof. The adjoint semigroup $(e^{t\mathcal{A}^*})_{t\geq 0} = (e^{t\mathcal{A}})^*\big|_{t\geq 0}$ of $(e^{t\mathcal{A}})_{t\geq 0}$ satisfies

$$\langle e^{t\mathcal{A}}v, w\rangle_X = \langle v, e^{t\mathcal{A}^*}w\rangle_X, \quad \forall v \in X_0^\perp,\ \forall w \in X_0, \tag{3.4}$$

see for instance [19, p. 41, Corollary 10.6]. We know from the assumption that $w \in \ker \mathcal{A}^*$. As a result, $\langle v, e^{t\mathcal{A}^*}w\rangle_X = \langle v, w\rangle_X = 0$ for every $v \in X_0^\perp$. Equation (3.4) now reads as $\langle e^{t\mathcal{A}}v, w\rangle_X = 0$. More precisely, $e^{t\mathcal{A}}v \in X_0^\perp$ for every $v \in X_0^\perp$ and $t \geq 0$, which proves semigroup invariance. □

The next step is to compute for the adjoint $\mathcal{A}^*$. Integrating by parts, we have

$$\langle \mathcal{A}(u,h), (v,g)\rangle_X = \langle (u,h), \tilde{\mathcal{A}}(v,g)\rangle_X, \quad \forall (u,h) \in D(\mathcal{A}), \forall (v,g) \in D(\tilde{\mathcal{A}}), \tag{3.5}$$

where $\tilde{\mathcal{A}} : D(\tilde{\mathcal{A}}) \subset X \to X$ is defined by

$$\tilde{\mathcal{A}}\begin{bmatrix} v \\ g \end{bmatrix} = \begin{bmatrix} \Lambda v_x - L^T v \\ -H^T g + Q^T \Lambda v(0) \end{bmatrix}, \tag{3.6}$$

with domain $D(\tilde{\mathcal{A}}) := \{(v, g) \in Y : v(\ell) = K^T v(0) - \Lambda^{-1} G^T g\}$. We deduce from identity (3.5) that $(v, g) \in D(\mathcal{A}^*)$, and that $\mathcal{A}^*$ is an extension of $\tilde{\mathcal{A}}$. As in the proof of Theorem 2.1, we can show that $\tilde{\mathcal{A}}$ generates a $\mathcal{C}_0$-semigroup on X. Thus, there exists $\tilde{\omega}_0 \in \mathbb{R}$ such that $(\tilde{\omega}_0, \infty) \subset \rho(\tilde{\mathcal{A}})$. Similarly, $\mathcal{A}^*$ generates a $\mathcal{C}_0$-semigroup on X, and there exists $\omega_0 \in \mathbb{R}$ such that $(\omega_0, \infty) \subset \rho(\mathcal{A}^*)$. It follows that $(\omega, \infty) \subset \rho(\tilde{\mathcal{A}}) \cap \rho(\mathcal{A}^*)$, where $\omega = \max\{\tilde{\omega}_0, \omega_0\}$. Invoking [15, Lemma 1.6.14], these operators and their domains coincide, i.e. $\mathcal{A}^* = \tilde{\mathcal{A}}$ and $D(\mathcal{A}^*) = D(\tilde{\mathcal{A}})$.

We next determine X_0 and $\ker \mathcal{A}^*$. If $\mathcal{A}$ is symmetric or skew symmetric, that is, $\mathcal{A} = \mathcal{A}^*$ or $\mathcal{A} = -\mathcal{A}^*$, respectively, then $X_0 = \ker \mathcal{A}^*$. In general, the equilibrium states of (1.2) satisfy the two-point boundary value problem:

$$\begin{cases} \Lambda u_x^* + L u^* = 0, & \text{in } (0, \ell), \\ H h^* + G u^*(\ell) = 0, \\ u^*(0) = K u^*(\ell) + Q h^*. \end{cases} \tag{3.7}$$

When $L = 0$, the state u is conserved on $[0, \ell]$. A simple computation yields

$$X_0 = \ker \begin{bmatrix} G & H \\ I - K & -Q \end{bmatrix}. \tag{3.8}$$

Similarly, we have

$$\ker \mathcal{A}^* = \ker \begin{bmatrix} -Q^T \Lambda & H^T \\ I - K^T & \Lambda^{-1} G^T \end{bmatrix}. \tag{3.9}$$

Hence, for conservation laws, the semigroup invariance of $X_0^\perp$ holds provided (3.8) is contained in (3.9). On the other hand, for $L \neq 0$, we need to verify that the solutions of (3.7) are also solutions of the adjoint problem:

$$\begin{cases} \Lambda v_x - L^T v = 0, & \text{in } (0, \ell), \\ H^T g - Q^T \Lambda v(0) = 0, \\ v(\ell) = K^T v(0) - \Lambda^{-1} G^T g. \end{cases}$$

After discussing a necessary condition for the semigroup invariance of $X_0^\perp$, let us examine the spectral properties of $\mathcal{A}_0$. Because $\mathcal{A}$ has compact resolvents, its spectrum and point spectrum coincide [10, 27]. Likewise, $\mathcal{A}_0$ has compact resolvents. It follows that $\mathbb{C} = \rho(\mathcal{A}_0) \cup \sigma_p(\mathcal{A}_0)$, where $\sigma_p(\cdot)$ denotes the point spectrum of an operator. Observe that taking the part of a semigroup generator on the orthogonal complement of its kernel eliminates the zero eigenvalue. In particular, $\rho(\mathcal{A}_0) = \rho(\mathcal{A}) \cup \{0\}$ and $\sigma_p(\mathcal{A}_0) = \sigma_p(\mathcal{A}) \setminus \{0\}$. An immediate consequence is the following lemma.

Lemma 3.3. *The linear operator $\mathcal{A}_0 : D(\mathcal{A}_0) \to X_0^\perp$ is invertible, and its inverse $\mathcal{A}_0^{-1}$ is bounded.*

We are now ready to state and prove the main result of this section.

Theorem 3.4. *Suppose that $X_0 \subset \ker \mathcal{A}^*$, and that there exist $C, r > 0$ such that*

$$\|u(t)\|_{H^1}^2 + |h(t)|^2 \leq C e^{-rt} (\|u_0\|_{H^1}^2 + |h_0|^2), \tag{3.10}$$

for all $t \geq 0$, *and* $(u_0, h_0) \in X_0^\perp \cap D(\mathcal{A})$. *Then, there exists* $\tilde{C} > 0$ *such that for every* $(u_0, h_0) \in X_0^\perp$,

$$\|u(t)\|_{L^2}^2 + |h(t)|^2 \leq \tilde{C} e^{-rt}(\|u_0\|_{L^2}^2 + |h_0|^2).$$

Proof. Let $(u_0, h_0) \in X_0^\perp$, and define $(v_0, g_0) \in D(\mathcal{A}_0)$ by $(v_0, g_0) := \mathcal{A}_0^{-1}(u_0, h_0)$. Applying (3.10) and Proposition 3.1 to $(v(t), g(t)) := e^{t\mathcal{A}_0}(v_0, g_0) \in C([0, T]; D(\mathcal{A}_0))$, there exists a constant $C_1 > 0$ such that for every $t \geq 0$,

$$\|(v(t), g(t))\|_{D(\mathcal{A}),\varepsilon,\delta}^2 \leq C_1 e^{-rt} \|(v_0, g_0)\|_{D(\mathcal{A}),\varepsilon,\delta}^2. \tag{3.11}$$

Since $(v_0, g_0) \in D(\mathcal{A}_0)$, we may replace $\mathcal{A}$ by $\mathcal{A}_0$. This allows us to estimate from above:

$$\|(v_0, g_0)\|_{D(\mathcal{A}),\varepsilon,\delta}^2 \leq C_{\varepsilon,\delta} \left(\|\mathcal{A}_0^{-1}(u_0, h_0)\|_X^2 + \|(u_0, h_0)\|_X^2 \right),$$

where $C_{\varepsilon,\delta} := \max\{\varepsilon, \varepsilon\delta\}$. Because $\mathcal{A}_0^{-1}$ is bounded (see Lemma 3.3), it follows that

$$\|(v_0, g_0)\|_{D(\mathcal{A}),\varepsilon,\delta}^2 \leq C_2 \|(u_0, h_0)\|_X^2, \tag{3.12}$$

where $C_2 := C_{\varepsilon,\delta}(\|\mathcal{A}_0^{-1}\|^2 + 1)$.

Similarly, we estimate (3.11) from below:

$$\|(v(t), g(t))\|_{D(\mathcal{A}),\varepsilon,\delta}^2 \geq C_3 \|\mathcal{A}_0(v(t), g(t))\|_X^2 = C_3 \|\mathcal{A}_0 e^{t\mathcal{A}_0}(v_0, g_0)\|_X^2,$$

where $C_3 = \min\{\varepsilon, \varepsilon\delta\}$. According to [19, Chapter 4, Theorem 1.3] and [10, Proposition 6.6], it holds that $\mathcal{A}_0 e^{t\mathcal{A}_0}(v_0, g_0) = e^{t\mathcal{A}_0}\mathcal{A}_0(v_0, g_0)$. Therefore,

$$\|(v(t), g(t))\|_{D(\mathcal{A}),\varepsilon,\delta}^2 \geq C_3 \|e^{t\mathcal{A}_0}\mathcal{A}_0(v_0, g_0)\|_X^2 = C_3 \|e^{t\mathcal{A}_0}(u_0, h_0)\|_X^2.$$

Moreover, $e^{t\mathcal{A}_0}(u_0, h_0) = e^{t\mathcal{A}}(u_0, h_0) = (u(t), h(t))$ since $(u_0, h_0) \in X_0^\perp$. The estimate above now reads

$$\|(v(t), g(t))\|_{D(\mathcal{A}),\varepsilon,\delta}^2 \geq C_3 \|(u(t), h(t))\|_X^2. \tag{3.13}$$

From estimates (3.11)-(3.13), we deduce $\|(u(t), h(t))\|_X^2 \leq \tilde{C} e^{-rt} \|(u_0, h_0)\|_X^2$, where $\tilde{C} := C_1 C_2 (C_3)^{-1}$. Conclusion follows. □

Finally, we extend the previous result from an initial data in $X_0^\perp$ to an initial data in X. We begin by writing X as a direct sum of its closed subspaces: $X = X_0 \oplus X_0^\perp$. If $\Pi_0 : X \to X_0$ is the orthogonal projection of X onto X_0, then every element $(u_0, h_0) \in X$ admits the unique decomposition

$$(u_0, h_0) = (u_0^\perp, h_0^\perp) + (I - \Pi_0)(u_0, h_0),$$

where $(u_0^\perp, h_0^\perp) = \Pi_0(u_0, h_0) \in X_0$, and $(I - \Pi_0)(u_0, h_0) \in X_0^\perp$. Applying the semigroup $(e^{t\mathcal{A}})_{t \geq 0}$ yields

$$(u(t), h(t)) := e^{t\mathcal{A}}(u_0, h_0) = e^{t\mathcal{A}}(u_0^\perp, h_0^\perp) + e^{t\mathcal{A}}(I - \Pi_0)(u_0, h_0).$$

Because $(u_0^\perp, h_0^\perp) \in X_0$, it holds that $e^{t\mathcal{A}}(u_0^\perp, h_0^\perp) = (u_0^\perp, h_0^\perp)$ for all $t \geq 0$. As a result, we have

$$(u(t) - u_0^\perp, h(t) - h_0^\perp) = e^{t\mathcal{A}}(I - \Pi_0)(u_0, h_0).$$

The stability of the steady states can therefore be derived from the stability at the origin of a projected initial data. Theorem 3.4 and the boundedness of projection operators yield the following result.

Theorem 3.5. *Let $\Pi_0 : X \to X_0$ be the orthogonal projection of X onto X_0, and define $(u_0^\perp, h_0^\perp) := \Pi_0(u_0, h_0)$. Under the assumptions of Theorem 3.4, there exists $\hat{C} > 0$ such that for all $t \geq 0$ and $(u_0, h_0) \in X$, it holds that*

$$\|u(t) - u_0^\perp\|_{L^2}^2 + |h(t) - h_0^\perp|^2 \leq \hat{C} e^{-rt}(\|u_0\|_{L^2}^2 + |h_0|^2).$$

4. Examples

In this section, we show that the energy $E(t)$ of each physical model decays exponentially. More precisely, we would like the energy to satisfy

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(t) + rE(t) \leq 0, \quad t > 0,$$

so that $E(t) \leq E(0)e^{-rt}$, with decay rate $r > 0$. However, we cannot directly derive this energy-dissipation inequality from the equations. Instead, we will derive energy and dissipation functionals, denoted by $\mathcal{E}(t)$ and $D(t)$, respectively, such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + D(t) \leq 0, \quad t > 0, \tag{4.1}$$

and show that $\mathcal{E}(t)$ and $D(t)$ are both equivalent to $E(t)$. The task requires derivations of several a priori estimates. In what follows, the constants, even with the same notations, may vary from one example to another.

4.1. A Low Loss Electrical Line Connecting an Inductive Power Supply to a Capacitive Load. The telegrapher equations, also known as transmission line equations, is a hyperbolic system of balance laws modeling the propagation of current and voltage along transmission lines. Developed in the 1800s by Oliver Heaviside [13], these equations are in the following form:

$$\begin{cases} I_t(t,x) + \dfrac{1}{L}V_x(t,x) + \dfrac{R}{L}I(t,x) = 0, & t > 0,\ 0 < x < \ell, \\ V_t(t,x) + \dfrac{1}{C}I_x(t,x) + \dfrac{G}{C}V(t,x) = 0, & t > 0,\ 0 < x < \ell, \end{cases} \tag{4.2}$$

where $I(t,x)$ and $V(t,x)$ are the current and voltage, respectively, at distance x along a transmission line of length ℓ. The constants L and C represent the line self-inductance and capacitance, respectively, per unit length. The distributed resistance of conductors per unit length is denoted by R, while G is the admittance per unit length of the dielectric material separating the conductors.

When the transmission line connects an inductive power supply to a capacitive load, system (4.2) can be subjected to the following dynamic boundary conditions, see for instance [1, Section 3.4.3]:

$$\begin{cases} L_0\dfrac{\mathrm{d}I(t,0)}{\mathrm{d}t} + R_0 I(t,0) + V(t,0) = U^*, & t > 0, \\ C_\ell\dfrac{\mathrm{d}V(t,\ell)}{\mathrm{d}t} + \dfrac{V(t,\ell)}{R_\ell} = I(t,\ell), & t > 0, \end{cases}$$

with a given constant input voltage U^*.

Now, we write system (4.2) around a steady state $I^*(x)$, $V^*(x)$. Due to the linearity of (4.2), we obtain a linear system with uniform coefficients even though the steady state may be nonuniform [1, Section 1.2]. For this linear model, we mention two results. First, the stability with respect to the L^∞-norm of a lossless line (i.e. $G = R = 0$) is established in [1, Section 2.1.5]. Second, for distortionless lines (lines satisfying the Heaviside condition $R/L = G/C$), the L^2-exponential stability easily follows.

In contrast to the considerations above, we shall consider a low loss electrical line, in particular, when $R = 0$ and $G > 0$. The steady state is then given by

$$I^* = \frac{U^*}{R_0 + R_\ell}, \quad V^* = \frac{R_\ell U^*}{R_0 + R_\ell}.$$

Introducing the variables $\varphi := 2(V - V^*)$ and $\psi := 2(I - I^*)/\sqrt{L/C}$, we transform the linear system to

$$\begin{cases} \varphi_t(t,x) + \lambda\psi_x(t,x) + 2\zeta\varphi(t,x) = 0, & t > 0,\ 0 < x < \ell, \\ \psi_t(t,x) + \lambda\varphi_x(t,x) = 0, & t > 0,\ 0 < x < \ell, \\ v'(t) + \gamma_1 v(t) + \delta_1\varphi(t,0) = 0, & t > 0, \\ w'(t) + \gamma_2 w(t) - \delta_2\psi(t,\ell) = 0, & t > 0, \\ \psi(t,0) = v(t), & t > 0, \\ \varphi(t,\ell) = w(t), & t > 0, \\ \varphi(0,x) = \varphi_0(x),\ \psi(0,x) = \psi_0(x), & 0 < x < \ell, \\ v(0) = v_0,\ w(0) = w_0, \end{cases} \tag{4.3}$$

with positive constant coefficients given by

$$\lambda = \frac{1}{\sqrt{LC}}, \quad \zeta = \frac{G}{2C}, \quad \delta_1 = \frac{1}{L_0}\sqrt{\frac{C}{L}}, \quad \delta_2 = \frac{1}{C_\ell}\sqrt{\frac{L}{C}}, \quad \gamma_1 = \frac{R_0}{L_0}, \quad \gamma_2 = \frac{1}{R_\ell C_\ell}.$$

The normalized first-order energy of (4.3) is the sum of the kinetic and potential energy acting on the transmission line:

$$E(t) := \int_0^\ell (\varphi^2 + \psi^2 + \varphi_x^2 + \psi_x^2)\,\mathrm{d}x + (v')^2 + (w')^2 + v^2 + w^2.$$

Let us equip the Hilbert space $\mathcal{X} := L^2(0,\ell;\mathbb{R}^2) \times \mathbb{R}^2$ with the inner product

$$\langle(\varphi_1,\psi_1,v_1,w_1),(\varphi_2,\psi_2,v_2,w_2)\rangle_{\mathcal{X}} := \frac{1}{\lambda}(\langle\varphi_1,\varphi_2\rangle_{L^2} + \langle\psi_1,\psi_2\rangle_{L^2}) + \frac{1}{\delta_1}v_1v_2 + \frac{1}{\delta_2}w_1w_2.$$

Define the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{A}\begin{bmatrix}\varphi \\ \psi \\ v \\ w\end{bmatrix} = \begin{bmatrix} -\lambda\psi_x - 2\zeta\varphi \\ -\lambda\varphi_x \\ -\gamma_1 v - \delta_1\varphi(0) \\ -\gamma_2 w + \delta_2\psi(\ell)\end{bmatrix},$$

with $D(\mathcal{A}) = \{(\varphi,\psi,v,w) \in \mathcal{X} : \varphi,\psi \in H^1(0,\ell), \psi(0) = v, \varphi(\ell) = w\}$. For each $(\varphi,\psi,v,w),(\phi,\xi,y,z) \in D(\mathcal{A})$, we have

$$\langle\mathcal{A}(\varphi,\psi,v,w),(\phi,\xi,y,z)\rangle_{\mathcal{X}} = \langle(\varphi,\psi,v,w),\mathcal{A}^*(\phi,\xi,y,z)\rangle_{\mathcal{X}},$$

where

$$\mathcal{A}^* \begin{bmatrix} \phi \\ \xi \\ y \\ z \end{bmatrix} = \begin{bmatrix} \lambda\xi_x - 2\zeta\phi \\ \lambda\phi_x \\ -\gamma_1 y + \delta_1\phi(0) \\ -\gamma_2 z - \delta_2\xi(\ell) \end{bmatrix}.$$

The operators $\mathcal{A}$ and $\mathcal{A}^*$ both have trivial kernels, and consequently, $(\ker\mathcal{A})^\perp = \mathcal{X}$. We present our stability result.

Theorem 4.1. *There exist constants $C, r > 0$ such that for all $(\varphi_0, \psi_0, v_0, w_0) \in D(\mathcal{A})$, the solution of system (4.3) satisfies for every $t \geq 0$ the estimate*

$$\|\varphi(t)\|_{H^1} + \|\psi(t)\|_{H^1} + |v(t)| + |w(t)| \leq Ce^{-rt}(\|\varphi_0\|_{H^1} + \|\psi_0\|_{H^1} + |v_0| + |w_0|).$$

Proof. We prove the theorem in two steps. First, we derive the functionals $\mathcal{E}(t)$ and $D(t)$ from the equations. Second, we prove that these functionals are equivalent to the energy $E(t)$ of system (4.3).

Step 1: Derivation of $\mathcal{E}(t)$ and $D(t)$. From system (4.3), we take the sum of the first equation multiplied by φ/λ and the second equation multiplied by ψ/λ. Integrating by parts, we compute that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\ell \frac{1}{\lambda}(\varphi^2 + \psi^2)\,\mathrm{d}x + \varphi(\ell)\psi(\ell) - \varphi(0)\psi(0) + \frac{2\zeta}{\lambda}\int_0^\ell \varphi^2\,\mathrm{d}x = 0.$$

Using the ODEs and boundary conditions, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^\ell \frac{1}{\lambda}(\varphi^2 + \psi^2)\,\mathrm{d}x + \frac{1}{\delta_1}v^2 + \frac{1}{\delta_2}w^2\right\} + \frac{2\zeta}{\lambda}\int_0^\ell \varphi^2\,\mathrm{d}x + \frac{\gamma_1}{\delta_1}v^2 + \frac{\gamma_2}{\delta_2}w^2 = 0. \tag{4.4}$$

Similarly, we take the time derivatives of the first two equations in (4.3) multiplied by φ_t/λ and ψ_t/λ, respectively. Again, we integrate their sum, and use the ODE and boundary conditions:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\left\{\int_0^\ell \frac{1}{\lambda}(\varphi_t^2 + \psi_t^2)\,\mathrm{d}x + \frac{1}{\delta_1}(v')^2 + \frac{1}{\delta_2}(w')^2\right\} \\ &+ \frac{2\zeta}{\lambda}\int_0^\ell \varphi_t^2\,\mathrm{d}x + \frac{\gamma_1}{\delta_1}(v')^2 + \frac{\gamma_2}{\delta_2}(w')^2 = 0.\end{aligned}$$

From the first equation in (4.3), we estimate $\varphi_t^2 \geq \lambda^2\psi_x^2 - c\varphi^2$ for some $c > 0$. This implies

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^\ell \frac{1}{\lambda}(\varphi_t^2 + \psi_t^2)\,\mathrm{d}x + \frac{1}{\delta_1}(v')^2 + \frac{1}{\delta_2}(w')^2\right\} + \frac{2\zeta}{\lambda}\int_0^\ell (\lambda^2\psi_x^2 - c\varphi^2)\,\mathrm{d}x \\ + \frac{\gamma_1}{\delta_1}(v')^2 + \frac{\gamma_2}{\delta_2}(w')^2 \leq 0.\end{aligned} \tag{4.5}$$

Adapting the derivation of (4.5), we replace time by space derivatives and get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\left\{\int_0^\ell \frac{1}{\lambda}(\varphi_x^2 + \psi_x^2)\,\mathrm{d}x + \frac{1}{\lambda^2}\left(\frac{1}{\delta_1}(v')^2 + \frac{1}{\delta_2}(w')^2 + 2\zeta\left(\frac{\gamma_1}{\delta_1}v^2 + \frac{\gamma_2}{\delta_2}w^2\right)\right)\right\} \\ &+ \frac{2\zeta}{\lambda}\int_0^\ell \varphi_x^2\,\mathrm{d}x + \frac{1}{\lambda^2}\left(\frac{1}{\delta_1}(\gamma_1 + 2\zeta)(v')^2 + \frac{\gamma_2}{\delta_2}(w')^2 + \frac{2\zeta}{\delta_2}w''w\right) = 0.\end{aligned}$$

Because $w''w = (w' \cdot w)' - (w')^2$, we further compute that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^\ell \frac{1}{\lambda}(\varphi_x^2+\psi_x^2)\,\mathrm{d}x + \frac{1}{\lambda^2}\left(2\zeta\left(\frac{\gamma_1}{\delta_1}v^2+\frac{\gamma_2}{\delta_2}w^2+\frac{2}{\delta_2}w'w\right)+\frac{1}{\delta_1}(v')^2 + \frac{1}{\delta_2}(w')^2\right)\right\} + \frac{1}{\lambda^2}\left(\frac{1}{\delta_1}(\gamma_1+2\zeta)(v')^2+\frac{1}{\delta_2}(\gamma_2-2\zeta)(w')^2\right) + \frac{2\zeta}{\lambda}\int_0^\ell \varphi_x^2\,\mathrm{d}x = 0. \tag{4.6}$$

Finally, we multiply the first equation in (4.3) by $(\ell-x)\psi/\lambda$ and integrate by parts. The boundary condition $\psi(0)=v$ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\ell \frac{2(\ell-x)}{\lambda}\varphi\psi\,\mathrm{d}x + \int_0^\ell\left(\frac{1}{2}\psi^2+(\ell-x)\varphi\varphi_x+\frac{2\zeta(\ell-x)}{\lambda}\varphi\psi\right)\mathrm{d}x - \frac{\ell}{2}v^2 = 0. \tag{4.7}$$

Now, let $\varepsilon,\eta>0$ (constants to be chosen later) and compute the following sum: (4.4)+ε(4.6)+η((4.5)+(4.7)). This yields the energy-dissipation inequality $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t)+D(t)\le 0$, where

$$\mathcal{E}(t) := \frac{1}{\lambda}\int_0^\ell\left(\varphi^2+\psi^2+\eta(\varphi_t^2+\psi_t^2)+\varepsilon(\varphi_x^2+\psi_x^2)+2\eta(\ell-x)\varphi\psi\right)\mathrm{d}x + \frac{1}{\delta_2}\left(\left(1+\frac{2\zeta\gamma_2\varepsilon}{\lambda^2}\right)w^2+\left(\eta+\frac{\varepsilon}{\lambda^2}\right)(w')^2+\frac{4\zeta\varepsilon}{\lambda^2}w'w\right) + \frac{1}{\delta_1}\left(\left(1+\frac{2\zeta\gamma_1\varepsilon}{\lambda^2}\right)v^2+\left(\eta+\frac{\varepsilon}{\lambda^2}\right)(v')^2\right),$$

and

$$D(t) := \int_0^\ell\left(\frac{2\zeta}{\lambda}\left((1-\eta c)\varphi^2+\eta\lambda^2\psi_x^2+\varepsilon\varphi_x^2+\eta(\ell-x)\varphi\psi\right)\mathrm{d}x + \left(\frac{\gamma_1}{\delta_1}-\frac{\eta\ell}{2}\right)v^2 + \frac{\gamma_2}{\delta_2}w^2 + \int_0^\ell \frac{\eta}{2}\left(\psi^2+2(\ell-x)\varphi\varphi_x\right)\mathrm{d}x + \frac{1}{\delta_1}\left(\eta\gamma_1+\frac{(\gamma_1+2\zeta)\varepsilon}{\lambda^2}\right)(v')^2 + \frac{1}{\delta_2}\left(\eta\gamma_2+\frac{(\gamma_2-2\zeta)\varepsilon}{\lambda^2}\right)(w')^2.$$

Step 2: Equivalence of $\mathcal{E}(t)$ and $D(t)$ to $E(t)$. The first two equations of (4.3) and the Cauchy-Schwarz inequality imply that for $0<\varepsilon<1$,

$$\mathcal{E}(t) \le \frac{1}{\lambda}\int_0^\ell\left(\left(1+8\eta\zeta^2+\eta\ell\right)\varphi^2+(1+\eta\ell)\psi^2+(\eta\lambda^2+1)\varphi_x^2+(2\eta\lambda^2+1)\psi_x^2\right)\mathrm{d}x + \frac{1}{\delta_1}\left(\left(1+\frac{2\zeta\gamma_1}{\lambda^2}\right)v^2+\left(\eta+\frac{1}{\lambda^2}\right)(v')^2\right) + \frac{1}{\delta_2}\left(\left(1+\frac{2\zeta}{\lambda^2}(\gamma_2+1)\right)w^2 + \left(\eta+\frac{1+2\zeta}{\lambda^2}\right)(w')^2\right).$$

This estimate implies $\mathcal{E}(t)\le c_2E(t)$, where $c_2>0$ is the maximum value among the coefficients on the right-hand side.

For the reverse inequality, recall from (4.5) that $\varphi_t^2 \geq \lambda^2\psi_x^2 - c\varphi^2$ for some $c > 0$. Again, we use Cauchy-Schwarz inequality and obtain $-4\zeta w'w \geq -\zeta(\gamma_2 w^2 + \hat{c}(w')^2)$ for some $\hat{c} > 0$. It follows that

$$\mathcal{E}(t) \geq \frac{1}{\lambda}\int_0^\ell \Big((1-\eta(c+\ell))\varphi^2 + (1-\eta\ell)\psi^2 + (\eta\lambda^2+\varepsilon)(\varphi_x^2+\psi_x^2)\Big)\,\mathrm{d}x + \frac{1}{\delta_1}\left(\left(1+\frac{2\zeta\gamma_1\varepsilon}{\lambda^2}\right)v^2 + \left(\eta+\frac{\varepsilon}{\lambda^2}\right)(v')^2\right) + \frac{1}{\delta_2}\left(\left(1+\frac{\zeta\gamma_2\varepsilon}{\lambda^2}\right)w^2 + \left(\eta - \left(\zeta\hat{c}+\frac{1}{\lambda^2}\right)\varepsilon\right)(w')^2\right). \tag{4.8}$$

Now, we choose $\eta > 0$ such that

$$\eta < \min\left\{\frac{1}{c+\ell},\ \zeta\hat{c}+\frac{1}{\lambda^2}\right\}. \tag{4.9}$$

We also take $\varepsilon = \eta\varepsilon_0$, where $0 < \varepsilon_0 < \zeta\hat{c} + \lambda^{-2}$. We can now conclude that $\mathcal{E}(t) \geq c_1 E(t)$, where $c_1 > 0$ is the minimum value among the coefficients on the right-hand side of (4.8).

Similarly, we estimate $D(t)$ using Cauchy-Schwarz inequality:

$$D(t) \leq \int_0^l \left(\left(\frac{2\zeta}{\lambda}(1-\eta c) + \eta\ell\left(\frac{\zeta}{\lambda}+\frac{1}{2}\right)\right)\varphi^2 + \eta\left(\frac{\zeta\ell}{\lambda}+\frac{1}{2}\right)\psi^2 + \left(\frac{2\zeta}{\lambda}+\frac{\eta\ell}{2}\right)\varphi_x^2 + 2\eta\zeta\lambda\psi_x^2\right)\mathrm{d}x + \frac{\gamma_1}{\delta_1}v^2 + \frac{\gamma_2}{\delta_2}w^2 + \frac{1}{\delta_1}\left(\eta\gamma_1 + \frac{\gamma_1+2\zeta}{\lambda^2}\right)(v')^2 + \frac{\gamma_2}{\delta_2}\left(\eta+\frac{1}{\lambda^2}\right)(w')^2. \tag{4.10}$$

Note that $(1-\eta c) > 0$ by our choice of $\eta > 0$ in (4.9). On the other hand, Young's inequality implies $\varphi\psi \leq c_{\hat{\varepsilon}}\varphi^2 + \hat{\varepsilon}\psi^2$, where $c_{\hat{\varepsilon}} > 0$ and $\hat{\varepsilon} > 0$ (to be chosen later). We arrive at the following estimate:

$$D(t) \geq \int_0^l \left(\left(\frac{2\zeta}{\lambda} - \eta\left(\frac{2\zeta}{\lambda}(c+\ell c_{\hat{\varepsilon}}) + \frac{\ell}{2}\right)\right)\varphi^2 + \frac{\eta}{2\lambda}(\lambda - 4\zeta\ell\hat{\varepsilon})\psi^2 + 2\eta\zeta\lambda\psi_x^2 + \frac{1}{2\lambda}(4\zeta-\eta\lambda\ell)\varphi_x^2\right)\mathrm{d}x + \frac{1}{\delta_1}\left(\eta\gamma_1 + \frac{(\gamma_1+2\zeta)\varepsilon}{\lambda^2}\right)(v')^2 + \frac{1}{\delta_2}\left(\eta\gamma_2 - \frac{(\gamma_2+2\zeta)\varepsilon}{\lambda^2}\right)(w')^2 + \left(\frac{\gamma_1}{\delta_1} - \frac{\eta\ell}{2}\right)v^2 + \frac{\gamma_2}{\delta_2}w^2. \tag{4.11}$$

Therefore, we choose $\hat{\varepsilon}, \eta > 0$ which satisfy

$$\hat{\varepsilon} < \frac{\lambda}{4\zeta\ell}, \quad \eta < \min\left\{\frac{4\zeta}{4\zeta(c+\ell c_\varepsilon)+\lambda\ell},\ \frac{4\zeta}{\lambda\ell},\ \frac{2\gamma_1}{\delta_1\ell}\right\},$$

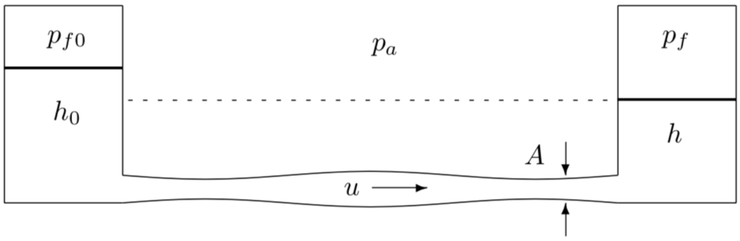
and take $\varepsilon = \eta\varepsilon_0$ with $0 < \varepsilon_0 < \gamma_2\lambda^2/(\gamma_2+2\zeta)$. These constants yield the inequality $\tilde{c}_1 E(t) \leq D(t) \leq \tilde{c}_2 E(t)$, where $\tilde{c}_1, \tilde{c}_2 > 0$ are the minimum and maximum values among the coefficients on the right-hand sides of (4.10) and (4.11), respectively. This completes the proof. □

We conclude this section with an immediate result from Theorems 3.5 and 4.1.

Theorem 4.2. *There exists $r, \hat{C} > 0$ such that for all $(\varphi_0, \psi_0, v_0, w_0) \in \mathcal{X}$ and $t \geq 0$, we have*

$$\|\varphi(t)\|_{L^2} + \|\psi(t)\|_{L^2} + |v(t)| + |w(t)| \leq \hat{C}e^{-rt}(\|\varphi_0\|_{L^2} + \|\psi_0\|_{L^2} + |v_0| + |w_0|).$$

4.2. Linearized Fluid Flow in an Elastic Tube Connecting Two Tanks. We consider a model of fluid flow [4, 18, 20, 22] in an elastic tube connecting two tanks [20]:



Euler's continuity equation and the law of balance of momentum determine the dynamics of the fluid velocity $u(t,x)$ and the vertical cross section $A(t,x)$ of the tube [18]. We couple these to the level heights h_0 and h in the tanks, both with horizontal cross section A_T, and starting at initial heights h_0^0 and h_0, respectively. We obtain the linear system:

$$\begin{cases} A_t(t,x) + A_e u_x(t,x) = 0, & t > 0,\ 0 < x < \ell, \\ u_t(t,x) + \alpha A_x(t,x) + \beta u(t,x) = 0, & t > 0,\ 0 < x < \ell, \\ h_0'(t) + \dfrac{A_e}{A_T} u(t,0) = 0, & t > 0, \\ h'(t) - \dfrac{A_e}{A_T} u(t,\ell) = 0, & t > 0, \\ A(t,0) = \gamma h_0(t),\ A(t,\ell) = \gamma h(t), & t > 0, \\ A(0,x) = A^0(x),\ u(0,x) = u^0(x), & 0 < x < \ell, \\ h_0(0) = h_0^0,\ h(0) = h^0, \end{cases} \tag{4.12}$$

where $\alpha > 0$, $\beta \geq 0$, and $\gamma > 0$ and $A_e > 0$ is the equilibrium cross section. We refer to [20] for the derivation of this linear model and the meaning of the involved parameters. In applications such as blood flow modeling, the elastic tube may be viewed as a blood vessel connecting two terminal compartments inside the human cardiovascular system.

We equip the Hilbert space $\mathcal{X} := L^2(0,\ell;\mathbb{R}^2) \times \mathbb{R}^2$ with the inner product

$$\langle (\varphi_1, \psi_1, a_1, b_1), (\varphi_2, \psi_2, a_2, b_2) \rangle_{\mathcal{X}} := \frac{1}{A_e}\langle \varphi_1, \psi_1 \rangle_{L^2} + \frac{1}{\alpha}\langle \varphi_2, \psi_2 \rangle_{L^2} + \frac{\gamma A_T}{A_e}(a_1 a_2 + b_1 b_2),$$

and define the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{A}\begin{bmatrix} A \\ u \\ h_0 \\ h \end{bmatrix} = \begin{bmatrix} -A_e u_x \\ -\alpha A_x - \beta u \\ -\frac{A_e}{A_T} u(0) \\ \frac{A_e}{A_T} u(\ell) \end{bmatrix}$$

with domain $D(\mathcal{A}) = \{(A, u, h_0, h) \in \mathcal{X} : A, u \in H^1(0, \ell), A(0) = \gamma h_0, A(\ell) = \gamma h\}$. For each $(A, u, h_0, h), (B, v, g_0, g) \in D(\mathcal{A})$, we have

$$\langle \mathcal{A}(A, u, h_0, h), (B, v, g_0, g)\rangle = \langle (A, u, h_0, h), \mathcal{A}^*(B, v, g_0, g)\rangle,$$

where

$$\mathcal{A}^*\begin{bmatrix} B \\ v \\ g_0 \\ g \end{bmatrix} = \begin{bmatrix} A_e v_x \\ \alpha B_x - \beta v \\ \frac{A_e}{A_T} v(0) \\ -\frac{A_e}{A_T} v(\ell) \end{bmatrix}.$$

In [20], the spectra of $\mathcal{A}$ and $\mathcal{A}^*$ were characterized completely. The uniform exponential stability of the model was also discussed, and optimal decay rates were provided using non-harmonic Fourier analysis. Here, we shall present an alternative proof of its exponential stability using Theorem 3.5.

The steady states of (4.12) are elements of $\ker \mathcal{A} = \ker \mathcal{A}^* = \{c(\gamma, 0, 1, 1) : c \in \mathbb{R}\}$. Its orthogonal complement corresponds to

$$(\ker \mathcal{A})^\perp = \left\{(A, u, h_0, h) \in \mathcal{X} : \int_0^\ell A(x)\,\mathrm{d}x + \frac{A_T}{\gamma} A(0) + \frac{A_T}{\gamma} A(\ell) = 0\right\}. \tag{4.13}$$

Finally, we define the first-order energy $E(t)$ of (4.12) as the sum of the kinetic and potential energy of the fluid:

$$E(t) := \int_0^\ell (A^2 + A_x^2 + u^2 + u_x^2)\,\mathrm{d}x + (h_0')^2 + (h')^2 + h_0^2 + h^2. \tag{4.14}$$

We are now ready to state our result.

Theorem 4.3. *There exist constants $C, r > 0$ such that for every initial data $(A^0, u^0, h_0^0, h^0) \in (\ker \mathcal{A})^\perp \cap D(\mathcal{A})$, the solution of (4.12) satisfies for $t \geq 0$ the inequality*

$$\|A(t)\|_{H^1} + \|u(t)\|_{H^1} + |h_0(t)| + |h(t)| \leq Ce^{-rt}(\|A_0\|_{H^1} + \|u_0\|_{H^1} + |h_0^0| + |h^0|).$$

Proof. We divide the proof into two steps. We derive $\mathcal{E}(t)$ and $D(t)$, and then show that both are equivalent to $E(t)$.

Step 1: Derivation of $\mathcal{E}(t)$ and $D(t)$. We multiply $A_e^{-1}A$ and $\alpha^{-1}u$ to the first and second equations in (4.12), respectively. Taking their sum and integrating by parts yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\ell \left(\frac{1}{A_e}A^2 + \frac{1}{\alpha}u^2\right)\mathrm{d}x + A(\ell)u(\ell) - A(0)u(0) + \frac{\beta}{\alpha}\int_0^\ell u^2\,\mathrm{d}x = 0.$$

Writing the boundary term $A(\ell)u(\ell) - A(0)u(0)$ in terms of h_0 and h, the equation above becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^\ell \left(\frac{1}{A_e}A^2 + \frac{1}{\alpha}u^2\right)\mathrm{d}x + \frac{\gamma A_T}{A_e}(h_0^2 + h^2)\right\} + \frac{\beta}{\alpha}\int_0^\ell u^2\,\mathrm{d}x = 0. \tag{4.15}$$

Next, we take the space derivatives of the first two equations in (4.12), utilize the multipliers $A_e^{-1}A_x$ and $\alpha^{-1}u_x$, and follow the steps from above. We have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^\ell \left(\frac{1}{A_e}A_x^2 + \frac{1}{\alpha}u_x^2\right)\mathrm{d}x + \frac{\gamma A_T}{\alpha A_e^2}((h_0')^2 + (h')^2)\right\} + \frac{\beta}{\alpha}\int_0^\ell u_x^2\,\mathrm{d}x + \frac{\beta\gamma A_T}{\alpha A_e^2}((h_0')^2 + (h')^2) = 0. \tag{4.16}$$

Now, with the identity $u_tA_x - A_tu_x = (uA_x)_t - (uA_t)_x$, and the ODEs and boundary conditions, it holds that

$$-u(\ell)A_t(\ell) + u(0)A_t(0) = -\frac{\gamma A_T}{A_e}((h_0')^2 + (h')^2).$$

We apply this equation to the difference between the continuity equation multiplied by u_x and the equation of balance of momentum multiplied by A_x. Proceeding as before, we compute

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^\ell 2uA_x\,\mathrm{d}x\right\} + \int_0^\ell (\alpha A_x^2 + \beta uA_x - A_eu_x^2)\,\mathrm{d}x - \frac{\gamma A_T}{A_e}((h_0')^2 + (h')^2) = 0. \tag{4.17}$$

Observe that we only lack dissipation terms for A, h_0, and h. Let us first derive one for A. We multiply the second equation in (4.12) by $(\ell - x)A$, and integrate by parts:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\ell 2(\ell - x)Au\,\mathrm{d}x + \int_0^\ell \left(\frac{\alpha}{2}A^2 + A_e(\ell - x)uu_x + \beta(\ell - x)Au\right)\mathrm{d}x - \frac{\alpha\ell}{2}A(0)^2 = 0. \tag{4.18}$$

Now, let us write $A(0)$ as follows:

$$A(0) = \left(\frac{2A_T}{\ell\gamma} + 1\right)^{-1}\left\{\left(\frac{1}{\ell}\int_0^\ell A(x)\,\mathrm{d}x + \frac{2A_T}{\ell\gamma}A(0)\right) + \left(A - \frac{1}{\ell}\int_0^\ell A(x)\,\mathrm{d}x\right) - (A - A(0))\right\}.$$

Recalling (4.13), it holds that $\int_0^\ell A(x)\,\mathrm{d}x = -\frac{A_T}{\gamma}\left(A(0) + \int_0^\ell A_x\,\mathrm{d}x\right)$. Dividing by ℓ and the applying Cauchy-Schwarz inequality yield

$$\left\|\frac{1}{\ell}\int_0^\ell A(x)\,\mathrm{d}x + \frac{2A_T}{\ell\gamma}A(0)\right\|_{L^2(0,\ell)}^2 \leq C_1\int_0^\ell A_x^2\,\mathrm{d}x,$$

where $C_1 = cA_T^2/(\ell\gamma)^2 > 0$ for a constant $c > 0$. This estimate, together with the Poincaré and Poincaré-Wirtinger inequalities, implies that there exists a $C_2 > 0$

such that

$$|A(0)|^2 \leq C_2 \int_0^\ell A_x^2 \,\mathrm{d}x. \tag{4.19}$$

Similarly, there exists $C_3 > 0$ such that $|A(\ell)|^2 \leq C_3 \int_0^\ell A_x^2 \,\mathrm{d}x$.

Because $h_0 = A(0)/\gamma$ and $h = A(\ell)/\gamma$, the last two estimates allow us to derive dissipation terms for h_0 and h:

$$|h_0|^2 + |h|^2 - C_4 \int_0^\ell A_x^2 \,\mathrm{d}x \leq 0, \tag{4.20}$$

where $C_4 = (C_2 + C_3)/\gamma^2 > 0$.

We are now ready to define $\mathcal{E}(t)$ and $D(t)$. Let $\varepsilon, \delta > 0$. Then the sum (4.15) + (4.16) + ε(4.17) + $\varepsilon\delta$(4.18) + $\frac{\varepsilon\delta\alpha\ell}{2}$((4.19) + (4.20)) reads as $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + D(t) \leq 0$, where

$$\begin{aligned}\mathcal{E}(t) := \int_0^\ell \left(\frac{1}{A_e}(A^2 + A_x^2) + \frac{1}{\alpha}(u^2 + u_x^2) + 2\varepsilon u A_x + 2\varepsilon\delta(\ell - x)Au \right) \mathrm{d}x \\ + \frac{\gamma A_T}{A_e} \left(\left(\frac{1}{\alpha A_e} \right) ((h_0')^2 + (h')^2) + (h_0^2 + h^2) \right),\end{aligned}$$

and

$$\begin{aligned}D(t) := \int_0^\ell \Bigg(\frac{\varepsilon\delta\alpha}{2}A^2 + \varepsilon\alpha \left(1 - \frac{\delta\ell}{2}(C_2 + C_4) \right) A_x^2 + \frac{\beta}{\alpha}u^2 + \left(\frac{\beta}{\alpha} - \varepsilon A_e \right) u_x^2 \\ + \varepsilon\beta u A_x + \varepsilon\delta\beta(\ell - x)Au + \varepsilon\delta A_e(\ell - x)u u_x \Bigg) \mathrm{d}x \\ + \frac{\gamma A_T}{A_e} \left(\frac{\beta}{\alpha A_e} - \varepsilon \right) ((h_0')^2 + (h')^2) + \frac{\varepsilon\delta\alpha\ell}{2}(h_0^2 + h^2).\end{aligned}$$

Step 2: Equivalence of $\mathcal{E}(t)$ and $D(t)$ to $E(t)$. We show that both $\mathcal{E}(t)$ and $D(t)$ are equivalent to the energy (4.14). Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}\mathcal{E}(t) \leq \int_0^\ell \Bigg(\left(\frac{1}{A_e} + \varepsilon\delta\ell \right) A^2 + \left(\frac{1}{A_e} + \varepsilon \right) A_x^2 + \left(\frac{1}{\alpha} + \varepsilon(1 + \delta\ell) \right) u^2 \\ + \frac{1}{\alpha}u_x^2 \Bigg) \mathrm{d}x + \frac{\gamma A_T}{A_e} \left(\left(\frac{1}{\alpha A_e} \right) ((h_0')^2 + (h')^2) + (h_0^2 + h^2) \right),\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}\mathcal{E}(t) \geq \int_0^\ell \Bigg(\left(\frac{1}{A_e} - \varepsilon\delta\ell \right) A^2 + \left(\frac{1}{A_e} - \varepsilon \right) A_x^2 + \left(\frac{1}{\alpha} - \varepsilon(1 + \delta\ell) \right) u^2 \\ + \frac{1}{\alpha}u_x^2 \Bigg) \mathrm{d}x + \frac{\gamma A_T}{A_e} \left(\left(\frac{1}{\alpha A_e} \right) ((h_0')^2 + (h')^2) + (h_0^2 + h^2) \right).\end{aligned} \tag{4.22}$$

Choosing $\varepsilon, \delta > 0$ such that

$$\varepsilon < \min\left\{ \frac{1}{A_e}, \frac{1}{\alpha} \right\} \quad \text{and} \quad \delta < \min\left\{ \frac{1}{\varepsilon\ell A_e}, \frac{1 - \varepsilon\alpha}{\varepsilon\alpha\ell} \right\}$$

yields the inequality $c_1 E(t) \leq \mathcal{E}(t) \leq c_2 E(t)$, where the constants $c_1, c_2 > 0$ are the minimum and maximum values among the coefficients on the right-hand sides of (4.22) and (4.21), respectively.

On the other hand, Cauchy-Schwarz and Young's inequalities invoke the following:

$$D(t) \leq \int_0^\ell \left(\frac{\varepsilon\delta}{2} (\alpha + \beta\ell) A^2 + \varepsilon \left(\alpha + \frac{\beta}{2} \right) A_x^2 + \left(\frac{\beta}{\alpha} + \frac{\varepsilon}{2} (\beta + \delta\ell (\beta + A_e)) \right) u^2 + \left(\frac{\beta}{\alpha} + \frac{\varepsilon\delta A_e \ell}{2} \right) u_x^2 \right) \mathrm{d}x + \frac{\gamma\beta A_T}{\alpha A_e^2}((h_0')^2 + (h')^2) + \frac{\varepsilon\delta\alpha\ell}{2}(h_0^2 + h^2), \tag{4.23}$$

and

$$D(t) \geq \int_0^\ell \left(\left(\frac{\beta}{\alpha} - \varepsilon \left(\beta c_{\varepsilon_1} + \delta\ell \left(\beta c_{\varepsilon_2} + \frac{A_e}{2} \right) \right) \right) u^2 + \left(\frac{\beta}{\alpha} - \varepsilon A_e \left(1 + \frac{\delta\ell}{2} \right) \right) u_x^2 + \frac{\varepsilon\delta}{2} (\alpha - 2\varepsilon_2\beta\ell) A^2 + \varepsilon \left(\alpha \left(1 - \frac{\delta\ell}{2}(C_2 + C_4) \right) - \varepsilon_1\beta \right) A_x^2 \right) \mathrm{d}x + \frac{\gamma A_T}{A_e} \left(\frac{\beta}{\alpha A_e} - \varepsilon \right) ((h_0')^2 + (h')^2) + \frac{\varepsilon\delta\alpha\ell}{2}(h_0^2 + h^2). \tag{4.24}$$

Now we choose $\varepsilon_1, \varepsilon_2, \delta, \varepsilon > 0$ such that

$$\varepsilon_1 < \frac{\alpha}{\beta}, \qquad \varepsilon_2 < \frac{\alpha}{2\beta\ell}, \qquad \delta < \frac{2}{C_2 + C_4} \left(\frac{\alpha - \varepsilon_1\beta}{\alpha\ell} \right),$$
$$\varepsilon < \frac{\beta}{\alpha} \min \left\{ (A_e)^{-1},\ (\beta c_{\varepsilon_1} + \delta\ell (\beta c_{\varepsilon_2} + A_e/2))^{-1},\ (A_e (1 + (\delta\ell)/2))^{-1} \right\}$$

These constants give the inequality $\tilde{c}_1 E(t) \leq D(t) \leq \tilde{c}_2 E(t)$, where $\tilde{c}_1, \tilde{c}_2 > 0$ are the minimum and maximum values among the coefficients on the left and right-hand sides of (4.24) and (4.23), respectively. Steps 1 and 2 yield the desired result. □

The next theorem follows immediately from Theorems 3.5 and 4.3.

Theorem 4.4. *There exists $\hat{C}, r > 0$ such that for all $(A^0, u^0, h_0^0, h^0) \in \mathcal{X}$, we have*

$$\|A(t) - A_0^\perp\|_{L^2} + \|u(t) - u_0^\perp\|_{L^2} + |h_0(t) - h_0^{0\perp}| + |h(t) - h^{0\perp}| \leq \hat{C}e^{-rt}(\|A_0\|_{L^2(0,\ell)} + \|u_0\|_{L^2(0,\ell)} + |h_0^0| + |h^0|),$$

for all $t \geq 0$, where $(A_0^\perp, u_0^\perp, h_0^{0\perp}, h^{0\perp}) = \Pi_0(A_0, u_0, h_0^0, h^0)$ with the orthogonal projection $\Pi_0 : X \to \ker(\mathcal{A})$ of $\mathcal{X}$ onto $\ker(\mathcal{A})$.

4.3. Damped Wave Equation with Oscillator Boundary Conditions. We consider a waveguide of length ℓ terminated by linear oscillators on both ends. The propagation of sound in the waveguide can be described by a damped wave equation coupled with the displacement dynamics of the oscillators at each end [2, 14]. In particular, we subject the velocity potential ψ of the wave to ODE boundary conditions describing the displacements δ_0 and δ_ℓ of the oscillators at $x = 0$ and

$x = \ell$, respectively. With damping coefficient equal to 1, this system reads

$$\begin{cases} \psi_{tt}(t,x) = \psi_{xx}(t,x) - \psi_t(t,x), & t>0,\ 0<x<\ell, \\ \psi_x(t,0) = -\delta_0'(t), & t>0, \\ \psi_x(t,\ell) = \delta_\ell'(t), & t>0, \\ m_0\delta_0''(t) + d_0\delta_0'(t) + k_0\delta_0(t) = -\rho\psi_t(t,0), & t>0, \\ m_\ell\delta_\ell''(t) + d_\ell\delta_\ell'(t) + k_\ell\delta_\ell(t) = -\rho\psi_t(t,\ell), & t>0, \\ \psi(0,x) = \psi_0(x), & 0<x<\ell, \\ \psi_t(0,x) = \psi_\ell(x), & 0<x<\ell, \\ \delta_i(0) = \delta_i^0, & i=0,\ell, \\ \delta_i'(0) = v_i^0, & i=0,\ell, \end{cases} \tag{4.25}$$

where ρ denotes fluid density. The properties of each oscillator i are encoded in the following constants: m_i denotes mass per unit area, d_i the resistivity, and k_i the spring constant. We assume that the surfaces of the oscillators are impenetrable by the fluid. For the physical interpretation of such phenomena, we refer to [16, page 263].

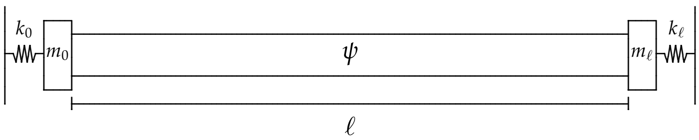


FIGURE 1. A waveguide terminated by oscillators

To facilitate the analysis, we decompose the waves into ϕ^- and ϕ^+, which denote the components propagating on the negative direction, and on the positive direction, respectively [14]. A first order hyperbolic PDE-ODE system is derived by introducing the variables $\phi^- := \frac{1}{2}(\partial_t\psi + \partial_x\psi)$, $\phi^+ := \frac{1}{2}(\partial_t\psi - \partial_x\psi)$, and $v_i := \delta_i'$, for $i = 0, \ell$. In particular, we have

$$\begin{cases} \phi_t^-(t,x) - \phi_x^-(t,x) + \frac{1}{2}(\phi^+(t,x) + \phi^-(t,x)) = 0, & t>0,\ 0<x<\ell, \\ \phi_t^+(t,x) + \phi_x^+(t,x) + \frac{1}{2}(\phi^+(t,x) + \phi^-(t,x)) = 0, & t>0,\ 0<x<\ell, \\ \phi^+(t,0) - \phi^-(t,0) = v_0(t), & t>0, \\ \phi^+(t,\ell) - \phi^-(t,\ell) = -v_\ell(t), & t>0, \\ \delta_0'(t) = v_0(t), \quad \delta_\ell'(t) = v_\ell(t), & t>0, \\ v_0'(t) + \frac{d_0}{m_0}v_0(t) + \frac{k_0}{m_0}\delta_0(t) + \frac{\rho}{m_0}(\phi^-(t,0) + \phi^+(t,0)) = 0, & t>0, \\ v_\ell'(t) + \frac{d_\ell}{m_\ell}v_\ell(t) + \frac{k_\ell}{m_\ell}\delta_\ell(t) + \frac{\rho}{m_\ell}(\phi^-(t,\ell) + \phi^+(t,\ell)) = 0, & t>0, \\ \phi^+(0,x) = \phi_0^+(x), & 0<x<\ell, \\ \phi^-(0,x) = \phi_0^-(x), & 0<x<\ell, \\ \delta_i(0) = \delta_i^0, & i=0,\ell, \\ v_i(0) = v_i^0, & i=0,\ell. \end{cases} \tag{4.26}$$

The *total energy* of (4.25), and hence of (4.26), is the sum of the kinetic and potential energy of the wave motion, and of the kinetic and potential energy of the oscillators.

It is defined as

$$\mathscr{E}(t) := \int_0^\ell ((\phi^+ - \phi^-)^2 + (\phi^+ + \phi^-)^2)\,\mathrm{d}x + \frac{1}{\rho}(m_\ell v_\ell^2 + k_\ell \delta_\ell^2 + m_0 v_0^2 + k_0 \delta_0^2).$$

A detailed discussion of the usage of the term can be found in [16, Section 1.3].

Now, let us equip the Hilbert space $\mathcal{X} := L^2(0,\ell;\mathbb{R}^2) \times \mathbb{R}^4$ with the inner product

$$\langle \varphi_1, \varphi_2 \rangle_{\mathcal{X}} := \langle \phi_1^-, \phi_2^- \rangle_{L^2} + \langle \phi_1^+, \phi_2^+ \rangle_{L^2} + \frac{1}{2\rho} \sum_{j=0,\ell} (k_j \delta_{j1}\delta_{j2} + m_j v_{j1} v_{j2}),$$

where $\varphi_i = (\phi_i^-, \phi_i^+, \delta_{0i}, \delta_{\ell i}, v_{0i}, v_{\ell i})$, for $i = 1, 2$. We define $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ as

$$\mathcal{A} \begin{bmatrix} \phi^- \\ \phi^+ \\ \delta_0 \\ \delta_\ell \\ v_0 \\ v_\ell \end{bmatrix} = \begin{bmatrix} \phi_x^- - \frac{1}{2}(\phi^+ + \phi^-) \\ -\phi_x^+ - \frac{1}{2}(\phi^+ + \phi^-) \\ v_0 \\ v_\ell \\ -\frac{d_0}{m_0} v_0 - \frac{k_0}{m_0}\delta_0 - \frac{\rho}{m_0}(\phi^-(0) + \phi^+(0)) \\ -\frac{d_\ell}{m_\ell} v_\ell - \frac{k_\ell}{m_\ell}\delta_\ell - \frac{\rho}{m_\ell}(\phi^-(\ell) + \phi^+(\ell)) \end{bmatrix},$$

with domain $D(\mathcal{A}) = \{(\phi^-, \phi^+, \delta_0, \delta_\ell, v_0, v_\ell) \in \mathcal{X} : \phi^-, \phi^+ \in H^1(0,\ell),\ v_0 = -\phi^-(0) + \phi^+(0),\ v_\ell = \phi^-(\ell) - \phi^+(\ell)\}$. Its adjoint $\mathcal{A}^* : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ is given by

$$\mathcal{A}^* \begin{bmatrix} \varphi^- \\ \varphi^+ \\ \gamma_0 \\ \gamma_\ell \\ w_0 \\ w_\ell \end{bmatrix} = \begin{bmatrix} -\varphi_x^- - \frac{1}{2}(\varphi^+ + \varphi^-) \\ \varphi_x^+ - \frac{1}{2}(\varphi^+ + \varphi^-) \\ -w_0 \\ -w_\ell \\ -\frac{d_0}{m_0} w_0 + \frac{k_0}{m_0}\gamma_0 + \frac{\rho}{m_0}(\varphi^-(0) + \varphi^+(0)) \\ -\frac{d_\ell}{m_\ell} w_\ell + \frac{k_\ell}{m_\ell}\gamma_\ell + \frac{\rho}{m_\ell}(\varphi^-(\ell) + \varphi^+(\ell)) \end{bmatrix}.$$

The equilibrium states of (4.26) lie in the set

$$\ker \mathcal{A} = \ker \mathcal{A}^* = \{c\,(k_0 k_\ell, k_0 k_\ell, -2\rho k_\ell, -2\rho k_0, 0, 0) : c \in \mathbb{R}\}.$$

Now, in theoretical acoustics, states corresponding to *hydrostatic pressure* occur when the acoustic pressure is equal to the sum of the displacement of the oscillators on the two ends. Removing these states, we find that

$$(\ker \mathcal{A})^\perp = \left\{ (\phi^-, \phi^+, \delta_0, \delta_\ell, v_0, v_\ell) \in \mathcal{X} : \int_0^\ell (\phi^- + \phi^+)(x)\,\mathrm{d}x - \delta_0 - \delta_\ell = 0 \right\},$$

see for instance [17, Section 6]. We now present our stability result.

Theorem 4.5. *There exist $C, r > 0$ such that for every initial data*

$$(\phi_0^-, \phi_0^+, \delta_0^0, \delta_\ell^0, v_0^0, v_\ell^0) \in (\ker \mathcal{A})^\perp \cap D(\mathcal{A}),$$

the solution of system (4.26) satisfies for every $t \geq 0$ the estimate

$$\begin{aligned} &\|\phi^-(t)\|_{H^1} + \|\phi^+(t)\|_{H^1} + |\delta_0(t)| + |\delta_\ell(t)| + |v_0(t)| + |v_\ell(t)| \\ &\quad \leq C e^{-rt}(\|\phi_0^-\|_{H^1} + \|\phi_0^+\|_{H^1} + |\delta_0^0| + |\delta_\ell^0| + |v_0^0| + |v_\ell^0|). \end{aligned}$$

Proof. To simplify the computations, we set $\psi^+ := \phi^+ + \phi^-$ and $\psi^- := \phi^+ - \phi^-$. Observe that stability in terms of $\phi^\pm$ is equivalent to stability in terms of $\psi^\pm$, thanks to the identity $\|\psi^-(t)\|_{H^1}^2 + \|\psi^+(t)\|_{H^1}^2 = 2(\|\phi^-(t)\|_{H^1}^2 + \|\phi^+(t)\|_{H^1}^2)$. We proceed as in the previous models, and for this purpose we define a normalized first-order energy of the system:

$$E_\kappa(t) = \int_0^\ell ((\psi^+)^2 + (\psi^-)^2 + (\psi_x^+)^2 + (\psi_x^-)^2)\,\mathrm{d}x + \kappa((v_0')^2 + (v_\ell')^2) + v_0^2 + v_\ell^2 + \delta_\ell^2 + \delta_0^2. \tag{4.27}$$

Using the equation for v_0' and v_ℓ' in (4.26), applying the trace theorem, and taking $\kappa > 0$ to be small enough, it is not hard to see that $E_\kappa(t)$ is equivalent to

$$E(t) = \int_0^\ell ((\psi^+)^2 + (\psi^-)^2 + (\psi_x^+)^2 + (\psi_x^-)^2)\,\mathrm{d}x + v_0^2 + v_\ell^2 + \delta_\ell^2 + \delta_0^2.$$

Step 1: Derivation of $\mathcal{E}(t)$ and $D(t)$. First, let us take the sum and difference of the first two equations in (4.26):

$$\psi_t^+ + \psi_x^- + \psi^+ = 0, \tag{4.28}$$

$$\psi_t^- + \psi_x^+ = 0. \tag{4.29}$$

We multiply ψ^+ to (4.28), and ψ^- to (4.29). Applying integration by parts to the sum of the resulting expressions, we compute that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\ell ((\psi^-)^2 + (\psi^+)^2)\,\mathrm{d}x + \int_0^\ell (\psi^+)^2\,\mathrm{d}x + \psi^-(\ell)\psi^+(\ell) - \psi^-(0)\psi^+(0) = 0.$$

From the ODEs and boundary conditions in (4.26), this identity now reads

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^\ell ((\psi^-)^2 + (\psi^+)^2)\,\mathrm{d}x + \frac{1}{\rho}(m_\ell v_\ell^2 + m_0 v_0^2 + k_\ell\delta_\ell^2 + k_0\delta_0^2)\right\} + \int_0^\ell (\psi^+)^2\,\mathrm{d}x + \frac{1}{\rho}(d_\ell v_\ell^2 + d_0 v_0^2) = 0. \tag{4.30}$$

We multiply the space derivative of (4.28) by ψ_x^+, and the space derivative of (4.29) by ψ_x^-. Following the steps above, we obtain the identity:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^\ell ((\psi_x^-)^2 + (\psi_x^+)^2)\,\mathrm{d}x + \frac{1}{\rho}(m_\ell(v_\ell')^2 + m_0(v_0')^2 + (d_\ell + k_\ell)v_\ell^2 + (d_0 + k_0)v_0^2 + 2k_\ell\delta_\ell v_\ell + 2k_0\delta_0 v_0)\right\} + \frac{1}{\rho}((d_\ell + m_\ell)(v_\ell')^2 + (d_0 + m_0)(v_0')^2 - k_\ell v_\ell^2 - k_0 v_0^2) + \int_0^\ell (\psi_x^+)^2\,\mathrm{d}x = 0. \tag{4.31}$$

Now, let us take the difference between the terms $(4.28)\times\psi_x^-$ and $(4.29)\times\psi_x^+$. We integrate by parts and use the identity $\psi_t^+\psi_x^- - \psi_t^-\psi_x^+ = (\psi^+\psi_x^-)_t - (\psi^+\psi_t^-)_x$. We compute

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^\ell 2\psi^+\psi_x^-\,\mathrm{d}x\right\} - \psi^+(\ell)\psi_t^-(\ell) + \psi^+(0)\psi_t^-(0)$$

$$+\int_0^\ell (\psi^+\psi_x^- + (\psi_x^-)^2 - (\psi_x^+)^2)\,\mathrm{d}x = 0.$$

Again for the boundary terms, we follow previous calculations, and get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^\ell 2\psi^+\psi_x^-\,\mathrm{d}x - \frac{1}{\rho}(d_\ell v_\ell^2 + d_0 v_0^2 + 2k_\ell\delta_\ell v_\ell + 2k_0\delta_0 v_0)\right\} + \int_0^\ell (\psi^+\psi_x^- + (\psi_x^-)^2 - (\psi_x^+)^2)\,\mathrm{d}x + \frac{1}{\rho}(k_\ell v_\ell^2 + k_0 v_0^2 - m_\ell(v_\ell')^2 - m_0(v_0')^2) = 0. \quad (4.32)$$

Let $\gamma_1 > 0$ (constant to be chosen later). The sum of γ_1(4.31) + $\frac{\gamma_1}{2}$(4.32) yields the inequality:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\gamma_1\int_0^\ell ((\psi_x^-)^2 + (\psi_x^+)^2 + \psi^+\psi_x^-)\,\mathrm{d}x + \frac{1}{\rho}\sum_{i=0,\ell}\left(\gamma_1 m_i(v_i')^2 + \gamma_1 k_i\delta_i v_i + \left(\gamma_1 k_i + \frac{\gamma_1}{2}d_i\right)v_i^2\right)\right\} + \frac{\gamma_1}{2}\int_0^\ell ((\psi_x^+)^2 + (\psi_x^-)^2 + \psi^+\psi_x^-)\,\mathrm{d}x + \frac{1}{\rho}\sum_{i=0,\ell}\left(\left(\gamma_1 d_i + \frac{\gamma_1}{2}m_i\right)(v_i')^2 - \frac{\gamma_1}{2}k_i v_i^2\right) \le 0. \quad (4.33)$$

Next, we wish to obtain a dissipation term for ψ^-. Let us multiply $(\ell - x)\psi^-$ to (4.28), and integrate by parts:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\ell 2(\ell - x)\psi^-\psi^+\,\mathrm{d}x + \frac{1}{2}\int_0^\ell \left((\psi^-)^2 + (\psi^+)^2 + 2(\ell - x)\psi^-\psi^+\right)\mathrm{d}x - \frac{\ell}{2}((\psi^-)^2(0) + (\psi^+)^2(0)) = 0.$$

The trace theorem is employed to estimate the boundary term: there exists $c > 0$ such that $-(\psi^+)^2(0) \ge -c\int_0^\ell ((\psi^+)^2 + (\psi_x^+)^2)\,\mathrm{d}x$. It follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\ell 2(\ell - x)\psi^-\psi^+\,\mathrm{d}x + \frac{1}{2}\int_0^\ell \left((\psi^-)^2 + (1 - c\ell)(\psi^+)^2 - c\ell(\psi_x^+)^2 + 2(\ell - x)\psi^-\psi^+\right)\mathrm{d}x - \frac{\ell}{2}v_0^2 \le 0, \quad (4.34)$$

where we used $(\psi^-)^2(0) = v_0^2$.

Finally, we multiply the corresponding δ_i (for $i = 0, \ell$) to the PDE-ODE coupling in (4.26):

$$\frac{1}{2}\sum_{i=0,\ell}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{d_i}{m_i}\delta_i^2 + 2\delta_i v_i\right\} - v_i^2 + \frac{k_i}{m_i}\delta_i^2 + \frac{\rho}{m_i}(\psi^+)(i)\delta_i\right) = 0.$$

Applying Young's inequality to each $\psi^+(i)\delta_i$ for $i = 0, \ell$, we derive the estimate

$$\sum_{i=0,\ell}\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{d_i}{m_i}\delta_i^2 + 2\delta_i v_i\right\} - v_i^2 + \frac{1}{m_i}(k_i - \rho\varepsilon_i)\,\delta_i^2 - \frac{\rho c_{\varepsilon_i}}{m_i}(\psi^+)^2(i)\right) \le 0, \quad (4.35)$$

with positive constants ε_i and c_{ε_i} for $i = 0, \ell$. We choose $\varepsilon_i > 0$ to satisfy

$$\varepsilon_i < \frac{k_i}{\rho}, \qquad \text{for} \quad i = 0, \ell, \quad (4.36)$$

and $\tilde{c} > 0$ such that $\tilde{c} = \bar{c}\rho \max\{c_{\varepsilon_\ell}/m_\ell, c_{\varepsilon_0}/m_0\}$ for some $\bar{c} > 0$. Once again, the trace theorem implies

$$-\frac{\rho c_{\varepsilon_\ell}}{m_\ell}(\psi^+)^2(\ell) - \frac{\rho c_{\varepsilon_0}}{m_0}(\psi^+)^2(0) \geq -\tilde{c}\int_0^\ell \left((\psi^+)^2 + (\psi_x^+)^2\right) \mathrm{d}x. \tag{4.37}$$

From inequalities (4.35) and (4.37), it holds that

$$\sum_{i=0,\ell}\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{d_i}{m_i}\delta_i^2 + 2\delta_i v_i\right\} - v_i^2 + \frac{1}{m_i}\left(k_i - \rho\varepsilon_i\right)\delta_i^2\right) \\ -\tilde{c}\int_0^\ell \left((\psi^+)^2 + (\psi_x^+)^2\right) \mathrm{d}x \leq 0. \tag{4.38}$$

We are ready to define the energy and dissipation functionals. Let $\gamma_2, \gamma_3 > 0$ (to be chosen later). The sum (4.30) + (4.33) + γ_2(4.34) + γ_3(4.38) yields the estimate $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + D(t) \leq 0$, where

$$\begin{aligned}\mathcal{E}(t) = \int_0^\ell \Big((\psi^+)^2 + (\psi^-)^2 + \gamma_1(\psi_x^+)^2 + \gamma_1(\psi_x^-)^2\gamma_1\psi^+\psi_x^- + 2\gamma_2(\ell - x)\psi^-\psi^+\Big)\,\mathrm{d}x \\ + \frac{1}{\rho}\sum_{i=0,\ell}\Big(\gamma_1 m_i(v_i')^2 + \Big(m_i + \frac{\gamma_1}{2}d_i + \gamma_1 k_i\Big)v_i^2 + \Big(k_i + \gamma_3\frac{\rho d_i}{m_i}\Big)\delta_i^2 \\ + \left(\gamma_1 k_i + 2\gamma_3\rho\right)\delta_i v_i\Big),\end{aligned}$$

$$\begin{aligned}D(t) = \int_0^\ell \Big(\Big(1 + \frac{\gamma_2}{2}(1 - c\ell) - \gamma_3\tilde{c}\Big)(\psi^+)^2 + \frac{\gamma_2}{2}(\psi^-)^2 + \Big(\frac{\gamma_1}{2} - \frac{\gamma_2}{2}c\ell - \gamma_3\tilde{c}\Big)(\psi_x^+)^2 \\ + \frac{\gamma_1}{2}(\psi_x^-)^2 + \frac{\gamma_1}{2}\psi^+\psi_x^- + \gamma_2(\ell - x)\psi^+\psi^-\Big)\,\mathrm{d}x + \frac{1}{\rho}\sum_{i=0,\ell}\Big(\Big(\gamma_1 d_i + \frac{\gamma_1}{2}m_i\Big)(v_i')^2 \\ + \Big(d_i - \frac{\gamma_1}{2}k_i - \gamma_3\rho\Big)v_i^2 + \frac{\gamma_3\rho}{m_i}(k_i - \rho\varepsilon_i)\delta_i^2\Big) - \frac{\gamma_2\ell}{2}v_0^2.\end{aligned}$$

Step 2: Equivalence of $\mathcal{E}(t)$ and $D(t)$ to $E_\kappa(t)$. We prove that $\mathcal{E}(t)$ and $D(t)$ are both equivalent to $E_\kappa(t)$ in (4.27). The Cauchy-Schwarz inequality implies

$$\begin{aligned}\mathcal{E}(t) \leq \int_0^\ell \Big(\Big(1 + \frac{\gamma_1}{2} + \gamma_2\ell\Big)(\psi^+)^2 + (1 + \gamma_2\ell)(\psi^-)^2 + \gamma_1(\psi_x^+)^2 + \frac{3\gamma_1}{2}(\psi_x^-)^2\Big)\,\mathrm{d}x \\ + \frac{1}{\rho}\sum_{i=0,l}\Big(\gamma_1 m_i(v_i')^2 + \Big(m_i + \frac{\gamma_1}{2}d_i + \frac{3\gamma_1}{2}k_i + \gamma_3\rho\Big)v_i^2 \\ + \Big(\Big(1 + \frac{\gamma_1}{2}\Big)k_i + \gamma_3\rho\Big(\frac{d_i}{m_i} + 1\Big)\Big)\delta_i^2\Big).\end{aligned}$$

Setting $c_2 > 0$ as the maximum value among the coefficients on the right-hand side above, it holds that $\mathcal{E}(t) \leq c_2 E_\kappa(t)$. We invoke by the Cauchy-Schwarz inequality that $-2\rho\delta_i v_i \geq -\rho(\frac{d_i}{2m_i}\delta_i^2 + \hat{c}v_i^2)$ for some constant $\hat{c} > 0$, for $i = 0, \ell$. The reverse inequality reads as

$$\mathcal{E}(t) \geq \int_0^\ell \left(\left(1 - \frac{\gamma_1}{2} - \gamma_2\ell\right) (\psi^+)^2 + (1-\gamma_2\ell)(\psi^-)^2 + \gamma_1(\psi_x^+)^2 + \frac{\gamma_1}{2}(\psi_x^-)^2 \right) \mathrm{d}x$$
$$+ \frac{1}{\rho} \sum_{i=0,\ell} \left(\gamma_1 m_i (v_i')^2 + \left(m_i + \frac{\gamma_1}{2}(d_i + k_i) - \gamma_3 \hat{c}\rho \right) v_i^2 \right.$$
$$\left. + \left(\left(1 - \frac{\gamma_1}{2}\right) k_i + \gamma_3 \left(\frac{d_i}{2m_i}\right) \rho \right) \delta_i^2 \right).$$

We further choose $\gamma_1, \gamma_2, \gamma_3 > 0$ successively such that

$$\gamma_1 < 2, \quad \gamma_2 < \min\left\{ \frac{2-\gamma_1}{2\ell}, \frac{1}{\ell} \right\},$$
$$\gamma_3 < \frac{1}{\rho} \min\left\{ \frac{1}{\hat{c}} \left(m_i + \frac{\gamma_1}{2}(d_i + k_i) \right), \frac{2m_i k_i}{d_i} \left(1 - \frac{\gamma_1}{2}\right) \right\},$$

for $i = 0, \ell$. With these constants, we take $c_1 > 0$ be the minimum value among the coefficients on the right-hand side of the estimate from below for $\mathcal{E}(t)$. We obtain $\mathcal{E}(t) \geq c_1 E_\kappa(t)$.

Similarly, we compute

$$\begin{aligned} D(t) \leq \int_0^\ell \Bigg(& \left(1 + \frac{\gamma_1}{4} + \frac{\gamma_2}{2}(1 - c\ell + \ell) - \gamma_3 \tilde{c}\right) (\psi^+)^2 + \frac{\gamma_2}{2}(1+\ell)(\psi^-)^2 \\ & + \frac{3\gamma_2}{4}(\psi_x^-)^2 + \left(\frac{\gamma_1 - \gamma_2 c\ell}{2} - \gamma_3\tilde{c} \right) (\psi_x^+)^2 \Bigg) \mathrm{d}x \\ & + \frac{1}{\rho} \Bigg(\left(d_\ell - \frac{\gamma_1}{2}k_\ell - \gamma_3\rho \right) v_\ell^2 + \left(d_0 - \frac{\gamma_1}{2}k_0 - \frac{\gamma_2\rho\ell}{2} - \gamma_3\rho \right) v_0^2 \Bigg) \\ & + \frac{1}{\rho} \sum_{i=0,\ell} \left(\left(\gamma_1 d_i + \frac{\gamma_1}{2} m_i \right) (v_i')^2 + \frac{\gamma_3\rho}{m_i}(k_i - \rho\varepsilon_i)\delta_i^2 \right). \end{aligned} \tag{4.39}$$

Now, let us choose $\varepsilon_\ell, \varepsilon_0, \gamma_1, \gamma_2, \gamma_3 > 0$ successively such that

$$\begin{cases} \varepsilon_\ell < \frac{k_l}{\rho}, \quad \varepsilon_0 < \frac{k_0}{\rho}, \quad \text{as in (4.36)}, \\ \gamma_1 < 2\min\left\{ \frac{d_\ell}{k_\ell}, \frac{d_0}{k_0} \right\}, \quad \gamma_2 < \min\left\{ \frac{\gamma_1}{c\ell}, \frac{2}{\rho\ell}\left(d_0 - \frac{\gamma_1}{2}k_0\right) \right\}, \\ \gamma_3 < \gamma := \min\Big\{ \frac{1}{\rho}\left(d_\ell - \frac{\gamma_1}{2}k_\ell\right), \frac{1}{\rho}(d_0 - \frac{\gamma_1}{2}k_0 - \frac{\gamma_2\rho\ell}{2}), \frac{1}{2\tilde{c}}(\gamma_1 - \gamma_2 c\ell), \\ \qquad \frac{1}{\tilde{c}}\left(1 + \frac{\gamma_1}{4} + \frac{\gamma_2}{2}(1 - c\ell + \ell)\right) \Big\}, \end{cases} \tag{4.40}$$

where we reduce $\gamma_1, \gamma_2, \gamma_3 > 0$ as needed. It follows that $D(t) \leq \tilde{c}_2 E_\kappa(t)$, where $\tilde{c}_2 > 0$ is the maximum value among the coefficients on the right-hand side of (4.39). On the other hand, the Cauchy-Schwarz and Young inequalities imply

$$D(t) \geq \int_0^\ell \left(\left(1 - \frac{\gamma_1}{4} + \gamma_2\left(\frac{1-c\ell}{2} - \ell\tilde{c}_\varepsilon\right) - \gamma_3\tilde{c}\right) (\psi^+)^2 + \frac{\gamma_1}{4}(\psi_x^-)^2 \right.$$
$$\left. + \left(\frac{\gamma_1}{2} - \frac{\gamma_2}{2}c\ell - \gamma_3\tilde{c}\right) (\psi_x^+)^2 + \gamma_2\left(\frac{1}{2} - \ell\tilde{\varepsilon}\right)(\psi^-)^2 \right) \mathrm{d}x$$

$$+\frac{1}{\rho}\left(\left(d_\ell-\frac{\gamma_1}{2}k_\ell-\gamma_3\rho\right)v_\ell^2+\left(d_0-\frac{\gamma_1}{2}k_0-\frac{\gamma_2\rho l}{2}-\gamma_3\rho\right)v_0^2\right)$$
$$+\frac{1}{\rho}\sum_{i=0,\ell}\left(\left(\gamma_1 d_i+\frac{\gamma_1}{2}m_i\right)(v_i')^2+\frac{\gamma_3\rho}{m_i}(k_i-\rho\varepsilon_i)\delta_i^2\right), \tag{4.41}$$

where $\tilde{\varepsilon},\tilde{c}_\varepsilon>0$. We choose $\varepsilon,\varepsilon_\ell,\varepsilon_0,\gamma_1,\gamma_2,\gamma_3>0$ such that

$$\begin{cases}\tilde{\varepsilon}<\frac{1}{2\ell}, \quad \varepsilon_0,\varepsilon_\ell,\gamma_1>0 \text{ as in (4.40)},\\ \gamma_2>0 \text{ small enough}, \quad \gamma_3<\min\left\{\gamma,\frac{1}{\tilde{c}}(1-\frac{\gamma_1}{4}+\gamma_2(\frac{1-c\ell}{2}-\ell\tilde{c}_\varepsilon))\right\},\end{cases}$$

where $\gamma>0$ is the constant from (4.40), and $\gamma_2>0$ is reduced (if needed) such that $(1-\gamma_1/4)>\gamma_2\left(\ell\tilde{c}_\varepsilon-(1-c\ell)/2\right)$. With these constants, we take $\tilde{c}_1>0$ to be the minimum value among the coefficients on the right-hand side of (4.41). Finally, $D(t)\geq\tilde{c}_1E_\kappa(t)$, and the proof is complete. □

An immediate consequence of Theorems 3.4 and 4.5 is stated below.

Theorem 4.6. *There exist constants $\hat{C},r>0$ such that for every initial data $(\phi_0^-,\phi_0^+,\delta_0^0,\delta_\ell^0,v_0^0,v_\ell^0)\in\mathcal{X}$, it holds that*

$$\|\phi^-(t)-\phi_0^{-\perp}\|_{L^2}+\|\phi^+(t)-\phi_0^{+\perp}\|_{L^2}+\sum_{i=0,\ell}(|\delta_i(t)-\delta_i^{0\perp}|+|v_i(t)-v_i^{0\perp}|)$$
$$\leq\hat{C}e^{-rt}(\|\phi_0^-\|_{L^2}+\|\phi_0^+\|_{L^2}+|\delta_0^0|+|\delta_\ell^0|+|v_0^0|+|v_\ell^0|),$$

for all $t\geq0$, where $(\phi_0^{-\perp},\phi_0^{+\perp},\delta_0^{0\perp},\delta_\ell^{0\perp},v_0^{0\perp},v_\ell^{0\perp})=\Pi_0(\phi_0^-,\phi_0^+,\delta_0^0,\delta_\ell^0,v_0^0,v_\ell^0)$ and $\Pi_0:X\to\ker(\mathcal{A})$ is the orthogonal projection of $\mathcal{X}$ onto $\ker(\mathcal{A})$.

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