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Error Estimates for Mixed and Hybrid FEM for Elliptic Optimal Control Problems with Penalizations

ERROR ESTIMATES FOR MIXED AND HYBRID FEM FOR ELLIPTIC OPTIMAL CONTROL PROBLEMS WITH PENALIZATIONS

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Abstract.
Mixed and hybrid finite element discretizations for distributed optimal control problems governed by an elliptic equation are analyzed. A cost functional keeping track of both the state and its gradient is studied. A priori error estimates and super-convergence properties for the continuous and discrete optimal states, adjoint states, and controls will be given. The approximating finite-dimensional systems will be solved by adding penalization terms for the state and the associated Lagrange multipliers. In general, performing optimization, discretization, hybridization, and penalization in any order lead to the same optimality system. Numerical examples based on the Raviart–Thomas finite elements will be presented.

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1. Introduction

In this paper, we analyze mixed and hybrid finite element discretizations for the distributed optimal control of a linear elliptic problem with a homogeneous Dirichlet boundary condition. For example, the state equation models stationary heat distribution on a two-dimensional medium. We consider the following linear-quadratic optimal control problem

$$\min_{q\in L^2(\Omega)} J(u,\nabla u,q) := \frac{1}{2}\int_\Omega \alpha|u-u_d|^2 + \beta|\nabla u - \boldsymbol{\sigma}_d|^2 + \gamma|q|^2 \,\mathrm{d}x \tag{1.1}$$

subject to the state equation

$$\left[\begin{aligned} -\Delta u &= f + q && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned}\right. \tag{1.2}$$

In (1.2), $u = u(x)$ is the temperature of a certain material at the point $x \in \Omega$. We assume that Ω is an open and convex polygonal domain in $\mathbb{R}^2$. The functions $u_d : \Omega \to \mathbb{R}$ and $\boldsymbol{\sigma}_d : \Omega \to \mathbb{R}^2$ are given desired temperature distribution and heat flux, the precise function spaces where they belong will be stated below. Moreover, $f : \Omega \to \mathbb{R}$ represents an external heat source or sink, while $q : \Omega \to \mathbb{R}$ is the control. The parameters in the cost functional J are assumed to satisfy α, $\beta \geq 0$ with $\alpha + \beta > 0$ and $\gamma > 0$. For simplicity of exposition, the thermal diffusivity is normalized to 1.

In the case $\beta = 0$, a typical discretization scheme for the optimal control problem (1.1) is the H^1-conforming scheme using piecewise Lagrange polynomials. However, if the gradient of the state variable is included in the objective functional, then mixed methods are advantageous in the sense that both the state variable and its gradient can be approximated at the same order of accuracy. If one wishes to obtain superconvergence for the gradient in the H^1-conforming scheme, then post-processing is necessary. Mixed and hybrid methods for approximating the solutions of partial differential equations and their applications to optimal control problems have been well studied in the literature. For instance, the reader may consult to [3, 9, 12, 26] for elliptic problems, [10, 27] for parabolic problems, and [5, 13, 14, 15, 17, 19, 25] for hyperbolic problems.

In this work, we want to extend the study under a post-processing method and penalization of the mixed and hybrid finite element methods. Specifically, by applying a post-processing strategy developed by Arnold and Brezzi [1], we prove the super-convergence properties of the optimal controls, as well as the corresponding optimal states and adjoint states. The advantage of the hybrid formulation is to simplify the construction of basis functions by introducing appropriate Lagrange multipliers relaxing the continuity requirement across the edges of the elements. At the theoretical level, the solutions of the mixed and hybrid formulations coincide, however, they differ with respect to the implementation aspect, for instance, the total degrees of freedom for the flux is different.

From the practical point of view, the disadvantage of the hybrid finite element method is the additional degrees of freedom. For example, in the case of the Raviart–Thomas finite elements, these additional unknowns correspond to the Lagrange multipliers on the interior edges of the subdivision of the domain. There are several methods in order to compute numerically the resulting saddle point problems, for instance, the mixed-Schur complement, mixed-Lagrangian, conjugate gradient, and Uzawa algorithms can be utilized.

We shall add regularization terms to the finite-dimensional system and by reduction, the resulting system will be in terms of the discretized scalar state only. Moreover, the associated matrix is symmetric and positive-definite, hence conjugate gradient methods are applicable in this case. This penalization strategy is widely used in the discretization of the Stokes equation. Of course, the additional error due to this penalization will be studied as well. Both at the continuous and discrete levels, the analysis of mixed variational problems under certain perturbations has been studied by Bercovier [6].

For the proposed numerical scheme, the order of performing optimization, discretization, hybridization, and penalization is immaterial, and they lead to the same optimality system. A more detailed explanation will be given in the succeeding sections.

The plan of the paper is as follows: In Section 2, we briefly discuss the mixed and hybrid formulations of the state equation and the corresponding discretizations by the Raviart–Thomas finite elements. A priori error estimates for the primal, adjoint, and control variables in the mixed, hybrid, and penalized discretizations will be developed in Sections 3 and 4. Finally, in Section 5 we present a gradient-based algorithm approximating the optimal control and provide numerical examples that illustrate the results of the paper.

2. Weak Formulation and Discretization of the State Equation

2.1. Weak Formulation.

In this section, we briefly discuss the mixed formulation of the state equation (1.2) and recall the standard existence, uniqueness, and stability of solutions with respect to the data. Also, the corresponding conforming finite element discretization through the Raviart–Thomas finite elements as well as its hybridized form will be presented. For more details, we refer the reader to [8, 22, 23].

First, let us define the appropriate functional spaces in the weak formulation. Given an open and convex two-dimensional polygonal domain Ω, we consider the Hilbert spaces $W = L^2(\Omega)$ and $\boldsymbol{V} = \boldsymbol{H}(\mathrm{div}, \Omega) := \{\boldsymbol{\sigma} \in L^2(\Omega)^2 : \mathrm{div}\,\boldsymbol{\sigma} \in L^2(\Omega)\}$, where the latter space is equipped with the graph norm

$$\|\boldsymbol{\sigma}\|_{\mathrm{div}} := (\|\boldsymbol{\sigma}\|^2 + \|\mathrm{div}\,\boldsymbol{\sigma}\|^2)^{\frac{1}{2}},$$

as the state spaces for the temperature and heat flux. We denote the space of controls by $Q = L^2(\Omega)$. The norm and inner product in $L^2(\Omega)$ will be denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. For convenience, we shall also use the same notation for the norm and inner product of L^2-spaces on arbitrary measurable domains. The typical notation for the Sobolev spaces $H^k(\Omega)$ and $H_0^k(\Omega)$ will be utilized here and $\|\cdot\|_k$ denotes the associated Sobolev norms. Furthermore, we let $\boldsymbol{L}^2(\Omega) = L^2(\Omega) \times L^2(\Omega)$ and $\boldsymbol{H}^k(\Omega) = H^k(\Omega) \times H^k(\Omega)$.

Introducing the temperature flux $\boldsymbol{\sigma} := \nabla u$ as a new state variable, we can recast the state equation (1.2) as follows

$$\left[\begin{array}{ll} \boldsymbol{\sigma} - \nabla u = 0 & \text{in } \Omega, \\ \mathrm{div}\,\boldsymbol{\sigma} = -(f+q) & \text{in } \Omega. \end{array}\right.$$

Define the continuous bilinear form $b : \boldsymbol{V} \times W \to \mathbb{R}$ by

$$b(\boldsymbol{\sigma}, u) = (\mathrm{div}\,\boldsymbol{\sigma}, u).$$

The weak formulation of the Poisson equation now reads as follows: Given $f \in W$ and $q \in Q$, find $(\boldsymbol{\sigma}, u) \in \boldsymbol{V} \times W$ that satisfies

$$\left[\begin{array}{ll} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, u) = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{V}, \\ b(\boldsymbol{\sigma}, v) = -(f+q, v) & \forall v \in W. \end{array}\right. \tag{2.1}$$

Observe that in this case, the homogeneous Dirichlet boundary condition now turns as a natural boundary condition in the mixed formulation. It is well-known that the pair $(\boldsymbol{V}, W)$ satisfies the inf-sup condition

$$\inf_{u \in W\setminus\{0\}} \sup_{\boldsymbol{\sigma} \in \boldsymbol{V}\setminus\{0\}} \frac{b(\boldsymbol{\sigma}, u)}{\|\boldsymbol{\sigma}\|_{\mathrm{div}}\|u\|} \geq c > 0. \tag{2.2}$$

In what follows, we shall consider the following general variational problem in order to accommodate also for the analysis of the adjoint equation. Given $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $g \in W$, find $(\boldsymbol{\sigma}, u) \in \boldsymbol{V} \times W$ such that

$$\left[\begin{array}{ll} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, u) = (\boldsymbol{f}, \boldsymbol{\tau}) & \forall \boldsymbol{\tau} \in \boldsymbol{V}, \\ b(\boldsymbol{\sigma}, v) = (g, v) & \forall v \in W. \end{array}\right. \tag{2.3}$$

This problem corresponds to the mixed formulation of the elliptic boundary value problem

$$\left[\begin{array}{ll} -\Delta u = \mathrm{div}\,\boldsymbol{f} - g & \text{in } \Omega, \\ u = 0 & \text{on } \Omega. \end{array}\right. \tag{2.4}$$

Using the continuous embedding $\boldsymbol{L}^2(\Omega) \subset \boldsymbol{V}^*$, where $\boldsymbol{V}^*$ denotes the dual of $\boldsymbol{V}$, we have the following existence, uniqueness, and stability of solutions to (2.3) in virtue of the Brezzi splitting theorem. For a proof, we refer the reader to [8]. Furthermore, by the divergence theorem and elliptic regularity theory, one can show further regularity of the component u.

Proposition 2.1. *Given $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $g \in W$, the variational system (2.3) has a unique solution $(\boldsymbol{\sigma}, u) \in \boldsymbol{V} \times W$ and there exists a constant $C > 0$ independent of the data and the solution such that*

$$\|\boldsymbol{\sigma}\|_{\mathrm{div}} + \|u\| \le C(\|\boldsymbol{f}\| + \|g\|). \tag{2.5}$$

Moreover, if $\boldsymbol{f} \in \boldsymbol{V}$, then $\nabla u = \boldsymbol{\sigma} - \boldsymbol{f}$ and $u \in H_0^1(\Omega) \cap H^2(\Omega)$ is the weak solution (2.4).

2.2. DISCRETIZATION. In this subsection, we present the mixed and hybrid finite element discretizations for the variational problem (2.3). Let $\{\mathcal{T}_h\}_{0<h<h_0}$ be a shape-regular family of triangulations of Ω parametrized by their mesh sizes $h = \max_{K\in\mathcal{T}_h} h_K$, where h_K is the length of the largest edge of K. This means that there exists a constant $C > 0$ such that $h \le C\varrho_K$ and $h_K \le C\vartheta_K$ for every $K \in \mathcal{T}_h$ and $0 < h < h_0$, where ϱ_K and ϑ_K are the radii of the largest inscribed and the smallest circumscribed balls of $\bar{K}$, respectively. In particular, this implies that $h \le Ch_K$ for every $K \in \mathcal{T}_h$. In other words, the length of the edges of the triangles in the mesh are equivalent to the mesh size.

Given a set S and a nonnegative integer k, we denote by $P_k(S)$ the space of all polynomials in S of degree at most k. For each triangle $K \in \mathcal{T}_h$, let $RT_k(K)$ be the kth-order Raviart–Thomas finite element on K, that is,

$$RT_k(K) = P_k(K)^2 \oplus \mathbf{x}P_k^\circ(K),$$

where $P_k^\circ(K)$ is the space of homogeneous polynomials of degree k in K.

Associated with a triangulation $\mathcal{T}_h$, we define the following standard finite element spaces

$$\boldsymbol{V}_h^k = \{\boldsymbol{\sigma}_h \in \boldsymbol{V} : \boldsymbol{\sigma}_h|_K \in RT_k(K)\ \forall K \in \mathcal{T}_h\}$$
$$W_h^k = \{u_h \in W : u_h|_K \in P_k(K)\ \forall K \in \mathcal{T}_h\}.$$

Define the Fortin projection operator $\boldsymbol{\Pi}_h^k : \boldsymbol{V} \to \boldsymbol{V}_h^k$ such that

$$\int_{\partial K} (\boldsymbol{\Pi}_h^k\boldsymbol{\sigma} \cdot \boldsymbol{\nu})\lambda_h \,\mathrm{d}s = \int_{\partial K} (\boldsymbol{\sigma} \cdot \boldsymbol{\nu})\lambda_h \,\mathrm{d}s \quad \forall \lambda_h \in P_k(\partial K),$$
$$\int_K \boldsymbol{\Pi}_h^k\boldsymbol{\sigma} \cdot \boldsymbol{u}_h \,\mathrm{d}x = \int_K \boldsymbol{\sigma} \cdot \boldsymbol{u}_h \,\mathrm{d}x \qquad \forall \boldsymbol{u}_h \in P_{k-1}(K) \times P_{k-1}(K),$$

for every $K \in \mathcal{T}_h$ and $\boldsymbol{\sigma} \in \boldsymbol{V}$, where we set $P_{-1}(K) = \{0\}$. For the existence of $\boldsymbol{\Pi}_h^k$, we refer to [8]. Also, define the L^2-projection operator $P_h^k : W \to W_h^k$ by

$$\int_K (P_h^k u)u_h \,\mathrm{d}x = \int_K uu_h \,\mathrm{d}x \quad \forall u_h \in W_h^k,$$

for every $u \in W$ and $K \in \mathcal{T}_h$. It is well-known that we have $P_h^k\,\mathrm{div} = \mathrm{div}\,\boldsymbol{\Pi}_h^k$ from $\boldsymbol{V}$ into W_h^k. Moreover, the following projection errors hold

$$\|P_h^k u - u\| \le Ch^{k+1}\|u\|_{k+1} \tag{2.6}$$
$$\|\boldsymbol{\Pi}_h^k\boldsymbol{\sigma} - \boldsymbol{\sigma}\| \le Ch^{k+1}\|\boldsymbol{\sigma}\|_{k+1} \tag{2.7}$$

$$\|\operatorname{div} \boldsymbol{\Pi}_h^k \boldsymbol{\sigma} - \operatorname{div} \boldsymbol{\sigma}\| \leq Ch^{k+1} \|\operatorname{div} \boldsymbol{\sigma}\|_{k+1}, \tag{2.8}$$

as long as the regularity requirements $u \in H^{k+1}(\Omega)$, $\boldsymbol{\sigma} \in \boldsymbol{H}^{k+1}(\Omega)$, and $\operatorname{div} \boldsymbol{\sigma} \in H^{k+1}(\Omega)$ are satisfied.

Similar to the continuous case (2.2), the pair $(\boldsymbol{V}_h^k, W_h^k)$ also satisfies the following discrete inf-sup condition

$$\inf_{u_h \in W_h^k \setminus \{0\}} \sup_{\boldsymbol{\sigma}_h \in \boldsymbol{V}_h^k \setminus \{0\}} \frac{b(\boldsymbol{\sigma}_h, u_h)}{\|\boldsymbol{\sigma}_h\|_{\operatorname{div}} \|u_h\|} \geq c \tag{2.9}$$

for some $c > 0$ independent of h.

The mixed finite element discretization of (2.3) is given as follows: Given $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $g \in W$, find $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{V}_h^k \times W_h^k$ such that

$$\left[\begin{array}{rl} (\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, u_h) = (\boldsymbol{f}, \boldsymbol{\tau}_h) & \forall \boldsymbol{\tau}_h \in \boldsymbol{V}_h^k, \\ b(\boldsymbol{\sigma}_h, v_h) = (g, v_h) & \forall v_h \in W_h^k. \end{array}\right. \tag{2.10}$$

In virtue of the definition of the discrete spaces, $\boldsymbol{V}_h^k \subset \boldsymbol{V}$ and $W_h^k \subset W$, thus (2.10) is a conforming approximation of (2.3). Moreover, thanks to the discrete inf-sup condition (2.9), we have the following well-posedness result. Again, we refer to [8] for a proof of this proposition.

Proposition 2.2. *Given $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $g \in W$, (2.10) has a unique solution $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{V}_h^k \times W_h^k$ and there exists a constant $C > 0$ independent of h, the data, and the solution such that*

$$\|\boldsymbol{\sigma}_h\|_{\operatorname{div}} + \|u_h\| \leq C(\|\boldsymbol{f}\| + \|g\|). \tag{2.11}$$

Now, let us consider the hybridization of the finite element approximation (2.10). For this purpose, we denote by $\mathcal{E}_h$ and $\mathcal{E}_h^i$ the set of all edges and interior edges in the triangulation $\mathcal{T}_h$, respectively. Define the kth order *discontinuous* Raviart–Thomas finite element space

$$\boldsymbol{Y}_h^k = \{\boldsymbol{\sigma}_h \in \boldsymbol{L}^2(\Omega) : \boldsymbol{\sigma}_h|_K \in RT_k(K) \ \forall K \in \mathcal{T}_h\}$$

and the space of Lagrange multipliers associated with the edges of the triangulation

$$L_h^k = \{\lambda_h \in L^2(\mathcal{E}_h) : \lambda_h|_e \in P_k(e) \ \forall e \in \mathcal{E}_h^i\}.$$

Let $M_h^k = \{\lambda_h \in L_h^k : \lambda_h|_e = 0 \ \forall e \in \mathcal{E}_h \setminus \mathcal{E}_h^i\}$ be the elements in L_h^k that vanish on the boundary edges. We denote by $\operatorname{div} \boldsymbol{\sigma}_h$ the piecewise divergence of $\boldsymbol{\sigma}_h \in \boldsymbol{Y}_h^k$, that is, $\operatorname{div} \boldsymbol{\sigma}_h|_K = \operatorname{div}(\boldsymbol{\sigma}_h|_K)$ for every $K \in \mathcal{T}_h$. Given $\lambda_h \in L_h^k$, consider the norm

$$\|\lambda_h\|_h^2 := \sum_{K \in \mathcal{T}_h} \int_{\partial K} h_K |\lambda_h|^2 \, \mathrm{d}s.$$

By shape-regularity of the triangulations, $\|\lambda_h\|_h$ is equivalent to $h^{\frac{1}{2}} \|\lambda_h\|$.

In addition, let us define the bilinear operators $b_h : \boldsymbol{Y}_h^k \times W_h^k \to \mathbb{R}$ and $d_h : \boldsymbol{Y}_h^k \times M_h^k \to \mathbb{R}$ according to

$$b_h(\boldsymbol{\sigma}_h, u_h) = \sum_{K \in \mathcal{T}_h} \int_K (\operatorname{div} \boldsymbol{\sigma}_h) u_h \, \mathrm{d}x$$

$$d_h(\boldsymbol{\sigma}_h, \lambda_h) = -\sum_{K\in\mathcal{T}_h}\int_{\partial K} \boldsymbol{\sigma}_h\cdot\boldsymbol{\nu}_K\lambda_h \, \mathrm{d}s,$$

where $\boldsymbol{\nu}_K$ is the unit normal vector on ∂K pointing outward from K. Likewise, define the projection operator $\pi_h^k : H^1(\Omega) \to L_h^k$ by

$$\int_e (\pi_h^k u)\lambda_h \, \mathrm{d}s = \int_e u\lambda_h \, \mathrm{d}s \quad \forall \lambda_h \in P_k(e),$$

for every $e \in \mathcal{E}_h$.

With the above notations, the hybridization of (2.10) is given as follows: Given $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $g \in W$, find $(\boldsymbol{\sigma}_h, u_h, \lambda_h) \in \boldsymbol{Y}_h^k \times W_h^k \times M_h^k$ such that

$$\left[\begin{aligned} (\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h, u_h) + d_h(\boldsymbol{\tau}_h, \lambda_h) &= (\boldsymbol{f}, \boldsymbol{\tau}_h) && \forall \boldsymbol{\tau}_h \in \boldsymbol{Y}_h^k, \\ b_h(\boldsymbol{\sigma}_h, v_h) &= (g, v_h) && \forall v_h \in W_h^k, \\ d_h(\boldsymbol{\sigma}_h, \mu_h) &= 0 && \forall \mu_h \in M_h^k. \end{aligned}\right. \tag{2.12}$$

Let us recall that $\boldsymbol{\sigma}_h \in \boldsymbol{Y}_h^k$ satisfies $d_h(\boldsymbol{\sigma}_h, \mu_h) = 0$ for every $\mu_h \in M_h^k$ if and only if $\boldsymbol{\sigma}_h \in \boldsymbol{V}_h^k$. Hence, it follows that if $(\boldsymbol{\sigma}_h, u_h, \lambda_h)$ is a solution of (2.12), then $(\boldsymbol{\sigma}_h, u_h)$ is a solution of (2.10). On the other hand, the existence and uniqueness of solution to (2.12) follows from the fact that the corresponding matrix for the finite-dimensional square system is injective. For the details, we refer to [1].

In particular, the solution of (2.12) satisfies the stability estimate (2.11). Furthermore, the Lagrange multiplier λ_h satisfies the stability estimate

$$\|\lambda_h\|_h \le C(h\|\boldsymbol{f}\| + h\|\boldsymbol{\sigma}_h\| + \|u_h\|). \tag{2.13}$$

To see this, let us first recall from [23, Sections 3 and 4] or [1, page 13] that there exists $\boldsymbol{\zeta}_h \in \boldsymbol{Y}_h^k$ such that $\boldsymbol{\zeta}_h \cdot \boldsymbol{\nu}|_e = \lambda_h|_e$ for every edge e in $\mathcal{T}_h$ and there exists a constant $C > 0$ independent of λ_h and h such that

$$h^2 \sum_{K\in\mathcal{T}_h}\int_K |\nabla\boldsymbol{\zeta}_h|^2 \, \mathrm{d}x + \|\boldsymbol{\zeta}_h\|^2 \le C\|\lambda_h\|_h^2. \tag{2.14}$$

Taking $\boldsymbol{\zeta}_h$ as the test function in (2.12), and utilizing (2.14) yields

$$\begin{aligned} \|\lambda_h\|^2 = -d_h(\boldsymbol{\zeta}_h, \lambda_h) &= (\boldsymbol{\sigma}_h, \boldsymbol{\zeta}_h) + b_h(\boldsymbol{\zeta}_h, u_h) - (\boldsymbol{f}, \boldsymbol{\zeta}_h) \\ &\le C(\|\boldsymbol{f}\| + \|\boldsymbol{\sigma}_h\| + h^{-1}\|u_h\|)\|\lambda_h\|_h \end{aligned}$$

and therefore, we have (2.13).

Let us recall the post-processing method described in [1]. Let k be an even integer. From [1, Lemma 2.1], we can deduce that for each $(\lambda_h, u_h) \in L_h^k \times W_h^k$, there exists a unique $\widetilde{u}_h \in W_h^{k+1}$ such that

$$\sum_{e\in\mathcal{E}_h}\int_e (\widetilde{u}_h - \lambda_h)\mu_h \, \mathrm{d}s = 0 \quad \forall \mu_h \in L_h^k,$$

$$\int_\Omega (\widetilde{u}_h - u_h)v_h \, \mathrm{d}x = 0 \quad \forall v_h \in W_h^{k-2},$$

where we set $W_h^{-2} = \{0\}$. Therefore, $R_h^{k+1} : (\lambda_h, u_h) \mapsto \widetilde{u}_h$ is a well-defined map from $L_h^k \times W_h^k$ into W_h^{k+1}. We shall call R_h^{k+1} as the *Arnold–Brezzi post-processing* *operator*. Moreover, it holds that

$$\|R_h^{k+1}(\lambda_h, u_h)\| \le C(\|\lambda_h\|_h + \|u_h\|). \tag{2.15}$$

If $k = 0$, then we simply write $R_h^1 \lambda_h$ for $R_h^1(\lambda_h, u_h)$ since the post-processing operator R_h^1 is independent of the second argument.

The assumption that k is even was imposed in order to have a unified proof for the above properties of the operator R_h^{k+1}. For odd k, one needs to construct *ad hoc* nonconforming approximation in order for such properties of R_h^{k+1} to hold. For example, the cases where $k = 1$ or $k = 2$ has been considered in [1].

To provide *a priori* error estimates for the discrete and continuous primal and dual variables, we shall often use the following general stability theorem. All throughout this paper, we shall assume additional regularity on the optimal primal states, dual states, and control. By classical elliptic regularity theory, these conditions can be achieved if the convex domain Ω is smooth enough and the desired states are also sufficiently regular. Note that it is also possible to *manufacture* solutions that satisfy such smoothness properties on rectangular domains, see for instance the discussion in Section 5 that involves eigenfunctions.

Theorem 2.3. *Suppose that g, $y \in W$, $\boldsymbol{f} \in \boldsymbol{V}$, $\boldsymbol{f}_h \in \boldsymbol{V}_h^k$, and $\operatorname{div} \boldsymbol{f}_h = P_h^k \operatorname{div} \boldsymbol{f}$. Let $(\boldsymbol{\sigma}, u) \in \boldsymbol{V} \times W$ be the solution of*

$$\left[\begin{array}{rll} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, u) = (\boldsymbol{f}, \boldsymbol{\tau}) & \forall \boldsymbol{\tau} \in \boldsymbol{V}, \\ b(\boldsymbol{\sigma}, v) = (g, v) & \forall v \in W, \end{array}\right. \tag{2.16}$$

and $(\boldsymbol{\sigma}_h, u_h, \lambda_h) \in \boldsymbol{Y}_h^k \times W_h^k \times M_h^k$ be the solution of

$$\left[\begin{array}{rll} (\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h, u_h) + d_h(\boldsymbol{\tau}_h, \lambda_h) = (\boldsymbol{f}_h, \boldsymbol{\tau}_h) & \forall \boldsymbol{\tau}_h \in \boldsymbol{Y}_h^k, \\ b_h(\boldsymbol{\sigma}_h, v_h) = (y, v_h) & \forall v_h \in W_h^k, \\ d_h(\boldsymbol{\sigma}_h, \mu_h) = 0 & \forall \mu_h \in M_h^k. \end{array}\right. \tag{2.17}$$

Suppose that $\boldsymbol{\sigma}, \boldsymbol{f} \in \boldsymbol{H}^{k+2}(\Omega)$ and $g \in H^{k+1}(\Omega)$. Then, there exists a constant $C > 0$ independent of h, the data, and on the continuous and discrete solutions such that

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\operatorname{div}} + \|u - u_h\| \le Ch^{k+1}\|\boldsymbol{\sigma}\|_{k+2} + C(\|\boldsymbol{f} - \boldsymbol{f}_h\| + \|g - y\|) \tag{2.18}$$

$$\begin{aligned} \|P_h^k u - u_h\| \le{}& Ch^{k+2}(\|g\|_{k+1} + \|\operatorname{div} \boldsymbol{f}\|_{k+1}) \\ &+ C\|P_h^k g - P_h^k y\| + Ch(\|\boldsymbol{f} - \boldsymbol{f}_h\| + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|). \end{aligned} \tag{2.19}$$

Proof. First, let us observe that the solution of (2.16) satisfies the following system of variational equations:

$$\left[\begin{array}{rll} (\boldsymbol{\Pi}_h^k \boldsymbol{\sigma}, \boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h, P_h^k u) + d_h(\boldsymbol{\tau}_h, \pi_h^k u) = (\boldsymbol{f} + \boldsymbol{\Pi}_h^k \boldsymbol{\sigma} - \boldsymbol{\sigma}, \boldsymbol{\tau}_h) & \forall \boldsymbol{\tau}_h \in \boldsymbol{Y}_h^k, \\ b_h(\boldsymbol{\Pi}_h^k \boldsymbol{\sigma}, v_h) = (g, v_h) & \forall v_h \in W_h^k, \\ d_h(\boldsymbol{\Pi}_h^k \boldsymbol{\sigma}, \mu_h) = 0 & \forall \mu_h \in M_h^k. \end{array}\right. \tag{2.20}$$

Consider the difference $(\delta\boldsymbol{\sigma}_h, \delta u_h, \delta\lambda_h) := (\boldsymbol{\Pi}_h^k \boldsymbol{\sigma} - \boldsymbol{\sigma}_h, P_h^k u - u_h, \pi_h^k u - \lambda_h)$ of the solutions for (2.20) and (2.17). By taking the difference of the variational formulations we obtain the following system:

$$\left[\begin{aligned} (\delta\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h)+b_h(\boldsymbol{\tau}_h,\delta u_h)+d_h(\boldsymbol{\tau}_h,\delta\lambda_h) &= (\boldsymbol{r}_h,\boldsymbol{\tau}_h) && \forall \boldsymbol{\tau}_h\in \boldsymbol{Y}_h^k,\\ b_h(\delta\boldsymbol{\sigma}_h,v_h) &= (g-y,v_h) && \forall v_h\in W_h^k,\\ d_h(\delta\boldsymbol{\sigma}_h,\mu_h) &= 0 && \forall \mu_h\in M_h^k, \end{aligned}\right. \tag{2.21}$$

where $\boldsymbol{r}_h=\boldsymbol{f}-\boldsymbol{f}_h+\boldsymbol{\Pi}_h^k\boldsymbol{\sigma}-\boldsymbol{\sigma}$. Due to the stability estimate (2.11), we have

$$\|\delta\boldsymbol{\sigma}_h\|_{\mathrm{div}}+\|\delta u\|\le C(\|\boldsymbol{f}-\boldsymbol{f}_h\|+\|\boldsymbol{\Pi}_h^k\boldsymbol{\sigma}-\boldsymbol{\sigma}\|+\|g-y\|). \tag{2.22}$$

By rewriting $\boldsymbol{\sigma}-\boldsymbol{\sigma}_h$ and $u-u_h$ as follows:

$$\begin{aligned} \boldsymbol{\sigma}-\boldsymbol{\sigma}_h &= (\boldsymbol{\sigma}-\boldsymbol{\Pi}_h^k\boldsymbol{\sigma})+\boldsymbol{\Pi}_h^k\boldsymbol{\sigma}-\boldsymbol{\sigma}_h=(\boldsymbol{\sigma}-\boldsymbol{\Pi}_h^k\boldsymbol{\sigma})+\delta\boldsymbol{\sigma}_h\\ u-u_h &= (u-P_h^ku)+(P_h^ku-u_h)=(u-P_h^ku)+\delta u_h, \end{aligned}$$

we can deduce (2.18) from (2.6), (2.7), and (2.22).

The proof of (2.19) is based on a standard duality argument. Let $z\in H_0^1(\Omega)\cap H^2(\Omega)$ be the weak solution of the elliptic boundary value problem $\Delta z=\delta u_h$ in Ω with homogeneous Dirichlet boundary condition $z=0$ on $\partial\Omega$, and define $\boldsymbol{\varphi}=\nabla z$. By standard regularity theory, it holds that

$$\|\boldsymbol{\varphi}\|_1+\|z\|_2\le C\|\delta u_h\|. \tag{2.23}$$

Taking $\boldsymbol{\Pi}_h^k\boldsymbol{\varphi}\in\boldsymbol{V}_h^k\subset\boldsymbol{Y}_h^k$ as the test function in (2.21), using the definition of the Fortin projection, and invoking the fact that $\boldsymbol{\varphi}$ changes signs on opposite sides of each interior edges, we have

$$d_h(\boldsymbol{\Pi}_h^k\boldsymbol{\varphi},\delta\lambda_h)=-\sum_{K\in\mathcal{T}_h}\int_{\partial K}\boldsymbol{\varphi}\cdot\boldsymbol{\nu}_K\delta\lambda_h\,\mathrm{d}s=0,$$

and thus we obtain the following:

$$\begin{aligned} \|\delta u_h\|^2 &= (\boldsymbol{f}-\boldsymbol{f}_h,\boldsymbol{\Pi}_h^k\boldsymbol{\varphi})-(\boldsymbol{\Pi}_h^k\boldsymbol{\varphi},\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\\ &= (\boldsymbol{f}-\boldsymbol{f}_h,\boldsymbol{\Pi}_h^k\boldsymbol{\varphi}-\boldsymbol{\varphi})-(\boldsymbol{\Pi}_h^k\boldsymbol{\varphi}-\boldsymbol{\varphi},\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\\ &\quad-(\nabla z,\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)+(\nabla z,\boldsymbol{f}-\boldsymbol{f}_h). \end{aligned} \tag{2.24}$$

From the divergence theorem, it follows that

$$-(\nabla z,\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)=(z,\operatorname{div}\boldsymbol{\sigma}-\operatorname{div}\boldsymbol{\sigma}_h)-\sum_{K\in\mathcal{T}_h}\int_{\partial K}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\cdot\boldsymbol{\nu}_K z\,\mathrm{d}s.$$

Note that $\boldsymbol{\sigma}_h\in\boldsymbol{V}_h^k$ according to [1, Lemma 1.2]. The above boundary terms vanish due to the fact that both $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_h$ are in $\boldsymbol{V}$. Hence, according to $\operatorname{div}\boldsymbol{\sigma}=g$ and $\operatorname{div}\boldsymbol{\sigma}_h=P_h^ky$, we deduce that

$$\begin{aligned} -(\nabla z,\boldsymbol{\sigma}-\boldsymbol{\sigma}_h) &= (g-P_h^ky,z)\\ &= (g-P_h^kg,z)+(P_h^kg-P_h^ky,z)\\ &= (g-P_h^kg,z-P_h^kz)+(P_h^kg-P_h^ky,z). \end{aligned}$$

As a result, the following estimate holds

$$|(\nabla z,\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)|\le C(h^{k+2}\|g\|_{k+1}+\|P_h^kg-P_h^ky\|)\|z\|_1. \tag{2.25}$$

Applying the same line of arguments and the assumption that $\operatorname{div} \boldsymbol{f}_h = P_h^k \operatorname{div} \boldsymbol{f}$,

$$-(\nabla z, \boldsymbol{f} - \boldsymbol{f}_h) = (\operatorname{div} \boldsymbol{f} - \operatorname{div} \boldsymbol{f}_h, z) = (\operatorname{div} \boldsymbol{f} - P_h^k \operatorname{div} \boldsymbol{f}, z - P_h^k z),$$

and therefore, it holds that

$$|(\nabla z, \boldsymbol{f} - \boldsymbol{f}_h)| \leq C h^{k+2} \|\operatorname{div} \boldsymbol{f}\|_{k+1} \|z\|_1. \tag{2.26}$$

Since $\|\boldsymbol{\Pi}_h^k \boldsymbol{\varphi} - \boldsymbol{\varphi}\| \leq Ch \|\boldsymbol{\varphi}\|_1$, we have

$$|(\boldsymbol{f} - \boldsymbol{f}_h, \boldsymbol{\Pi}_h^k \boldsymbol{\varphi} - \boldsymbol{\varphi})| + |(\boldsymbol{\Pi}_h^k \boldsymbol{\varphi} - \boldsymbol{\varphi}, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)| \leq Ch(\|\boldsymbol{f} - \boldsymbol{f}_h\| + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|)\|\boldsymbol{\varphi}\|_1. \tag{2.27}$$

Using (2.25)–(2.27) and (2.23) in (2.24) yields (2.19). □

The following theorem deals with the error between the post-processed state and the component u of the solution. We would like to emphasize that the proofs of the estimates below are independent of the proofs of the estimates given in the previous theorem.

Theorem 2.4. *Suppose that $\boldsymbol{f}$, $\boldsymbol{f}_h \in \boldsymbol{L}^2(\Omega)$ and g, $y \in W$. Let $(\boldsymbol{\sigma}, u)$ and $(\boldsymbol{\sigma}_h, u_h, \lambda_h)$ be the solutions of the variational equations in Theorem 2.3. Then there exists a constant $C > 0$ independent of h, the data, and on the solutions such that*

$$\|\pi_h^k u - \lambda_h\|_h \leq C(\|P_h^k u - u_h\| + h\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| + h\|\boldsymbol{f} - \boldsymbol{f}_h\|). \tag{2.28}$$

Moreover, if k is an even integer, then it holds that

$$\|u - R_h^{k+1}(\lambda_h, u_h)\| \leq C(\|\pi_h^k u - \lambda_h\|_h + \|P_h^k u - u_h\|) + Ch^{k+2}\|u\|_{k+2}. \tag{2.29}$$

Proof. We utilize the notations in the proof of the preceeding theorem. Similar to the proof of (2.13), choose $\boldsymbol{\zeta}_h \in \boldsymbol{Y}_h^k$ such that $\boldsymbol{\zeta}_h \cdot \boldsymbol{\nu} = \delta\lambda_h$ on each interior edge of the triangulation and such that

$$h^2 \sum_{K \in \mathcal{T}_h} \int_K |\nabla \boldsymbol{\zeta}_h|^2 \,\mathrm{d}x + \|\boldsymbol{\zeta}_h\|^2 \leq C\|\delta\lambda_h\|_h^2.$$

Taking $\boldsymbol{\zeta}_h$ as the test function in (2.21), we have

$$\|\delta\lambda_h\|^2 = -d_h(\boldsymbol{\zeta}_h, \delta\lambda_h) = (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\zeta}_h) + b_h(\boldsymbol{\zeta}_h, \delta u_h) - (\boldsymbol{f} - \boldsymbol{f}_h, \boldsymbol{\zeta}_h),$$

and by the Cauchy–Schwarz inequality and the above estimate for $\boldsymbol{\zeta}_h$, this implies (2.28).

Consider the nonconforming approximation $\widehat{u}_h = R_h^{k+1}(\pi_h^k u, P_h^k u) \in W_h^{k+1}$. By standard scaling argument, see [16] for instance, we have

$$\|u - \widehat{u}_h\| \leq Ch^{k+2}\|u\|_{k+2}. \tag{2.30}$$

According to the linearity of the Arnold–Brezzi post-processing operator and the fact that $P_h^{k-2} u = P_h^{k-2}(P_h^k u)$, we have $\widehat{u}_h - R_h^{k+1}(\lambda_h, u_h) = R_h^{k+1}(\delta\lambda_h, \delta u_h)$, and consequently, utilizing the estimate (2.30) along with the boundedness of the operator R_h^{k+1} given in (2.15), we obtain (2.29). □

Remark 2.5. Combining (2.28) and (2.29), we obtain

$$\|u - R_h^{k+1}(\lambda_h, u_h)\| \leq C(\|P_h^k u - u_h\| + h\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| + h\|\boldsymbol{f} - \boldsymbol{f}_h\|) + Ch^{k+2}\|u\|_{k+2},$$

provided that $u \in H^{k+2}(\Omega)$. Therefore, in order to prove super-convergence of the post-processed state $R_h^{k+1}(\lambda_h, u_h)$ to u, it is enough to establish super-convergence of the discrete solution u_h to the projection $P_h^k u$ of u.

3. Error Estimates for the Primal, Adjoint, and Control Variables

The goal of the current section is to recast the optimal control problem (1.1) in its mixed and hybrid formulations given in the previous section. We then address the well-posedness of the optimal control problem. Finally, we shall prove *a priori* error estimates for the continuous and discrete optimal states, adjoint states, and controls.

With the mixed formulation of the Poisson equation, the optimal control problem (1.1) can be expressed as

$$\min_{q\in Q} J(u, \boldsymbol{\sigma}, q) \quad \text{subject to (2.1).} \tag{3.1}$$

Introducing the control-to-state map $q \mapsto (\boldsymbol{\sigma}, u) = (\boldsymbol{\sigma}(q), u(q)) : L^2(\Omega) \to \boldsymbol{V} \times L^2(\Omega)$, where $(\boldsymbol{\sigma}(q), u(q))$ is the solution of (2.1) for a given control q, as well as the reduced cost $j : Q \to \mathbb{R}$ by $j(q) = J(u(q), \boldsymbol{\sigma}(q), q)$, the constrained optimization problem (3.1) can be equivalently formulated as an unconstrained minimization in Q as

$$\min_{q\in Q} j(q). \tag{3.2}$$

The derivative of j at $q \in Q$ in the direction $\delta q \in Q$ is given by

$$j'(q)\delta q = \alpha(u(q) - u_d, u(\delta q)) + \beta(\boldsymbol{\sigma}(q) - \boldsymbol{\sigma}_d, \boldsymbol{\sigma}(\delta q)) + \gamma(q, \delta q).$$

Introducing the adjoint variable $(\boldsymbol{\varphi}(q), w(q)) = (\boldsymbol{\varphi}, w) \in \boldsymbol{V} \times W$ solving the problem

$$\left[\begin{aligned} (\boldsymbol{\varphi}, \boldsymbol{\psi}) + b(\boldsymbol{\psi}, w) &= -\beta(\boldsymbol{\sigma}(q) - \boldsymbol{\sigma}_d, \boldsymbol{\psi}) \quad && \forall \boldsymbol{\psi} \in \boldsymbol{V}, \\ b(\boldsymbol{\varphi}, \phi) &= -\alpha(u(q) - u_d, \phi) && \forall \phi \in W, \end{aligned}\right. \tag{3.3}$$

we can express the above directional derivative as

$$j'(q)\delta q = (\gamma q + w(q), \delta q).$$

Take note that the solution of the variational system (3.3) satisfies $\operatorname{div} \boldsymbol{\varphi}(q) = -\alpha(u(q) - u_d)$, and if $\boldsymbol{\sigma}_d \in \boldsymbol{V}$, then $w(q) \in H_0^1(\Omega) \cap H^2(\Omega)$ is the weak solution of the following boundary value problem:

$$\left[\begin{aligned} -\Delta w(q) &= \alpha(u(q) - u_d) - \beta \operatorname{div}(\boldsymbol{\sigma}(q) - \boldsymbol{\sigma}_d) \quad && \text{in } \Omega, \\ w(q) &= 0 && \text{on } \partial\Omega. \end{aligned}\right. \tag{3.4}$$

The following well-posedness result can be established using standard methods in linear-quadratic optimal control problems, see [24]. Moreover, the first order necessary optimality condition $j'(\bar{q})\delta q = 0$ for all $\delta q \in Q$ for the optimal control $\bar{q}$ is also sufficient.

Theorem 3.1. *Given $f \in L^2(\Omega)$, $u_d \in L^2(\Omega)$, and $\boldsymbol{\sigma}_d \in \boldsymbol{L}^2(\Omega)$, the optimal control problem (3.1) has a unique solution $(\bar{q}, \bar{\boldsymbol{\sigma}}, \bar{u}) \in Q \times \boldsymbol{V} \times W$, where $(\bar{\boldsymbol{\sigma}}, \bar{u}) = (\boldsymbol{\sigma}(\bar{q}), u(\bar{q}))$ is the corresponding optimal state. Moreover, if $(\bar{\boldsymbol{\varphi}}, \bar{w}) = (\boldsymbol{\varphi}(\bar{q}), w(\bar{q}))$ is the associated optimal adjoint state, then $\bar{q} = -\gamma^{-1}\bar{w}$.*

Now we discuss the semidiscretization of (3.1), that is, the optimal control problem where the state equation as well as the desired states are discretized, while the control space is still retained. For the state equation, we have the following mixed finite element semidiscretization: Given $q \in Q$, find $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{V}_h^k \times W_h^k$ such that

$$\left[\begin{array}{ll} (\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, u_h) = 0 & \forall \boldsymbol{\tau}_h \in \boldsymbol{V}_h^k, \\ \qquad b(\boldsymbol{\sigma}_h, v_h) = -(f_h + q, v_h) & \forall v_h \in W_h^k, \end{array}\right. \tag{3.5}$$

where $f_h \in W$ is a certain approximation of f.

Given appropriate approximations $u_{dh} \in W$ and $\boldsymbol{\sigma}_{dh} \in \boldsymbol{V}$ of the desired states u_d and $\boldsymbol{\sigma}_d$, to be specified concretely below, consider the discretized cost functional $J_h : W \times \boldsymbol{V} \times Q \to \mathbb{R}$ defined by

$$J_h(u, \boldsymbol{\sigma}, q) := \frac{\alpha}{2}\|u - u_{dh}\|^2 + \frac{\beta}{2}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{dh}\|^2 + \frac{\gamma}{2}\|q\|^2$$

and the semidiscrete reduced cost functional $j_h : Q \to \mathbb{R}$ given by

$$j_h(q) = J_h(u_h(q), \boldsymbol{\sigma}_h(q), q),$$

where $q \mapsto (\boldsymbol{\sigma}_h, u_h) := (\boldsymbol{\sigma}_h(q), u_h(q)) : Q \to \boldsymbol{V}_h^k \times W_h^k$ is the operator that maps a control $q \in Q$ to the solution of (3.5). The reduced semidiscrete control problem is now given by

$$\min_{q \in Q} j_h(q). \tag{3.6}$$

As in the continuous case, the directional derivative of j_h at $q \in Q$ in the direction $\delta q \in Q$ is given by

$$j_h'(q)\delta q = (\gamma q + w_h(q), \delta q),$$

where $w_h(q)$ is the second component of the pair $(\boldsymbol{\varphi}_h, w_h) = (\boldsymbol{\varphi}_h(q), w_h(q)) \in \boldsymbol{V}_h^k \times W_h^k$ solving the semidiscrete adjoint equation

$$\left[\begin{array}{ll} (\boldsymbol{\varphi}_h, \boldsymbol{\psi}_h) + b(\boldsymbol{\psi}_h, w_h) = -\beta(\boldsymbol{\sigma}_h(q) - \boldsymbol{\sigma}_{dh}, \boldsymbol{\psi}_h) & \forall \boldsymbol{\psi}_h \in \boldsymbol{V}_h^k, \\ \qquad b(\boldsymbol{\varphi}_h, \phi_h) = -\alpha(u_h(q) - u_{dh}, \phi_h) & \forall \phi_h \in W_h^k. \end{array}\right. \tag{3.7}$$

Observe that (3.7) is the mixed finite element discretization of the continuous adjoint equation (3.3). Hence, the process of optimization and discretization commute for the finite element scheme discussed above. In other words, the discretized optimality system of the continuous control problem is the optimality system of the discretized control problem. Analogous to the continuous case, we have the following existence theorem.

Theorem 3.2. *Suppose that $f_h \in L^2(\Omega)$, $u_{dh} \in L^2(\Omega)$, and $\boldsymbol{\sigma}_{dh} \in \boldsymbol{L}^2(\Omega)$. Then, the optimal control problem (3.1) has a unique solution $(\bar{q}_h, \bar{\boldsymbol{\sigma}}_h, \bar{u}_h) \in Q \times \boldsymbol{V}_h^k \times W_h^k$, where $(\bar{\boldsymbol{\sigma}}_h, \bar{u}_h) = (\boldsymbol{\sigma}_h(\bar{q}_h), u_h(\bar{q}_h))$ is the corresponding optimal state. Moreover, if $(\bar{\boldsymbol{\varphi}}_h, \bar{w}_h) = (\boldsymbol{\varphi}_h(\bar{q}_h), w_h(\bar{q}_h))$ is the optimal adjoint state, then $\bar{q}_h = -\gamma^{-1}\bar{w}_h$.*

The hybrid formulation of the semidiscrete state equation (3.5) is

$$\left[\begin{array}{ll} (\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h, u_h) + d_h(\boldsymbol{\tau}_h, \lambda_h) = 0 & \forall \boldsymbol{\tau}_h \in \boldsymbol{Y}_h^k, \\ \qquad b_h(\boldsymbol{\sigma}_h, v_h) = -(f_h + q, v_h) & \forall v_h \in W_h^k, \\ \qquad d_h(\boldsymbol{\sigma}_h, \mu_h) = 0 & \forall \mu_h \in M_h^k. \end{array}\right. \tag{3.8}$$

For a given control $q \in Q$, we denote the solution of (3.8) by $(\boldsymbol{\sigma}_h(q), u_h(q), \lambda_h(q)) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$. By the remarks on the previous section, the solution of the optimal control problem (3.6) is the same if we replace the solution operator determined by (3.5) with the solution operator determined by the hybrid form (3.8).

On the other hand, if (3.8) is utilized as the state equation in (3.6), the corresponding adjoint equation will be

$$\left[\begin{array}{lll} (\boldsymbol{\varphi}_h, \boldsymbol{\psi}_h) + b_h(\boldsymbol{\psi}_h, w_h) + d_h(\boldsymbol{\psi}_h, \mu_h) = -\beta(\boldsymbol{\sigma}_h(q) - \boldsymbol{\sigma}_{dh}, \boldsymbol{\psi}_h) & \forall \boldsymbol{\psi}_h \in \boldsymbol{Y}_h^k, \\ b_h(\boldsymbol{\varphi}_h, \phi_h) = -\alpha(u_h(q) - u_{dh}, \phi_h) & \forall \phi_h \in W_h^k, \\ d_h(\boldsymbol{\varphi}_h, \theta_h) = 0 & \forall \theta_h \in M_h^k. \end{array}\right. \tag{3.9}$$

For a given control $q \in Q$, we denote by $(\boldsymbol{\varphi}_h(q), w_h(q), \mu_h(q)) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ the solution of (3.9).

Take note that (3.9) is the hybrid formulation of the adjoint equation (3.7). Therefore, with the proposed numerical scheme, the process of optimization and hybridization commute at the discrete level. For this type of approximation, we denote the optimal state by $(\bar{\boldsymbol{\sigma}}_h, \bar{u}_h, \bar{\lambda}_h) = (\boldsymbol{\sigma}_h(\bar{q}_h), u_h(\bar{q}_h), \lambda_h(\bar{q}_h))$ and the optimal adjoint state by $(\bar{\boldsymbol{\varphi}}_h, \bar{w}_h, \bar{\mu}_h) = (\boldsymbol{\varphi}_h(\bar{q}_h), w_h(\bar{q}_h), \mu_h(\bar{q}_h))$.

We now consider the fully discrete optimal control problem, that is, the control space is also discretized. Given a finite-dimensional subspace Q_ρ of Q, let $(\boldsymbol{\sigma}_h(q_\rho), u_h(q_\rho))$ be the solution of (3.5) with q replaced by q_ρ. For example, one may take $Q_\rho = W_h^k$ in the mixed case and $Q_\rho = W_h^{k+1}$ in the hybrid case. As in the continuous and semidiscrete case, consider the fully discrete reduced cost functional $j_{h\rho} = j_h|_{Q_\rho} : Q_\rho \to \mathbb{R}$. The fully discrete finite-dimensional approximation of (3.2) is

$$\min_{q_\rho \in Q_\rho} j_{h\rho}(q_\rho) := J_h(u_h(q_\rho), \boldsymbol{\sigma}_h(q_\rho), q_\rho). \tag{3.10}$$

The directional derivative of $j_{h\rho}$ is $j'_{h\rho}(q_\rho)\delta q_\rho = (\gamma q_\rho + w_h(q_\rho), \delta q_\rho)$. Similar to the above discussions, the unique optimal control of (3.10), denoted by $\bar{q}_{h\rho}$, is given by $\bar{q}_{h\rho} = -\gamma^{-1} w_h(\bar{q}_{h\rho})$. Likewise, if the fully discrete state equation (3.5) with $q = q_\rho$ is replaced by its hybridized form (3.8) with $q = q_\rho$, then we denote the corresponding optimal state and adjoint state by $(\bar{\boldsymbol{\sigma}}_{h\rho}, \bar{u}_{h\rho}, \bar{\lambda}_{h\rho}) = (\boldsymbol{\sigma}_h(\bar{q}_{h\rho}), u_h(\bar{q}_{h\rho}), \lambda_h(\bar{q}_{h\rho}))$ and $(\bar{\boldsymbol{\varphi}}_{h\rho}, \bar{w}_{h\rho}, \bar{\mu}_{h\rho}) = (\boldsymbol{\varphi}_h(\bar{q}_{h\rho}), w_h(\bar{q}_{h\rho}), \mu_h(\bar{q}_{h\rho}))$, respectively. Again, at the discrete level, the process of optimization and hybridization commute.

The first *a priori* estimate we will establish is concerned on the discretization errors between the continuous and semidiscrete state and adjoint equations with a given fixed control. In the following and for the remaining parts of the paper, we assume that the primal, dual, and control variables are sufficiently smooth.

Theorem 3.3. *Let $\ell \geq k$, $f_h = P_h^\ell f$, $u_{dh} = P_h^\ell u_d$, and $\boldsymbol{\sigma}_{dh} = \boldsymbol{\Pi}_h^k \boldsymbol{\sigma}_d$. Suppose that $\boldsymbol{\sigma}(q), \boldsymbol{\sigma}_d \in \boldsymbol{H}^{k+2}(\Omega)$, $f, q, u_d \in H^{k+1}(\Omega)$, and $u(q) \in H^{k+2}(\Omega)$. Given a control $q \in Q$, there exists a constant $C > 0$ that depends on $\|\boldsymbol{\sigma}(q)\|_{k+2}$, $\|f\|_{k+1}$, $\|q\|_{k+1}$, and $\|u(q)\|_{k+2}$ but independent of h and on the continuous and semidiscrete solutions such that*

$$\|\boldsymbol{\sigma}(q) - \boldsymbol{\sigma}_h(q)\|_{\mathrm{div}} + \|u(q) - u_h(q)\| \leq Ch^{k+1} \tag{3.11}$$
$$\|P_h^k u(q) - u_h(q)\| + \|\pi_h^k u(q) - \lambda_h(q)\|_h \leq Ch^{k+2}. \tag{3.12}$$

Similarly, for the adjoint equations, assuming that $\boldsymbol{\varphi}(q) \in \boldsymbol{H}^{k+2}(\Omega)$ and $w(q) \in H^{k+2}(\Omega)$, we have

$$\|\boldsymbol{\varphi}(q) - \boldsymbol{\varphi}_h(q)\|_{\text{div}} + \|w(q) - w_h(q)\| \leq Ch^{k+1} \tag{3.13}$$

$$\|P_h^k w(q) - w_h(q)\| + \|\pi_h^k w(q) - \mu_h(q)\|_h \leq Ch^{k+2} \tag{3.14}$$

for some positive contant C depending only of $\|\boldsymbol{\sigma}(q)\|_{k+2}$, $\|\boldsymbol{\sigma}_d\|_{k+2}$, $\|\boldsymbol{\varphi}(q)\|_{k+2}$, $\|f\|_{k+1}$, $\|q\|_{k+1}$, $\|u_d\|_{k+1}$, $\|u(q)\|_{k+2}$, and $\|w(q)\|_{k+2}$. Moreover, if k is an even integer, then it holds that

$$\|u(q) - R_h^{k+1}(\lambda_h(q), u_h(q))\| \leq Ch^{k+2} \tag{3.15}$$

$$\|w(q) - R_h^{k+1}(\mu_h(q), w_h(q))\| \leq Ch^{k+2}. \tag{3.16}$$

These estimates also hold if we replace the control q in the discrete variables $\boldsymbol{\sigma}_h$, u_h, λ_h, $\boldsymbol{\varphi}_h$, w_h, and μ_h by the projection $P_h^k q$ of q.

Proof. The estimates (3.11) and (3.14) involving the state variables can be obtained from Theorem 2.3 by taking $\boldsymbol{f} = \boldsymbol{f}_h = \boldsymbol{0}$, $g = -(f+q)$, and $y = -(P_h^\ell f + q)$. Indeed, (2.18) implies that

$$\|\boldsymbol{\sigma}(q) - \boldsymbol{\sigma}_h(q)\|_{\text{div}} + \|u(q) - u_h(q)\| \leq Ch^{k+1}(\|\boldsymbol{\sigma}(q)\|_{k+2} + \|f\|_{k+1})$$

while (2.19), (2.28), and $P_h^k g = -(P_h^k f + P_h^k q) = -(P_h^k P_h^\ell f + P_h^k q) = P_k^k y$, since $\ell \geq k$, give us the estimate

$$\|P_h^k u(q) - u_h(q)\| + \|\pi_h^k u(q) - \lambda_h(q)\|_h \leq Ch^{k+2}\|f + q\|_{k+1} + Ch\|\boldsymbol{\sigma}(q) - \boldsymbol{\sigma}_h(q)\|.$$

Furthermore, (2.5) and (3.11) imply (3.12).

With regards to the adjoint variables, we take $\boldsymbol{f} = -\beta(\boldsymbol{\sigma}(q) - \boldsymbol{\sigma}_d)$, $\boldsymbol{f}_h = -\beta(\boldsymbol{\sigma}_h(q) - \boldsymbol{\sigma}_{dh})$, $g = -\alpha(u(q) - u_d)$, and $y = -\alpha(u_h(q) - u_{dh})$. Observe that $P_h^k \operatorname{div} \boldsymbol{\sigma}(q) = -P_h^k(f+q) = \operatorname{div} \boldsymbol{\sigma}_h(q)$ and $P_h^k \operatorname{div} \boldsymbol{\sigma}_d = \operatorname{div} \boldsymbol{\Pi}_h^k \sigma_d = \operatorname{div} \boldsymbol{\sigma}_{dh}$. Hence, $P_h^k \operatorname{div} \boldsymbol{f} = \operatorname{div} \boldsymbol{f}_h$. Also, one can see that $P_h^k g - P_h^k y = -\alpha(P_h^k u(q) - u_h(q))$. Applying Theorem 2.3, we obtain

$$\begin{aligned}&\|\boldsymbol{\varphi}(q) - \boldsymbol{\varphi}_h(q)\|_{\text{div}} + \|w(q) - w_h(q)\| \\ &\quad \leq Ch^{k+1} + C(\|\boldsymbol{\sigma}(q) - \boldsymbol{\sigma}_h(q)\| + \|\boldsymbol{\sigma}_d - \boldsymbol{\sigma}_{dh}\| + \|u(q) - u_h(q)\| + \|u_d - u_{dh}\|),\end{aligned}$$

and this implies (3.13) using (3.11) and the definition of the discretizations $\boldsymbol{\sigma}_{dh}$ and u_{dh}. From Theorems 2.3 and 2.4, we also have

$$\begin{aligned}&\|P_h^k w(q) - w_h(q)\| + \|\pi_h^k w(q) - \mu_h(q)\|_h \\ &\quad \leq Ch^{k+2} + C\|P_h^k u(q) - u_h(q)\| + Ch(\|\boldsymbol{\sigma}(q) - \boldsymbol{\sigma}_h(q)\| + \|\boldsymbol{\sigma}_d - \boldsymbol{\sigma}_{dh}\|) \\ &\qquad + Ch\|\boldsymbol{\varphi}(q) - \boldsymbol{\varphi}_h(q)\|.\end{aligned}$$

Utilizing the error estimates (3.11) and (3.12), we obtain (3.14). Moreover, if k is even, then we also obtain from Theorem 2.4 the corresponding super-convergence error estimate (3.16).

For the last statement of the theorem, it is enough to observe that

$$(\boldsymbol{\sigma}_h(q), u_h(q), \lambda_h(q)) = (\boldsymbol{\sigma}_h(P_h^k q), u_h(P_h^k q), \lambda_h(P_h^k q))$$

and as a consequence, we also have the equality

$$(\boldsymbol{\varphi}_h(q), w_h(q), \mu_h(q)) = (\boldsymbol{\varphi}_h(P_h^k q), w_h(P_h^k q), \mu_h(P_h^k q)).$$

This completes the proof of the theorem. □

In the following lemma, we establish an error estimate between the directional derivative of the reduced cost and reduced semidiscrete cost, as well as the Lipschitz continuity of the reduced semidiscrete cost functional, see [21].

Lemma 3.4. *There exists a constant $C > 0$ such that for every $p, q, \delta q \in Q$ we have*

$$|j'(q)\delta q - j_h'(q)\delta q| \leq \|w(q) - w_h(q)\| \|\delta q\|$$
$$|j_h'(p)\delta q - j_h'(q)\delta q| \leq C\|p - q\| \|\delta q\|$$

Proof. The first estimate follows from $j'(q)\delta q - j_h'(q)\delta q = (w(q) - w_h(q), \delta q)$. On the other hand, the second one is a direct consequence of

$$j_h'(p)\delta q - j_h'(q)\delta q = \gamma(p - q, \delta q) + (w_h(p) - w_h(p), \delta q)$$

and the stability estimate

$$\|w_h(p) - w_h(q)\| \leq C(\|\boldsymbol{\sigma}_h(p) - \boldsymbol{\sigma}_h(q)\| + \|u_h(p) - u_h(q)\|) \leq C\|p - q\|,$$

where the last inequality is based on the discrete stability estimate obtained from Proposition 2.2. □

The following result states that the error between the optimal controls of the continuous and fully discrete control problems can be bounded by the sum of the discretization error and the approximation error between the discretized control space and the solution of the semidiscrete optimal control problem.

Theorem 3.5. *Assume that $\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\varphi}}, \boldsymbol{\sigma}_d \in \boldsymbol{H}^{k+2}(\Omega)$, $f, \bar{q}, u_d \in H^{k+1}(\Omega)$, and $\bar{u}, \bar{w} \in H^{k+2}(\Omega)$. Let $\bar{q}$, $\bar{q}_h$, and $\bar{q}_{h\rho}$ be the optimal controls to the continuous (3.2), semidiscrete (3.6), and fully discrete (3.10) control problems, respectively. Then, there exists a positive constant C independent of h and ρ such that*

$$\|\bar{q}_{h\rho} - \bar{q}\| \leq C \inf_{p_\rho \in Q_\rho} \|\bar{q}_h - p_\rho\| + Ch^{k+1}. \tag{3.17}$$

In particular, if $W_h^k \subset Q_\rho$, then $\bar{q}_h = \bar{q}_{h\rho} = P_h^k \bar{q}_{h\rho}$ and

$$\|\bar{q}_{h\rho} - \bar{q}\| \leq Ch^{k+1}. \tag{3.18}$$

Proof. We adapt the proof in [21]. Fix an element $p_\rho \in Q_\rho$. Let us split the error $\bar{q}_{h\rho} - \bar{q}$ in three parts as follows:

$$\bar{q}_{h\rho} - \bar{q} = (\bar{q}_{h\rho} - p_\rho) + (p_\rho - \bar{q}_h) + (\bar{q}_h - \bar{q}). \tag{3.19}$$

According to the linear-quadratic structure of the reduced cost functional, we have for every $q, \delta q, \delta p \in Q$

$$j_h''(q)(\delta q, \delta p) = \alpha(u_h(\delta q), u_h(\delta p)) + \beta(\boldsymbol{\sigma}_h(\delta q), \boldsymbol{\sigma}_h(\delta p)) + \gamma(\delta q, \delta p).$$

In particular, $j_h''(q)$ is independent of q. Using this, invoking the fact that $j_h'(\bar{q}_{h\rho})(\bar{q}_{h\rho} - p_\rho) = j_h'(\bar{q}_h)(\bar{q}_{h\rho} - p_\rho) = 0$, and applying the previous lemma, we obtain

$$\begin{aligned}\gamma\|\bar{q}_{h\rho} - p_\rho\|^2 &\leq j_h''(\bar{q})(\bar{q}_{h\rho} - p_\rho, \bar{q}_{h\rho} - p_\rho)\\ &= j_h'(\bar{q}_{h\rho})(\bar{q}_{h\rho} - p_\rho) - j_h'(p_\rho)(\bar{q}_{h\rho} - p_\rho)\end{aligned}$$

$$= j_h'(\bar{q}_h)(\bar{q}_{h\rho} - p_\rho) - j_h'(p_\rho)(\bar{q}_{h\rho} - p_\rho)$$
$$\leq C\|\bar{q}_h - p_\rho\|\|\bar{q}_{h\rho} - p_\rho\|.$$

Thus, $\gamma\|\bar{q}_{h\rho} - p_\rho\| \leq C\|\bar{q}_h - p_\rho\|$.

Similarly, from the optimality conditions, we have $j'(\bar{q})(\bar{q}_h - \bar{q}) = 0$ and $j_h'(\bar{q}_h)(\bar{q}_h - \bar{q}) = 0$, and therefore from the previous lemma

$$\begin{aligned}\gamma\|\bar{q}_h - \bar{q}\|^2 &\leq j_h''(\bar{q})(\bar{q}_h - \bar{q}, \bar{q}_h - \bar{q})\\ &= j_h'(\bar{q}_h)(\bar{q}_h - \bar{q}) - j_h'(\bar{q})(\bar{q}_h - \bar{q})\\ &= j'(\bar{q})(\bar{q}_h - \bar{q}) - j_h'(\bar{q})(\bar{q}_h - \bar{q})\\ &\leq \|w(q) - w_h(q)\|\|\bar{q}_h - \bar{q}\|.\end{aligned}$$

Consequently, it follows from the stability estimate for the solution of the adjoint system in Theorem 3.3 that

$$\gamma\|\bar{q}_h - \bar{q}\| \leq \|w(\bar{q}) - w_h(\bar{q})\| \leq Ch^{k+1}.$$

Combining the above estimates in (3.19), we obtain (3.17).

For the remaining part, it is enough to note that if Q_ρ contains W_h^k, then $\bar{q}_h = -\gamma^{-1}\bar{w}_h \in W_h^k$. As a result, the above infimum in (3.17) vanishes, and thus (3.18) is satisfied. Furthermore, $\bar{q}_h$ and $P_h^k\bar{q}_{h\rho}$ satisfy the same optimality system for the fully discrete optimal control problem, hence by uniqueness, it follows that we have $\bar{q}_h = \bar{q}_{h\rho} = P_h^k\bar{q}_{h\rho}$. □

The following is concerned with the error estimates for the optimal state and adjoint state. We would like to point out that these are valid both in the mixed and hybrid formulations.

Corollary 3.6. *Let $\bar{q}$ and $\bar{q}_{h\rho}$ be the optimal controls to the continuous (3.2) and fully discrete (3.10) control problems, respectively. If $(\bar{\boldsymbol{\sigma}}, \bar{u})$ and $(\bar{\boldsymbol{\sigma}}_{h\rho}, \bar{u}_{h\rho})$ are the corresponding optimal states, then*

$$\|\bar{\boldsymbol{\sigma}}_{h\rho} - \bar{\boldsymbol{\sigma}}\|_{\mathrm{div}} + \|\bar{u}_{h\rho} - \bar{u}\| \leq Ch^{k+1} + C\|\bar{q}_{h\rho} - \bar{q}\|. \tag{3.20}$$

Also, if $(\bar{\boldsymbol{\varphi}}, \bar{w})$ and $(\bar{\boldsymbol{\varphi}}_{h\rho}, \bar{w}_{h\rho})$ are the corresponding optimal adjoint states, then

$$\|\bar{\boldsymbol{\varphi}}_{h\rho} - \bar{\boldsymbol{\varphi}}\|_{\mathrm{div}} + \|\bar{w}_{h\rho} - \bar{w}\| \leq Ch^{k+1} + C\|\bar{q}_{h\rho} - \bar{q}\|. \tag{3.21}$$

Proof. Decompose the error in two parts according to $\bar{\boldsymbol{\sigma}}_{h\rho} - \bar{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}_h(\bar{q}_{h\rho}) - \boldsymbol{\sigma}_h(\bar{q})) + (\boldsymbol{\sigma}_h(\bar{q}) - \boldsymbol{\sigma}(\bar{q}))$ and $\bar{u}_{h\rho} - \bar{u} = (u_h(\bar{q}_{h\rho}) - u_h(\bar{q})) + (u_h(\bar{q}) - u(\bar{q}))$. Then, applying the stability estimates in Proposition 2.1 and Proposition 2.2, we obtain (3.20). Using similar decompositions for the adjoint variables yields (3.21). □

The above corollary together with (3.18) implies that if Q_ρ contains W_h^k, then

$$\|\bar{\boldsymbol{\sigma}}_{h\rho} - \bar{\boldsymbol{\sigma}}\|_{\mathrm{div}} + \|\bar{u}_{h\rho} - \bar{u}\| + \|\bar{\boldsymbol{\varphi}}_{h\rho} - \bar{\boldsymbol{\varphi}}\|_{\mathrm{div}} + \|\bar{w}_{h\rho} - \bar{w}\| \leq Ch^{k+1}. \tag{3.22}$$

Now we prove super-convergence of the discrete optimal control to the projection of the continuous optimal control. As a result, we have the super-convergence of the scalar state and adjoint state in terms of the Arnold–Brezzi post-processing operator.

Theorem 3.7. *Suppose that $\bar{q}$ and $\bar{q}_{h\rho}$ are the optimal controls to the continuous (3.2) and fully discrete (3.10) control problems, respectively. Let $(\bar{\boldsymbol{\sigma}}, \bar{u})$ and $(\bar{\boldsymbol{\sigma}}_{h\rho}, \bar{u}_{h\rho}, \bar{\lambda}_{h\rho})$ be the corresponding optimal states. Also, let $(\bar{\boldsymbol{\varphi}}, \bar{w})$ and $(\bar{\boldsymbol{\varphi}}_{h\rho}, \bar{w}_{h\rho}, \bar{\mu}_{h\rho})$ be the optimal adjoint states. If $W_h^k \subset Q_\rho$, then there exists a constant $C > 0$ such that*

$$\|\bar{q}_{h\rho} - P_h^k \bar{q}\| \leq Ch^{k+2}. \tag{3.23}$$

In addition, if k is an even integer, then it holds that

$$\|R_h^{k+1}(\bar{\lambda}_{h\rho}, \bar{u}_{h\rho}) - \bar{u}\| + \|R_h^{k+1}(\bar{\mu}_{h\rho}, \bar{w}_{h\rho}) - \bar{w}\| \leq Ch^{k+2}. \tag{3.24}$$

Proof. By applying a similar strategy as in the proof of Theorem 3.5, one can deduce that

$$\begin{aligned}
\gamma\|\bar{q}_{h\rho} - P_h^k \bar{q}\|^2 &\leq j_h'(\bar{q}_{h\rho})(\bar{q}_{h\rho} - P_h^k \bar{q}) - j_h'(P_h^k \bar{q})(\bar{q}_{h\rho} - P_h^k \bar{q}) \\
&= j'(\bar{q})(\bar{q}_{h\rho} - P_h^k \bar{q}) - j_h'(P_h^k \bar{q})(\bar{q}_{h\rho} - P_h^k \bar{q}) \\
&= \gamma(\bar{q} - P_h^k \bar{q}, \bar{q}_{h\rho} - P_h^k \bar{q}) + (P_h^k w(\bar{q}) - w_h(P_h^k \bar{q}), \bar{q}_{h\rho} - P_h^k \bar{q}) \\
&\leq \|P_h^k w(\bar{q}) - w_h(P_h^k \bar{q})\| \|\bar{q}_{h\rho} - P_h^k \bar{q}\|.
\end{aligned}$$

The first term on the third line vanishes since $\bar{q}_{h\rho} - P_h^k \bar{q} \in W_h^k$. Thus, $\gamma\|\bar{q}_{h\rho} - P_h^k \bar{q}\| \leq \|P_h^k w(\bar{q}) - w_h(P_h^k \bar{q})\|$.

According to the last statement of Theorem 2.3, it holds that

$$\|P_h^k w(\bar{q}) - w_h(P_h^k \bar{q})\| \leq Ch^{k+2},$$

and therefore (3.23) is satisfied. Next, we decompose the following difference as follows

$$P_h^k u(\bar{q}) - u_h(\bar{q}_{h\rho}) = (P_h^k u(\bar{q}) - u_h(P_h^k \bar{q})) + (u_h(P_h^k \bar{q}) - u_h(\bar{q}_{h\rho})).$$

The first difference on the right-hand side can be estimated using the last statement of Theorem 2.3, while the second difference can be estimated by invoking Proposition 2.2 and (3.23). Hence,

$$\|P_h^k u(\bar{q}) - u_h(\bar{q}_{h\rho})\| \leq Ch^{k+2}.$$

Therefore, from Remark 2.5, we have $\|R_h^{k+1}(\bar{\lambda}_{h\rho}, \bar{u}_{h\rho}) - \bar{u}\| \leq Ch^{k+2}$, which proves the first part of (3.24).

For the case of adjoint variables, we also write the error $P_h^k w(\bar{q}) - w_h(\bar{q}_{h\rho})$ as

$$P_h^k w(\bar{q}) - w_h(\bar{q}_{h\rho}) = (P_h^k w(\bar{q}) - w_h(P_h^k \bar{q})) + (w_h(P_h^k \bar{q}) - w_h(\bar{q}_{h\rho}))$$

and use the same reasoning as above to establish that $\|R_h^{k+1}(\bar{\mu}_{h\rho}, \bar{w}_{h\rho}) - \bar{w}\| \leq Ch^{k+2}$. This verifies the other part of (3.24). □

Let us analyze the error between the fully discrete post-processed optimal control and the continuous control. Likewise, we also prove error estimates if this new control is used on the fully discrete state equation and on the fully discrete adjoint equation with the state variable u_h replaced by the associated Arnold–Brezzi post-processed state.

Theorem 3.8. *Let k be even and $W_h^k \subset Q_\rho$. Consider the post-processed control*

$$q_{h\rho}^* = -\gamma^{-1} R_h^{k+1}(\bar{\mu}_{h\rho}, \bar{w}_{h\rho})$$

and let $(\boldsymbol{\sigma}_{h\rho}^*, u_{h\rho}^*, \lambda_{h\rho}^*) = (\boldsymbol{\sigma}_h(q_{h\rho}^*), u_h(q_{h\rho}^*), \lambda_h(q_{h\rho}^*))$ *and* $(\boldsymbol{\varphi}_{h\rho}^*, w_{h\rho}^*, \mu_{h\rho}^*)$ *be the solution of the modified discrete hybrid adjoint system*

$$\left[\begin{aligned} (\boldsymbol{\varphi}_{h\rho}^*, \boldsymbol{\psi}_h) + b_h(\boldsymbol{\psi}_h, w_{h\rho}^*) + d_h(\boldsymbol{\psi}_h, \mu_{h\rho}^*) &= -\beta(\boldsymbol{\sigma}_{h\rho}^* - \boldsymbol{\sigma}_{dh}, \boldsymbol{\psi}_h) && \forall \boldsymbol{\psi}_h \in \boldsymbol{Y}_h^k, \\ b_h(\boldsymbol{\varphi}_{h\rho}^*, \phi_h) &= -\alpha(R_h^{k+1}(\lambda_{h\rho}^*, u_{h\rho}^*) - u_{dh}, \phi_h) && \forall \phi_h \in W_h^k, \\ d_h(\boldsymbol{\varphi}_{h\rho}^*, \theta_h) &= 0 && \forall \theta_h \in M_h^k. \end{aligned}\right.$$

Then, there exists a constant $C > 0$ *independent of* h *such that*

$$\|q_{h\rho}^* - \bar{q}\| \le Ch^{k+2} \tag{3.25}$$

$$\|R_h^{k+1}(\lambda_{h\rho}^*, u_{h\rho}^*) - \bar{u}\| + \|R_h^{k+1}(\mu_{h\rho}^*, w_{h\rho}^*) - \bar{w}\| \le Ch^{k+2} \tag{3.26}$$

$$\|\boldsymbol{\sigma}_{h\rho}^* - \bar{\boldsymbol{\sigma}}\|_{\text{div}} + \|\boldsymbol{\varphi}_{h\rho}^* - \bar{\boldsymbol{\varphi}}\|_{\text{div}} + \|u_{h\rho}^* - \bar{u}\| + \|w_{h\rho}^* - \bar{w}\| \le Ch^{k+1}. \tag{3.27}$$

Proof. The estimate (3.25) follows directly from the following equation

$$q_{h\rho}^* - \bar{q} = -\frac{1}{\gamma}(R_h^{k+1}(\bar{\mu}_{h\rho}, \bar{w}_{h\rho}) - \bar{w})$$

and the error estimate for the post-processed adjoint state given by (3.24) in the previous theorem. According to Theorem 2.3, we have

$$\begin{aligned} &\|\boldsymbol{\sigma}_{h\rho}^* - \bar{\boldsymbol{\sigma}}\|_{\text{div}} + \|u_{h\rho}^* - \bar{u}\| \\ &\quad \le \|\boldsymbol{\sigma}_{h\rho}^* - \bar{\boldsymbol{\sigma}}_{h\rho}\|_{\text{div}} + \|\bar{\boldsymbol{\sigma}}_{h\rho} - \bar{\boldsymbol{\sigma}}\|_{\text{div}} + \|u_{h\rho}^* - \bar{u}_{h\rho}\| + \|\bar{u}_{h\rho} - \bar{u}\| \\ &\quad \le C\|q_{h\rho}^* - \bar{q}_{h\rho}\| + Ch^{k+1} \le Ch^{k+1}. \end{aligned} \tag{3.28}$$

Next, we split error $P_h^k\bar{u} - u_{h\rho}^*$ as

$$P_h^k\bar{u} - u_{h\rho}^* = (P_h^k u(\bar{q}) - u_h(\bar{q})) + (u_h(\bar{q}) - u_h(q_{h\rho}^*)).$$

We then apply Theorem 2.3, Proposition 2.2, and (3.25) to deduce that $\|P_h^k\bar{u} - u_{h\rho}^*\| \le Ch^{k+2}$. As a consequence of Remark 2.5, it holds that

$$\|R_h^{k+1}(\lambda_{h\rho}^*, u_{h\rho}^*) - \bar{u}\| \le Ch^{k+2}. \tag{3.29}$$

Therefore, (3.26) and (3.27) are verified in the case of the state variables.

For the adjoint variables, we shall write $\boldsymbol{\varphi}_{h\rho}^* - \bar{\boldsymbol{\varphi}} = (\boldsymbol{\varphi}_{h\rho}^* - \bar{\boldsymbol{\varphi}}_{h\rho}) + (\bar{\boldsymbol{\varphi}}_{h\rho} - \bar{\boldsymbol{\varphi}})$ and $w_{h\rho}^* - \bar{w} = (w_{h\rho}^* - \bar{w}_{h\rho}) + (\bar{w}_{h\rho} - \bar{w})$. The second terms can be bounded from above thanks to (3.22). Also, the first terms can be estimated as follows in virtue of Proposition 2.2

$$\|\boldsymbol{\varphi}_{h\rho}^* - \bar{\boldsymbol{\varphi}}_{h\rho}\|_{\text{div}} + \|w_{h\rho}^* - \bar{w}_{h\rho}\| \le C(\|\boldsymbol{\sigma}_{h\rho}^* - \bar{\boldsymbol{\sigma}}_{h\rho}\| + \|R_h^{k+1}(\lambda_{h\rho}^*, u_{h\rho}^*) - \bar{u}_{h\rho}\|). \tag{3.30}$$

By further writing $\boldsymbol{\sigma}_{h\rho}^* - \bar{\boldsymbol{\sigma}}_{h\rho} = (\boldsymbol{\sigma}_{h\rho}^* - \bar{\boldsymbol{\sigma}}) + (\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{\sigma}}_{h\rho})$ and $R_h^{k+1}(\lambda_{h\rho}^*, u_{h\rho}^*) - \bar{u}_{h\rho} = (R_h^{k+1}(\lambda_{h\rho}^*, u_{h\rho}^*) - \bar{u}) + (\bar{u} - \bar{u}_{h\rho})$, and utilizing (3.22), (3.28), (3.29), and Theorem 3.7, one has

$$\|\boldsymbol{\varphi}_{h\rho}^* - \bar{\boldsymbol{\varphi}}\|_{\text{div}} + \|w_{h\rho}^* - \bar{w}\| \le Ch^{k+1}. \tag{3.31}$$

Moreover, according to the decomposition

$$P_h^k\bar{w} - w_{h\rho}^* = (P_h^k w(\bar{q}) - w_h(\bar{q})) + (w_h(\bar{q}) - w_h(q_{h\rho}^*))$$

along with the same argument as in the case of the state equation, we have

$$\|R_h^{k+1}(\mu_{h\rho}^*, w_{h\rho}^*) - \bar{w}\| \le Ch^{k+2}.$$

These show (3.26) and (3.27) in the case of the adjoint variables. □

4. Penalization of the Optimal Control Problems

To compute numerically the finite-dimensional systems corresponding to the discrete state and adjoint equations, we shall add penalty terms for the second and third equations in the hybrid formulation. This is to reduce the size of the system matrix via elimination and substitution but at the expense of an additional error, see (5.1) and (5.2) in the succeeding section.

Before going to the discrete case, let us discuss the situation of adding a penalty term at the continuous level. Given $\varepsilon > 0$, let us consider the optimal control problem

$$\min_{q\in Q} j_\varepsilon(q) := J(u_\varepsilon(q), \boldsymbol{\sigma}_\varepsilon(q), q) \tag{4.1}$$

where given $q \in Q$, the pair $(\boldsymbol{\sigma}_\varepsilon(q), u_\varepsilon(q)) = (\boldsymbol{\sigma}_\varepsilon, u_\varepsilon) \in \boldsymbol{V} \times W$ is the solution of the penalized state equation

$$\left[\begin{array}{ll} (\boldsymbol{\sigma}_\varepsilon, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, u_\varepsilon) = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{V}, \\ b(\boldsymbol{\sigma}_\varepsilon, v) - \varepsilon(u_\varepsilon, v) = -(f + q, v) & \forall v \in W. \end{array}\right. \tag{4.2}$$

For this state equation, the corresponding bilinear form on $\boldsymbol{V} \times W$ is coercive, hence, existence and uniqueness of solutions to (4.2) follows immediately from the Lax–Milgram Lemma. Moreover, we have $\operatorname{div} \boldsymbol{\sigma}_\varepsilon = \varepsilon u_\varepsilon - (f + q)$ and $u_\varepsilon \in H_0^1(\Omega) \cap H^2(\Omega)$ is the weak solution of

$$\left[\begin{array}{ll} -\Delta u_\varepsilon + \varepsilon u_\varepsilon = f + q & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{array}\right. \tag{4.3}$$

which is a linear perturbation of (2.4).

The directional derivative of j_ε at q in the direction $\delta q \in Q$ is similarly given by $j_\varepsilon'(q)\delta q = (\gamma q + w_\varepsilon(q), \delta q)$, where $w_\varepsilon(q)$ is the second component of $(\boldsymbol{\varphi}_\varepsilon(q), w_\varepsilon(q)) = (\boldsymbol{\varphi}_\varepsilon, w_\varepsilon) \in \boldsymbol{V} \times W$, the solution to

$$\left[\begin{array}{ll} (\boldsymbol{\varphi}_\varepsilon, \boldsymbol{\psi}) + b(\boldsymbol{\psi}, w_\varepsilon) = -\beta(\boldsymbol{\sigma}_\varepsilon(q) - \boldsymbol{\sigma}_d, \boldsymbol{\psi}) & \forall \boldsymbol{\psi} \in \boldsymbol{V}, \\ b(\boldsymbol{\varphi}_\varepsilon, \phi) - \varepsilon(w_\varepsilon, \phi) = -\alpha(u_\varepsilon(q) - u_d, \phi) & \forall \phi \in W. \end{array}\right. \tag{4.4}$$

Again, the strong form of the equation for $w_\varepsilon(q)$ in (4.4) is the following linear perturbation of the elliptic boundary value problem (3.4)

$$\left[\begin{array}{ll} -\Delta w_\varepsilon(q) + \varepsilon w_\varepsilon(q) = \alpha(u_\varepsilon(q) - u_d) - \beta \operatorname{div}(\boldsymbol{\sigma}_\varepsilon(q) - \boldsymbol{\sigma}_d) & \text{in } \Omega, \\ w_\varepsilon(q) = 0 & \text{on } \Omega. \end{array}\right.$$

Thus, we can see that the two approaches penalize-then-optimize and optimize-then-penalize lead to the same optimality system.

Theorem 4.1. *Let $\bar{q}_\varepsilon$ be the optimal control for (4.1), $(\bar{\boldsymbol{\sigma}}_\varepsilon, \bar{u}_\varepsilon)$ the optimal state, and $(\bar{\boldsymbol{\varphi}}_\varepsilon, \bar{w}_\varepsilon)$ the optimal adjoint state. Then, there exists a constant $C > 0$ independent of ε such that*

$$\|\bar{q} - \bar{q}_\varepsilon\| + \|\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{\sigma}}_\varepsilon\|_{\text{div}} + \|\bar{u} - \bar{u}_\varepsilon\| + \|\bar{\boldsymbol{\varphi}} - \bar{\boldsymbol{\varphi}}_\varepsilon\|_{\text{div}} + \|\bar{w} - \bar{w}_\varepsilon\| \le C\varepsilon.$$

Proof. Applying the idea of the proof in Theorem 3.5, one can deduce that

$$\gamma\|\bar{q}_\varepsilon - \bar{q}\| \le \|w(\bar{q}) - w_\varepsilon(\bar{q})\|. \tag{4.5}$$

According to [6, Theorem 3.1], we know that

$$\|\boldsymbol{\sigma}(\bar{q}) - \boldsymbol{\sigma}_\varepsilon(\bar{q})\|_{\text{div}} + \|u(\bar{q}) - u_\varepsilon(\bar{q})\| \le C\varepsilon. \tag{4.6}$$

Let us write the difference of the solutions to the adjoint equations by $w(\bar{q}) - w_\varepsilon(\bar{q}) = (w(\bar{q}) - w^\varepsilon(\bar{q})) + (w^\varepsilon(\bar{q}) - w_\varepsilon(\bar{q}))$ and $\boldsymbol{\varphi}(\bar{q}) - \boldsymbol{\varphi}_\varepsilon(\bar{q}) = (\boldsymbol{\varphi}(\bar{q}) - \boldsymbol{\varphi}^\varepsilon(\bar{q})) + (\boldsymbol{\varphi}^\varepsilon(\bar{q}) - \boldsymbol{\varphi}_\varepsilon(\bar{q}))$, where $(\boldsymbol{\varphi}^\varepsilon(\bar{q}), w^\varepsilon(\bar{q})) \in \boldsymbol{V} \times W$ is the solution of

$$\left[\begin{aligned} (\boldsymbol{\varphi}^\varepsilon(\bar{q}), \boldsymbol{\psi}) + b(\boldsymbol{\psi}, w^\varepsilon(\bar{q})) &= -\beta(\boldsymbol{\sigma}_\varepsilon(\bar{q}) - \boldsymbol{\sigma}_d, \boldsymbol{\psi}) \quad && \forall \boldsymbol{\psi} \in \boldsymbol{V}, \\ b(\boldsymbol{\varphi}^\varepsilon(\bar{q}), \phi) &= -\alpha(u_\varepsilon(\bar{q}) - u_d, \phi) && \forall \phi \in W. \end{aligned}\right.$$

From the stability estimate in Proposition 2.2 and (4.6), we have

$$\|\boldsymbol{\varphi}(\bar{q}) - \boldsymbol{\varphi}^\varepsilon(\bar{q})\|_{\text{div}} + \|w(\bar{q}) - w^\varepsilon(\bar{q})\| \le C(\|\boldsymbol{\sigma}(\bar{q}) - \boldsymbol{\sigma}_\varepsilon(\bar{q})\|) + \|u(\bar{q}) - u_\varepsilon(\bar{q})\| \le C\varepsilon. \tag{4.7}$$

For the other terms, we again apply [6, Theorem 3.1] to deduce that

$$\|\boldsymbol{\varphi}^\varepsilon(\bar{q}) - \boldsymbol{\varphi}_\varepsilon(\bar{q})\|_{\text{div}} + \|w^\varepsilon(\bar{q}) - w_\varepsilon(\bar{q})\| \le C\varepsilon. \tag{4.8}$$

Utilizing (4.7) and (4.8) in the above decomposition and invoking (4.5), we obtain the desired estimate for the error in optimal controls.

The error estimates for the optimal states can now be established from $\bar{u} - \bar{u}_\varepsilon = (u(\bar{q}) - u(\bar{q}_\varepsilon)) + (u(\bar{q}_\varepsilon) - u_\varepsilon(\bar{q}_\varepsilon))$ and $\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{\sigma}}_\varepsilon = (\boldsymbol{\sigma}(\bar{q}) - \boldsymbol{\sigma}(\bar{q}_\varepsilon)) + (\boldsymbol{\varphi}(\bar{q}_\varepsilon) - \boldsymbol{\varphi}_\varepsilon(\bar{q}_\varepsilon))$, while the case of optimal adjoint states can be handled in a similar way. □

Now, we discuss the case of the fully discrete problem. Let $\varepsilon = (\varepsilon_1, \varepsilon_2)$ be a pair of nonnegative numbers such that $|\varepsilon| := \varepsilon_1 + \varepsilon_2 > 0$. The penalized semidiscrete hybridized optimal control problem is

$$\min_{q \in Q} j_{h\varepsilon}(q) := J_h(u_{h\varepsilon}(q), \boldsymbol{\sigma}_{h\varepsilon}(q), q) \tag{4.9}$$

where $q \mapsto (u_{h\varepsilon}(q), \boldsymbol{\sigma}_{h\varepsilon}(q), \lambda_{h\varepsilon}(q))$ is the solution operator which maps a control $q \in Q$ into the solution $(\boldsymbol{\sigma}_{h\varepsilon}, u_{h\varepsilon}, \lambda_{h\varepsilon}) = (u_{h\varepsilon}(q), \boldsymbol{\sigma}_{h\varepsilon}(q), \lambda_{h\varepsilon}(q)) \in \boldsymbol{Y}_h^k \times W_h^k \times M_h^k$ of the penalized discrete state equation

$$\left[\begin{aligned} (\boldsymbol{\sigma}_{h\varepsilon}, \boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h, u_{h\varepsilon}) + d_h(\boldsymbol{\tau}_h, \lambda_{h\varepsilon}) &= 0 && \forall \boldsymbol{\tau}_h \in \boldsymbol{Y}_h^k, \\ b_h(\boldsymbol{\sigma}_{h\varepsilon}, v_h) - \varepsilon_1(u_{h\varepsilon}, v_h) &= -(f_h + q, v_h) && \forall v_h \in W_h^k, \\ d_h(\boldsymbol{\sigma}_{h\varepsilon}, \mu_h) - \varepsilon_2(\lambda_{h\varepsilon}, \mu_h)_h &= 0 && \forall \mu_h \in M_h^k. \end{aligned}\right. \tag{4.10}$$

Here $(\cdot,\cdot)_h$ is the inner product corresponding to the norm $\|\cdot\|_h$ in L_h^k.

If $\varepsilon_2 > 0$, then this is a nonconforming approximation of (2.1) since $\boldsymbol{Y}_h^k$ is not contained in $\boldsymbol{V}$. On the other hand, if $\varepsilon_2 = 0$, then the solution of (4.10) corresponds to the mixed finite element discretization of (4.2). The existence and uniqueness of solutions follows from the fact that the corresponding finite-dimensional square system is injective. In fact, if $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, then the corresponding form is coercive.

The directional derivative of $j_{h\varepsilon}$ at $q \in Q$ in the direction $\delta q \in Q$ is given by $j_{h\varepsilon}'(q)\delta q = (\gamma q + w_{h\varepsilon}(q), \delta q)$, where $w_{h\varepsilon}$ is the second component for the triple

$(\boldsymbol{\varphi}_{h\varepsilon}, w_{h\varepsilon}, \mu_{h\varepsilon}) = (\boldsymbol{\varphi}_{h\varepsilon}(q), w_{h\varepsilon}(q), \mu_{h\varepsilon}(q)) \in \boldsymbol{Y}_h^k \times W_h^k \times M_h^k$ solving the associated adjoint equation

$$\left[\begin{aligned} (\boldsymbol{\varphi}_{h\varepsilon}, \boldsymbol{\psi}_h) + b_h(\boldsymbol{\psi}_h, w_{h\varepsilon}) + d_h(\boldsymbol{\psi}_h, \mu_{h\varepsilon}) &= -\beta(\boldsymbol{\sigma}_{h\varepsilon}(q) - \boldsymbol{\varphi}_{dh}, \boldsymbol{\psi}_h) & \forall \boldsymbol{\psi}_h \in \boldsymbol{Y}_h^k,\\ b_h(\boldsymbol{\varphi}_{h\varepsilon}, \phi_h) - \varepsilon_1(w_{h\varepsilon}, \phi_h) &= -\alpha(u_{h\varepsilon}(q) - u_{dh}, \phi_h) & \forall \phi_h \in W_h^k,\\ d_h(\boldsymbol{\varphi}_{h\varepsilon}, \theta_h) - \varepsilon_2(\mu_{h\varepsilon}, \theta_h)_h &= 0 & \forall \theta_h \in M_h^k. \end{aligned}\right. \tag{4.11}$$

In an analogous way, we consider the fully discrete penalized hybrid optimal control problem

$$\min_{q_\rho \in Q_\rho} j_{h\varepsilon\rho}(q_\rho) := J_h(u_{h\varepsilon}(q_\rho), \boldsymbol{\sigma}_{h\varepsilon}(q_\rho), q_\rho). \tag{4.12}$$

We have $j'_{h\varepsilon\rho}(q_\rho)\delta q_\rho = (\gamma q_\rho + w_{h\varepsilon}(q_\rho), \delta q_\rho)$ for every $q_\rho, \delta q_\rho \in Q_\rho$. Let us denote the optimal controls of (4.9) and (4.12) by $\bar{q}_{h\varepsilon}$ and $\bar{q}_{h\varepsilon\rho}$, respectively.

At this point, we have four processes namely optimization, discretization, hybridization, and penalization. Since discretization comes first before hybridization, there are 12 possible ways of doing these processes in succession. In the event where hybridization is performed before penalization, the optimality system consists of the state equation (4.10) with $q = \bar{q}_{h\varepsilon\rho}$, the adjoint equation (4.11) with $q = \bar{q}_{h\varepsilon\rho}$, and the optimality condition $j'_{h\varepsilon\rho}(\bar{q}_{h\varepsilon\rho})\delta q_\rho = 0$ for every $\delta q_\rho \in Q_\rho$. On the other hand, in the approaches where penalization is performed before hybridization, the resulting optimality system is almost the same as in the above approaches, the main difference is that $\varepsilon_2 = 0$. We can view the former optimality system as a penalization of the latter optimality system. Therefore, loosely speaking, the processes of performing optimization, discretization, hybridization, and penalization commute.

In the following, we establish *a priori* estimates for the discrete equations with penalizations.

Theorem 4.2. *Let $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, $g \in W$, and $\varepsilon = (\varepsilon_1, \varepsilon_2)$ with $\varepsilon_1, \varepsilon_2 \geq 0$. Suppose that $(\boldsymbol{\sigma}_h, u_h, \lambda_h) \in \boldsymbol{V}_h^k \times W_h^k \times M_h^k$ is the solution of*

$$\left[\begin{aligned} (\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h, u_h) + d_h(\boldsymbol{\tau}_h, \lambda_h) &= (\boldsymbol{f}, \boldsymbol{\tau}_h) & \forall \boldsymbol{\tau}_h \in \boldsymbol{Y}_h^k,\\ b_h(\boldsymbol{\sigma}_h, v_h) &= (g, v_h) & \forall v_h \in W_h^k,\\ d_h(\boldsymbol{\sigma}_h, \mu_h) &= 0 & \forall \mu_h \in M_h^k, \end{aligned}\right. \tag{4.13}$$

and let $(\boldsymbol{\sigma}_{h\varepsilon}, u_{h\varepsilon}, \lambda_{h\varepsilon}) \in \boldsymbol{Y}_h^k \times W_h^k \times M_h^k$ be the solution of

$$\left[\begin{aligned} (\boldsymbol{\sigma}_{h\varepsilon}, \boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h, u_{h\varepsilon}) + d_h(\boldsymbol{\tau}_h, \lambda_{h\varepsilon}) &= (\boldsymbol{f}, \boldsymbol{\tau}_h) & \forall \boldsymbol{\tau}_h \in \boldsymbol{Y}_h^k,\\ b_h(\boldsymbol{\sigma}_{h\varepsilon}, v_h) - \varepsilon_1(u_{h\varepsilon}, v_h) &= (g, v_h) & \forall v_h \in W_h^k,\\ d_h(\boldsymbol{\sigma}_{h\varepsilon}, \mu_h) - \varepsilon_2(\lambda_{h\varepsilon}, \mu_h)_h &= 0 & \forall \mu_h \in M_h^k. \end{aligned}\right. \tag{4.14}$$

Suppose that $0 < h < h_0$ and $|\varepsilon| < 1$. Then, for some constant $C = C(h_0) > 0$ independent of the h, ε, and on the solutions we have

$$\|\boldsymbol{\sigma}_{h\varepsilon} - \boldsymbol{\sigma}_h\|_{\mathrm{div}} + \|u_{h\varepsilon} - u_h\| + \|\lambda_{h\varepsilon} - \lambda_h\|_h \leq C|\varepsilon|(\|\boldsymbol{f}\| + \|g\|) \tag{4.15}$$

$$\|R_h^{k+1}(\lambda_{h\varepsilon}, u_{h\varepsilon}) - R_h^{k+1}(\lambda_h, u_h)\| \leq C|\varepsilon|(\|\boldsymbol{f}\| + \|g\|). \tag{4.16}$$

Proof. Recall the stability estimate from Proposition 2.2 and (2.13)

$$\|\boldsymbol{\sigma}_h\| + \|u_h\| + \|\lambda_h\|_h \leq C(\|\boldsymbol{f}\| + \|g\|). \tag{4.17}$$

Let us define the difference of solutions as

$$(\delta\boldsymbol{\sigma}_{h\varepsilon}, \delta u_{h\varepsilon}, \delta\lambda_{h\varepsilon}) := (\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_{h\varepsilon}, u_h - u_{h\varepsilon}, \lambda_h - \lambda_{h\varepsilon}) \in \boldsymbol{Y}_h^k \times W_h^k \times M_h^k,$$

which satisfies the following variational system

$$\left[\begin{array}{ll} (\delta\boldsymbol{\sigma}_{h\varepsilon}, \boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h, \delta u_{h\varepsilon}) + d_h(\boldsymbol{\tau}_h, \delta\lambda_{h\varepsilon}) = 0 & \forall \boldsymbol{\tau}_h \in \boldsymbol{Y}_h^k, \\ b_h(\delta\boldsymbol{\sigma}_{h\varepsilon}, v_h) - \varepsilon_1(\delta u_{h\varepsilon}, v_h) = -\varepsilon_1(u_h, v_h) & \forall v_h \in W_h^k, \\ d_h(\delta\boldsymbol{\sigma}_{h\varepsilon}, \mu_h) - \varepsilon_2(\delta\lambda_{h\varepsilon}, \mu_h)_h = -\varepsilon_2(\lambda_h, \mu_h)_h & \forall \mu_h \in M_h^k. \end{array}\right.$$

With $(\delta\boldsymbol{\sigma}_{h\varepsilon}, \delta u_{h\varepsilon}, \delta\lambda_{h\varepsilon})$ as the test function in this system, we have

$$\|\delta\boldsymbol{\sigma}_{h\varepsilon}\|^2 + \varepsilon_1\|\delta u_{h\varepsilon}\|^2 + \varepsilon_2\|\delta\lambda_{h\varepsilon}\|_h^2 \le C(\varepsilon_1\|u_h\|\|\delta u_{h\varepsilon}\| + \varepsilon_2\|\lambda_h\|_h\|\delta\lambda_{h\varepsilon}\|_h).$$

Applying the uniform boundedness in h of the discrete solution given by (4.17), we obtain

$$\|\delta\boldsymbol{\sigma}_{h\varepsilon}\|^2 \le C\varepsilon(\|\boldsymbol{f}\|\|\delta u_{h\varepsilon}\| + \|g\|\|\delta\lambda_{h\varepsilon}\|_h). \tag{4.18}$$

Since $\operatorname{div}\delta\boldsymbol{\sigma}_{h\varepsilon} \in W_h^k$, it follows that

$$\|\operatorname{div}\delta\boldsymbol{\sigma}_h\| = \sup_{\|v_h\|=1} |b_h(\delta\boldsymbol{\sigma}_h, v_h)| \le C\varepsilon(\|\delta u_{h\varepsilon}\| + \|u_h\|). \tag{4.19}$$

Let $\boldsymbol{\varphi}$ and $\boldsymbol{\zeta}$ be as in the proof of Theorem 2.3 with δu_h and $\delta\lambda_h$ replaced by $\delta u_{h\varepsilon}$ and $\delta\lambda_{h\varepsilon}$, respectively. Then, by applying the same argument as in the proof of the said theorem, one obtains

$$\|\delta u_{h\varepsilon}\|^2 = b_h(\boldsymbol{\Pi}_h^k\boldsymbol{\varphi}, \delta u_{h\varepsilon}) = -(\delta\boldsymbol{\sigma}_{h\varepsilon}, \boldsymbol{\Pi}_h^k\boldsymbol{\varphi}),$$

and thus, $\|\delta u_{h\varepsilon}\| \le C\|\delta\boldsymbol{\sigma}_{h\varepsilon}\|$. Likewise, $\|\delta\lambda_{h\varepsilon}\| \le Ch\|\delta\boldsymbol{\sigma}_{h\varepsilon}\| + \|\delta u_{h\varepsilon}\| \le C(h_0)\|\delta\boldsymbol{\sigma}_{h\varepsilon}\|$ since

$$\|\delta\lambda_{h\varepsilon}\|^2 = -d_h(\zeta_h, \delta\lambda_{h\varepsilon}) = (\delta\boldsymbol{\sigma}_{h\varepsilon}, \boldsymbol{\Pi}_h^k\boldsymbol{\zeta}) + b_h(\boldsymbol{\Pi}_h^k\boldsymbol{\zeta}, \delta u_{h\varepsilon}).$$

Using these in (4.18) and (4.19) yields (4.15). The second estimate (4.16) follows from (4.15) together with the linearity and boundedness of the Arnold–Brezzi post-processing operator R_h^{k+1} given by (2.15). □

Next we prove *a priori* error estimates for the above discretizations with the additional penalty terms in the optimal control problem.

Theorem 4.3. *Let $\bar{q}$ and $\bar{q}_{h\varepsilon\rho}$ be the optimal controls to the continuous (3.2) and fully discrete penalized (4.12) control problems, respectively. Suppose that $W_h^k \subset Q_\rho$, $|\varepsilon| < 1$, and $0 < h < h_0$. Then, there exists a constant $C(h_0) > 0$ independent of h and ε such that*

$$\|\bar{q}_{h\varepsilon\rho} - P_h^k\bar{q}\| \le C(h^{k+2} + |\varepsilon|) \tag{4.20}$$

$$\|\bar{q}_{h\varepsilon\rho} - \bar{q}\| \le C(h^{k+1} + |\varepsilon|). \tag{4.21}$$

If $(\bar{\boldsymbol{\sigma}}, \bar{u})$ and $(\bar{\boldsymbol{\sigma}}_{h\varepsilon\rho}, \bar{u}_{h\varepsilon\rho}, \bar{\lambda}_{h\varepsilon\rho})$ are the corresponding optimal states, then

$$\|\bar{\boldsymbol{\sigma}}_{h\varepsilon\rho} - \bar{\boldsymbol{\sigma}}\|_{\operatorname{div}} + \|\bar{u}_{h\varepsilon\rho} - \bar{u}\| \le C(h^{k+1} + |\varepsilon|) \tag{4.22}$$

$$\|R_h^{k+1}(\bar{\lambda}_{h\varepsilon\rho}, \bar{u}_{h\varepsilon\rho}) - \bar{u}\| \le C(h^{k+2} + |\varepsilon|). \tag{4.23}$$

Also, if $(\bar{\boldsymbol{\varphi}}, \bar{w})$ *and* $(\bar{\boldsymbol{\varphi}}_{h\varepsilon\rho}, \bar{w}_{h\varepsilon\rho}, \bar{\mu}_{h\varepsilon\rho})$ *are the corresponding optimal adjoint states, then*

$$\|\bar{\boldsymbol{\varphi}}_{h\varepsilon\rho} - \bar{\boldsymbol{\varphi}}\|_{\mathrm{div}} + \|\bar{w}_{h\varepsilon\rho} - \bar{w}\| \leq C(h^{k+1} + |\varepsilon|) \tag{4.24}$$

$$\|R_h^{k+1}(\bar{\mu}_{h\varepsilon\rho}, \bar{w}_{h\varepsilon\rho}) - \bar{w}\| \leq C(h^{k+2} + |\varepsilon|). \tag{4.25}$$

Proof. Following the arguments in the proof of Theorem 3.5, one can deduce that

$$\gamma\|\bar{q}_{h\varepsilon\rho} - P_h^k\bar{q}\| \leq \|P_h^k w(\bar{q}) - w_{h\varepsilon}(P_h^k\bar{q})\|. \tag{4.26}$$

Let $(\widetilde{\boldsymbol{\varphi}}_h, \widetilde{w}_h, \widetilde{\mu}_h)$ be the solution of the adjoint hybrid system (3.9) with $\boldsymbol{\sigma}_h(q)$ and $u_h(q)$ replaced by the discrete penalized counterparts $\boldsymbol{\sigma}_{h\varepsilon}(P_h^k\bar{q})$ and $u_{h\varepsilon}(P_h^k\bar{q})$, respectively. We separate the norm of the error $P_h^k w(\bar{q}) - w_{h\varepsilon}(P_h^k\bar{q})$ as follows:

$$\begin{aligned}&\|P_h^k w(\bar{q}) - w_{h\varepsilon}(P_h^k\bar{q})\| \\ &\quad\leq \|P_h^k w(\bar{q}) - w_h(P_h^k\bar{q})\| + \|w_h(P_h^k\bar{q}) - \widetilde{w}_h\| + \|\widetilde{w}_h - w_{h\varepsilon}(P_h^k\bar{q})\|.\end{aligned} \tag{4.27}$$

According to Theorem 2.3, we have $\|P_h^k w(\bar{q}) - w_h(P_h^k\bar{q})\| \leq Ch^{k+2}$. From the stability estimate in Proposition 2.2

$$\|w_h(P_h^k\bar{q}) - \widetilde{w}_h\| \leq C(\|\boldsymbol{\sigma}_h(P_h^k\bar{q}) - \boldsymbol{\sigma}_{h\varepsilon}(P_h^k\bar{q})\| + \|u_h(P_h^k\bar{q}) - u_{h\varepsilon}(P_h^k\bar{q})\|). \tag{4.28}$$

Applying Theorem 4.2 to the right-hand side of (4.28) and the fact that $\|\alpha(f_h + P_h^k\bar{q})\| \leq C\alpha(\|f\| + \|\bar{q}\|)$, we have

$$\|\boldsymbol{\sigma}_h(P_h^k\bar{q}) - \boldsymbol{\sigma}_{h\varepsilon}(P_h^k\bar{q})\| + \|u_h(P_h^k\bar{q}) - u_{h\varepsilon}(P_h^k\bar{q})\| \leq C|\varepsilon|. \tag{4.29}$$

Similarly, since $0 < \varepsilon < 1$, we obtain that

$$\|\widetilde{w}_h - w_{h\varepsilon}(P_h^k\bar{q})\| \leq C|\varepsilon|. \tag{4.30}$$

The estimate (4.20) now follows from (4.26)–(4.30). Moreover, using the projection error (2.6), this also implies (4.21). In particular, from Theorem 3.7, we have

$$\|\bar{q}_{h\varepsilon\rho} - \bar{q}_{h\rho}\| \leq \|\bar{q}_{h\varepsilon\rho} - P_h^k\bar{q}\| + \|P_h^k\bar{q} - \bar{q}_{h\rho}\| \leq C(h^{k+2} + |\varepsilon|). \tag{4.31}$$

The error estimates involving the state variables can be established by writing the difference in solutions as

$$\begin{aligned}\|\bar{u}_{h\varepsilon\rho} - \bar{u}\| &\leq \|\bar{u}_{h\varepsilon\rho} - \bar{u}_{h\rho}\| + \|\bar{u}_{h\rho} - \bar{u}\| \\ \|\bar{\boldsymbol{\sigma}}_{h\varepsilon\rho} - \bar{\boldsymbol{\sigma}}\|_{\mathrm{div}} &\leq \|\bar{\boldsymbol{\sigma}}_{h\varepsilon\rho} - \bar{\boldsymbol{\sigma}}_{h\rho}\|_{\mathrm{div}} + \|\bar{\boldsymbol{\sigma}}_{h\rho} - \bar{\boldsymbol{\sigma}}\|_{\mathrm{div}}.\end{aligned}$$

One can estimate the second terms on the right-hand sides of these inequalities by using (3.22). On the other hand, the first terms and the corresponding Lagrange multipliers can be estimated in two parts according to

$$\begin{aligned}\|\bar{u}_{h\varepsilon\rho} - \bar{u}_{h\rho}\| &\leq \|u_{h\varepsilon}(\bar{q}_{h\varepsilon\rho}) - u_h(\bar{q}_{h\varepsilon\rho})\| + \|u_h(\bar{q}_{h\varepsilon\rho}) - u_h(\bar{q}_{h\rho})\| \\ \|\bar{\lambda}_{h\varepsilon\rho} - \bar{\lambda}_{h\rho}\|_h &\leq \|\lambda_{h\varepsilon}(\bar{q}_{h\varepsilon\rho}) - \lambda_h(\bar{q}_{h\varepsilon\rho})\|_h + \|\lambda_h(\bar{q}_{h\varepsilon\rho}) - \lambda_h(\bar{q}_{h\rho})\|_h \\ \|\bar{\boldsymbol{\sigma}}_{h\varepsilon\rho} - \bar{\boldsymbol{\sigma}}_{h\rho}\|_{\mathrm{div}} &\leq \|\boldsymbol{\sigma}_{h\varepsilon}(\bar{q}_{h\varepsilon\rho}) - \boldsymbol{\sigma}_h(\bar{q}_{h\varepsilon\rho})\|_{\mathrm{div}} + \|\boldsymbol{\sigma}_h(\bar{q}_{h\varepsilon\rho}) - \boldsymbol{\sigma}_h(\bar{q}_{h\rho})\|_{\mathrm{div}}.\end{aligned}$$

Note that $\|\bar{q}_{h\varepsilon\rho}\| \leq \|\bar{q}_{h\varepsilon\rho} - \bar{q}\| + \|\bar{q}\| \leq C$, for some constant $C > 0$ independent of ε and h, whenever $0 < h < h_0$ and $|\varepsilon| < 1$. This implies that Theorem 4.2 can be utilized to bound the first terms on the right-hand sides. For the remaining terms, we can apply Proposition 2.2 and (4.31). Hence,

$$\|\bar{u}_{h\varepsilon\rho} - \bar{u}_{h\rho}\| + \|\bar{\boldsymbol{\sigma}}_{h\varepsilon\rho} - \bar{\boldsymbol{\sigma}}_{h\rho}\|_{\mathrm{div}} + \|\bar{\lambda}_{h\varepsilon\rho} - \bar{\lambda}_{h\rho}\|_h \leq C(h^{k+2} + |\varepsilon|). \tag{4.32}$$

Utilizing the above estimates yields (4.22).

With regards to the post-processing operator, we write

$$R_h^{k+1}(\bar{\lambda}_{h\varepsilon\rho}, \bar{u}_{h\varepsilon\rho}) - \bar{u} = R_h^{k+1}(\bar{\lambda}_{h\varepsilon\rho} - \bar{\lambda}_{h\rho}, \bar{u}_{h\varepsilon\rho} - \bar{u}_{h\rho}) + (R_h^{k+1}(\bar{\lambda}_{h\rho}, \bar{u}_{h\rho}) - \bar{u})$$

and apply the boundedness of R_h^{k+1}, the inequality (4.32), and Theorem 3.7 to obtain (4.23).

Finally, let us consider the case of the adjoint equations. Denote by $(\widetilde{\boldsymbol{\sigma}}_{h\varepsilon\rho}, \widetilde{w}_{h\varepsilon\rho}, \widetilde{\mu}_{h\varepsilon\rho})$ the solution of (3.9) with $\boldsymbol{\sigma}_h(q)$ and $u_h(q)$ replaced by the penalized counterparts $\bar{\boldsymbol{\sigma}}_{h\varepsilon\rho}$ and $\bar{u}_{h\varepsilon\rho}$. We shall also bound the error on the adjoint variables as

$$\begin{aligned}\|\bar{w}_{h\varepsilon\rho} - \bar{w}\| &\le \|\bar{w}_{h\varepsilon\rho} - \bar{w}_{h\rho}\| + \|\bar{w}_{h\rho} - \bar{w}\|\\ \|\bar{\boldsymbol{\varphi}}_{h\varepsilon\rho} - \bar{\boldsymbol{\varphi}}\|_{\mathrm{div}} &\le \|\bar{\boldsymbol{\varphi}}_{h\varepsilon\rho} - \bar{\boldsymbol{\varphi}}_{h\rho}\|_{\mathrm{div}} + \|\bar{\boldsymbol{\varphi}}_{h\rho} - \bar{\boldsymbol{\varphi}}\|_{\mathrm{div}}.\end{aligned}$$

The second terms on the right-hand sides are again estimated from (3.22). We decompose the first terms along with their corresponding Lagrange multipliers as follows

$$\begin{aligned}\|\bar{w}_{h\varepsilon\rho} - \bar{w}_{h\rho}\| &\le \|w_{h\varepsilon}(\bar{q}_{h\varepsilon\rho}) - \widetilde{w}_{h\varepsilon\rho}\| + \|\widetilde{w}_{h\varepsilon\rho} - w_h(\bar{q}_{h\rho})\|\\ \|\bar{\lambda}_{h\varepsilon\rho} - \bar{\lambda}_{h\rho}\|_h &\le \|\lambda_{h\varepsilon}(\bar{q}_{h\varepsilon\rho}) - \widetilde{\lambda}_{h\varepsilon\rho}\|_h + \|\widetilde{\lambda}_{h\varepsilon\rho} - \lambda_h(\bar{q}_{h\rho})\|_h\\ \|\bar{\boldsymbol{\varphi}}_{h\varepsilon\rho} - \bar{\boldsymbol{\varphi}}_{h\rho}\|_{\mathrm{div}} &\le \|\boldsymbol{\varphi}_{h\varepsilon}(\bar{q}_{h\varepsilon\rho}) - \widetilde{\boldsymbol{\varphi}}_{h\varepsilon\rho}\|_{\mathrm{div}} + \|\widetilde{\boldsymbol{\varphi}}_{h\varepsilon\rho} - \boldsymbol{\varphi}_h(\bar{q}_{h\rho})\|_{\mathrm{div}}.\end{aligned}$$

The first terms of these inequalities can be estimated from above with the help of Theorem 4.2, while for the second terms we apply Proposition 2.2 and (4.32). Doing these yields

$$\|\bar{w}_{h\varepsilon\rho} - \bar{w}_{h\rho}\| + \|\bar{\boldsymbol{\varphi}}_{h\varepsilon\rho} - \bar{\boldsymbol{\varphi}}_{h\rho}\|_{\mathrm{div}} + \|\bar{\lambda}_{h\varepsilon\rho} - \bar{\lambda}_{h\rho}\|_h \le C(h^{k+2} + |\varepsilon|)$$

and consequently (4.24). By the same argument as in (4.23), one can also deduce the estimate (4.25). □

To close this section, we present error estimates with the post-processed optimal control for the penalized hybrid system similar to that in Theorem 3.8.

Theorem 4.4. *Suppose that k is even, $W_h^k \subset Q_\rho$, $|\varepsilon| < 1$, and $0 < h < h_0$. Define*

$$q^*_{h\varepsilon\rho} = -\gamma^{-1} R_h^{k+1}(\bar{\mu}_{h\varepsilon\rho}, \bar{w}_{h\varepsilon\rho})$$

and let $(\boldsymbol{\sigma}^*_{h\varepsilon\rho}, u^*_{h\varepsilon\rho}, \lambda^*_{h\varepsilon\rho}) = (\boldsymbol{\sigma}_{h\varepsilon}(q^*_{h\varepsilon\rho}), u_{h\varepsilon}(q^*_{h\varepsilon\rho}), \lambda_{h\varepsilon}(q^*_{h\rho}))$ and $(\boldsymbol{\varphi}^*_{h\varepsilon\rho}, w^*_{h\varepsilon\rho}, \mu^*_{h\varepsilon\rho})$ be the solution of the modified hybrid adjoint system with penalization

$$\left[\begin{aligned} (\boldsymbol{\varphi}^*_{h\varepsilon\rho}, \boldsymbol{\psi}_h) + b_h(\boldsymbol{\psi}_h, w^*_{h\varepsilon\rho}) + d_h(\boldsymbol{\psi}_h, \mu^*_{h\varepsilon\rho}) &= -\beta(\boldsymbol{\sigma}^*_{h\varepsilon\rho} - \boldsymbol{\sigma}_{dh}, \boldsymbol{\psi}_h) \ \forall \boldsymbol{\psi}_h \in \boldsymbol{Y}_h^k,\\ b_h(\boldsymbol{\varphi}^*_{h\varepsilon\rho}, \phi_h) - \varepsilon_1(w^*_{h\varepsilon\rho}, \phi_h) &= -\alpha(R_h^{k+1}(\lambda^*_{h\varepsilon\rho}, u^*_{h\varepsilon\rho}) - u_{dh}, \phi_h) \ \forall \phi_h \in W_h^k,\\ d_h(\boldsymbol{\varphi}^*_{h\varepsilon\rho}, \theta_h) - \varepsilon_2(\mu^*_{h\varepsilon\rho}, \theta_h)_h &= 0 \ \ \forall \theta_h \in M_h^k.\end{aligned}\right.$$

Then, there exists a constant $C > 0$ independent of h and ε such that

$$\|q^*_{h\varepsilon\rho} - \bar{q}\| \le C(h^{k+2} + |\varepsilon|) \tag{4.33}$$

$$\|R_h^{k+1}(\lambda^*_{h\varepsilon\rho}, u^*_{h\varepsilon\rho}) - \bar{u}\| + \|R_h^{k+1}(\mu^*_{h\varepsilon\rho}, w^*_{h\varepsilon\rho}) - \bar{w}\| \le C(h^{k+2} + |\varepsilon|) \tag{4.34}$$

$$\|\boldsymbol{\sigma}^*_{h\varepsilon\rho} - \bar{\boldsymbol{\sigma}}\|_{\mathrm{div}} + \|\boldsymbol{\varphi}^*_{h\varepsilon\rho} - \bar{\boldsymbol{\varphi}}\|_{\mathrm{div}} + \|u^*_{h\varepsilon\rho} - \bar{u}\| + \|w^*_{h\varepsilon\rho} - \bar{w}\| \le C(h^{k+1} + |\varepsilon|). \tag{4.35}$$

Proof. The first estimate (4.33) follows immediately from (4.25) in the previous theorem. Next, we shall write

$$\begin{aligned} u^*_{h\varepsilon\rho} - \bar{u} &= (u_{h\varepsilon}(q^*_{h\varepsilon\rho}) - u^*_{h\rho}) + (u^*_{h\rho} - \bar{u}) \\ \boldsymbol{\sigma}^*_{h\varepsilon\rho} - \bar{\boldsymbol{\sigma}} &= (\boldsymbol{\sigma}_{h\varepsilon}(q^*_{h\varepsilon\rho}) - \boldsymbol{\sigma}^*_{h\rho}) + (\boldsymbol{\sigma}^*_{h\rho} - \bar{\boldsymbol{\sigma}}), \end{aligned}$$

where $\boldsymbol{\sigma}^*_{h\rho}$ and $u^*_{h\rho}$ are those that are given in Theorem 3.8. We further decompose the first terms as follows:

$$\begin{aligned} u^*_{h\varepsilon\rho} - u^*_{h\rho} &= (u^*_{h\varepsilon\rho} - u_h(q^*_{h\varepsilon\rho})) + (u_h(q^*_{h\varepsilon\rho}) - u_h(q^*_{h\rho})) \\ \lambda^*_{h\varepsilon\rho} - \lambda^*_{h\rho} &= (\lambda^*_{h\varepsilon\rho} - \lambda_h(q^*_{h\varepsilon\rho})) + (\lambda_h(q^*_{h\varepsilon\rho}) - \lambda_h(q^*_{h\rho})) \\ \boldsymbol{\sigma}^*_{h\varepsilon\rho} - \boldsymbol{\sigma}^*_{h\rho} &= (\boldsymbol{\sigma}^*_{h\varepsilon\rho} - \boldsymbol{\sigma}_h(q^*_{h\varepsilon\rho})) + (\boldsymbol{\sigma}_h(q^*_{h\varepsilon\rho}) - \boldsymbol{\sigma}_h(q^*_{h\rho})). \end{aligned}$$

For the post-processing, we write the error as follows:

$$R_h^{k+1}(\lambda^*_{h\varepsilon\rho}, u^*_{h\varepsilon\rho}) - \bar{u} = R_h^{k+1}(\lambda^*_{h\varepsilon\rho} - \lambda^*_{h\rho}, u^*_{h\varepsilon\rho} - u^*_{h\rho}) + (R_h^{k+1}(\lambda^*_{h\rho}, u^*_{h\rho}) - \bar{u}).$$

These decomposition along with the same arguments as in the proof of previous theorem can be used to prove the required *a priori* error estimates involving the state equations in (4.34) and (4.35). The main difference is the use of Theorem 3.8 instead of Theorem 3.7.

With regards to the adjoint equations, we also split the error into two parts as

$$\begin{aligned} w^*_{h\varepsilon\rho} - \bar{w} &= (w^*_{h\varepsilon\rho} - w^*_{h\rho}) + (w^*_{h\rho} - \bar{w}) \\ \boldsymbol{\varphi}^*_{h\varepsilon\rho} - \bar{\boldsymbol{\varphi}} &= (\boldsymbol{\varphi}^*_{h\varepsilon\rho} - \boldsymbol{\varphi}^*_{h\rho}) + (\boldsymbol{\varphi}^*_{h\rho} - \bar{\boldsymbol{\varphi}}). \end{aligned}$$

Let $(\widetilde{\boldsymbol{\varphi}}_{h\varepsilon\rho}, \widetilde{w}_{h\varepsilon\rho}, \widetilde{\mu}_{h\varepsilon\rho})$ be the solution of the discrete hybrid adjoint equation (3.9) with $u_h(q)$ and $\boldsymbol{\sigma}_h(q)$ replaced by $R_h^{k+1}(\lambda^*_{h\varepsilon\rho}, u^*_{h\varepsilon\rho})$ and $\boldsymbol{\sigma}^*_{h\varepsilon\rho}$, respectively. With this definition, we further split the first terms as follows:

$$\begin{aligned} \boldsymbol{\varphi}^*_{h\varepsilon\rho} - \boldsymbol{\varphi}^*_{h\rho} &= (\boldsymbol{\varphi}^*_{h\varepsilon\rho} - \widetilde{\boldsymbol{\varphi}}_{h\varepsilon\rho}) + (\widetilde{\boldsymbol{\varphi}}_{h\varepsilon\rho} - \boldsymbol{\varphi}_h(q^*_{h\rho})) \\ w^*_{h\varepsilon\rho} - w^*_{h\rho} &= (w^*_{h\varepsilon\rho} - \widetilde{w}_{h\varepsilon\rho}) + (\widetilde{w}_{h\varepsilon\rho} - w_h(q^*_{h\rho})) \\ \mu^*_{h\varepsilon\rho} - \mu^*_{h\rho} &= (\mu^*_{h\varepsilon\rho} - \widetilde{\mu}_{h\varepsilon\rho}) + (\widetilde{\mu}_{h\varepsilon\rho} - \mu_h(q^*_{h\rho})). \end{aligned}$$

Applying the same reasoning as in the proof of the previous theorem to these decompositions, one can obtain the *a priori* error estimates for the adjoint variables stipulated in the inequalities (4.34) and (4.35). □

Now, let us introduce more practical penalty terms. For this purpose, we define the symmetric discrete positive-definite bilinear forms $s_h : W_h^k \times W_h^k \to \mathbb{R}$ and $r_h : M_h^k \times M_h^k \to \mathbb{R}$ as follows:

$$\begin{aligned} s_h(u_h, v_h) &= h^2 \sum_{K \in \mathcal{T}_h} \frac{1}{|K|} \int_K u_h v_h \, \mathrm{d}x \\ r_h(\lambda_h, \mu_h) &= h^2 \sum_{e \in \mathcal{E}_h^i} \frac{1}{|e|} \int_e \lambda_h \mu_h \, \mathrm{d}s \end{aligned}$$

where $|K|$ and $|e|$ denote the area of the triangle K and the length of the edge e, respectively. In the case where $k = 0$, the corresponding mass matrices for r_h and s_h

will be scalar multiples of the identity. This is in fact the usual strategy in practice. The penalized discrete state equation can be alternatively replaced by the problem

$$\left[\begin{array}{rl} (\boldsymbol{\sigma}_{h\varepsilon}, \boldsymbol{\tau}_h) + b_h(\boldsymbol{\tau}_h, u_{h\varepsilon}) + d_h(\boldsymbol{\tau}_h, \lambda_{h\varepsilon}) = 0 & \forall \boldsymbol{\tau}_h \in \boldsymbol{Y}_h^k, \\ b_h(\boldsymbol{\sigma}_{h\varepsilon}, v_h) - \varepsilon_1 r_h(u_{h\varepsilon}, v_h) = -(f_h + q_\rho, v_h) & \forall v_h \in W_h^k, \\ d_h(\boldsymbol{\sigma}_{h\varepsilon}, \mu_h) - \varepsilon_2 s_h(\lambda_{h\varepsilon}, \mu_h)_h = 0 & \forall \mu_h \in M_h^k. \end{array}\right. \tag{4.36}$$

The factor h^2 is used in the above bilinear forms so that r_h and s_h will be equivalent to the norms on W_h^k and M_h^k, respectively. Indeed, due to the shape-regularity of the triangulations there exist constants c, $C > 0$ independent of h such that $s_h(u_h, u_h) \geq c\|u_h\|^2$ and $|s_h(u_h, v_h)| \leq C\|u_h\|\|v_h\|$ for every $u_h, v_h \in W_h^k$. In a similar way, we have $r_h(\lambda_h, \lambda_h) \geq c\|\lambda_h\|_h^2$ and $|r_h(\lambda_h, \mu_h)| \leq C\|\lambda_h\|_h\|\mu_h\|_h$ whenever $\lambda_h, \mu_h \in M_h^k$. Using the same methodologies as above, the error estimates in the previous theorem also hold if we consider the alternative state equation (4.36) with the given mesh-dependent penalty terms. The formulation (4.36) will be utilized in the implementation given in the succeeding section.

5. NUMERICAL EXAMPLES

In this section, we give numerical examples illustrating the results of the paper. First, we shall write the corresponding algebraic form for the penalized discrete and adjoint equations. We shall utilize the lowest order Raviart Thomas finite element space. The space of controls will be discretized using the space $Q_h := Q_\rho = W_h^1$ in the hybrid formulation and $Q_h := Q_\rho = W_h^0$ in the mixed formulation. In the following discussion, we only present the case of the hybrid formulation, the case of mixed formulation can be treated in a similar manner.

Let N_{Kh} and N_{eh} be the number of triangles and interior edges of the triangulation $\mathcal{T}_h$. Let $\{\boldsymbol{\psi}_h^k\}_{k=1}^{3N_{Kh}}$, $\{\phi_h^j\}_{j=1}^{N_{Kh}}$, $\{v_h^k\}_{k=1}^{3N_{Kh}}$, and $\{\theta_h^\ell\}_{\ell=1}^{N_{eh}}$ be bases for $\boldsymbol{Y}_h^0$, W_h^0, W_h^1, and M_h^0, respectively. Define the matrices A_h, B_h, D_h, E_h, and G_h having sizes of $3N_{Kh} \times 3N_{Kh}$, $3N_{Kh} \times N_{Kh}$, $3N_{Kh} \times N_{eh}$, $3N_{Kh} \times N_{Kh}$, and $3N_{Kh} \times 3N_{Kh}$, respectively, with the corresponding entries

$$(A_h)_{kl} = (\boldsymbol{\psi}_h^k, \boldsymbol{\psi}_h^l), \qquad (B_h)_{kj} = b_h(\boldsymbol{\psi}_h^k, \phi_h^j),$$
$$(D_h)_{k\ell} = d_h(\boldsymbol{\psi}_h^k, \theta_h^\ell), \qquad (E_h)_{kj} = (v_h^k, \phi_h^j), \qquad (G_h)_{kl} = (v_h^k, v_h^l).$$

In the implementation, we shall take the penalty parameters to be of the form $\varepsilon_1 = \varepsilon_0 h^{-2}$ and $\varepsilon_2 = \varepsilon_0 h^{-2}$ for a fixed $0 < \varepsilon_0 \ll 1$. Also, we shall use $P_h^1 u_d$ and $P_h^1 f$ as the approximations of the desired state u_d and the external source f. Every element $q_h \in W_h^1$ can be written uniquely as

$$q_h = \sum_{k=1}^{3N_{Kh}} q_k v_h^k.$$

Similarly, every element of $\boldsymbol{\sigma}_h \in \boldsymbol{Y}_h^0$, $u_h \in W_h^0$, and $\lambda_h \in M_h^0$ has the following unique representations

$$\boldsymbol{\sigma}_h = \sum_{k=1}^{3N_{Kh}} \sigma_k \boldsymbol{\psi}_h^k, \qquad u_h = \sum_{j=1}^{N_{Kh}} u_j \phi_h^j, \qquad \lambda_h = \sum_{\ell=1}^{N_{eh}} \lambda_\ell \theta_h^\ell.$$

With slight abuse of notation, we identify an element of a vector space with the vector of coefficients with respect to a given basis. For instance, we let $q_h = (q_k)_{k=1,\dots,3N_{Kh}}$, $\boldsymbol{\sigma}_h = (\sigma_k)_{k=1,\dots,3N_{Kh}}$, $u_h = (u_j)_{j=1,\dots,N_{Kh}}$, and $\lambda_h = (\lambda_\ell)_{\ell=1,\dots,N_{eh}}$. Furthermore, we shall use the same notation R_h^1 for the matrix determined by the post-processing operator R_h^1.

The algebraic form of the penalized hybrid discrete system (4.36) is now given as follows: Given $q_h \in Q_h$, find $(\boldsymbol{\sigma}_{h\varepsilon}, u_{h\varepsilon}, \lambda_{h\varepsilon}) = (\boldsymbol{\sigma}_{h\varepsilon}(q_h), u_{h\varepsilon}(q_h), \lambda_{h\varepsilon}(q_h)) \in \mathbb{R}^{3N_{Kh}} \times \mathbb{R}^{N_{Kh}} \times \mathbb{R}^{N_{eh}}$ such that

$$\left[\begin{aligned} A_h\boldsymbol{\sigma}_{h\varepsilon} + B_h^T u_{h\varepsilon} + D_h^T\lambda_{h\varepsilon} &= 0 \\ B_h\boldsymbol{\sigma}_{h\varepsilon} - \varepsilon_0 u_{h\varepsilon} &= -E_h(f_h + q_h) \\ D_h\boldsymbol{\sigma}_{h\varepsilon} - \varepsilon_0\lambda_{h\varepsilon} &= 0. \end{aligned}\right.$$

Here, the superscript T denotes transposition. This system is equivalent to the following

$$\left[\begin{aligned} F_h\boldsymbol{\sigma}_{h\varepsilon} &= -\frac{1}{\varepsilon_0}B_h^T E_h(f_h + q_h), \\ u_{h\varepsilon} &= \frac{1}{\varepsilon_0}B_h\boldsymbol{\sigma}_{h\varepsilon} + \frac{1}{\varepsilon_0}E_h(f_h + q_h), \\ \lambda_{h\varepsilon} &= \frac{1}{\varepsilon_0}D_h\boldsymbol{\sigma}_{h\varepsilon} \end{aligned}\right. \tag{5.1}$$

where

$$F_h = A_h + \frac{1}{\varepsilon_0}B_h^T B_h + \frac{1}{\varepsilon_0}D_h^T D_h,$$

which is symmetric and positive-definite. Similarly, for the solution $(\boldsymbol{\varphi}_{h\varepsilon}, w_{h\varepsilon}, \mu_{h\varepsilon}) = (\boldsymbol{\varphi}_{h\varepsilon}(q_h), w_{h\varepsilon}(q_h), \mu_{h\varepsilon}(q_h)) \in \mathbb{R}^{3N_{Kh}} \times \mathbb{R}^{N_{Kh}} \times \mathbb{R}^{N_{eh}}$ of the modified hybrid adjoint equation with penalization, we have the system

$$\left[\begin{aligned} F_h\boldsymbol{\varphi}_{h\varepsilon} &= -\beta A_h(\boldsymbol{\sigma}_{h\varepsilon} - \boldsymbol{\sigma}_{dh}) - \frac{1}{\varepsilon_0}\alpha B_h^T E_h(R_h^1\lambda_{h\varepsilon} - u_{dh}) \\ w_{h\varepsilon} &= \frac{1}{\varepsilon_0}B_h\boldsymbol{\varphi}_{h\varepsilon} + \frac{1}{\varepsilon_0}\alpha E_h(R_h^1\lambda_{h\varepsilon} - u_{dh}) \\ \mu_{h\varepsilon} &= \frac{1}{\varepsilon_0}D_h\boldsymbol{\varphi}_{h\varepsilon}. \end{aligned}\right. \tag{5.2}$$

After solving the above adjoint equation, we shall post-process the component $\mu_{h\varepsilon}$ of the adjoint state and consider the following control

$$q_{h\varepsilon}^* = -\gamma^{-1}R_h^1\mu_{h\varepsilon}(q_h). \tag{5.3}$$

The corresponding modified discrete cost functional where the Lagrange multipliers associated to the primal and dual states are post-processed, which is still denoted by $j_{h\varepsilon}$, is given by

$$\begin{aligned} j_{h\varepsilon}(q_h) = {} & \frac{\alpha}{2}(R_h^1\lambda_{h\varepsilon} - u_{dh})^T G_h(R_h^1\lambda_{h\varepsilon} - u_{dh}) \\ & + \frac{\beta}{2}(\boldsymbol{\sigma}_{h\varepsilon} - \boldsymbol{\sigma}_{dh})^T A_h(\boldsymbol{\sigma}_{h\varepsilon} - \boldsymbol{\sigma}_{dh}) + \frac{\gamma}{2}q_{h\varepsilon}^{*T}G_h q_{h\varepsilon}^*. \end{aligned} \tag{5.4}$$

We present the gradient-based algorithm utilized in this paper to approximate the solution of the optimal control problem. The reduced optimization problem is solved by the Barzilai–Borwein gradient method where the steplength is selected alternately [4]. The second iterate in the gradient method is computed by backtracking with the Armijo rule as a steplength selection criterion. We refer the reader to [2] for the analysis of the Barzilai–Borwein method when applied to strictly convex quadratic optimization problems in infinite-dimensional Hilbert spaces.

In the following algorithm, given a control q_h^k at the kth iteration, the variables $(\boldsymbol{\sigma}_{h\varepsilon}^k, u_{h\varepsilon}^k, \lambda_{h\varepsilon}^k)$, $(\boldsymbol{\varphi}_{h\varepsilon}^k, w_{h\varepsilon}^k, \mu_{h\varepsilon}^k)$, $q_{h\varepsilon}^{*k}$, and $j_{h\varepsilon}^k$ are the solutions of (5.1), (5.2), (5.3), and the value of (5.4), respectively, with $q_h = q_h^k$.

Algorithm: GRADIENT METHOD FOR PENALIZED HYBRID OPTIMAL CONTROL

Initialize α, β, γ, τ, ε_0, and q_h^0.
Construct the mesh $\mathcal{T}_h$ and calculate the edge and element data structures.
Assemble the matrices A_h, B_h, D_h, F_h, G_h, and R_h^1.
Compute discretizations u_{dh} and $\boldsymbol{\sigma}_{dh}$ of the desired states.
Solve for $(\boldsymbol{\sigma}_{h\varepsilon}^0, u_{h\varepsilon}^0, \lambda_{h\varepsilon}^0)$, $(\boldsymbol{\varphi}_{h\varepsilon}^0, w_{h\varepsilon}^0, \mu_{h\varepsilon}^0)$, $q_{h\varepsilon}^{*0}$, and $j_{h\varepsilon}^0$.
Determine a second control q_h^1 by inexact line search.
Solve for $(\boldsymbol{\sigma}_{h\varepsilon}^1, u_{h\varepsilon}^1, \lambda_{h\varepsilon}^1)$, $(\boldsymbol{\varphi}_{h\varepsilon}^1, w_{h\varepsilon}^1, \mu_{h\varepsilon}^1)$, $q_{h\varepsilon}^{*1}$, and $j_{h\varepsilon}^1$.
Set $k = 1$.
while $|j_{h\varepsilon}^k - j_{h\varepsilon}^{k-1}|/j_{h\varepsilon}^k > \tau$ **do**
  $\delta q_{h\varepsilon}^* = q_{h\varepsilon}^{*k} - q_{h\varepsilon}^{*(k-1)}$
  $\delta w_{h\varepsilon} = R_h^1\mu_{h\varepsilon}^k - R_h^1\mu_{h\varepsilon}^{k-1}$
  $\delta j_{h\varepsilon}' = \gamma\delta q_{h\varepsilon}^* + \delta w_{h\varepsilon}$
  if k is odd **then**
    $s = (\delta q_{h\varepsilon}^*)^T j_{h\varepsilon}'/|\delta j_{h\varepsilon}'|^2$
  else
    $s = |\delta q_{h\varepsilon}^*|^2/(\delta q_{h\varepsilon}^*)^T\delta j_{h\varepsilon}'$
  end if
  $k = k + 1$
  $q_h^k = q_{h\varepsilon}^{*(k-1)} - s(\gamma q_{h\varepsilon}^{*(k-1)} + R_h^1\mu_{h\varepsilon}^{k-1})$
  Solve for $(\boldsymbol{\sigma}_{h\varepsilon}^k, u_{h\varepsilon}^k, \lambda_{h\varepsilon}^k)$, $(\boldsymbol{\varphi}_{h\varepsilon}^k, w_{h\varepsilon}^k, \mu_{h\varepsilon}^k)$, $q_{h\varepsilon}^{*k}$, and $j_{h\varepsilon}^k$.
end while

In system (5.1), the flux component $\boldsymbol{\sigma}_{h\varepsilon}$ is calculated first using the conjugate gradient method, and then the result is substituted to the second and third equations to obtain the other components $u_{h\varepsilon}$ and $\lambda_{h\varepsilon}$. The same strategy will be employed in the case of the adjoint equation (5.2) after post-processing the Lagrange multiplier $\lambda_{h\varepsilon}$. Aside from (5.2) and (5.3), the Arnold–Brezzi post-processing operator was also utilized in the steplength selection of the Barzilai–Borwein method and the derivative of the reduced cost functional. An alternative stopping criterion is $\|\gamma q_{h\varepsilon}^{*k} + R_h^1\mu_{h\varepsilon}^k\| < \tau$, that is, when the optimality residual is less than the prescribed tolerance.

We shall construct an analytical solution of (1.1) based on the eigenfunction

$$e(x, y) = \sin(2\pi x)\sin(2\pi y) \tag{5.5}$$

of the Laplacian on $\Omega = (0,1)^2$. Define the following state, adjoint, control, desired states, and source term

$$\bar{u} = \bar{w} = e, \qquad \bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\varphi}} = \nabla e, \qquad \bar{q} = -\gamma^{-1} e,$$
$$f = -\Delta e + \gamma^{-1} e, \qquad u_d = e + \alpha^{-1}\Delta e, \qquad \boldsymbol{\sigma}_d = \nabla e.$$

One can easily verify that these satisfy the optimality system for the control problem (1.1) for any given positive parameters α, β, and γ.

Now we verify the error estimates given in the previous sections by starting with a uniform triangulation of the domain Ω and successively refine the mesh by bisection. For the parameters appearing in the cost, we shall use $\alpha = 1$, $\beta = 1$, $\gamma = 10^{-1}$, and $\varepsilon_0 = 10^{-10}$. The mesh sizes for the triangulations are $h = \sqrt{2}/2^k$ for $2 \le k \le 9$. On the other hand, in order to examine the behavior of the errors due to the penalization, we shall use a fixed triangulation with mesh size $h = \sqrt{2}/2^6$ and vary the penalty parameter using the values $\varepsilon_0 = 10^{-k}$ for $2 \le k \le 10$. We terminate the optimization algorithm once the relative error between two consecutive cost values is less than the tolerance 10^{-6}.

The algorithm presented above for the reduced optimal control problem was implemented in Python 3.9.7 (Python Software Foundation, http://www.python.org) on a 2.3 GHz Intel Core i5 with 8GB RAM. The source code and the numerical values for the discretization errors are available at https://github.com/grperalta/rtpoisson.

k	N_{Kh}	N_{eh}	dof	reduced ssm	reduction
2	32	40	168	96	42.86%
3	128	176	688	384	44.19%
4	512	736	2,784	1,536	44.83%
5	2,048	3,008	11,200	6,144	45.14%
6	8,192	12,160	44,928	24,576	45.30%
7	32,768	48,896	179,968	98,304	45.38%
8	131,072	19,6096	720,384	393,216	45.42%
9	524,288	785,408	2,882,560	1,572,864	45.54%

TABLE 1. Number of elements (N_{Kh}), interior edges (N_{eh}), total degrees of freedom (dof $= 4N_{Kh} + N_{eh}$), reduced ssm (size of system matrix $3N_{Kh}$), and the reduction percentage for the discrete state variable with decreasing mesh sizes in the hybrid formulation.

We report in Table 1 the number of elements and interior edges corresponding to the triangulations with mesh size $h = \sqrt{2}/2^k$. The total dof (degrees of freedom) is computed by $4N_{Kh} + N_{eh}$, which corresponds to the dimension of the approximating space $\boldsymbol{Y}_h^0 \times W_h^0 \times M_h^0$. The reduction percentage, that is, the ratio of the eliminated components in the linear system to the total number of unknowns, is calculated by

$$\text{reduction} = \frac{\text{dof} - \text{reduced ssm}}{\text{dof}} = \frac{N_{Kh} + N_{eh}}{4N_{Kh} + N_{eh}}.$$

Overall, we can observe an approximate 45% reduction for the size of the linear systems when the penalty method is applied to compute the solution of the discrete saddle point problems associated with the state and adjoint equations.

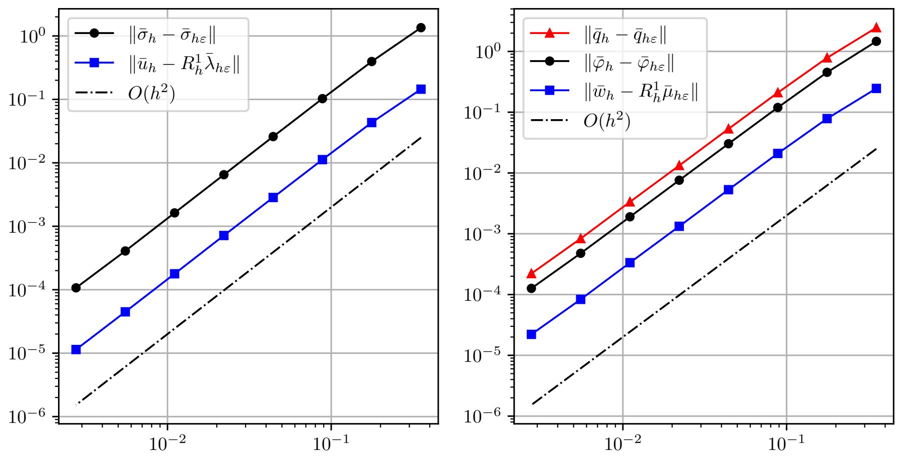

Figure 1. Discretization errors for the primal, dual, and control variables with respect to the mesh size in the hybrid finite element method with fixed penalization parameter $\varepsilon_0 = 10^{-10}$.

In Figure 1, we plot the discretization errors for the state variables (left) and the adjoint and control variables (right) by reducing the mesh sizes. We can see the expected order of convergence $\mathcal{O}(h^2)$ according to Theorem 4.3. In fact, we also have a quadratic order of convergence for the Fortin projection of the components $\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{\varphi}}$ of the state and adjoint variables to their numerical counterparts, which is better than the one expected. This is due to the fact that our triangulations are uniform, see for instance [7].

In Figure 2, we plot the discretization errors for the state variables (left) and the adjoint and control variables (right) by successively reducing the penalty parameter. It can be seen that the errors for the state and flux are large, unless we take the penalization parameter $\varepsilon_0 = 10^{-10}$. Although penalization reduces the dimension of the system matrix, one drawback is that the resulting reduced system can have huge condition numbers that may result to large errors. Nevertheless, we can observe an approximate order of convergence $\mathcal{O}(\varepsilon)$ for the majority of the error between the continuous and penalized discrete state, adjoint state, and control as stated by Theorem 4.3. Also, the error decreased faster at $\varepsilon_0 = 10^{-10}$. We would like to mention that for $\varepsilon_0 = 10^{-11}$, the corresponding error increases. Therefore, in practice, there is a threshold for the penalty parameter due to round-off errors. Moreover, appropriate preconditioners can be applied for larger penalty parameters due to large condition number of the system matrix.

We note that the solutions of the penalized mixed method differs from those that are given by the penalized hybrid method. In fact, as mentioned in the previous section, the optimality system of the latter optimal control problem is a penalization

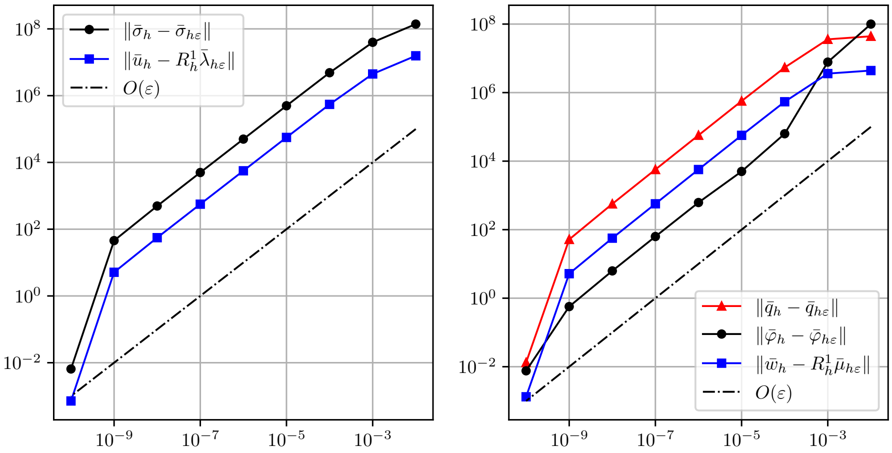

FIGURE 2. Penalization errors for the primal, dual and control variables in the hybrid finite element method for a fixed triangulation with mesh size $h = \sqrt{2}/2^6$.

of the former problem, where the penalty term appears in the Lagrange multipliers. In Table 2, we compare the norms of the difference between the computed solutions obtained from the hybrid and mixed formulations. To distinguish the solutions of the two methods, we append the superscripts H and M corresponding to the hybrid and mixed methods, respectively.

The penalization parameter $\varepsilon_0 = 10^{-10}$ was utilized in the comparison. As the mesh size decreases, we can observe that the difference between the solutions provided by the two methods decreases as well. The errors in the computed optimal controls are comparably larger than those in the state and adjoint variables due to the different discretization of the control space. Furthermore, one can see a linear order of reduction of the difference in the optimal controls, which is consistent from the one obtained in the theory, see for instance the *a priori* estimate (3.18).

To compare the norms of the fluxes in the state and adjoint variables, they are expressed in terms of the Lagrange shape functions on each triangle corresponding to the discontinuous P^1 finite element space. This is to compensate the different representations of the flux variables used in the approximations. In particular, the dimensions of the coefficient vectors of each method differs, for which the number of interior edges corresponds to the dimension in the mixed method while thrice the number of elements to that of the hybrid method. In general, the mixed formulation requires more gradient iterations in contrast with the hybrid one.

Let us compare the performance of the penalized hybrid formulation with post-processing and the typical H^1-conforming method. Here, we utilized the same test example based on the eigenfunction (5.5). The Lagrange interpolation of the exact solutions are denoted by $\widetilde{u}_h, \widetilde{q}_h, \widetilde{w}_h$, while the approximations obtained from the usual method are given by $\widehat{u}_h, \widehat{q}_h, \widehat{w}_h$.

k	$\lVert\bar{q}^H_{h\varepsilon} - \bar{q}^M_{h\varepsilon}\rVert$	$\lVert\bar{\boldsymbol{\sigma}}^H_{h\varepsilon} - \bar{\boldsymbol{\sigma}}^M_{h\varepsilon}\rVert$	$\lVert\bar{u}^H_{h\varepsilon} - \bar{u}^M_{h\varepsilon}\rVert$	$\lVert\bar{\boldsymbol{\varphi}}^H_{h\varepsilon} - \bar{\boldsymbol{\varphi}}^M_{h\varepsilon}\rVert$	$\lVert\bar{w}^H_{h\varepsilon} - \bar{w}^M_{h\varepsilon}\rVert$
2	7.759187e–1	2.612056e–2	3.485176e–3	2.080860e–2	4.932468e–3
3	9.768092e–1	1.449179e–2	2.367384e–3	1.562847e–2	2.436102e–3
4	6.100684e–1	4.309833e–3	5.442836e–4	4.206953e–4	5.393033e–4
5	3.217493e–1	2.396436e–3	4.918615e–4	3.177990e–4	1.402689e–3
6	1.629187e–1	3.054530e–4	6.175866e–5	1.010845e–4	9.919974e–5
7	8.172378e–2	7.018624e–5	8.740210e–6	1.287241e–5	1.324480e–5
8	4.089508e–2	1.813897e–5	2.278971e–6	1.089335e–5	2.972948e–6
9	2.045165e–2	7.612611e–6	8.190436e–7	1.150863e–5	9.860226e–7

TABLE 2. Norms of differences between the solutions of the mixed and hybrid finite element approximations for decreasing mesh sizes.

If $\widetilde{A}_h$ and $\widetilde{M}_h$ are the stiffness and mass matrices for the H^1-method, then the coupled primal-dual system is given by

$$\left[\begin{array}{l} \widetilde{A}_h\widehat{u}_h = \widetilde{M}_h\widetilde{f}_h - \gamma^{-1}\widehat{w}_h \\ \widetilde{A}_h\widehat{w}_h = \alpha\widetilde{M}_h(\widehat{u}_h - \widetilde{u}_{dh}) + \beta\widetilde{A}_h(\widehat{u}_h - \widetilde{e}_h). \end{array}\right. \tag{5.6}$$

where $\widetilde{f}_h$, $\widetilde{u}_{dh}$, and $\widetilde{e}_h$ are the Lagrange interpolations of the source, desired state, and eigenfunction, respectively. Unlike for the hybrid method where we used a gradient algorithm to compute for the solutions of the optimal control problem, the solutions to the coupled system (5.6) were calculated using a sparse solver. We can observe from Figure 3 that the usual method has quadratic orders of convergence for the errors, however, the hybrid method yields smaller errors in comparison to the conforming one.

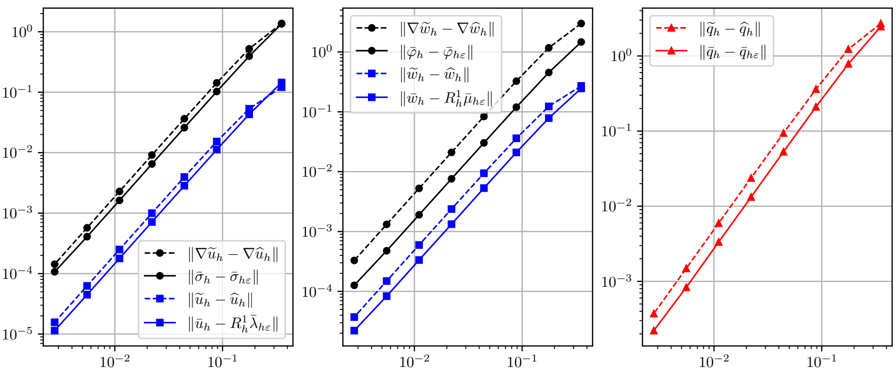

FIGURE 3. Discretization errors for the primal, dual, and control variables obtained from the usual H^1-conforming FEM (dashed lines) and the penalized hybrid FEM with post-processing (solid lines).

A disadvantage of the mixed method is the larger number of degrees of freedom in relation to the usual H^1-conforming method. Also, the treatments of the resulting saddle point problems are not trivial tasks. In this work, additional penalty terms have been introduced to reduce the size of the corresponding linear systems, leading to symmetric and positive-definite equations.

To circumvent the difficulties arising from mixed methods, hybridizable discontinuous Galerkin (HDG) methods have been studied and successfully applied to the discretization of optimal control problems with PDE constraints. For example, in the case of elliptic PDEs, we refer to [18] and [20] with Dirichlet and Neumann controls, respectively. Other related discretization schemes that can be explored are the so-called hybrid high-order (HHO) methods, see [11] for instance. It would be interesting to compare the performance of mixed and hybrid formulations to other hybridizable methods. Since these are out of the scope of the current manuscript, we recommend such topics for future theoretical and numerical investigations.

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