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Distributed Optimal Control of the 2D Cahn–Hilliard–Oberbeck–Boussinesq System for Nonisothermal Viscous Two-Phase Flows

DISTRIBUTED OPTIMAL CONTROL OF THE 2D CAHN–HILLIARD–OBERBECK–BOUSSINESQ SYSTEM FOR NONISOTHERMAL VISCOUS TWO-PHASE FLOWS

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Abstract.
We analyze a distributed optimal control problem where the state equation is governed by the coupling of the two-dimensional Cahn-Hilliard and Oberbeck-Boussinesq systems modelling incompressible viscous two-phase flows with convective heat transfer. Pointwise constraints are imposed on the controls that act as external sources in the fluid and convection-diffusion equations. The objective functional is of tracking-type that consists of a weighted energy of the difference between the state and a desired target. We establish the existence of optimal controls, the differentiability of the control-to-state operator, and the necessary and sufficient optimality conditions. For initial and target data with finite energy norms, limited space-time regularity of the adjoint states arises due to convection and surface tension.

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Cahn-Hilliard equation, Oberbeck-Boussinesq system, two-phase flows, Galerkin method, optimal control, optimality system.

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1. Introduction

Phase-field models aim to provide quantitative and qualitative descriptions for the dynamics of multiphase flows and phase transitions such as solidification, segregation, crystallization, and precipitation to name a few. One of the classical problems is to determine both the reduced temperature and the interface or boundary separating two bulk phases, often called the Stefan problem [50]. In this formulation, the interface is defined as the level set of temperature at some critical value. For example, the equilibrium melting temperature in solidification processes. The phases are then characterized by the sign difference of the temperatures.

In [9], Caginalp proposed an approach to the above free boundary value problem by using mean field theories from statistical mechanics and condensed matter physics. Starting with the enthalpy or H-method in [46] that combines the heat and latent heat as a single equation, the method is to replace the step phase function, also called the order parameter or concentration, by a continuous one. In the evolutionary case, the rate of change of the order parameter must be proportional to the minimizer of a free energy functional, an extension of the Landau-Ginzburg theory for equilibrium phase transitions. When the difference between the chemical potentials of the phases is also taken into account, one has the Cahn-Hilliard equation for spinodal decomposition [12, 11]. The asymptotic analysis in [10] showed that this equation can be obtained from the phase-field system when the latent heat vanishes. Other recent developments in the field include hydrodynamic properties for which the Navier-Stokes equation is used to model the mean velocity [4, 15, 27, 33], stress diffusion for non-Newtonian fluids [16], and Cattaneo-Maxwell law for finite speed heat propagation in place of the usual Fourier law of heat conduction [3].

In this paper, we study a distributed optimal control problem governed by a Cahn-Hilliard-Oberbeck-Boussinesq phase-field system describing the dynamics of a binary viscous and incompressible fluid mixture incorporating thermal effects. For the sake of the reader, we shall outline the essential parts of the model formulation. Further details on the modelling aspect and other related papers are referred to [7, 15, 16, 27, 35, 47, 48] and the references therein.

Let $T > 0$ be a fixed final time and $\Omega \subset \mathbb{R}^2$ be the region occupied by the binary mixture. For technical reasons, we assume that Ω is an open, connected, and bounded domain that is either of class C^2 or convex polygonal with boundary Γ. Denote by $\boldsymbol{u} : (0,T)\times\Omega \to \mathbb{R}^2$ the mean velocity of the fluid mixture, $\mathfrak{p} : (0,T) \to \mathbb{R}$ the pressure, $\theta : (0,T)\times\Omega \to \mathbb{R}$ the relative temperature around some critical value θ_c, and $\phi : (0,T)\times\Omega \to \mathbb{R}$ the order parameter or concentration describing the normalized fractional part of one fluid in the mixture. We consider without loss of generality that $\theta_c = 0$. Typically, $\phi = 1$ represents one phase while $\phi = -1$ designates the other phase.

Following [46, 9], let us introduce the function $H(\theta,\phi) := \theta - l_{\rm h}\varphi$, where $l_{\rm h} > 0$ is a constant related to the latent heat and $\boldsymbol{q}$ be the heat flux. Then the evolution of the temperature is governed by the equation

$$\rho_0 c_{\rm p} {\rm D}_t H(\theta,\phi) + \operatorname{div} \boldsymbol{q} = \alpha_0 \boldsymbol{g}\cdot\boldsymbol{u} + s,$$

where ${\rm D}_t = \partial_t + \boldsymbol{u}\cdot\nabla$ denotes the material derivative, ρ_0 the reference density, $c_{\rm p}$ the specific heat at constant pressure, and s an external heat source or sink. The second term on the right hand side expresses linearized adiabatic effects at some reference temperature, where $\boldsymbol{g}$ is the gravitational constant, see also [36] and [55, Section 9.3] in the context of the Bènard problem. Assuming Fourier's law of thermal conduction, the heat flux can be expressed as $\boldsymbol{q} = -\kappa\nabla\theta$, where $\kappa > 0$ is the thermal conductivity, and one obtains the convection-diffusion equation

$$\rho_0 c_{\rm p}[\partial_t\theta - l_{\rm h}\partial_t\phi + \boldsymbol{u}\cdot\nabla(\theta - l_{\rm h}\phi)] - \kappa\Delta\theta = \alpha_0\boldsymbol{g}\cdot\boldsymbol{u} + s. \tag{1.1}$$

From the mass conservation law, the evolution of the order parameter is given by

$$\rho_0 {\rm D}_t\phi + \operatorname{div}\boldsymbol{j} = 0$$

where $\boldsymbol{j}$ is the mass flux. A typical assumption is Fick's law $\boldsymbol{j} = -m\nabla\mu$, where $m > 0$ is the diffusive mobility and $\mu : (0,T)\times\Omega \to \mathbb{R}$ is the chemical potential. The Cahn-Hilliard description for the chemical potential μ is a minimizer to the following Gibbs free energy incorporating the temperature

$$\mathcal{G}(\phi,\theta) := \int_\Omega \left(\frac{\alpha}{2}|\nabla\phi|^2 + F(\phi) + l_{\rm c}\theta\phi\right) {\rm d}x,$$

where $l_{\rm c} > 0$ is a constant related to the latent heat, see [15, 38, 39, 47] for instance. Here, the parameter $\alpha > 0$ characterizes the thickness of the boundary layer or interface that separates the two phases. In this work, we take the Ginzburg-Landau-Wilson free energy functional corresponding to the double-well potential $F(\phi) := \frac{1}{4}(1-\phi^2)^2$. This is an approximation of the logarithmic-type potential in [21]. Taking formally the variational derivative of $\mathcal{G}$ with respect to ϕ and under suitable boundary conditions, one obtains the following Cahn-Hilliard equation with temperature:

$$\rho_0(\partial_t\phi + \boldsymbol{u}\cdot\nabla\phi) = m\Delta\mu \tag{1.2}$$

$$\mu = \partial_\phi \mathcal{G}(\phi,\theta) = -\alpha\Delta\phi + \phi^3 - \phi + l_c\theta. \tag{1.3}$$

Ignoring the latent heat and without convection ($l_c = 0$ and $\boldsymbol{u} = \boldsymbol{0}$), these equations reduce to the standard Cahn-Hilliard model for non-equilibrium phase separation.

The description of the mean velocity starts with the momentum balance equation

$$\rho_0 \mathrm{D}_t\boldsymbol{u} - \operatorname{div}\boldsymbol{T} = \rho_0\ell(\phi,\theta)\boldsymbol{g} + \boldsymbol{f}.$$

Here, $\boldsymbol{T}$ is the stress tensor, $\rho_0\ell(\phi,\theta) := \rho_0(\alpha_1 + \alpha_2\phi + \alpha_3\theta)$ having the constant parameters α_1, α_2, and α_3 is the linearized equation of state, and $\boldsymbol{f}$ is an external body force, see [35, Chapter 8] for instance. By assuming that the relative momentum and kinetic energy of each phase is small compared to net fluid flow, the stress tensor can be written as a sum $\boldsymbol{T} = \boldsymbol{T}_{cs} + \boldsymbol{T}_{st}$ of two second-order tensors [27]. The first component $\boldsymbol{T}_{cs} = \nu(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^\top) - \mathfrak{p}\boldsymbol{I}$ is the classical Cauchy stress tensor for Newtonian incompressible viscous fluids, with $\nu > 0$ the kinematic viscosity and $\boldsymbol{I}$ the identity tensor, while the other component $\boldsymbol{T}_{st} = \mathcal{K}\alpha(\frac{1}{3}|\nabla\phi|^2\boldsymbol{I} - \nabla\phi\otimes\nabla\phi)$ accounts for the capillary forces due to surface tension, where $\mathcal{K} > 0$ is the capillarity stress coefficient. Such formulation already appeared in the work of Korteweg for gradient fluids where the density is utilized instead of the concentration [57, Section 124].

From the equation of the chemical potential in (1.3), we have the identity

$$\mathcal{K}(\mu - l_c\theta)\nabla\phi = \mathcal{K}\nabla\left(\frac{\alpha}{2}|\nabla\phi|^2 + F(\phi)\right) - \mathcal{K}\alpha\operatorname{div}(\nabla\phi\otimes\nabla\phi).$$

By setting $p := \mathfrak{p} + \mathcal{K}(\frac{\alpha}{6}|\nabla\phi|^2 + F(\phi))$, the above considerations lead to the following modified incompressible Navier-Stokes equation

$$\rho_0[\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}] - \nu\Delta\boldsymbol{u} + \nabla p = \mathcal{K}(\mu - l_c\theta)\nabla\phi + \rho_0\ell(\phi,\theta)\boldsymbol{g} + \boldsymbol{f} \tag{1.4}$$

$$\operatorname{div}\boldsymbol{u} = 0. \tag{1.5}$$

Neglecting the gravitational force and the latent heat ($\boldsymbol{g} = \boldsymbol{0}$ and $l_c = 0$) in (1.2)-(1.5), we end up with the coupled Cahn-Hilliard-Navier-Stokes system in [27]. On the other hand, without surface tension ($\mathcal{K} = 0$) and ignoring the latent heat ($l_h = 0$), equations (1.1), (1.4) and (1.5) comprise the Oberbeck-Boussinesq system [7, 45] in thermohydraulics. Now for simplicity of exposition, we set ρ_0, c_p and α_0 all equal to 1, and assume that the remaining parameters appearing in (1.1)-(1.4) to be constant.

The present paper is devoted to the study of a nonlinear infinite-dimensional optimization problem:

$$\min_{(\boldsymbol{y},z)\in Q_{ad}} J(\phi,\mu,\boldsymbol{u},\theta,\boldsymbol{y},z), \tag{1.6}$$

where the objective function J is given by

$$J(\phi,\mu,\boldsymbol{u},\theta,\boldsymbol{y},z) := G(\phi,\mu,\boldsymbol{u},\theta) + \frac{\gamma_f}{2}\int_0^T\int_{\omega_f}|\boldsymbol{y}(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\gamma_h}{2}\int_0^T\int_{\omega_h}|z(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t.$$

Here, $\boldsymbol{y}$ and z are the controls that act as external body force and heat source on certain parts of the domain, respectively. The quadruple $(\phi, \mu, \boldsymbol{u}, \theta)$ is a suitable weak solution of the two-dimensional coupled Cahn-Hilliard-Oberbeck-Boussinesq system (1.1)-(1.5). More precisely, the equation of the state with the application of the controls is governed by the system

$$\begin{cases} \partial_t \phi + \boldsymbol{u} \cdot \nabla \phi - m\Delta \mu = 0 \\ \mu = -\alpha \Delta \phi + \phi^3 - \phi + l_{\mathrm{c}} \theta \\ \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = \mathcal{K}(\mu - l_{\mathrm{c}}\theta)\nabla \phi + \ell(\phi, \theta)\boldsymbol{g} + \chi_{\omega_{\mathrm{f}}} \boldsymbol{y} \\ \operatorname{div} \boldsymbol{u} = 0 \\ \partial_t \theta - l_{\mathrm{h}} \partial_t \phi + \boldsymbol{u} \cdot \nabla(\theta - l_{\mathrm{h}}\phi) - \kappa \Delta \theta = \boldsymbol{g} \cdot \boldsymbol{u} + \chi_{\omega_{\mathrm{h}}} z \end{cases} \tag{1.7}$$

in $(0, T) \times \Omega$, and supplied with the initial conditions

$$\phi(0) = \phi_0, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \theta(0) = \theta_0 \quad \text{in } \Omega. \tag{1.8}$$

The set Q_{ad} of admissible controls and the tracking-type functional G will be discussed in detail below. In (1.6), $\gamma_{\mathrm{f}} > 0$ and $\gamma_{\mathrm{h}} > 0$ are Tikhonov regularization parameters, the control regions ω_{f} and ω_{h} are nonempty open subsets of Ω, and χ_ω denotes the characteristic function of a set $\omega \subset \Omega$ in (1.7).

The boundary conditions that we are interested in are the following:

$$\partial_{\boldsymbol{n}} \phi = \partial_{\boldsymbol{n}} \Delta \phi = 0, \quad \boldsymbol{u} = 0, \quad \partial_{\boldsymbol{n}} \theta = 0, \quad \text{on } (0, T) \times \Gamma, \tag{1.9}$$

where $\boldsymbol{n}$ is the unit normal vector outward to Γ and $\partial_{\boldsymbol{n}} h = \nabla h \cdot \boldsymbol{n}$ is the derivative of a function h in the direction of $\boldsymbol{n}$. The second condition is the no-slip boundary condition for the fluid, while the third equation imposes that there is no heat flux through the boundary. On the other hand, the first condition requires that locally the interface is orthogonal to the boundary and there is no diffusion across it.

The set of admissible controls $Q_{\mathrm{ad}} \subset Q := L^2((0, T); L^2(\omega_{\mathrm{f}})^2) \times L^2((0, T); L^2(\omega_{\mathrm{h}}))$ in (1.6) is defined by

$$Q_{\mathrm{ad}} := \{(\boldsymbol{y}, z) \in Q : \boldsymbol{a}_{\mathrm{f}} \le \boldsymbol{y} \le \boldsymbol{b}_{\mathrm{f}} \text{ a.e. } (0, T) \times \omega_{\mathrm{f}}, \ a_{\mathrm{h}} \le z \le b_{\mathrm{h}} \text{ a.e. } (0, T) \times \omega_{\mathrm{h}}\},$$

where $\boldsymbol{a}_{\mathrm{f}} = (a_{\mathrm{f1}}, a_{\mathrm{f2}})$, $\boldsymbol{b}_{\mathrm{f}} = (b_{\mathrm{f1}}, b_{\mathrm{f2}})$, $-\infty \le a_{\mathrm{h}} < b_{\mathrm{h}} \le \infty$ and $-\infty \le a_{\mathrm{f}i} < b_{\mathrm{f}i} \le \infty$ for $i = 1, 2$. Here and throughout the rest of the paper, "a.e." stands for the measure-theoretic terminology *almost everywhere*. For vectors and functions, the usual notation for order relations in $\mathbb{R}$ are to be understood componentwise and pointwise, respectively.

Let us define the *modified* Ginzburg-Landau free energy functional

$$\mathcal{E}(\phi(t), \theta(t)) := \frac{1}{2} \int_\Omega \alpha |\nabla \phi(t, x)|^2 + \frac{1}{2}(1 - \phi(t, x)^2)^2 + l_{\mathrm{c}} |\theta(t, x)|^2 \, \mathrm{d}x.$$

Formal calculations lead us to the following energy identity for the solutions of the system (1.7)-(1.9):

$$\begin{aligned} &E(\phi(t), \boldsymbol{u}(t), \theta(t)) + \int_0^t \int_\Omega \frac{m}{l_{\mathrm{c}}} |\nabla \mu(s, x)|^2 + \frac{\nu}{\mathcal{K} l_{\mathrm{c}}} |\nabla \boldsymbol{u}(s, x)|^2 + \frac{\kappa}{l_{\mathrm{h}}} |\nabla \theta(s, x)|^2 \, \mathrm{d}s \, \mathrm{d}x \\ &= E(\phi(0), \boldsymbol{u}(0), \theta(0)) + \int_0^t \int_\Omega \frac{1}{\mathcal{K} l_{\mathrm{c}}} [\ell(\phi(s, x), \theta(s, x))\boldsymbol{g} + \chi_{\omega_{\mathrm{f}}}(x) \boldsymbol{y}(s, x)] \cdot \boldsymbol{u}(s, x) \, \mathrm{d}x \, \mathrm{d}s \end{aligned}$$

$$+\int_0^t\int_\Omega \frac{1}{l_\mathrm{h}}[\boldsymbol{g}\cdot\boldsymbol{u}(s,x)+\chi_{\omega_\mathrm{h}}(x)z(s,x)]\theta(s,x)\,\mathrm{d}x\,\mathrm{d}s$$

for every $t\in[0,T]$, where E denotes the total energy of the system given by

$$E(\phi(t),\boldsymbol{u}(t),\theta(t)) := \frac{1}{2}\left(\frac{1}{l_\mathrm{c}}\mathcal{E}(\phi(t),l_\mathrm{h}^{-1/2}\theta(t)) + \int_\Omega \frac{1}{\mathcal{K}l_\mathrm{c}}|\boldsymbol{u}(t,x)|^2\,\mathrm{d}x\right). \tag{1.10}$$

Indeed, this energy identity can be derived by using the test functions $l_\mathrm{c}^{-1}\mu$, $l_\mathrm{c}^{-1}\partial_t\phi$, $(\mathcal{K}l_\mathrm{c})^{-1}\boldsymbol{u}$, and $l_\mathrm{h}^{-1}\theta$ in (1.7), see also Section 3. Ignoring gravitational effects and without the controls, we see that the energy decreases through time due to diffusion in the fluid, heat, and chemical potential. Moreover, for each $t\in[0,T]$ we have the conservation law

$$\frac{1}{|\Omega|}\int_\Omega \phi(t,x)\,\mathrm{d}t = \frac{1}{|\Omega|}\int_\Omega \phi_0(x)\,\mathrm{d}x,$$

where $|\Omega|$ is the Lebesgue measure of Ω. This follows by integrating the first equation in (1.7), using the divergence theorem, and invoking the boundary conditions (1.9) and as well as

$$\partial_{\boldsymbol{n}}\mu = -\alpha\partial_{\boldsymbol{n}}\Delta\phi + (3\phi^2-1)\partial_{\boldsymbol{n}}\phi + l_\mathrm{c}\partial_{\boldsymbol{n}}\theta = 0 \quad \text{on } (0,T)\times\Gamma,$$

at least for sufficiently smooth solutions.

We consider a cost functional G that incorporates various goals of steering at least one of the velocity, vorticity, temperature, order parameter, chemical potential and as well as their fluxes to a given set of desired targets. More precisely, the objective G is suppose to be separable in the sense that

$$G(\phi,\mu,\boldsymbol{u},\theta) = G_1(\phi)+G_2(\mu)+G_3(\boldsymbol{u})+G_4(\theta), \tag{1.11}$$

where the terms on the right hand side are given by

$$\begin{aligned}
G_1(\phi) &:= \frac{1}{2}\int_0^T \alpha_\mathrm{o}\|\phi-\phi_d\|^2 + \delta_\mathrm{o}\|\nabla\phi-\boldsymbol{\psi}_d\|^2\,\mathrm{d}t \\
&\qquad + \frac{\beta_\mathrm{o}}{2}\|\phi(T)-\phi_T\|^2 + \frac{\omega_\mathrm{o}}{2}\|\nabla\phi(T)-\boldsymbol{\psi}_T\|^2 \\
G_2(\mu) &:= \frac{1}{2}\int_0^T \alpha_\mathrm{c}\|\mu-\mu_d\|^2 + \delta_\mathrm{c}\|\nabla\mu-\boldsymbol{\xi}_d\|^2\,\mathrm{d}t \\
G_3(\boldsymbol{u}) &:= \frac{1}{2}\int_0^T \alpha_\mathrm{f}\|\boldsymbol{u}-\boldsymbol{u}_d\|^2 + \delta_\mathrm{f}\|\nabla\times\boldsymbol{u}\|^2\,\mathrm{d}t + \frac{\beta_\mathrm{f}}{2}\|\boldsymbol{u}(T)-\boldsymbol{u}_T\|^2 \\
G_4(\theta) &:= \frac{1}{2}\int_0^T \alpha_\mathrm{h}\|\theta-\theta_d\|^2 + \delta_\mathrm{h}\|\nabla\theta-\boldsymbol{\zeta}_d\|^2\,\mathrm{d}t + \frac{\beta_\mathrm{h}}{2}\|\theta(T)-\theta_T\|^2.
\end{aligned} \tag{1.12}$$

In (1.12), $\|\cdot\|$ denotes either the norm of the Lebesgue space $L^2(\Omega)$ or $L^2(\Omega)\times L^2(\Omega)$, where it is suitable. The given structure of G is motivated from the energy identity discussed above, for which the norms appearing in G are precisely those that are involved in the energy E. Also, $\alpha_\mathrm{o},\delta_\mathrm{o},\beta_\mathrm{o},\omega_\mathrm{o},\alpha_\mathrm{c},\delta_\mathrm{c},\alpha_\mathrm{f},\delta_\mathrm{f},\beta_\mathrm{f},\alpha_\mathrm{h},\delta_\mathrm{h},\beta_\mathrm{h}\in[0,\infty)$ are fixed nonnegative parameters, where at least one of them is nonzero in order to have a nontrivial solution to (1.6). These parameters signify on which parts of the energy are to be prioritized. The subscripts o, c, f, and h stand for order parameter, chemical potential, fluid velocity, and heat. Furthermore, the functions ϕ_d, $\boldsymbol{\psi}_d$, μ_d,

$\boldsymbol{\xi}_d$, $\boldsymbol{u}_d$, θ_d, $\boldsymbol{\zeta}_d$, ϕ_T, $\boldsymbol{\psi}_T$, $\boldsymbol{u}_T$, and θ_T are given target states, having the appropriate regularity conditions that will be discussed precisely in Section 6.

For the past decades, there are numerous contributions that deal with the analysis of optimal control problems for time-dependent fluid flows with either distributed or boundary controls: Navier-Stokes equation [1, 31, 59], Allen-Cahn equation [20], Cahn-Hilliard equation [17, 19, 18, 25, 32, 28, 62, 63, 64], Boussinesq system [1, 6, 34, 40], coupled Cahn-Hilliard-Navier-Stokes system [23, 24, 30, 29], and phase-field systems [41, 53]. This is of course an incomplete list and we refer the reader to the literature provided in these works. For the coupling of the Cahn-Hilliard and inviscid Boussinesq systems, the global well-posedness, regularity, and blow-up criteria have been discussed in [44, 61, 65], respectively.

Most of the works presented above deal with smooth enough initial data, for which the method of transposition can be applied to successfully derive the first order necessary condition characterizing the solutions of the optimal control problem. In this paper, we shall consider initial data that are at the very least have finite energies, that is, $E(\phi_0, \boldsymbol{u}_0, \theta_0) < \infty$ with E given by (1.10). In the case of instationary Navier-Stokes equation with the tracking type functional G_3 as defined above, this direction has been investigated thoroughly in [31, 59]. It has been shown that the time derivative of the optimal adjoint velocity admits lower integrability compared to that of the optimal velocity. In this case, the solutions of the state equation are not admissible test functions to the adjoint system. To circumvent the difference in regularity, duality methods were utilized.

The limited regularity stems from the convection term. Following the methods in [31], we will also achieve this property for the solutions of (1.6). Due to the presence of the order parameter flux and the chemical potential in the cost function, one can also expect even less regularity in space for the adjoint states corresponding to these state variables. This makes the analysis of the control problem more involved. Nevertheless, additional regularity on the initial and desired data is expected to result in more regular adjoint states, and we shall take advantage of this in order to establish the second order sufficient conditions. As in [13, 14], the gap between the necessary and sufficient conditions is the usual one as in the context of finite-dimensional optimization problems with box constraints.

It will be shown in terms of PDEs (see Sections 5 and 6) that the optimal adjoint state is either an appropriate weak or very weak solution, depending on the regularity of the data, of the following system that is posed backward in time:

$$
\begin{cases}
-\partial_t \varphi + l_{\mathrm{h}} \partial_t \vartheta - \boldsymbol{u} \cdot \nabla(\varphi - l_{\mathrm{h}} \vartheta) + \alpha \Delta \eta \\
\qquad = f'(\phi)\eta + \alpha_2 \boldsymbol{g} \cdot \boldsymbol{v} - \mathcal{K} \boldsymbol{v} \cdot \nabla(\mu - l_{\mathrm{c}} \theta) + \alpha_{\mathrm{o}}(\phi - \phi_d) - \delta_{\mathrm{o}} \mathrm{Div}(\nabla \phi - \boldsymbol{\psi}_T) \\
\quad -\eta = -m \Delta \varphi - \mathcal{K} \boldsymbol{v} \cdot \nabla \phi - \alpha_{\mathrm{c}}(\mu - \mu_d) + \delta_{\mathrm{c}} \mathrm{Div}(\nabla \mu - \boldsymbol{\xi}_d) \\
-\partial_t \boldsymbol{v} - (\boldsymbol{u} \cdot \nabla) \boldsymbol{v} + (\nabla \boldsymbol{u})^\top \boldsymbol{v} - \nu \Delta \boldsymbol{v} + \nabla \pi \\
\qquad = \vartheta \boldsymbol{g} - \varphi \nabla \phi - \vartheta \nabla(\theta - l_{\mathrm{h}} \phi) + \alpha_{\mathrm{f}}(\boldsymbol{u} - \boldsymbol{u}_d) + \delta_{\mathrm{f}} \nabla \times (\nabla \times \boldsymbol{u}) \\
\mathrm{div}\, \boldsymbol{v} = 0 \\
-\partial_t \vartheta - \boldsymbol{u} \cdot \nabla \vartheta + \mathcal{K} l_{\mathrm{c}} \boldsymbol{v} \cdot \nabla \phi - \kappa \Delta \vartheta \\
\qquad = \alpha_3 \boldsymbol{g} \cdot \boldsymbol{v} + l_{\mathrm{c}} \eta + \alpha_{\mathrm{h}}(\theta - \theta_d) - \delta_{\mathrm{h}} \mathrm{Div}(\nabla \theta - \boldsymbol{\zeta}_d)
\end{cases}
$$

in $(0,T)\times\Omega$, with the boundary conditions $\partial_{\boldsymbol{n}}\varphi = \partial_{\boldsymbol{n}}\eta = 0$, $\boldsymbol{v} = 0$, $\partial_{\boldsymbol{n}}\vartheta = 0$ on $(0,T)\times\Gamma$, and the terminal conditions

$$\begin{aligned} \varphi(T) - l_{\mathrm{h}}\vartheta(T) &= \beta_{\mathrm{o}}(\phi(T)-\phi_T) - \omega_{\mathrm{o}}\mathrm{Div}(\nabla\phi(T)-\boldsymbol{\psi}_T) && \text{in } \Omega,\\ \boldsymbol{v}(T) = \beta_{\mathrm{f}}(\boldsymbol{u}(T)-\boldsymbol{u}_T),&\quad \vartheta(T) = \beta_{\mathrm{h}}(\theta(T)-\theta_T) && \text{in } \Omega. \end{aligned}$$

Here, Div is an extension of the distributional divergence with test functions in the Sobolev space $H^1(\Omega)$, see (6.2) for the precise definition. The curl of a vector-valued function $\boldsymbol{u} = (u_1,u_2)$ is given by $\nabla\times\boldsymbol{u} = \partial_{x_2}u_1 - \partial_{x_1}u_2$, while the curl of a scalar-valued function h is defined by $\nabla\times h := (-\partial_{x_2}h, \partial_{x_1}h)$, provided that the derivatives exist, see [59]. The above linear system can be readily obtained by a formal Lagrangian approach. Such a formalism will be justified rigorously in this paper. We would like to point out that a first step towards the development of efficient gradient-based numerical schemes for the approximation of the controls is by identifying a dual problem to the state equation.

The plan of the paper is as follows: In Section 2, we recall the relevant function spaces and operators involved in the weak formulation of (1.7)-(1.9) and write the equivalent evolution equations in suitable Bochner spaces. The well-posedness of the state, linearized state, and adjoint systems are the concerns of Sections 3, 4, and 5, respectively. Finally, we discuss the analysis of the optimal control problem (1.6), including the first and second order necessary and sufficient optimality conditions in Section 6.

2. Preliminaries

Given $1\le p\le\infty$ and $s\in\mathbb{R}$, $L^p(\Omega)$ and $H^s(\Omega)$ are the usual Lebesgue and Sobolev spaces equipped with the norms denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^s}$. A subscript will be indicated to emphasize the space where the norm or inner product is defined. Let $H_0^s(\Omega)$ be the closure of the space $C_0^\infty(\Omega)$, the set of infinitely differentiable functions that are compactly supported in Ω, with respect to the norm of $H^s(\Omega)$. We refer to the classical text [2] for more details.

Let $I = (0,T)$ be the time interval and $\bar{I} = [0,T]$ be its closure. Given a Banach space Y with norm $\|\cdot\|_Y$, $C(\bar{I};Y)$ and $L^p(I;Y)$ are the space of continuous functions and Bochner spaces with values in Y endowed with the norms $\|u\|_{C(Y)} := \sup_{t\in\bar{I}}\|u(t)\|_Y$, $\|w\|_{L^\infty(Y)} := \operatorname{ess\,sup}_{t\in I}\|w(t)\|_Y$,

$$\|v\|_{L^p(Y)} := \left(\int_0^T \|v(t)\|_Y^p\,\mathrm{d}t\right)^{1/p} \quad (1\le p<\infty).$$

For each positive integer k, $W^{k,p}(I;Y)$ is the Banach space of all elements $u\in L^p(I;Y)$ having derivatives $\partial_t^j u\in L^p(I;Y)$ for every $1\le j\le k$ in the sense of vector-valued distributions, and set $H^k(I;Y) := W^{k,2}(I;Y)$. The dual of Y will be denoted by Y^* and $\langle y^*, y\rangle_{Y^*\times Y}$ represents the duality pairing between $y^*\in Y^*$ and $y\in Y$. For Banach spaces Y and Z, the norm of the intersection $Y\cap Z$ will be given by $\|u\|_{Y\cap Z} := \max\{\|u\|_Y, \|u\|_Z\}$.

In the following, all Hilbert spaces are assumed to be separable. Given $1\le p\le\infty$ and two Hilbert spaces Y and Z such that $Y\subset Z$ continuously, let

$$W^p(I;Y,Z) := \{u\in L^2(I;Y) : \partial_t u\in L^p(I;Z)\}.$$

This is a Banach space with respect to the graph norm

$$\|u\|_{W^p(Y,Z)} := \|u\|_{L^2(Y)} + \|\partial_t u\|_{L^p(Z)}.$$

In the case where the larger space is the dual of Y, we simply write $W^p(I;Y)$ instead of $W^p(I;Y,Y^*)$ and $\|u\|_{W^p(Y)} = \|u\|_{L^2(Y)} + \|\partial_t u\|_{L^p(Y^*)}$.

Note that $W^p(I;Y,Z) \subset C(\bar{I};Z)$ continuously for every $1 \le p \le \infty$. In studying the linearized and adjoint systems, the following closed subspace of $W^p(I;Y,Z)$ will be utilized

$$W_0^p(I;Y,Z) := \{u \in W^p(I;Y,Z) : u(0) = 0\},$$

and we set $W_0^p(I;Y) := W_0^p(I;Y,Y^*)$. If there is another Hilbert space X such that $Y \subset X$ is compact and $X \subset Z$ is continuous, then by the well-known Aubin-Lions-Simon Lemma, the compact embedding $W^p(I;Y,Z) \subset L^2(I;X)$ holds. For the interpolation space $[Y,Z]_{1/2}$ between Y and Z, we have the continuous embedding

$$W^2(I;Y,Z) \subset C(\bar{I};[Y,Z]_{1/2}). \tag{2.1}$$

If $Y \subset X$ is dense, then $X^* \subset Y^*$ is also dense and $W^2(I;Y) \subset C(\bar{I};X)$ continuously. The space of linear and bounded operators from Y into Z will be denoted by $\mathcal{L}(Y,Z)$. For more details on these topics, we refer the reader to [43] and [51].

In the remaining parts of the paper, we let $X := L^2(\Omega)$, $Y := H^1(\Omega)$, $\boldsymbol{X} := X \times X$ and $\boldsymbol{Y} := Y \times Y$. The classical function spaces for square-integrable and divergence-free vector fields with the no-slip boundary condition will be denoted by

$$\boldsymbol{H} := \{\boldsymbol{u} \in \boldsymbol{X} : \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}, \qquad \boldsymbol{V} := \boldsymbol{H} \cap H_0^1(\Omega)^2.$$

These are endowed with the norms $\|\boldsymbol{u}\|_{\boldsymbol{H}} := \|\boldsymbol{u}\|_{\boldsymbol{X}}$ and $\|\boldsymbol{u}\|_{\boldsymbol{V}} := \|\nabla \boldsymbol{u}\|_{\boldsymbol{X}\times\boldsymbol{X}}$, respectively. Then the Helmholtz decomposition $\boldsymbol{X} = \boldsymbol{H} \oplus \nabla X$ holds and denote by $\boldsymbol{P_H} : \boldsymbol{X} \to \boldsymbol{H}$ the orthogonal projection of $\boldsymbol{X}$ onto $\boldsymbol{H}$. Thanks to this, one can eliminate the pressure in the weak formulation.

We now introduce the notation for the Laplace operators associated with the system (1.7). Let $\boldsymbol{A}_S : D(\boldsymbol{A}_S) \subset \boldsymbol{H} \to \boldsymbol{H}$ be the Stokes operator with domain $D(\boldsymbol{A}_S) = \boldsymbol{V} \cap H^2(\Omega)^2$ and defined by $\boldsymbol{A}_S\boldsymbol{u} = -\boldsymbol{P_H}\Delta\boldsymbol{u}$, see [37] in the case of convex polygonal domains and [52, Theorem III.2.1.1] in the case of C^2-domains. The linear operator $\boldsymbol{A}_S$ is a positive self-adjoint operator with compact resolvents. The norms $\|\cdot\|_{H^2\times H^2}$ and $\|\boldsymbol{A}_S\cdot\|_{\boldsymbol{X}}$ are equivalent on $D(\boldsymbol{A}_S)$ and there exist constants $c_1, c_2 > 0$ such that $\|\boldsymbol{u}\|_{\boldsymbol{H}} \le c_1\|\boldsymbol{u}\|_{\boldsymbol{V}}$ for every $\boldsymbol{u} \in \boldsymbol{V}$ and $\|\boldsymbol{u}\|_{\boldsymbol{V}} \le c_2\|\boldsymbol{A}_S\boldsymbol{u}\|_{\boldsymbol{X}}$ for every $\boldsymbol{u} \in D(\boldsymbol{A}_S)$. The first inequality is the Poincaré inequality while the second is a consequence of the equivalence just mentioned. Moreover, $\boldsymbol{A}_S$ admits an extension $\boldsymbol{A}_S : \boldsymbol{V} \to \boldsymbol{V}^*$ that is also linear and bounded.

The Neumann map $A_N : D(A_N) \subset X \to X$ is defined by $A_N\phi = -\Delta\phi$ with domain $D(A_N) = \{\phi \in H^2(\Omega) : \partial_{\boldsymbol{n}}\phi = 0 \text{ on } \Gamma\}$, see [26] for instance. Let us extend this definition to $A_N : Y \to Y^*$ by $\langle A_N\phi, \psi\rangle_{Y^*\times Y} := (\nabla\phi, \nabla\psi)_{\boldsymbol{X}}$ for $\phi, \psi \in Y$. Integrating by parts and using the density of Y in X, this extension coincides with the earlier definition of $A_N\phi$ for $\phi \in D(A_N)$.

Given $\phi \in L^1(\Omega)$, the average of ϕ over Ω is given by $\langle\phi\rangle := |\Omega|^{-1}(\phi, 1)_X$. By the Poincaré-Wirtinger inequality, there is a constant $c > 0$ such that

$$\|\phi - \langle\phi\rangle\|_X \le c\|\nabla\phi\|_{\boldsymbol{X}} \quad \forall \phi \in Y. \tag{2.2}$$

Therefore the usual norm $\|\cdot\|_X + \|\nabla\cdot\|_{\boldsymbol{X}}$ of the Sobolev space Y is equivalent to $|\langle\cdot\rangle| + \|\nabla\cdot\|_{\boldsymbol{X}}$. Also note that the norm $|\langle\cdot\rangle| + \|A_N\cdot\|_X$ is equivalent to $\|\cdot\|_{H^2}$ in $D(A_N)$. From the inequality $|\langle\phi\rangle| \le |\Omega|^{1/2}\|\phi\|_X$, we obtain for a constant $c>0$ that

$$\|\phi\|_{H^2} \le c(\|\Delta\phi\|_X + \|\phi\|_X) \quad \forall\phi\in D(A_N). \tag{2.3}$$

Let $\widehat{X} := \{\phi\in X : \langle\phi\rangle = 0\}$ and consider the restriction $\widehat{A}_N : D(\widehat{A}_N)\subset\widehat{X}\to\widehat{X}$ of A_N to square-integrable functions with zero average, that is, $\widehat{A}_N\phi = A_N\phi$ for $\phi\in D(\widehat{A}_N) = D(A_N)\cap\widehat{X}$. Notice that $D(A_N) = D(\widehat{A}_N)\oplus\mathbb{R}$. It follows that $\widehat{A}_N$ is a positive self-adjoint operator having compact resolvents. For each $\phi\in D(A_N)$, it holds that

$$\|\nabla\phi\|_{\boldsymbol{Y}} = \|\nabla(\phi-\langle\phi\rangle)\|_{\boldsymbol{Y}} \le \|\phi-\langle\phi\rangle\|_{H^2} \le c\|\Delta\phi\|_X \tag{2.4}$$

for a constant $c>0$ independent of ϕ, and consequently $\|\nabla\phi\|_{\boldsymbol{X}} \le c\|\Delta\phi\|_X$.

By using Fourier spectral decompositions, the positive powers $\boldsymbol{A}_S^r$ and $\widehat{A}_N^r$ are well-defined for every $r>0$. In this way, for $r>0$ and $s\ge 2$ we shall set

$$Y^s := \{\phi\in D(A_N) : A_N\phi\in D(\widehat{A}_N^{(s-2)/2})\}, \quad \boldsymbol{V}^r := D(\boldsymbol{A}_S^{r/2}),$$

where $\widehat{A}_N^0 := I$ is the identity operator in $\widehat{X}$. Particular cases are $\boldsymbol{V}^1 = \boldsymbol{V}$, $Y^2 = D(A_N)$ and $Y^4 = D(A_N^2)$. If $\phi\in Y^s$ for an $s\ge 3$, then $\langle A_N\phi\rangle = 0$ by Green's identity, and from (2.2) we get

$$\|\Delta\phi\|_Y \le c\|\nabla\Delta\phi\|_{\boldsymbol{X}} \quad \forall\phi\in Y^3, \quad \|\Delta\phi\|_{H^2} \le c\|\Delta^2\phi\|_X \quad \forall\phi\in Y^4. \tag{2.5}$$

We shall equip Y^2, Y^3 and Y^4 with the norms

$$\|\phi\|_{Y^2} := \|\phi\|_X + \|\Delta\phi\|_X,\ \|\psi\|_{Y^3} := \|\psi\|_X + \|\nabla\Delta\psi\|_{\boldsymbol{X}},\ \|\varphi\|_{Y^4} := \|\varphi\|_X + \|\Delta^2\varphi\|_X$$

for $\phi\in Y^2$, $\psi\in Y^3$, and $\varphi\in Y^4$. These are Hilbert spaces with the inner products associated with the given norms. The dual spaces Y^{s*} and $\boldsymbol{V}^{r*}$ of Y^s and $\boldsymbol{V}^r$ shall be taken with respect to the pivot spaces X and $\boldsymbol{H}$, respectively. For further details regarding these topics, the reader is referred to [5, 52, 54].

We shall often use the Sobolev embedding $Y\subset L^p(\Omega)$ for every $1\le p<\infty$, which is valid for two-dimensional bounded Lipschitz domains Ω. Also, the Gagliardo-Nirenberg inequality

$$\|\phi\|_{L^4} \le c_{\mathrm{GN}}\|\phi\|_X^{1/2}\|\phi\|_Y^{1/2} \quad \forall\phi\in Y \tag{2.6}$$

and Agmon's inequality combined with (2.3)

$$\|\phi\|_{L^\infty} \le c_{\mathrm{A}}\|\phi\|_X^{1/2}\|\phi\|_{Y^2}^{1/2} \quad \forall\phi\in Y^2 \tag{2.7}$$

will be often utilized. The positive constants c_{GN} and c_{A} depend on the domain Ω, but are independent of ϕ.

In situations where the context is clear, we shall adopt the common notation $-\Delta$ for the operators $\boldsymbol{A}_S$, A_N, and $\widehat{A}_N$. All throughout this paper, c will denote a generic positive constant that depends on the parameters in the state equation, the domain Ω and the terminal time T. A subscript will be used to emphasize the dependence of this constant. Likewise, $\mathfrak{C} : \mathbb{R}^k\to(0,\infty)$ for $k\ge 1$ will denote a generic positive continuous function.

3. Analysis of the State Equation

In this section, we shall specify the notion of weak solutions to (1.7) and formulate the equivalent abstract evolution system. First, let us define the trilinear forms arising from the convection and surface tension terms. Let $b : \boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ and $r : \boldsymbol{V} \times Y \times Y \to \mathbb{R}$ be defined by $b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) = ((\boldsymbol{u} \cdot \nabla)\boldsymbol{w}, \boldsymbol{v})_{\boldsymbol{X}}$ and $r(\boldsymbol{v}, \phi, \varphi) = (\boldsymbol{v}, \varphi\nabla\phi)_{\boldsymbol{X}} = (\boldsymbol{v} \cdot \nabla\phi, \varphi)_X$ for $\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{V}$ and $\phi, \varphi \in Y$. Integrating by parts and using the fact that elements of V are divergence-free and vanish on the boundary Γ, the following identities hold:

$$b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) = -b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}), \quad b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) = ((\nabla \boldsymbol{w})^\top \boldsymbol{v}, \boldsymbol{u})_{\boldsymbol{X}}, \quad r(\boldsymbol{v}, \phi, \varphi) = -r(\boldsymbol{v}, \varphi, \phi).$$

In particular, $b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{w}) = r(\boldsymbol{v}, \phi, \phi) = 0$ for every $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$ and $\phi \in Y$.

We set $f(\phi) := \phi^3 - \phi$ for the nonlinear term in the equation for the chemical potential. Let us abbreviate the weak solution space by

$$\mathcal{W} := W^2(I; Y^3, Y^*) \times L^2(I; Y) \times W^2(I; \boldsymbol{V}) \times W^2(I; Y).$$

For now, we shall ignore the characteristic functions appearing on (1.7).

Definition 3.1. *Let $\boldsymbol{y} \in L^2(I; \boldsymbol{V}^*)$, $z \in L^2(I; Y^*)$, $\phi_0 \in Y$, $\boldsymbol{u}_0 \in \boldsymbol{H}$, and $\theta_0 \in X$. A quadruple $(\phi, \mu, \boldsymbol{u}, \theta) \in \mathcal{W}$ is called a weak solution of (1.7) if the following variational equations hold:*

$$\begin{cases} \langle \partial_t \phi, \varphi \rangle_{Y^* \times Y} + r(\boldsymbol{u}, \phi, \varphi) + m(\nabla\mu, \nabla\varphi)_{\boldsymbol{X}} = 0 & \forall \varphi \in Y, \text{ a.e. in } I, \\ \mu = -\alpha\Delta\phi + f(\phi) + l_{\mathrm{c}}\theta & \text{a.e. in } I \times \Omega, \\ \langle \partial_t \boldsymbol{u}, \boldsymbol{v} \rangle_{\boldsymbol{V}^* \times \boldsymbol{V}} + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + \nu(\nabla\boldsymbol{u}, \nabla\boldsymbol{v})_{\boldsymbol{X} \times \boldsymbol{X}} & \\ \quad = \mathcal{K} r(\boldsymbol{v}, \phi, \mu - l_{\mathrm{c}}\theta) + (\ell(\phi, \theta)\boldsymbol{g}, \boldsymbol{v})_{\boldsymbol{X}} + \langle \boldsymbol{y}, \boldsymbol{v} \rangle_{\boldsymbol{V}^* \times \boldsymbol{V}} & \forall \boldsymbol{v} \in \boldsymbol{V}, \text{ a.e. in } I, \\ \langle \partial_t \theta, \vartheta \rangle_{Y^* \times Y} - l_{\mathrm{h}} \langle \partial_t \phi, \vartheta \rangle_{Y^* \times Y} + r(\boldsymbol{u}, \theta - l_{\mathrm{h}}\phi, \vartheta) & \\ \quad + \kappa(\nabla\theta, \nabla\vartheta)_{\boldsymbol{X}} = (\boldsymbol{g} \cdot \boldsymbol{u}, \vartheta)_X + \langle z, \vartheta \rangle_{Y^* \times Y} & \forall \vartheta \in Y, \text{ a.e. in } I, \end{cases} \tag{3.1}$$

as well as the initial conditions $\phi(0) = \phi_0$ in Y, $\boldsymbol{u}(0) = \boldsymbol{u}_0$ in $\boldsymbol{H}$, and $\theta(0) = \theta_0$ in X.

Note that the initial conditions in the above definition are meaningful due to the continuity of the embeddings $W^2(I; Y^3, Y^*) \subset C(\bar{I}; Y)$, $W^2(I; \boldsymbol{V}) \subset C(\bar{I}; \boldsymbol{H})$, and $W^2(I; Y) \subset C(\bar{I}; X)$. If $(\phi, \mu, \boldsymbol{u}, \theta) \in \mathcal{W}$ is a weak solution of (1.7), then the following energy identity is satisfied:

$$\begin{aligned} E(\phi(t), \boldsymbol{u}(t), \theta(t)) &+ \int_0^t \frac{m}{l_{\mathrm{c}}} \|\nabla\mu(s)\|_{\boldsymbol{X}}^2 + \frac{\nu}{\mathcal{K} l_{\mathrm{c}}} \|\boldsymbol{u}(s)\|_{\boldsymbol{V}}^2 + \frac{\kappa}{l_{\mathrm{h}}} \|\nabla\theta(s)\|_{\boldsymbol{X}}^2 \,\mathrm{d}s \\ &= E(\phi_0, \boldsymbol{u}_0, \theta_0) + \int_0^t \frac{1}{\mathcal{K} l_{\mathrm{c}}} [(\ell(\phi(s), \theta(s))\boldsymbol{g}, \boldsymbol{u}(s))_{\boldsymbol{X}} + \langle \boldsymbol{y}(s), \boldsymbol{u}(s) \rangle_{\boldsymbol{V}^* \times \boldsymbol{V}}] \,\mathrm{d}s \\ &+ \int_0^t \frac{1}{l_{\mathrm{h}}} [(\boldsymbol{g} \cdot \boldsymbol{u}(s), \theta(s))_X + \langle z(s), \theta(s) \rangle_{Y^* \times Y}] \,\mathrm{d}s \end{aligned}$$

for almost every $t \in [0, T]$, where E is given by (1.10). As mentioned in the introduction, this follows by choosing the test function $(\varphi, \boldsymbol{v}, \vartheta) = (l_{\mathrm{c}}^{-1}\mu(s), (\mathcal{K} l_{\mathrm{c}})^{-1}\boldsymbol{u}(s), l_{\mathrm{h}}^{-1}\theta(s))$ in (3.1), taking the duality pairing of the second equation in (3.1) with $l_{\mathrm{c}}^{-1}\partial_t\phi(s)$ and then integrating over $[0, t]$.

3.1. Analysis of state equation. Let us convert the variational equations (3.1) in the framework of Bochner spaces. To do this, we extend the definitions of the Laplace operators defined in the preliminary section to the time-dependent case, and for simplicity adapt the same notations. Define the linear operators $\boldsymbol{A}_S : L^2(I;\boldsymbol{V}) \to L^2(I;\boldsymbol{V}^*)$ and $A_N : L^2(I;Y) \to L^2(I;Y^*)$ according to $(\boldsymbol{A}_S\boldsymbol{u})(t) := \boldsymbol{A}_S\boldsymbol{u}(t)$ and $(A_N\phi)(t) := A_N\phi(t)$ for a.e. $t \in I$, $\boldsymbol{u} \in L^2(I;\boldsymbol{V})$, and $\phi \in L^2(I;Y)$. These operators are bounded, that is, they satisfy the estimates

$$\|\boldsymbol{A}_S\boldsymbol{u}\|_{L^2(\boldsymbol{V}^*)} \le c\|\boldsymbol{u}\|_{L^2(\boldsymbol{V})}, \quad \|A_N\phi\|_{L^2(Y^*)} \le c\|\phi\|_{L^2(Y)}. \tag{3.2}$$

With regard to the terms corresponding to convection and surface tension, we introduce the bilinear operators $B_1 : W^2(I;\boldsymbol{V}) \times W^2(I;Y) \to L^2(I;Y^*)$ and $\boldsymbol{B}_2 : L^2(I;Y) \times L^\infty(I;Y) \to L^2(I;\boldsymbol{V}^*)$ defined respectively by

$$\langle B_1(\boldsymbol{u},\phi),\varphi\rangle_{L^2(Y^*)\times L^2(Y)} = \int_0^T r(\boldsymbol{u}(t),\phi(t),\varphi(t))\,\mathrm{d}t$$

$$\langle \boldsymbol{B}_2(\mu,\psi),\boldsymbol{v}\rangle_{L^2(\boldsymbol{V}^*)\times L^2(\boldsymbol{V})} = \int_0^T r(\boldsymbol{v}(t),\psi(t),\mu(t))\,\mathrm{d}t$$

for $\boldsymbol{u} \in W^2(I;\boldsymbol{V})$, $\boldsymbol{v} \in L^2(I;\boldsymbol{V})$, $\phi \in W^2(I;Y)$, $\psi \in L^\infty(I;Y)$, and φ, $\mu \in L^2(I;Y)$. For the convection term in the Navier-Stokes equation, let us introduce the bilinear operator $\boldsymbol{B} : W^2(I;\boldsymbol{V}) \times W^2(I;\boldsymbol{V}) \to L^2(I;\boldsymbol{V}^*)$ given by

$$\langle \boldsymbol{B}(\boldsymbol{u},\boldsymbol{w}),\boldsymbol{v}\rangle_{L^2(\boldsymbol{V}^*)\times L^2(\boldsymbol{V})} = \int_0^T b(\boldsymbol{u}(t),\boldsymbol{w}(t),\boldsymbol{v}(t))\,\mathrm{d}t \tag{3.3}$$

for $\boldsymbol{u},\boldsymbol{w} \in W^2(I;\boldsymbol{V})$, $\boldsymbol{v} \in L^2(I;\boldsymbol{V})$, and set $\boldsymbol{B}(\boldsymbol{u}) := \boldsymbol{B}(\boldsymbol{u},\boldsymbol{u})$. These maps are well-defined according to the Gagliardo-Nirenberg and Hölder inequalities. Indeed, we have the following:

$$\|\boldsymbol{B}_2(\mu,\psi)\|_{L^2(\boldsymbol{V}^*)} \le c\|\mu\|_{L^2(Y)}\|\psi\|_{L^\infty(Y)} \tag{3.4}$$

$$\|B_1(\boldsymbol{u},\phi)\|_{L^2(Y^*)} \le c(\|\boldsymbol{u}\|_{L^\infty(\boldsymbol{H})}\|\phi\|_{L^2(Y)} + \|\phi\|_{L^\infty(X)}\|\boldsymbol{u}\|_{L^2(\boldsymbol{V})}) \tag{3.5}$$

$$\|\boldsymbol{B}(\boldsymbol{u},\boldsymbol{w})\|_{L^2(\boldsymbol{V}^*)} \le c(\|\boldsymbol{u}\|_{L^\infty(\boldsymbol{H})}\|\boldsymbol{w}\|_{L^2(\boldsymbol{V})} + \|\boldsymbol{w}\|_{L^\infty(\boldsymbol{H})}\|\boldsymbol{u}\|_{L^2(\boldsymbol{V})}) \tag{3.6}$$

for every $\boldsymbol{u},\boldsymbol{w} \in W^2(I;\boldsymbol{V})$, $\phi \in W^2(I;Y)$, $\mu \in L^2(I;Y)$, and $\psi \in L^\infty(I;Y)$. The inequalities on the right hand sides of (3.5) and (3.6) are valid according to the continuity of the embeddings $W^2(I;\boldsymbol{V}) \subset L^\infty(I;\boldsymbol{H})$ and $W^2(I;Y) \subset L^\infty(I;X)$.

Finally, let us define $f : L^\infty(I;Y) \cap L^2(I;Y^2) \to L^2(I;Y)$ by $f(\phi) = \phi^3 - \phi$. Applying the Hölder inequality and Sobolev embedding, there is a constant $c > 0$ such that for every $\phi \in L^\infty(I;Y) \cap L^2(I;Y^2)$

$$\|f(\phi)\|_{L^2(Y)} \le c(\|\phi\|_{L^\infty(Y)}^3 + \|\phi\|_{L^\infty(Y)}^2\|\phi\|_{L^2(Y^2)} + \|\phi\|_{L^\infty(Y)}). \tag{3.7}$$

The weak formulation in Definition 3.1 can now be written equivalently as follows:

$$\begin{cases} \partial_t\phi + B_1(\boldsymbol{u},\phi) + mA_N\mu = 0 & \text{in } L^2(I;Y^*), \\ \mu = \alpha A_N\phi + f(\phi) + l_c\theta & \text{in } L^2(I;Y), \\ \partial_t\boldsymbol{u} + \boldsymbol{B}(\boldsymbol{u}) + \nu\boldsymbol{A}_S\boldsymbol{u} = \mathcal{K}\boldsymbol{B}_2(\mu - l_c\theta,\phi) + \ell(\phi,\theta)\boldsymbol{g} + \boldsymbol{y} & \text{in } L^2(I;\boldsymbol{V}^*), \\ \partial_t\theta - l_h\partial_t\phi + B_1(\boldsymbol{u},\theta - l_h\phi) + \kappa A_N\theta = \boldsymbol{g}\cdot\boldsymbol{u} + z & \text{in } L^2(I;Y^*), \\ \phi(0) = \phi_0 \ \text{ in } Y, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0 \ \text{ in } \boldsymbol{H}, \quad \theta(0) = \theta_0 \ \text{ in } X. \end{cases} \tag{3.8}$$

Indeed, multiplying the variational equations (3.1) by functions in $C_0^\infty(I)$ and using the density of the linear span of the set $\{\chi f : \chi \in C_0^\infty(I), f \in Z\}$ in $L^2(I;Z)$, where Z is either Y or $\boldsymbol{V}$, we see that the equations in (3.8) are valid. The converse is analogous by using smooth test functions. Take note that the second equation in (3.8) holds in $L^2(I;Y)$ according to the continuous embedding $W^2(I;Y^3,Y^*) \subset L^\infty(I;Y) \cap L^2(I;Y^3)$ and the estimate (3.7). All throughout, we shall use the more convenient system (3.8) as the definition of weak solutions.

The existence of a weak solution is established by a standard spectral Galerkin method, and we provide the details for future reference, especially in the context of regularity of solutions, the linearized system, and the existence of optimal controls. For the Galerkin method applied to the Navier-Stokes and the Cahn-Hilliard-Navier-Stokes system, we refer to [8, 16, 54].

Let $\{\boldsymbol{v}_j\}_{j=1}^\infty$ and $\{\varphi_j\}_{j=2}^\infty$ be orthonormal bases for $\boldsymbol{H}$ and $\widehat{X}$ consisting of eigenfunctions of $\boldsymbol{A}_S$ and $\widehat{A}_N$, respectively. Define the constant function $\varphi_1 := |\Omega|^{-1/2}$. Then $\{\varphi_j\}_{j=1}^\infty$ is an orthonormal basis for X. Let $\boldsymbol{H}_k$ and X_k be the subspaces generated by $\{\boldsymbol{v}_j\}_{j=1}^k$ and $\{\varphi_j\}_{j=1}^k$, respectively, and set the orthogonal projections $\boldsymbol{P}_{\boldsymbol{H}_k} : \boldsymbol{H} \to \boldsymbol{H}_k$ and $P_{X_k} : X \to X_k$ by

$$\boldsymbol{P}_{\boldsymbol{H}_k}\boldsymbol{u} := \sum_{j=1}^k (\boldsymbol{u}, \boldsymbol{v}_j)_{\boldsymbol{H}} \boldsymbol{v}_j, \quad P_{X_k}\phi := \sum_{j=1}^k (\phi, \varphi_j)_X \varphi_j = \langle \phi \rangle + \sum_{j=2}^k (\phi, \varphi_j)_X \varphi_j,$$

for $u \in \boldsymbol{H}$ and $\phi \in X$. Note that $\boldsymbol{P}_{\boldsymbol{H}_k} \in \mathcal{L}(\boldsymbol{H}_k, \boldsymbol{V})$ and $P_{X_k} \in \mathcal{L}(X_k, Y)$, hence for the adjoint operators, we have $\boldsymbol{P}_{\boldsymbol{H}_k}^* \in \mathcal{L}(\boldsymbol{V}^*, \boldsymbol{H}_k)$ and $P_{X_k}^* \in \mathcal{L}(Y^*, X_k)$, where $\boldsymbol{H}_k^*$ and X_k^* are identified with $\boldsymbol{H}_k$ and X_k, respectively.

Theorem 3.2. *Suppose that $\boldsymbol{y} \in L^2(I;\boldsymbol{V}^*)$, $z \in L^2(I;Y^*)$, $\phi_0 \in Y$, $\boldsymbol{u}_0 \in \boldsymbol{H}$, and $\theta_0 \in X$. Then the nonlinear system (3.8) has a unique solution $(\phi, \mu, \boldsymbol{u}, \theta) \in \mathcal{W}$. Furthermore, there exists a continuous function $\mathfrak{C} > 0$, independent of $(\phi, \mu, \boldsymbol{u}, \theta)$, such that*

$$\|(\phi, \mu, \boldsymbol{u}, \theta)\|_{\mathcal{W}} \le \mathfrak{C}(\|\phi_0\|_Y, \|\boldsymbol{u}_0\|_{\boldsymbol{H}}, \|\theta_0\|_X, \|\boldsymbol{y}\|_{L^2(\boldsymbol{V}^*)}, \|z\|_{L^2(Y^*)}). \tag{3.9}$$

In particular, $\phi \in C(\bar{I};Y)$, $\boldsymbol{u} \in C(\bar{I};\boldsymbol{H})$, and $\theta \in C(\bar{I};X)$.

Proof. Let us divide the proof in several steps for ease of reading.

Step 1. *Finite-dimensional approximation.* Given a fix positive integer k, let $\phi_{k0} = P_{X_k}\phi_0$, $\boldsymbol{u}_{k0} = \boldsymbol{P}_{\boldsymbol{H}_k}\boldsymbol{u}_0$, $\theta_{k0} = P_{X_k}\theta_0$ and consider the following ansatz

$$\phi_k(t) = \sum_{j=1}^k \alpha_j(t)\varphi_k, \quad \boldsymbol{u}_k(t) = \sum_{j=1}^k \beta_j(t)\boldsymbol{v}_k, \quad \theta_k(t) = \sum_{j=1}^k \gamma_j(t)\varphi_k, \tag{3.10}$$

where $\alpha_j, \beta_j, \gamma_j \in H^1(I)$ for $j = 1, \ldots, k$. From the Cauchy-Lipschitz Theorem, the nonlinear finite-dimensional system of differential equations

$$\begin{cases} \partial_t \phi_k + P^*_{X_k}(B_1(\boldsymbol{u}_k, \phi_k) + mA_N\mu_k) = 0 & \text{in } L^2(I; X_k), \\ \mu_k = \alpha A_N \phi_k + P_{X_k} f(\phi_k) + l_{\mathrm{c}}\theta_k & \text{in } H^1(I; X_k), \\ \partial_t \boldsymbol{u}_k + \boldsymbol{P}^*_{\boldsymbol{H}_k}(\boldsymbol{B}(\boldsymbol{u}_k) + \nu \boldsymbol{A}_S \boldsymbol{u}_k) & \\ \qquad = \boldsymbol{P}^*_{\boldsymbol{H}_k}(\mathcal{K}\boldsymbol{B}_2(\mu_k - l_{\mathrm{c}}\theta_k, \phi_k) + \ell(\phi_k, \theta_k)\boldsymbol{g} + \boldsymbol{y}) & \text{in } L^2(I; \boldsymbol{H}_k), \\ \partial_t \theta_k - l_{\mathrm{h}}\partial_t\phi_k + P^*_{X_k}(B_1(\boldsymbol{u}_k, \theta_k - l_{\mathrm{h}}\phi_k) + \kappa A_N \theta_k) & \\ \qquad = P^*_{X_k}(\boldsymbol{g}\cdot\boldsymbol{u}_k + z) & \text{in } L^2(I; X_k), \end{cases} \tag{3.11}$$

with the initial conditions $\phi_k(0) = \phi_{k0}$, $\boldsymbol{u}_k(0) = \boldsymbol{u}_{k0}$, and $\theta_k(0) = \theta_{k0}$, admits a local solution on $I_k := [0, t_k)$ for some $t_k \in (0, T]$. Moreover, $\phi_k, \mu_k, \theta_k \in H^1(I_k; X_k)$ and $\boldsymbol{u}_k \in H^1(I_k; \boldsymbol{H}_k)$. The a priori estimates below will show that in fact we have $t_k = T$ for every k.

STEP 2. *Energy-type estimates.* Integrating the first equation of (3.11), we have $\langle \partial_t \phi_k(t)\rangle = 0$, and hence the conservation property $\langle \phi_k(t)\rangle = \langle \phi_{k0}\rangle = \langle P_{X_k}\phi_0\rangle = \langle \phi_0\rangle$ for every $t \in I_k$. On the other hand, taking the inner product of the said equation with $l_{\mathrm{c}}^{-1}\mu_k$ in X yields

$$\frac{1}{l_{\mathrm{c}}}(\partial_t\phi_k, \mu_k)_X + \frac{m}{l_{\mathrm{c}}}\|\nabla\mu_k\|^2_{\boldsymbol{X}} = -\frac{1}{l_{\mathrm{c}}} r(\boldsymbol{u}_k, \phi_k, \mu_k). \tag{3.12}$$

Likewise, taking the inner product of the second equation of (3.11) with $l_{\mathrm{c}}^{-1}\partial_t\phi_k$ in X leads to

$$\frac{1}{l_{\mathrm{c}}}(\mu_k, \partial_t\phi_k)_X = \frac{1}{2l_{\mathrm{c}}}\frac{d}{dt}\left(\|\nabla\phi_k\|^2_{\boldsymbol{X}} + \frac{1}{2}\|1-\phi_k^2\|^2_X\right) + (\theta_k, \partial_t\phi_k)_X. \tag{3.13}$$

Combining (3.12) and (3.13), and using the fact that $\langle\phi_k\rangle$ is constant on I, we get

$$\begin{aligned} \frac{1}{2l_{\mathrm{c}}}\frac{d}{dt}\left(\|\nabla\phi_k\|^2_{\boldsymbol{X}} + |\langle\phi_k\rangle|^2 + \frac{1}{2}\|1-\phi_k^2\|^2_X\right) + \frac{m}{l_{\mathrm{c}}}\|\nabla\mu_k\|^2_{\boldsymbol{X}} & \\ = -\frac{1}{l_{\mathrm{c}}} r(\boldsymbol{u}_k, \phi_k, \mu_k) - (\partial_t\phi_k, \theta_k)_X. & \end{aligned} \tag{3.14}$$

For the inner product of the third equation of (3.11) and $(l_{\mathrm{c}}\mathcal{K})^{-1}\boldsymbol{u}_k$ in $\boldsymbol{H}$, we have

$$\begin{aligned} &\frac{1}{2\mathcal{K}l_{\mathrm{c}}}\frac{d}{dt}\|\boldsymbol{u}_k\|^2_{\boldsymbol{H}} + \frac{\nu}{\mathcal{K}l_{\mathrm{c}}}\|\boldsymbol{u}_k\|^2_{\boldsymbol{V}} \\ &\quad = \frac{1}{l_{\mathrm{c}}} r(\boldsymbol{u}_k, \phi_k, \mu_k - l_{\mathrm{c}}\theta_k) + \frac{1}{\mathcal{K}l_{\mathrm{c}}}(\ell(\phi_k, \theta_k)\boldsymbol{g}, \boldsymbol{u}_k)_{\boldsymbol{X}} + \frac{1}{\mathcal{K}l_{\mathrm{c}}}\langle \boldsymbol{y}, \boldsymbol{u}_k\rangle_{\boldsymbol{V}^*\times\boldsymbol{V}}. \end{aligned} \tag{3.15}$$

The second and third terms on the right hand side of (3.15) can be estimated from above by the Cauchy-Schwarz inequality as follows:

$$\begin{aligned} \frac{1}{\mathcal{K}l_{\mathrm{c}}}|\langle \boldsymbol{y}, \boldsymbol{u}_k\rangle_{\boldsymbol{V}^*\times\boldsymbol{V}}| &\le \frac{\nu}{2\mathcal{K}l_{\mathrm{c}}}\|\boldsymbol{u}_k\|^2_{\boldsymbol{V}} + c\|\boldsymbol{y}\|^2_{\boldsymbol{V}^*} \\ \frac{1}{\mathcal{K}l_{\mathrm{c}}}|(\ell(\phi_k, \theta_k)\boldsymbol{g}, \boldsymbol{u}_k)_{\boldsymbol{X}}| &\le c\left(1 + \frac{1}{l_{\mathrm{c}}}\|\phi_k\|^2_X + \frac{1}{l_{\mathrm{h}}}\|\theta_k\|^2_X + \frac{1}{\mathcal{K}l_{\mathrm{c}}}\|\boldsymbol{u}_k\|^2_{\boldsymbol{H}}\right). \end{aligned}$$

Using these estimates in (3.15), and then applying the Poincaré-Wirtinger inequality (2.2), we obtain

$$\frac{1}{2\mathcal{K}l_c}\frac{d}{dt}\|\boldsymbol{u}_k\|_{\boldsymbol{H}}^2 + \frac{\nu}{2\mathcal{K}l_c}\|\boldsymbol{u}_k\|_{\boldsymbol{V}}^2 \leq \frac{1}{l_c}r(\boldsymbol{u}_k,\phi_k,\mu_k - l_c\theta_k) + c\left(1 + \frac{1}{l_c}\|\nabla\phi_k\|_{\boldsymbol{X}}^2 + \frac{1}{l_c}|\langle\phi_k\rangle|^2 + \frac{1}{\mathcal{K}l_c}\|\boldsymbol{u}_k\|_{\boldsymbol{H}}^2 + \frac{1}{l_h}\|\theta_k\|_X^2 + \|\boldsymbol{y}\|_{\boldsymbol{V}^*}^2\right). \quad (3.16)$$

Finally, taking the inner product of the fourth equation of (3.11) with $l_h^{-1}\theta_k$ in X and using the Cauchy-Schwarz inequality once more, we have the following estimate:

$$\frac{1}{2l_h}\frac{d}{dt}\|\theta_k\|_X^2 + \frac{\kappa}{2l_h}\|\nabla\theta_k\|_{\boldsymbol{X}}^2 \leq c\left(\frac{1}{\mathcal{K}l_c}\|\boldsymbol{u}_k\|_{\boldsymbol{H}}^2 + \frac{1}{l_h}\|\theta_k\|_X^2 + \|z_k\|_{Y^*}^2\right) + (\partial_t\phi_k,\theta_k)_X + r(\boldsymbol{u}_k,\phi_k,\theta_k). \quad (3.17)$$

Let us introduce the following dissipation and energy functionals defined on the interval I_k

$$D_k := \frac{1}{2}\left(\frac{2m}{l_c}\|\nabla\mu_k\|_{\boldsymbol{X}}^2 + \frac{\nu}{\mathcal{K}l_c}\|\boldsymbol{u}_k\|_{\boldsymbol{V}}^2 + \frac{\kappa}{l_h}\|\nabla\theta_k\|_{\boldsymbol{X}}^2\right)$$

$$E_k := \frac{1}{2}\left(\frac{1}{l_c}\|\nabla\phi_k\|_{\boldsymbol{X}}^2 + \frac{1}{l_c}|\langle\phi_k\rangle|^2 + \frac{1}{2l_c}\|1-\phi_k^2\|_X^2 + \frac{1}{\mathcal{K}l_c}\|\boldsymbol{u}_k\|_{\boldsymbol{H}}^2 + \frac{1}{l_h}\|\theta_k\|_X^2\right).$$

Taking the sum of (3.14), (3.16), and (3.17), and then integrating over $[0,t]$, we deduce that

$$E_k(t) + \int_0^t D_k(s)\,\mathrm{d}s \leq c\left(1 + E_k(0) + \int_0^t E_k(s) + \|\boldsymbol{y}(s)\|_{\boldsymbol{V}^*}^2 + \|z(s)\|_{Y^*}^2\,\mathrm{d}s\right)$$

for every $t \in I_k$. By the Gronwall Lemma, there is a constant $c > 0$ independent on k such that

$$\sup_{t\in I_k} E_k(t) + \int_0^{t_k} D_k(s)\,\mathrm{d}s \leq ce^{cT}(1 + E_k(0) + \|\boldsymbol{y}\|_{L^2(\boldsymbol{V}^*)}^2 + \|z\|_{L^2(Y^*)}^2). \quad (3.18)$$

We will estimate the initial energy $E_k(0)$. First, the Sobolev embedding yields the inequality $\|1-\phi_{k0}^2\|_X \leq c(1 + \|\phi_{k0}\|_X + \|\phi_{k0}\|_Y^2)$. According to $\|\boldsymbol{P}_{\boldsymbol{H}_k}\|_{\mathcal{L}(\boldsymbol{H},\boldsymbol{H})} \leq 1$, $\|P_{X_k}\|_{\mathcal{L}(X,X)} \leq 1$, and $\|P_{X_k}\|_{\mathcal{L}(Y,Y)} \leq 1$ for every k, we have $\|\phi_{k0}\|_Y \leq \|\phi_0\|_Y$, $\|\boldsymbol{u}_{k0}\|_{\boldsymbol{H}} \leq \|\boldsymbol{u}_0\|_{\boldsymbol{H}}$, and $\|\theta_{k0}\|_X \leq \|\theta_0\|_X$. Thus, we obtain

$$E_k(0) \leq c(1 + \|\phi_0\|_Y^2 + \|\phi_0\|_Y^4 + \|\boldsymbol{u}_0\|_{\boldsymbol{H}}^2 + \|\theta_0\|_X^2). \quad (3.19)$$

Therefore, from (3.18) and (3.19), we deduce the following inequality after taking square roots

$$\|\phi_k\|_{L^\infty(Y)} + \|\boldsymbol{u}_k\|_{L^\infty(\boldsymbol{H})} + \|\theta_k\|_{L^\infty(X)} + \|\nabla\mu_k\|_{L^2(\boldsymbol{X})} + \|\boldsymbol{u}_k\|_{L^2(\boldsymbol{V})} + \|\nabla\theta_k\|_{L^2(\boldsymbol{X})} \leq c(1 + \|\phi_0\|_Y^2 + \|\phi_0\|_Y + \|\boldsymbol{u}_0\|_{\boldsymbol{H}} + \|\theta_0\|_X + \|\boldsymbol{y}\|_{L^2(\boldsymbol{V}^*)} + \|z\|_{L^2(Y^*)}) \quad (3.20)$$

where $c > 0$ is a constant independent on the initial data and the source terms. By a standard continuation argument, it follows from (3.20) that the finite-dimensional system (3.11) has a solution on the whole interval I.

Step 3. *Additional a priori estimates.* From the boundary condition $\partial_{\boldsymbol{n}}\phi_k = 0$ on $I \times \Gamma$ and Green's identity, we have $\langle A_N\phi_k\rangle = 0$ in I. Hence, for the average of μ_k, one has

$$|\langle\mu_k\rangle| \leq \frac{1}{|\Omega|}\int_\Omega |\phi_k - \phi_k^3|\,\mathrm{d}x + \frac{l_\mathrm{c}}{|\Omega|}\int_\Omega |\theta_k|\,\mathrm{d}x \leq c(\|\phi_k\|_X + \|\phi_k\|_Y^3 + \|\theta_k\|_X).$$

Taking the square and then integrating over I, this inequality leads to

$$|\langle\mu_k\rangle|_{L^2(I)} \leq c(\|\phi_k\|_{L^2(X)} + \|\phi_k\|^3_{L^\infty(Y)} + \|\theta_k\|_{L^2(X)}). \tag{3.21}$$

From the equation $\Delta\phi_k = -\alpha^{-1}P_{X_k}(\mu_k - \phi_k^3 + \phi_k - l_\mathrm{c}\theta_k)$, we deduce that

$$\|\Delta\phi_k\|_{L^2(X)} \leq c(\|\mu_k\|_{L^2(X)} + \|\phi_k\|^3_{L^\infty(Y)} + \|\phi_k\|_{L^2(X)} + \|\theta_k\|_{L^2(X)}). \tag{3.22}$$

On the other hand, from $\nabla(\mu_k - \phi_k^3 + \phi_k - l_\mathrm{c}\theta_k) = \nabla\mu_k - (3\phi_k^2 - 1)\nabla\phi_k - l_\mathrm{c}\nabla\theta_k$ and the fact that $\|\nabla P_{X_k}\varphi\|_{\boldsymbol{X}} \leq \|\nabla\varphi\|_{\boldsymbol{X}}$ for every $\varphi \in Y$, we obtain the estimate

$$\|\nabla\Delta\phi_k\|_{\boldsymbol{X}} \leq c(\|\nabla\mu_k\|_{\boldsymbol{X}} + \|\phi_k\|_Y^2\|\Delta\phi_k\|_X + \|\nabla\phi_k\|_{\boldsymbol{X}} + \|\nabla\theta_k\|_{\boldsymbol{X}})$$

by (2.4) and the Sobolev embedding. Thus, we have the inequality

$$\|\nabla\Delta\phi_k\|_{L^2(\boldsymbol{X})} \leq c(\|\nabla\mu_k\|_{L^2(\boldsymbol{X})} + (\|\phi_k\|^2_{L^\infty(Y)} + 1)\|\Delta\phi_k\|_{L^2(X)} + \|\nabla\theta_k\|_{L^2(\boldsymbol{X})}). \tag{3.23}$$

Using the estimates (3.20)-(3.23) and the Poincaré-Wirtinger inequality, we see that the sequences $\{\phi_k\}_{k=1}^\infty$, $\{\mu_k\}_{k=1}^\infty$, $\{\boldsymbol{u}_k\}_{k=1}^\infty$, and $\{\theta_k\}_{k=1}^\infty$ are bounded in the function spaces $L^\infty(I;Y)\cap L^2(I;Y^3)$, $L^2(I;Y)$, $L^\infty(I;\boldsymbol{H})\cap L^2(I;\boldsymbol{V})$, and $L^\infty(I;X)\cap L^2(I;Y)$, respectively.

Step 4. *A priori estimates on time derivatives.* Utilizing (3.5) and $\|P^*_{X_k}\|_{\mathcal{L}(Y^*,Y^*)} \leq 1$ for every k in the system of differential equations (3.11), we have

$$\|\partial_t\phi_k\|_{L^2(Y^*)} \leq c(\|\boldsymbol{u}_k\|_{L^\infty(\boldsymbol{H})}\|\phi_k\|_{L^2(Y)} + \|\phi_k\|_{L^\infty(X)}\|\boldsymbol{u}_k\|_{L^2(\boldsymbol{V})} + \|\mu_k\|_{L^2(Y)}) \tag{3.24}$$

$$\begin{aligned}\|\partial_t\theta_k\|_{L^2(Y^*)} \leq c(&\|\partial_t\phi_k\|_{L^2(Y^*)} + (1 + \|\boldsymbol{u}_k\|_{L^\infty(\boldsymbol{H})})\|\theta_k\|_{L^2(Y)}\\ &+ \|\theta_k\|_{L^\infty(X)}\|\boldsymbol{u}_k\|_{L^2(\boldsymbol{V})} + \|\boldsymbol{u}_k\|_{L^\infty(\boldsymbol{H})}\|\phi_k\|_{L^2(Y)}\\ &+ \|\phi_k\|_{L^\infty(X)}\|\boldsymbol{u}_k\|_{L^2(\boldsymbol{V})} + \|\boldsymbol{u}_k\|_{L^2(\boldsymbol{H})} + \|z\|_{L^2(Y^*)}).\end{aligned} \tag{3.25}$$

In a similar way, from (3.4), (3.6), and $\|\boldsymbol{P}^*_{\boldsymbol{H}_k}\|_{\mathcal{L}(\boldsymbol{V}^*,\boldsymbol{V}^*)} \leq 1$ for every k, we obtain

$$\begin{aligned}\|\partial_t\boldsymbol{u}_k\|_{L^2(\boldsymbol{V}^*)} \leq c(&1 + (1 + \|\boldsymbol{u}_k\|_{L^\infty(\boldsymbol{H})})\|\boldsymbol{u}_k\|_{L^2(\boldsymbol{V})} + \|\phi_k\|_{L^2(X)}\\ &+ \|\theta_k\|_{L^2(X)} + (\|\mu_k\|_{L^2(Y)} + \|\theta_k\|_{L^2(Y)})\|\phi_k\|_{L^\infty(Y)} + \|\boldsymbol{y}\|_{L^2(\boldsymbol{V}^*)}).\end{aligned} \tag{3.26}$$

From Step 3 and these estimates, it follows that $\{\phi_k\}_{k=1}^\infty$, $\{\boldsymbol{u}_k\}_{k=1}^\infty$, and $\{\theta_k\}_{k=1}^\infty$ are respectively bounded in $W^2(I;Y^3,Y^*)$, $W^2(I;\boldsymbol{V})$, and $W^2(I;Y)$.

Step 5. *Passage to limit.* According to Steps 3 and 4, one can extract subsequences, still denoted by the same indices for simplicity, so that in the weak and weak-star topologies we have $\mu_k \rightharpoonup \mu$ in $L^2(I;Y)$,

$$\begin{aligned}
&\phi_k \overset{*}{\rightharpoonup} \phi \text{ in } L^\infty(I;Y), && \phi_k \rightharpoonup \phi \text{ in } L^2(I;Y^3), && \partial_t\phi_k \rightharpoonup \partial_t\phi \text{ in } L^2(I;Y^*),\\
&\boldsymbol{u}_k \overset{*}{\rightharpoonup} \boldsymbol{u} \text{ in } L^\infty(I;\boldsymbol{H}), && \boldsymbol{u}_k \rightharpoonup \boldsymbol{u} \text{ in } L^2(I;\boldsymbol{V}), && \partial_t\boldsymbol{u}_k \rightharpoonup \partial_t\boldsymbol{u} \text{ in } L^2(I;\boldsymbol{V}^*),\\
&\theta_k \overset{*}{\rightharpoonup} \theta \text{ in } L^\infty(I;X), && \theta_k \rightharpoonup \theta \text{ in } L^2(I;Y), && \partial_t\theta_k \rightharpoonup \partial_t\theta \text{ in } L^2(I;Y^*),
\end{aligned}$$

for some $(\phi,\mu,\boldsymbol{u},\theta)\in\mathcal{W}$. By the Aubin-Lions-Simon Lemma, after extracting possibly another subsequence, we have $\phi_k\to\phi$ in $L^2(I;Y^2)$, $\boldsymbol{u}_k\to\boldsymbol{u}$ in $L^2(I;\boldsymbol{H})$, and $\theta_k\to\theta$ in $L^2(I;X)$ strongly.

Now, let us pass to the limit in (3.11). Since the linear terms are straightforward, it is enough to consider the nonlinear terms. For the surface tension term, if $\boldsymbol{w}\in L^\infty(I;\boldsymbol{V})$ then

$$\langle \boldsymbol{B}_2(\mu_k,\phi_k)-\boldsymbol{B}_2(\mu,\phi),\boldsymbol{w}\rangle_{L^2(\boldsymbol{V}^*)\times L^2(\boldsymbol{V})}=(\phi\boldsymbol{w},\nabla\mu)_{L^2(\boldsymbol{X})}-(\phi_k\boldsymbol{w},\nabla\mu_k)_{L^2(\boldsymbol{X})}\to 0$$

since $\nabla\mu_k\rightharpoonup\nabla\mu$ in $L^2(I;\boldsymbol{X})$ and $\phi_k\boldsymbol{w}\to\phi\boldsymbol{w}$ in $L^2(I;\boldsymbol{X})$. Given $\boldsymbol{v}\in L^2(I;\boldsymbol{V})$ and $\varepsilon>0$, there exists $\boldsymbol{w}_\varepsilon\in L^\infty(I;\boldsymbol{V})$ such that $\|\boldsymbol{v}-\boldsymbol{w}_\varepsilon\|_{L^2(\boldsymbol{V})}<\varepsilon$ by density of $L^\infty(I;\boldsymbol{V})$ in $L^2(I;\boldsymbol{V})$. By (3.4) and the triangle inequality

$$\begin{aligned}&|\langle \boldsymbol{B}_2(\mu_k,\phi_k)-\boldsymbol{B}_2(\mu,\phi),\boldsymbol{v}\rangle_{L^2(\boldsymbol{V}^*)\times L^2(\boldsymbol{V})}|\\&\qquad\le|\langle \boldsymbol{B}_2(\mu_k,\phi_k)-\boldsymbol{B}_2(\mu,\phi),\boldsymbol{w}_\varepsilon\rangle_{L^2(\boldsymbol{V}^*)\times L^2(\boldsymbol{V})}|\\&\qquad\quad+c(\|\mu_k\|_{L^2(Y)}\|\phi_k\|_{L^\infty(Y)}+\|\mu\|_{L^2(Y)}\|\phi\|_{L^\infty(Y)})\|\boldsymbol{v}-\boldsymbol{w}_\varepsilon\|_{L^2(\boldsymbol{V})}.\end{aligned}$$

Taking the limit superior and recalling that $\{\mu_k\}_{k=1}^\infty$ and $\{\phi_k\}_{k=1}^\infty$ are bounded in $L^2(I;Y)$ and $L^\infty(I;Y)$, respectively, there exists a constant $c>0$ independent on k and ε such that

$$\limsup_{k\to\infty}|\langle \boldsymbol{B}_2(\mu_k,\phi_k)-\boldsymbol{B}_2(\mu,\phi),\boldsymbol{v}\rangle_{L^2(\boldsymbol{V}^*)\times L^2(\boldsymbol{V})}|\le c\varepsilon.$$

Since $\varepsilon>0$ and $\boldsymbol{v}\in L^2(I;\boldsymbol{V})$ are arbitrary, this implies that $\boldsymbol{B}_2(\mu_k,\phi_k)\rightharpoonup\boldsymbol{B}_2(\mu,\phi)$ in $L^2(I;\boldsymbol{V}^*)$. In the same way, replacing μ_k by θ_k gives us $\boldsymbol{B}_2(\theta_k,\phi_k)\rightharpoonup\boldsymbol{B}_2(\theta,\phi)$ in $L^2(I;\boldsymbol{V}^*)$. The remaining trilinear terms associated to convection can be handled in a similar fashion, see also [54, Lemma III.3.2], so that we have the weak convergences $\boldsymbol{B}(\boldsymbol{u}_k)\rightharpoonup\boldsymbol{B}(\boldsymbol{u})$ in $L^2(I;\boldsymbol{V}^*)$, $B_1(\boldsymbol{u}_k,\phi_k)\rightharpoonup B_1(\boldsymbol{u},\phi)$ in $L^2(I;Y^*)$, and $B_1(\boldsymbol{u}_k,\theta_k)\rightharpoonup B_1(\boldsymbol{u},\theta)$ in $L^2(I;Y^*)$.

Using the fact that $A_N\mu_k\rightharpoonup A_N\mu$ in $L^2(I;Y^*)$ and $P^*_{X_k}\psi_k\rightharpoonup\psi$ in $L^2(I;Y^*)$ whenever $\psi_k\rightharpoonup\psi$ in $L^2(I;Y^*)$, we obtain that

$$\partial_t\phi_k+P^*_{X_k}(B_1(\boldsymbol{u}_k,\phi_k)+mA_N\mu_k)\rightharpoonup\partial_t\phi+B_1(\boldsymbol{u},\phi)+mA_N\mu\quad\text{ in } L^2(I;Y^*).$$

Analogous arguments allow us to pass to the weak limit in the third and fourth equations in (3.11) thanks to the convergences discussed above.

For the second equation in (3.11), we write $f(\phi_k)-f(\phi)=(\phi_k^2+\phi_k\phi+\phi^2-1)(\phi_k-\phi)$, and apply the Hölder inequality and Sobolev embedding in order to obtain

$$\|f(\phi_k)-f(\phi)\|_{L^2(X)}\le c(\|\phi_k\|^2_{L^\infty(Y)}+\|\phi\|^2_{L^\infty(Y)}+1)\|\phi_k-\phi\|_{L^2(Y)}\to 0.$$

This implies that $P_{X_k}f(\phi_k)-f(\phi)=P_{X_k}(f(\phi_k)-f(\phi))+(P_{X_k}-I)f(\phi)\to 0$ in $L^2(I;X)$. Thus, $\mu_k-\alpha A_N\phi_k-P_{X_k}f(\phi_k)-l_c\theta_k\rightharpoonup\mu-\alpha A_N\phi-f(\phi)-l_c\theta$ in $L^2(I;X)$.

Next, we pass to the limit in the initial conditions. First, note that the map $\psi\mapsto\psi(0)$ is continuous from $W^2(I;Y^3,Y^*)$ into Y. As a consequence, $\phi_k(0)\rightharpoonup\phi(0)$ in Y, and since $\phi_{k0}\to\phi_0$ in Y, this implies that $\phi(0)=\phi_0$. In a similar way, $\boldsymbol{u}(0)=\boldsymbol{u}_0$ in $\boldsymbol{H}$ and $\theta(0)=\theta_0$ in X.

Therefore, we have verified that $(\phi,\mu,\boldsymbol{u},\theta)$ is a solution to (3.8). The estimate (3.9) follows by taking the limit inferior in (3.20)-(3.25) and applying the lower sequential semicontinuity of norms in the weak and weak-star topologies. Finally, the uniqueness of the weak solution follows from the local Lipschitz continuity of the corresponding solution operator, see Theorem 3.3 below. This completes the proof of the theorem. □

Theorem 3.3. *Given $R > 0$, there exists a constant $c_R > 0$ such that for every $(\boldsymbol{y}_i, z_i) \in L^2(I; \boldsymbol{V}^*) \times L^2(I; Y^*)$ and $(\phi_{0i}, \boldsymbol{u}_{0i}, \theta_{0i}) \in Y \times \boldsymbol{H} \times X$ with norms less than R for $i = 1, 2$,*

$$\|(\phi_1, \mu_1, \boldsymbol{u}_1, \theta_1) - (\phi_2, \mu_2, \boldsymbol{u}_2, \theta_2)\|_{\mathcal{W}} \le c_R(\|\phi_{01} - \phi_{02}\|_Y \\ + \|\boldsymbol{u}_{01} - \boldsymbol{u}_{02}\|_{\boldsymbol{H}} + \|\theta_{01} - \theta_{02}\|_X + \|\boldsymbol{y}_1 - \boldsymbol{y}_2\|_{L^2(\boldsymbol{V}^*)} + \|z_1 - z_2\|_{L^2(Y^*)}),$$

where $(\phi_i, \mu_i, \boldsymbol{u}_i, \theta_i) \in \mathcal{W}$ is the solution of (3.8) with source term $(\boldsymbol{y}_i, z_i)$ and initial data $(\phi_{0i}, \boldsymbol{u}_{0i}, \theta_{0i})$ for $i = 1, 2$.

Proof. The proof is similar to that of the linearized system provided in Theorem 4.1. We shall skip the details to avoid repetition. □

3.2. Regularity of solutions. For the remaining part of this section, we will establish the existence of more regular solutions to the state equation. The following theorem deals with improved regularity of the solution to (3.8) under additional assumptions on the data and the source terms. Let us define the following strong solution space

$$\mathcal{V} := W^2(I; Y^4, X) \times W^2(I; Y^2) \times W^2(I; \boldsymbol{V}^2, \boldsymbol{H}) \times W^2(I; Y^2, X).$$

Theorem 3.4. *Suppose that $\boldsymbol{y} \in L^2(I; \boldsymbol{X})$, $z \in L^2(I; X)$, $\phi_0 \in Y^2$, $\boldsymbol{u}_0 \in \boldsymbol{V}$, and $\theta_0 \in Y$. Then the solution of (3.8) satisfy $(\phi, \mu, \boldsymbol{u}, \theta) \in \mathcal{V}$, and there exists a unique $p \in L^2(I; Y/\mathbb{R})$ such that*

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \mathcal{K}(\mu - l_c\theta)\nabla\phi + \ell(\phi, \theta)\boldsymbol{g} + \boldsymbol{y} \quad \text{in } L^2(I, \boldsymbol{X}). \tag{3.27}$$

Furthermore, there is a continuous function $\mathfrak{C} > 0$ such that

$$\|(\phi, \mu, \boldsymbol{u}, \theta)\|_{\mathcal{V}} + \|p\|_{L^2(I;Y/\mathbb{R})} \le \mathfrak{C}(\|\phi_0\|_{Y^2}, \|\boldsymbol{u}_0\|_{\boldsymbol{V}}, \|\theta_0\|_Y, \|\boldsymbol{y}\|_{L^2(\boldsymbol{X})}, \|z\|_{L^2(X)}). \tag{3.28}$$

In particular, $\phi \in C(\bar{I}; Y^2)$, $\mu \in C(\bar{I}; X)$, $\boldsymbol{u} \in C(\bar{I}; \boldsymbol{V})$, and $\theta \in C(\bar{I}; Y)$.

Proof. We proceed by deriving a priori estimates for the Galerkin approximations $(\phi_k, \mu_k, \boldsymbol{u}_k, \theta_k)$, constructed from the proof of Theorem 3.2, with respect to the norm of $\mathcal{V}$. To simplify the a priori estimates, we shall write

$$\mathfrak{C}(\phi_0, \boldsymbol{u}_0, \theta_0, \boldsymbol{y}, z) := \mathfrak{C}(\|\phi_0\|_{Y^2}, \|\boldsymbol{u}_0\|_{\boldsymbol{V}}, \|\theta_0\|_Y, \|\boldsymbol{y}\|_{L^2(\boldsymbol{X})}, \|z\|_{L^2(X)}),$$

where $\mathfrak{C} : \mathbb{R}^5 \to (0, \infty)$ is a generic continuous function. From the continuity of the embeddings $Y^2 \subset Y \subset X$, $\boldsymbol{V} \subset \boldsymbol{H}$, $L^2(I; \boldsymbol{X}) \subset L^2(I; \boldsymbol{V}^*)$, and $L^2(I; X) \subset L^2(I; Y^*)$, the stability estimate (3.9) immediately implies that

$$\|\phi_k\|_{W^2(Y^3,Y^*)} + \|\mu_k\|_{L^2(Y)} + \|\boldsymbol{u}_k\|_{W^2(\boldsymbol{V})} + \|\theta_k\|_{W^2(Y)} \le \mathfrak{C}(\phi_0, \boldsymbol{u}_0, \theta_0, \boldsymbol{y}, z). \tag{3.29}$$

Let us split the derivation of the a priori estimates into five steps. In the following, $\varepsilon > 0$ will be a constant whose value varies in each step.

Step 1. *$L^\infty(Y)$ and $L^2(Y^2)$ estimates for θ_k.* Taking the inner product with $-(\Delta\theta_k - l_{\rm h}\Delta\phi_k)$ to the fourth equation in (3.11) in X and applying Green's identity, one has

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta_k - l_{\rm h}\nabla\phi_k\|_{\boldsymbol{X}}^2 + \kappa\|\Delta\theta_k\|_X^2 = r(\boldsymbol{u}_k, \theta_k - l_{\rm h}\phi_k, \Delta\theta_k) - l_{\rm h} r(\boldsymbol{u}_k, \theta_k - l_{\rm h}\phi_k, \Delta\phi_k) + (\boldsymbol{g}\cdot\boldsymbol{u}_k + z_k, l_{\rm h}\Delta\phi_k - \Delta\theta_k)_X. \tag{3.30}$$

By the Cauchy-Schwarz inequality, one can estimate the last term on the right hand side by

$$|(\boldsymbol{g}\cdot\boldsymbol{u}_k + z_k, l_{\rm h}\Delta\phi_k - \Delta\theta_k)_X| \le \varepsilon\|\Delta\theta_k\|_X^2 + c_\varepsilon(\|\boldsymbol{u}_k\|_{\boldsymbol{H}}^2 + \|\Delta\phi_k\|_X^2 + \|z_k\|_X^2). \tag{3.31}$$

For the trilinear terms, we apply the Hölder and Gagliardo-Nirenberg inequalities to obtain

$$|r(\boldsymbol{u}_k, \theta_k - l_{\rm h}\phi_k, \Delta\theta_k)| \le \varepsilon\|\Delta\theta_k\|_X^2 + c_\varepsilon\|\boldsymbol{u}_k\|_{\boldsymbol{H}}^2\|\boldsymbol{u}_k\|_{\boldsymbol{V}}^2(\|\nabla\theta_k\|_{\boldsymbol{X}}^2 + \|\Delta\phi_k\|_X^2) \tag{3.32}$$

$$|l_{\rm h} r(\boldsymbol{u}_k, \theta_k - l_{\rm h}\phi_k, \Delta\phi_k)| \le \varepsilon\|\Delta\theta_k\|_X^2 + c_\varepsilon\|\Delta\phi_k\|_X^2 + c_\varepsilon\|\boldsymbol{u}_k\|_{\boldsymbol{H}}^2\|\boldsymbol{u}_k\|_{\boldsymbol{V}}^2(\|\nabla\theta_k\|_{\boldsymbol{X}}^2 + \|\Delta\phi_k\|_X^2). \tag{3.33}$$

Substituting (3.31)-(3.33) in (3.30) and choosing $6\varepsilon = \kappa$, we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta_k - l_{\rm h}\nabla\phi_k\|_{\boldsymbol{X}}^2 + \frac{\kappa}{2}\|\Delta\theta_k\|_X^2 \le K_{1k}(\|\nabla\theta_k\|_{\boldsymbol{X}}^2 + \|\Delta\phi_k\|_X^2) + K_{2k}, \tag{3.34}$$

where $K_{1k} := c\|\boldsymbol{u}_k\|_{\boldsymbol{H}}^2\|\boldsymbol{u}_k\|_{\boldsymbol{V}}^2$ and $K_{2k} := c(\|\Delta\phi_k\|_X^2 + \|\boldsymbol{u}_k\|_{\boldsymbol{H}}^2 + \|z_k\|_X^2)$. Integrating (3.34) over $[0,t]$, and then using the triangle inequality to separate the term involving ϕ_k, we have

$$\|\nabla\theta_k(t)\|_{\boldsymbol{X}}^2 + \frac{\kappa}{2}\int_0^t \|\Delta\theta_k(\tau)\|_X^2 \,\mathrm{d}\tau \le c(\|\nabla\theta_{k0}\|_{\boldsymbol{X}}^2 + \|\nabla\phi_{k0}\|_{\boldsymbol{X}}^2 + \|\nabla\phi_k\|_{L^\infty(\boldsymbol{X})}^2) + c\int_0^t K_{1k}(\tau)(\|\nabla\theta_k(\tau)\|_{\boldsymbol{X}}^2 + \|\Delta\phi_k(\tau)\|_X^2) + K_{2k}(\tau)\,\mathrm{d}\tau.$$

Using Gronwall Lemma, $\|P_{X_k}\|_{\mathcal{L}(Y,Y)} \le 1$ for every k, and the estimate (3.29), we obtain

$$\|\nabla\theta_k\|_{L^\infty(\boldsymbol{X})} + \|\Delta\theta_k\|_{L^2(X)} \le \mathfrak{C}(\phi_0, \boldsymbol{u}_0, \theta_0, \boldsymbol{y}, z). \tag{3.35}$$

Step 2. *$L^\infty(Y^2)$ and $L^2(Y^4)$ estimates for ϕ_k.* For the inner product of the first equation in (3.11) with $\Delta^2\phi_k$ in X, one has

$$\frac{1}{2}\frac{d}{dt}\|\Delta\phi_k\|_X^2 - m(\Delta\mu_k, \Delta^2\phi_k)_X = -r(\boldsymbol{u}_k, \phi_k, \Delta^2\phi_k). \tag{3.36}$$

We use the Gagliardo-Nirenberg and Hölder inequalities to the right hand side so that

$$|r(\boldsymbol{u}_k, \phi_k, \Delta^2\phi_k)| \le \varepsilon\|\Delta^2\phi_k\|_X^2 + c_\varepsilon\|\boldsymbol{u}_k\|_{\boldsymbol{H}}^2\|\boldsymbol{u}_k\|_{\boldsymbol{V}}^2\|\nabla\phi_k\|_{\boldsymbol{X}}^2\|\Delta\phi_k\|_X^2. \tag{3.37}$$

Let us estimate from below the second term on the left hand side of (3.36). By the chain rule, $\Delta f(\phi_k) = f''(\phi_k)|\nabla\phi_k|^2 + f'(\phi_k)\Delta\phi_k = 6\phi_k|\nabla\phi_k|^2 + (3\phi_k^2 - 1)\Delta\phi_k$.

From the Sobolev embedding, Agmon inequality, (2.5), and $\|\Delta P_{X_k}\varphi\|_X \le \|\Delta\varphi\|_X$ for every $\varphi \in Y^2$, we get

$$\begin{aligned}\|\Delta P_{X_k} f(\phi_k)\|_X^2 &\le c(\|\phi_k\|_{L^6}^2\|\nabla\phi_k\|_{L^6}^4 + (\|\phi_k\|_{L^4}^4+1)\|\Delta\phi_k\|_{L^\infty}^2)\\ &\le \varepsilon\|\Delta^2\phi_k\|_X^2 + c_\varepsilon(\|\phi_k\|_Y^2\|\Delta\phi_k\|_X^2 + \|\phi_k\|_Y^8+1)\|\Delta\phi_k\|_X^2. \quad (3.38)\end{aligned}$$

Using (3.38), the Young inequality and the equation $\Delta\mu_k = P_{X_k}(-\alpha\Delta^2\phi_k + \Delta f(\phi_k) + l_c\Delta\theta_k)$, we obtain that

$$\begin{aligned}&-m(\Delta\mu_k, \Delta^2\phi_k)_X \ge \frac{m\alpha}{2}\|\Delta^2\phi_k\|_X^2 - c\|\Delta P_{X_k} f(\phi_k)\|_X^2 - c\|\Delta\theta_k\|_X^2 \quad (3.39)\\ &\ge \left(\frac{m\alpha}{2} - c\varepsilon\right)\|\Delta^2\phi_k\|_X^2 - c_\varepsilon((\|\phi_k\|_Y^2\|\Delta\phi_k\|_X^2 + \|\phi_k\|_Y^8+1)\|\Delta\phi_k\|_X^2 + \|\Delta\theta_k\|_X^2).\end{aligned}$$

Substituting the estimates (3.37) and (3.39) in the equation (3.36), and then taking $\varepsilon > 0$ such that $4(c+1)\varepsilon = m\alpha$, one obtains

$$\frac{1}{2}\frac{d}{dt}\|\Delta\phi_k\|_X^2 + \frac{m\alpha}{4}\|\Delta^2\phi_k\|_X^2 \le K_{3k}\|\Delta\phi_k\|_X^2 + c\|\Delta\theta_k\|_X^2, \quad (3.40)$$

where $K_{3k} := c(\|\boldsymbol{u}_k\|_{\boldsymbol{H}}^2\|\boldsymbol{u}_k\|_{\boldsymbol{V}}^2\|\nabla\phi_k\|_{\boldsymbol{X}}^2 + \|\phi_k\|_Y^2\|\Delta\phi_k\|_X^2 + \|\phi_k\|_Y^8 + 1)$. Integrate (3.40) in time and then use Gronwall lemma, (3.29), (3.35), and $\|\Delta\phi_{k0}\|_X \le \|\Delta\phi_0\|_X$ so that

$$\|\Delta\phi_k\|_{L^\infty(X)} + \|\Delta^2\phi_k\|_{L^2(X)} \le \mathfrak{C}(\phi_0, \boldsymbol{u}_0, \theta_0, \boldsymbol{y}, z). \quad (3.41)$$

Step 3. *$L^2(Y^2)$ estimate for μ_k.* From (3.35), (3.38) and (3.41), we immediately obtain

$$\begin{aligned}\|\Delta\mu_k\|_{L^2(X)} &\le c(\|\Delta^2\phi_k\|_{L^2(X)} + \|\Delta P_{X_k} f(\phi_k)\|_{L^2(X)} + \|\Delta\theta_k\|_{L^2(X)})\\ &\le \mathfrak{C}(\phi_0, \boldsymbol{u}_0, \theta_0, \boldsymbol{y}, z). \quad (3.42)\end{aligned}$$

Step 4. *$L^\infty(\boldsymbol{V})$ and $L^2(\boldsymbol{V}^2)$ estimates for $\boldsymbol{u}_k$.* By taking the inner product of the third equation of (3.11) with $-\boldsymbol{P_H}\Delta\boldsymbol{u}_k$ in $\boldsymbol{H}$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_k\|_{\boldsymbol{V}}^2 + \nu\|\Delta\boldsymbol{u}_k\|_{\boldsymbol{H}}^2 &= b(\boldsymbol{u}_k, \boldsymbol{u}_k, \boldsymbol{P_H}\Delta\boldsymbol{u}_k)\\ &\quad - (\mathcal{K}(\mu_k - l_c\theta_k)\nabla\phi_k + \ell(\phi_k,\theta_k)\boldsymbol{g} + \boldsymbol{y}, \boldsymbol{P_H}\Delta\boldsymbol{u}_k)_{\boldsymbol{X}}. \quad (3.43)\end{aligned}$$

Using the Cauchy-Schwarz inequality for the second term on the right hand side

$$\begin{aligned}|(\mathcal{K}(\mu_k - l_c\theta_k)\nabla\phi_k + \ell(\phi_k,\theta_k)\boldsymbol{g} + \boldsymbol{y}, \boldsymbol{P_H}\Delta\boldsymbol{u}_k)_{\boldsymbol{X}}| &\le \frac{\nu}{4}\|\Delta\boldsymbol{u}_k\|_{\boldsymbol{H}}^2\\ &+ c(1 + (\|\mu_k\|_Y^2 + \|\theta_k\|_Y^2)\|\Delta\phi_k\|_X^2 + \|\phi_k\|_X^2 + \|\theta_k\|_X^2 + \|\boldsymbol{y}\|_{\boldsymbol{X}}^2).\end{aligned}$$

Also, by the Gagliardo-Nirenberg inequality, we can estimate the trilinear term as

$$|b(\boldsymbol{u}_k, \boldsymbol{u}_k, \boldsymbol{P_H}\Delta\boldsymbol{u}_k)| \le \frac{\nu}{4}\|\Delta\boldsymbol{u}_k\|_{\boldsymbol{H}}^2 + c\|\boldsymbol{u}_k\|_{\boldsymbol{H}}^2\|\boldsymbol{u}_k\|_{\boldsymbol{V}}^4.$$

Substitution of the previous two inequalities to (3.43) leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_k\|_{\boldsymbol{V}}^2 + \frac{\nu}{2}\|\Delta\boldsymbol{u}_k\|_{\boldsymbol{H}}^2 &\le c\|\boldsymbol{u}_k\|_{\boldsymbol{H}}^2\|\boldsymbol{u}_k\|_{\boldsymbol{V}}^4\\ &+ c(1 + (\|\mu_k\|_Y^2 + \|\theta_k\|_Y^2)\|\Delta\phi_k\|_X^2 + \|\phi_k\|_X^2 + \|\theta_k\|_X^2 + \|\boldsymbol{y}\|_{\boldsymbol{X}}^2).\end{aligned}$$

By the Gronwall Lemma and the estimates (3.29), (3.41), and $\|\boldsymbol{u}_{k0}\|_{\boldsymbol{V}} \leq \|\boldsymbol{u}_0\|_{\boldsymbol{V}}$, we have

$$\|\boldsymbol{u}_k\|_{L^\infty(\boldsymbol{V})} + \|\Delta \boldsymbol{u}_k\|_{L^2(\boldsymbol{H})} \leq \mathfrak{C}(\phi_0, \boldsymbol{u}_0, \theta_0, \boldsymbol{y}, z). \tag{3.44}$$

Step 5. *Estimates on the time derivatives.* The differential equation for ϕ_k in (3.11) together with (3.29), (3.41), (3.42), and $\|P^*_{X_k}\|_{\mathcal{L}(X,X)} \leq 1$ for all k yields

$$\|\partial_t \phi_k\|_{L^2(X)} \leq c(\|\boldsymbol{u}_k\|_{L^2(\boldsymbol{V})}\|\Delta\phi_k\|_{L^\infty(X)} + \|\Delta\mu_k\|_{L^2(X)}) \leq \mathfrak{C}(\phi_0, \boldsymbol{u}_0, \theta_0, \boldsymbol{y}, z). \tag{3.45}$$

On a similar note, the differential equation for θ_k and the inequalities (3.29), (3.35), (3.44), and (3.45) imply that

$$\begin{aligned}\|\partial_t\theta_k\|_{L^2(X)} &\leq c(\|\partial_t\phi_k\|_{L^2(X)} + \|\boldsymbol{u}_k\|_{L^\infty(\boldsymbol{V})}(\|\Delta\theta_k\|_{L^2(X)} + \|\Delta\phi_k\|_{L^2(X)})) \\ &\quad + c(\|\boldsymbol{u}_k\|_{L^2(\boldsymbol{H})} + \|z_k\|_{L^2(X)}) \leq \mathfrak{C}(\phi_0, \boldsymbol{u}_0, \theta_0, \boldsymbol{y}, z). \end{aligned} \tag{3.46}$$

Taking the time derivative of both sides of the second equation in (3.8), we obtain that $\partial_t\mu_k = -\alpha\Delta\partial_t\phi_k + P_{X_k}(3\phi_k^2 - 1)\partial_t\phi_k + l_c\partial_t\theta_k$. Hence, by Agmon's inequality, (3.29), (3.45) and (3.46), we deduce that

$$\begin{aligned}\|\partial_t\mu_k\|_{L^2(Y^{2*})} &\leq c(1 + \|\phi_k\|^2_{L^\infty(Y^2)})\|\partial_t\phi_k\|_{L^2(X)} + \|\partial_t\theta_k\|_{L^2(X)} \\ &\leq \mathfrak{C}(\phi_0, \boldsymbol{u}_0, \theta_0, \boldsymbol{y}, z).\end{aligned} \tag{3.47}$$

Lastly, using (3.29), (3.41) and (3.44), the time derivative of $\boldsymbol{u}_k$ can be estimated from above by

$$\begin{aligned}\|\partial_t\boldsymbol{u}_k\|_{\boldsymbol{H}} &\leq c(1 + \|\boldsymbol{u}_k\|_{L^\infty(\boldsymbol{V})}\|\Delta\boldsymbol{u}_k\|_{L^2(\boldsymbol{H})} + (\|\mu_k\|_{L^2(Y)} + \|\theta_k\|_{L^2(Y)})\|\Delta\phi_k\|_{L^\infty(X)} \\ &\quad + \|\phi_k\|_{L^2(X)} + \|\theta_k\|_{L^2(X)} + \|\boldsymbol{y}\|_{L^2(\boldsymbol{X})}) \leq \mathfrak{C}(\phi_0, \boldsymbol{u}_0, \theta_0, \boldsymbol{y}, z).\end{aligned} \tag{3.48}$$

Getting the sum of (3.45)-(3.48), one has

$$\|\phi_k\|_{W^2(Y^4,X)} + \|\mu_k\|_{W^2(Y^2)} + \|\boldsymbol{u}_k\|_{W^2(\boldsymbol{V}^2,\boldsymbol{H})} + \|\theta_k\|_{W^2(Y^2,X)} \leq \mathfrak{C}(\phi_0, \boldsymbol{u}_0, \theta_0, \boldsymbol{y}, z).$$

This proves that, up to a subsequence, $(\phi_k, \mu_k, \boldsymbol{u}_k, \theta_k) \rightharpoonup (\phi, \mu, \boldsymbol{u}, \theta)$ in $\mathcal{V}$, and the estimate (3.28) holds without the pressure term. The existence of a unique pressure $p \in L^2(I; Y/\mathbb{R})$ satisfying (3.27) and the estimate $\|p\|_{L^2(I;Y/\mathbb{R})} \leq \mathfrak{C}(\phi_0, \boldsymbol{u}_0, \theta_0, \boldsymbol{y}, z)$ follows directly from the well-known de Rham's Theorem, see [54, Proposition I.2.3] for instance. The last statement of the theorem is a consequence of the continuous embedding (2.1), applied to the interpolation spaces $[Y^4, X]_{1/2} = Y^2$, $[Y^2, Y^{2*}]_{1/2} = X$, $[\boldsymbol{V}^2, \boldsymbol{H}]_{1/2} = \boldsymbol{V}$, and $[Y^2, X]_{1/2} = Y$. □

We close this section by establishing regularity theorems that deal with the time derivatives of the solution. We provide a proof based on the linearization of the system.

Theorem 3.5. *Assume that $\phi_0 \in Y^4$, $\boldsymbol{u}_0 \in \boldsymbol{V}^2$, and $\theta_0 \in Y^2$. Let $z \in W^2(I; X, Y^{2*})$ and $\boldsymbol{y} \in W^2(I; \boldsymbol{H}, \boldsymbol{V}^*)$ be such that $\boldsymbol{y}(0) \in \boldsymbol{H}$. Then we have*

$$(\partial_t\phi, \partial_t\mu, \partial_t\boldsymbol{u}, \partial_t\theta) \in W^2(I; Y^2) \times L^2(I; X) \times W^2(I; \boldsymbol{V}) \times W^2(I; X, Y^{2*}).$$

Proof. Due to the available time derivatives of $\boldsymbol{y}$ and z, it is permissible to take the time derivative of the finite-dimensional system satisfied by the Galerkin approximations $(\phi_k, \mu_k, \boldsymbol{u}_k, \theta_k)$. The derivatives then satisfy the linearized equation

$$\begin{aligned}\mathcal{A}_k(\phi_k, \mu_k, \boldsymbol{u}_k, \theta_k)&(\partial_t\phi_k, \partial_t\mu_k, \partial_t\boldsymbol{u}_k, \partial_t\theta_k)\\ &= (0, 0, \boldsymbol{P}^*_{\boldsymbol{H}_k}\partial_t\boldsymbol{y}, P^*_{X_k}\partial_t z, \partial_t\phi_k(0), \partial_t\boldsymbol{u}_k(0), \partial_t\theta_k(0)),\end{aligned}$$

where $\mathcal{A}_k$ is the Galerkin approximation of the linear operator $\mathcal{A}$ defined by (4.1) below. More precisely, $(\partial_t\phi_k, \partial_t\mu_k, \partial_t\boldsymbol{u}_k, \partial_t\theta_k)$ satisfies (4.5), with $(\phi, \mu, \boldsymbol{u}, \theta)$ being replaced by $(\phi_k, \mu_k, \boldsymbol{u}_k, \theta_k)$. Thus, to establish the theorem, it suffices to verify that $(\partial_t\phi_k(0), \partial_t\boldsymbol{u}_k(0), \partial_t\theta_k(0))$ is uniformly bounded in $X \times \boldsymbol{H} \times Y^*$ according to Theorem 4.2 below.

Evaluating the first equation in (3.11) at $t = 0$, and using the Hölder and Agmon inequalities

$$\|\partial_t\phi_k(0)\|_X \le \|\boldsymbol{u}_{k0}\cdot\nabla\phi_{k0}\|_X + m\|\Delta\mu_k(0)\|_X \le c(\|\boldsymbol{u}_0\|_{\boldsymbol{V}}\|\phi_0\|_{Y^2} + \|\Delta\mu_k(0)\|_X).$$

From the second equation in (3.11), the approximate initial chemical potential satisfies the following inequalities

$$\begin{aligned}\|\mu_k(0)\|_X &\le c(\|\Delta\phi_{k0}\|_X + \|\phi_{k0}^3 - \phi_{k0}\|_X + \|\theta_{k0}\|_X)\\ &\le c(\|\phi_0\|_{Y^2} + \|\phi_0\|_Y^3 + \|\phi_0\|_X + \|\theta_0\|_X)\\ \|\Delta\mu_k(0)\|_X &\le c(\|\Delta^2\phi_{k0}\|_X + \|6\phi_{k0}|\nabla\phi_{k0}|^2 + (3\phi_{k0}^2 - 1)\Delta\phi_{k0}\|_X + \|\Delta\theta_{k0}\|_X)\\ &\le c(\|\phi_0\|_{Y^4} + \|\phi_0\|_{Y^2}^3 + \|\phi_0\|_{Y^2} + \|\theta_0\|_{Y^2}).\end{aligned}$$

Evaluating the third equation in (3.11) at $t = 0$, we obtain the following bound

$$\begin{aligned}\|\partial_t\boldsymbol{u}_k(0)\|_{\boldsymbol{H}} \le c(1 &+ \|\boldsymbol{u}_0\|_{\boldsymbol{V}}\|\boldsymbol{u}_0\|_{\boldsymbol{V}^2} + \|\boldsymbol{u}_0\|_{\boldsymbol{V}^2}\\ &+ (\|\mu_k(0)\|_{Y^2} + \|\theta_0\|_{Y^2})\|\phi_0\|_Y + \|\phi_0\|_X + \|\theta_0\|_X + \|\boldsymbol{y}(0)\|_{\boldsymbol{H}}).\end{aligned}$$

Finally, using the approximate convection-diffusion equation in (3.11), one has

$$\begin{aligned}\|\partial_t\theta_k(0)\|_{Y^*} &\le c(\|\partial_t\phi_k(0)\|_{Y^*} + \|\boldsymbol{u}_0\|_{\boldsymbol{V}}(\|\theta_0\|_Y + \|\phi_0\|_Y))\\ &\quad + c(\|\theta_0\|_Y + \|\boldsymbol{u}_0\|_{\boldsymbol{H}} + \|z(0)\|_{Y^*}).\end{aligned}$$

The last term of this inequality is valid due to the continuity of the embedding $W^2(I; X, Y^{2*}) \subset C(\bar{I}; Y^*)$. For the first term, note that $\|\partial_t\phi_k(0)\|_{Y^*} \le c\|\partial_t\phi_k(0)\|_X$. From these estimates, we deduce that indeed $(\partial_t\phi_k(0), \partial_t\boldsymbol{u}_k(0), \partial_t\theta_k(0))$ is bounded in $X \times \boldsymbol{H} \times Y^*$. □

Theorem 3.6. *Suppose that $\phi_0 \in Y^5$, $\boldsymbol{u}_0 \in \boldsymbol{V}^2$, $\theta_0 \in Y^3$, $\boldsymbol{y} \in W^2(I; \boldsymbol{H}, \boldsymbol{V}^*)$, and $z \in W^2(I; X, Y^*)$ where $\boldsymbol{y}(0) \in \boldsymbol{H}$ and $z(0) \in X$. Then it holds that $(\partial_t\phi, \partial_t\mu, \partial_t\boldsymbol{u}, \partial_t\theta) \in \mathcal{W}$.*

Proof. The proof is similar to the one provided in the previous theorem, but now in this case, one utilizes Theorem 4.1 rather than Theorem 4.2. Here, we note that the Hilbert space Y^5 is endowed with the norm $\|\phi_0\|_{Y^5} = \|\phi_0\|_X + \|\nabla\Delta^2\phi_0\|_{\boldsymbol{X}}$. □

4. Linearized System and Differentiability of the Solution Operator

The goal of this section is to study the linearization of (3.8) at a fixed element $(\phi, \mu, \boldsymbol{w}, \theta) \in \mathcal{W}$. The corresponding solution operator of this linearization determines the directional derivative of the so-called control-to-state map.

4.1. Linearized state equation.

First, let us discuss the existence, uniqueness and stability of solutions to the linearized system. For this purpose, we introduce the following predual space for the source terms

$$\mathcal{Q} := L^2(I; Y) \times L^2(I; Y^*) \times L^2(I; \boldsymbol{V}) \times L^2(I; Y).$$

Consider the nonlinear operator

$$\mathcal{A} : \mathcal{W} \to \mathcal{L}(\mathcal{W}, \mathcal{Q}^* \times Y \times \boldsymbol{H} \times X) \tag{4.1}$$

defined by $\mathcal{A} = (A, A_0)$, where the component $A : \mathcal{W} \to \mathcal{L}(\mathcal{W}, \mathcal{Q}^*)$ is given by

$$A(\phi, \mu, \boldsymbol{u}, \theta)(\psi, \xi, \boldsymbol{w}, \zeta) := \tag{4.2}$$
$$\begin{bmatrix} \partial_t \psi + B_1(\boldsymbol{u}, \psi) + B_1(\boldsymbol{w}, \phi) + m A_N \xi \\ \xi - \alpha A_N \psi - f'(\phi)\psi - l_c \zeta \\ \partial_t \boldsymbol{w} + \mathrm{D}\boldsymbol{B}(\boldsymbol{u})\boldsymbol{w} + \nu \boldsymbol{A}_S \boldsymbol{w} - \mathcal{K}(\boldsymbol{B}_2(\xi - l_c\zeta, \phi) - \boldsymbol{B}_2(\mu - l_c\theta, \psi)) - (\alpha_2 \psi + \alpha_3 \zeta)\boldsymbol{g} \\ \partial_t \zeta - l_h \partial_t \psi + B_1(\boldsymbol{u}, \zeta - l_h \psi) + B_1(\boldsymbol{w}, \theta - l_h \phi) + \kappa A_N \zeta - \boldsymbol{g} \cdot \boldsymbol{w} \end{bmatrix}$$

while the component

$$A_0 : \mathcal{W} \to \mathcal{L}(\mathcal{W}, Y \times \boldsymbol{H} \times X)$$

is defined by $A_0(\phi, \mu, \boldsymbol{u}, \theta)(\psi, \xi, \boldsymbol{w}, \zeta) := (\psi(0), \boldsymbol{w}(0), \zeta(0))$. Here, $\mathrm{D}\boldsymbol{B}(\boldsymbol{u})\boldsymbol{w} = \boldsymbol{B}(\boldsymbol{u}, \boldsymbol{w}) + \boldsymbol{B}(\boldsymbol{w}, \boldsymbol{u})$ is the Frechét derivative of $\boldsymbol{B}$ at $\boldsymbol{u}$ in the direction $\boldsymbol{w}$, see Lemma 4.5 below. It is easy to see that $\mathcal{A}$ is well-defined.

Theorem 4.1. *Given $(\phi, \mu, \boldsymbol{u}, \theta) \in \mathcal{W}$, $(s, \sigma, \boldsymbol{y}, z) \in \mathcal{Q}^*$, and $(\phi_0, \boldsymbol{w}_0, \zeta_0) \in Y \times \boldsymbol{H} \times X$, there exists a unique $(\psi, \xi, \boldsymbol{w}, \zeta) \in \mathcal{W}$ such that*

$$\mathcal{A}(\phi, \mu, \boldsymbol{u}, \theta)(\psi, \xi, \boldsymbol{w}, \zeta) = (s, \sigma, \boldsymbol{y}, z, \phi_0, \boldsymbol{w}_0, \zeta_0). \tag{4.3}$$

Furthermore, there is a continuous function $\mathfrak{C} > 0$, independent of $(\psi, \xi, \boldsymbol{w}, \zeta)$, such that

$$\|(\psi, \xi, \boldsymbol{w}, \zeta)\|_{\mathcal{W}} \le \mathfrak{C}(\|(\phi, \mu, \boldsymbol{u}, \theta)\|_{\mathcal{W}}) \|(s, \sigma, \boldsymbol{y}, z, \phi_0, \boldsymbol{w}_0, \zeta_0)\|_{\mathcal{Q}^* \times Y \times \boldsymbol{H} \times X}. \tag{4.4}$$

Proof. The proof of existence is again based on the Galerkin method. Suppose that $(\psi_k, \xi_k, \boldsymbol{w}_k, \zeta_k) \in \boldsymbol{H}^1(I; X_k \times X_k \times \boldsymbol{H}_k \times X_k)$, with components having similar representations as in (3.10), is the solution of the following finite-dimensional linear system

$$\begin{cases} \partial_t\psi_k + P^*_{X_k}(B_1(\boldsymbol{u},\psi_k) + B_1(\boldsymbol{w}_k,\phi) + mA_N\xi_k) = P^*_{X_k}s & \text{in } L^2(I;X_k),\\ \xi_k = \alpha A_N\psi_k + P_{X_k}(f'(\phi)\psi_k) + l_c\zeta_k + P_{X_k}\sigma & \text{in } H^1(I;X_k),\\ \partial_t\boldsymbol{w}_k + \boldsymbol{P}^*_{\boldsymbol{H}_k}(\mathrm{D}\boldsymbol{B}(\boldsymbol{u})\boldsymbol{w}_k + \nu\boldsymbol{A}_S\boldsymbol{w}_k) & \\ \qquad = \boldsymbol{P}^*_{\boldsymbol{H}_k}(\mathcal{K}\boldsymbol{B}_2(\xi_k - l_c\zeta_k,\phi) + \mathcal{K}\boldsymbol{B}_2(\mu - l_c\theta,\psi_k)) & \\ \qquad + \boldsymbol{P}^*_{\boldsymbol{H}_k}((\alpha_2\psi_k + \alpha_3\zeta_k)\boldsymbol{g} + \boldsymbol{y}) & \text{in } L^2(I;\boldsymbol{H}_k),\\ \partial_t\zeta_k - l_h\partial_t\psi_k + P^*_{X_k}(B_1(\boldsymbol{u},\zeta_k - l_h\psi_k) + B_1(\boldsymbol{w}_k,\theta - l_h\phi)) & \\ \qquad + P^*_{X_k}(\kappa A_N\zeta_k - \boldsymbol{g}\cdot\boldsymbol{w}_k - z) = 0 & \text{in } L^2(I;X_k), \end{cases} \tag{4.5}$$

with initial conditions $\psi_k(0) = P_{X_k}\psi_0$, $\boldsymbol{w}_k(0) = \boldsymbol{P}_{\boldsymbol{H}_k}\boldsymbol{w}_0$, and $\zeta_k(0) = P_{X_k}\zeta_0$. In what follows, $\mathfrak{C}$ will denote a generic positive continuous function.

STEP 1. *Estimate for ψ_k.* Take the test function ψ_k in the first equation of (4.5) so that

$$\frac{1}{2}\frac{d}{dt}\|\psi_k\|_X^2 - m(\xi_k,\Delta\psi_k)_X = -r(\boldsymbol{w}_k,\phi,\psi_k) + \langle s,\psi_k\rangle_{Y^*\times Y}. \tag{4.6}$$

Using (2.3) and the Agmon inequality, the terms on the right hand side can be estimated from above according to

$$|\langle s,\psi_k\rangle_{Y^*\times Y}| \le \varepsilon\|\Delta\psi_k\|_X^2 + c_\varepsilon(\|s\|_{Y^*}^2 + \|\psi_k\|_X^2) \tag{4.7}$$

$$|r(\boldsymbol{w}_k,\phi,\psi_k)| \le \varepsilon\|\Delta\psi_k\|_X^2 + c_\varepsilon(\|\nabla\phi\|_{\boldsymbol{X}}^2\|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2 + \|\psi_k\|_X^2). \tag{4.8}$$

For the term on the left hand side of (4.6), let us first estimate the L^2-norm of $P_{X_k}(f'(\phi)\psi_k) = P_{X_k}((3\phi^2-1)\psi_k)$ using the Gagliardo-Nirenberg inequality (2.6) by

$$\begin{aligned} \|P_{X_k}(f'(\phi)\psi_k)\|_X^2 &\le c(\|\phi\|_{L^8}^4\|\psi_k\|_X\|\psi_k\|_Y + \|\psi_k\|_X^2)\\ &\le \varepsilon\|\Delta\psi_k\|_X^2 + c_\varepsilon(\|\phi\|_Y^8 + 1)\|\psi_k\|_X^2. \end{aligned} \tag{4.9}$$

Therefore, from $\xi_k = -\alpha\Delta\psi_k + P_{X_k}(f'(\phi)\psi_k) + l_c\zeta_k + P_{X_k}\sigma$, we obtain for $\varepsilon = 1$ that

$$\|\xi_k\|_X^2 \le c_\alpha\|\Delta\psi_k\|_X^2 + K_0\|\psi_k\|_X^2 + c(\|\zeta_k - l_h\psi_k\|_X^2 + \|\sigma\|_X^2), \tag{4.10}$$

where $K_0 := c(1 + \|\phi\|_Y^8)$. Moreover, the second term on the left hand side of (4.6) can be estimated from below by

$$\begin{aligned} -m(\xi_k,\Delta\psi_k)_X &\ge \frac{m\alpha}{2}\|\Delta\psi_k\|_X^2 - c(\|P_{X_k}(f'(\phi)\psi_k)\|_X^2 + \|\zeta_k\|_X^2 + \|\sigma\|_X^2)\\ &\ge \left(\frac{m\alpha}{2} - c\varepsilon\right)\|\Delta\psi_k\|_X^2 - c_\varepsilon K_0\|\psi_k\|_X^2 - c(\|\zeta_k\|^2 + \|\sigma\|_X^2). \end{aligned} \tag{4.11}$$

One can now apply the estimates (4.7), (4.8), and (4.11) in (4.6) so that for $4(c+2)\varepsilon = m\alpha$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\psi_k\|_X^2 + \frac{m\alpha}{4}\|\Delta\psi_k\|_X^2\\ &\qquad \le K_1(\|\psi_k\|_X^2 + \|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2) + c(\|\zeta_k - l_h\psi_k\|_X^2 + \|s\|_{Y^*}^2 + \|\sigma\|_X^2), \end{aligned} \tag{4.12}$$

where $K_1 := c(K_0 + \|\nabla\phi\|_{\boldsymbol{X}}^2)$. Here, we used $\|\zeta_k\|_X \le \|\zeta_k - l_h\psi_k\|_X + l_h\|\psi_k\|_X$.

STEP 2. *Estimate for ζ_k.* Testing by $\zeta_k - l_h\psi_k$ the fourth equation in (4.5) leads to

$$\frac{1}{2}\frac{d}{dt}\|\zeta_k - l_h\psi_k\|_X^2 + \kappa\|\nabla\zeta_k\|_{\boldsymbol{X}}^2 = -\, l_h\kappa(\zeta_k,\Delta\psi_k)_X - r(\boldsymbol{w}_k,\theta - l_h\phi,\zeta_k)$$

$$+ l_{\rm h} r(\boldsymbol{w}_k, \theta - l_{\rm h}\phi, \psi_k) + (\boldsymbol{g}\cdot\boldsymbol{w}_k, \zeta_k - l_{\rm h}\psi_k)_X + \langle z, \zeta_k - l_{\rm h}\psi_k\rangle_{Y^*\times Y}. \quad (4.13)$$

Using the Cauchy-Schwarz and Gagliardo-Nirenberg inequalities to the first, fourth and fifth terms on the right hand side of the previous equation, one has

$$|l_{\rm h}\kappa(\zeta_k, \Delta\psi_k)_X| \le \varepsilon\|\Delta\psi_k\|_X^2 + c_\varepsilon\|\zeta_k\|_X^2 \quad (4.14)$$

$$|(\boldsymbol{g}\cdot\boldsymbol{w}_k, \zeta_k - l_{\rm h}\psi_k)_X| \le c(\|\psi_k\|_X^2 + \|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2 + \|\zeta_k\|_X^2) \quad (4.15)$$

$$|\langle z, \zeta_k - l_{\rm h}\psi_k\rangle_{Y^*\times Y}| \le \frac{\kappa}{4}\|\nabla\zeta_k\|_{\boldsymbol{X}}^2 + \varepsilon\|\Delta\psi_k\|_X^2 + c_\varepsilon(\|\psi_k\|_X^2 + \|z\|_{Y^*}^2). \quad (4.16)$$

Similarly, for the trilinear terms on the right hand side of (4.13), we obtain

$$|l_{\rm h} r(\boldsymbol{w}_k, \theta - l_{\rm h}\phi, \psi_k)|$$
$$\le \varepsilon\|\Delta\psi_k\|_X^2 + c_\varepsilon((\|\nabla\theta\|_{\boldsymbol{X}}^2 + \|\nabla\phi\|_{\boldsymbol{X}}^2)\|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2 + \|\psi_k\|_X^2) \quad (4.17)$$

$$|r(\boldsymbol{w}_k, \theta - l_{\rm h}\phi, \zeta_k)|$$
$$\le \frac{\kappa}{4}\|\nabla\zeta_k\|_{\boldsymbol{X}}^2 + \frac{\nu}{4}\|\boldsymbol{w}_k\|_{\boldsymbol{V}}^2 + c(\|\nabla\theta\|_{\boldsymbol{X}}^2 + \|\nabla\phi\|_{\boldsymbol{X}}^2)(\|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2 + \|\zeta_k\|_X^2). \quad (4.18)$$

Let $K_2 := c(1 + \|\nabla\theta\|_{\boldsymbol{X}}^2 + \|\nabla\phi\|_{\boldsymbol{X}}^2)$. By applying (4.14)-(4.18) in (4.13), taking $24\varepsilon = m\alpha$, and using the inequality $\|\zeta_k\|_X \le \|\zeta_k - l_{\rm h}\psi_k\|_X + l_{\rm h}\|\psi_k\|_X$ once more, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\zeta_k - l_{\rm h}\psi_k\|_X^2 + \frac{\kappa}{2}\|\nabla\zeta_k\|_{\boldsymbol{X}}^2 - \frac{\nu}{4}\|\boldsymbol{w}_k\|_{\boldsymbol{V}}^2 - \frac{m\alpha}{8}\|\Delta\psi_k\|_X^2$$
$$\le K_2(\|\psi_k\|_X^2 + \|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2 + \|\zeta_k - l_{\rm h}\psi_k\|_X^2) + c\|z\|_{Y^*}^2. \quad (4.19)$$

Step 3. *Estimate for* $\boldsymbol{w}_k$. We test the third equation in (4.5) by $\boldsymbol{w}_k$ so that

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2 + \nu\|\boldsymbol{w}_k\|_{\boldsymbol{V}}^2 = b(\boldsymbol{w}_k, \boldsymbol{w}_k, \boldsymbol{u}) + \mathcal{K}r(\boldsymbol{w}_k, \phi, \xi_k - l_{\rm c}\zeta_k)$$
$$+ \mathcal{K}(\boldsymbol{w}_k, \psi_k, \mu - l_{\rm c}\theta) + ((\alpha_2\psi_k + \alpha_3\zeta_k)\boldsymbol{g}, \boldsymbol{w}_k)_{\boldsymbol{X}} + \langle \boldsymbol{y}, \boldsymbol{w}_k\rangle_{\boldsymbol{V}^*\times\boldsymbol{V}}. \quad (4.20)$$

We apply the Cauchy-Schwarz inequality to the last two terms on the right hand side

$$|\langle \boldsymbol{y}, \boldsymbol{w}_k\rangle_{\boldsymbol{V}^*\times\boldsymbol{V}}| \le \frac{\nu}{6}\|\boldsymbol{w}_k\|_{\boldsymbol{V}}^2 + c\|\boldsymbol{y}\|_{\boldsymbol{V}^*}^2 \quad (4.21)$$

$$|((\alpha_2\psi_k + \alpha_3\zeta_k)\boldsymbol{g}, \boldsymbol{w}_k)_{\boldsymbol{X}}| \le c(\|\psi_k\|_X^2 + \|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2 + \|\zeta_k\|_X^2). \quad (4.22)$$

To deal with the trilinear terms, we again utilize the Gagliardo-Nirenberg, Holdër and Young inequalities to obtain the following estimates

$$|b(\boldsymbol{w}_k, \boldsymbol{w}_k, \boldsymbol{u})| \le \frac{\nu}{6}\|\boldsymbol{w}_k\|_{\boldsymbol{V}}^2 + c\|\boldsymbol{u}\|_{\boldsymbol{H}}^2\|\boldsymbol{u}\|_{\boldsymbol{V}}^2\|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2 \quad (4.23)$$

$$|\mathcal{K}r(\boldsymbol{w}_k, \psi_k, \mu - l_{\rm c}\theta)| \le \varepsilon\|\Delta\psi_k\|_X^2 + c_\varepsilon(\|\mu\|_Y^2 + \|\theta\|_Y^2)\|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2 \quad (4.24)$$

$$|\mathcal{K}r(\boldsymbol{w}_k, \phi, \xi_k - l_{\rm c}\zeta_k)| \le \frac{\nu}{6}\|\boldsymbol{w}_k\|_{\boldsymbol{V}}^2 + \varepsilon\|\xi_k\|_X^2 + \varepsilon\|\zeta_k\|_X^2$$
$$+ c_\varepsilon\|\nabla\phi\|_{\boldsymbol{X}}^2\|\Delta\phi\|_X^2\|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2. \quad (4.25)$$

Substituting the estimates (4.21)-(4.25) in equation (4.20) yields

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2 + \frac{\nu}{2}\|\boldsymbol{w}_k\|_{\boldsymbol{V}}^2 - \varepsilon\|\Delta\psi_k\|_X^2 - \varepsilon\|\xi_k\|_X^2$$
$$\le K_{3\varepsilon}(\|\psi_k\|_X^2 + \|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2 + \|\zeta_k - l_{\rm h}\psi_k\|_X^2) + c(\|\zeta_k - l_{\rm h}\psi_k\|_X^2 + \|\boldsymbol{y}\|_{\boldsymbol{V}^*}^2) \quad (4.26)$$

where $K_{3\varepsilon} := c_\varepsilon \mathfrak{C}(\|\phi\|_{L^\infty(Y)}, \|\boldsymbol{u}\|_{L^\infty(\boldsymbol{H})})(1 + \|\Delta\phi\|_X^2 + \|\mu\|_Y^2 + \|\theta\|_Y^2 + \|\boldsymbol{u}\|_{\boldsymbol{V}}^2)$.

Step 4. *Estimate for* $\nabla\psi_k$. With the test function $-\Delta\psi_k$ in the first equation of (4.5), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\psi_k\|_{\boldsymbol{X}}^2 - m(\nabla\xi_k, \nabla\Delta\psi_k)_{\boldsymbol{X}} = -r(\boldsymbol{u}, \Delta\psi_k, \psi_k) - r(\boldsymbol{w}_k, \Delta\psi_k, \phi) - \langle s, \Delta\psi_k\rangle_{Y^*\times Y}. \tag{4.27}$$

Applying the Hölder, Gagliardo-Nirenberg and Young inequalities, the terms on the right hand side satisfy the following inequalities

$$|\langle s, \Delta\psi_k\rangle_{Y^*\times Y}| \le \varepsilon\|\nabla\Delta\psi_k\|_{\boldsymbol{X}}^2 + c_\varepsilon\|s\|_{Y^*}^2 \tag{4.28}$$

$$|r(\boldsymbol{u}, \Delta\psi_k, \psi_k)| \le \varepsilon\|\nabla\Delta\psi_k\|_{\boldsymbol{X}}^2 + c_\varepsilon(\|\boldsymbol{u}\|_{\boldsymbol{H}}^2\|\boldsymbol{u}\|_{\boldsymbol{V}}^2\|\nabla\psi_k\|_{\boldsymbol{X}}^2 + \|\psi_k\|_X^2) \tag{4.29}$$

$$|r(\boldsymbol{w}_k, \Delta\psi_k, \phi)| \le \varepsilon\|\nabla\Delta\psi_k\|_{\boldsymbol{X}}^2 + \|\boldsymbol{w}_k\|_{\boldsymbol{V}}^2 + c_\varepsilon\|\phi\|_X^2\|\nabla\phi\|_{\boldsymbol{X}}^2\|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2. \tag{4.30}$$

By the chain rule $\nabla(f'(\phi)\psi_k) = 6\phi\psi_k\nabla\phi + (3\phi^2 - 1)\nabla\psi_k$, and estimating the L^2-norm yields

$$\begin{aligned}\|\nabla P_{X_k}(f'(\phi)\psi_k)\|_{\boldsymbol{X}}^2 &\le c(\|\phi\|_{L^6}^2\|\psi_k\|_{L^6}^2\|\nabla\phi\|_{L^6}^2 + (\|\phi\|_{L^8}^4 + 1)\|\nabla\psi_k\|_{\boldsymbol{X}}\|\nabla\Delta\psi_k\|_{\boldsymbol{X}}) \\ &\le \varepsilon\|\nabla\Delta\psi_k\|_{\boldsymbol{X}}^2 + c_\varepsilon(\|\phi\|_Y^2\|\Delta\phi\|_X^2 + \|\phi\|_Y^8 + 1)(\|\nabla\psi_k\|_{\boldsymbol{X}}^2 + \|\psi_k\|_X^2).\end{aligned}$$

Let $K_4 := c(1 + \|\phi\|_Y^2\|\Delta\phi\|_X^2 + \|\phi\|_Y^8)$. In particular, for $\varepsilon = 1$ one obtains that

$$\|\nabla\xi_k\|_{\boldsymbol{X}}^2 \le c_\alpha\|\nabla\Delta\psi_k\|_{\boldsymbol{X}}^2 + K_4(\|\nabla\psi_k\|_{\boldsymbol{X}}^2 + \|\psi_k\|_X^2) + c(\|\nabla\zeta_k\|_{\boldsymbol{X}}^2 + \|\nabla\sigma\|_{\boldsymbol{X}}^2). \tag{4.31}$$

Likewise, we have the following estimate for the term on the left hand side of (4.27)

$$\begin{aligned}&- m(\nabla\xi_k, \nabla\Delta\psi_k)_{\boldsymbol{X}} \ge \frac{m\alpha}{2}\|\nabla\Delta\psi_k\|_{\boldsymbol{X}}^2 - c(\|\nabla P_{X_k}(f'(\phi)\psi_k)\|_{\boldsymbol{X}}^2 + \|\nabla\zeta_k\|_{\boldsymbol{X}}^2 + \|\nabla\sigma\|_{\boldsymbol{X}}^2) \\ &\ge \left(\frac{m\alpha}{2} - c\varepsilon\right)\|\nabla\Delta\psi_k\|_{\boldsymbol{X}}^2 - c_\varepsilon K_4(\|\nabla\psi_k\|_{\boldsymbol{X}}^2 + \|\psi_k\|_X^2) - c(\|\nabla\zeta_k\|_{\boldsymbol{X}}^2 + \|\nabla\sigma\|_{\boldsymbol{X}}^2).\end{aligned} \tag{4.32}$$

Take $\varepsilon > 0$ such that $4(c+3)\varepsilon = m\alpha$. Plugging the estimates (4.28), (4.29), (4.30), and (4.32) in (4.27), we deduce by putting $K_5 := c(1 + K_4 + \|\boldsymbol{u}\|_{\boldsymbol{H}}^2\|\boldsymbol{u}\|_{\boldsymbol{V}}^2 + \|\phi\|_X^2\|\nabla\phi\|_{\boldsymbol{X}}^2)$ that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\psi_k\|_{\boldsymbol{X}}^2 + \frac{m\alpha}{4}\|\nabla\Delta\psi_k\|_{\boldsymbol{X}}^2 - c(\|\boldsymbol{w}_k\|_{\boldsymbol{V}}^2 + \|\nabla\zeta_k\|_{\boldsymbol{X}}^2) \\ &\qquad \le K_5(\|\nabla\psi_k\|_{\boldsymbol{X}}^2 + \|\psi_k\|_X^2 + \|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2) + c(\|s\|_{Y^*}^2 + \|\nabla\sigma\|_{\boldsymbol{X}}^2).\end{aligned} \tag{4.33}$$

Step 5. *Energy-type estimate.* Multiplying (4.10) and (4.33) by $\tilde\varepsilon > 0$, (4.31) by $\tilde\varepsilon^2$ and then taking the sum of the resulting inequalities with (4.12), (4.19), and (4.26) we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}E_{k,\tilde\varepsilon} + \left(\frac{m\alpha}{8} - c_\alpha\tilde\varepsilon - \varepsilon\right)\|\Delta\psi_k\|_X^2 + \left(\frac{m\alpha}{4} - c_\alpha\tilde\varepsilon\right)\tilde\varepsilon\|\nabla\Delta\psi_k\|_{\boldsymbol{X}}^2 + (\tilde\varepsilon - \varepsilon)\|\xi_k\|_X^2 \\ &\quad + \tilde\varepsilon^2\|\nabla\xi_k\|_{\boldsymbol{X}}^2 + \left(\frac{\nu}{4} - c\tilde\varepsilon\right)\|\boldsymbol{w}_k\|_{\boldsymbol{V}}^2 + \left(\frac{\kappa}{2} - c\tilde\varepsilon - c\tilde\varepsilon^2\right)\|\nabla\zeta_k\|_{\boldsymbol{X}}^2 \\ &\quad \le c_{\tilde\varepsilon}K_{\varepsilon,\tilde\varepsilon}E_{k\tilde\varepsilon} + c_{\varepsilon,\tilde\varepsilon}(\|s\|_{Y^*}^2 + \|\sigma\|_Y^2 + \|\boldsymbol{y}\|_{\boldsymbol{V}^*}^2 + \|z\|_{Y^*}^2),\end{aligned}$$

where $E_{k,\tilde\varepsilon} := \|\psi_k\|_X^2 + \tilde\varepsilon\|\nabla\psi_k\|_{\boldsymbol{X}}^2 + \|\zeta_k - l_{\mathrm{h}}\psi_k\|_X^2 + \|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2$ and $K_{\varepsilon,\tilde\varepsilon} = \tilde\varepsilon K_0 + \tilde\varepsilon K_5 + \tilde\varepsilon^2 K_4 + K_1 + K_2 + K_{3\varepsilon}$.

By a straightforward calculation, $\|K_{\varepsilon,\tilde{\varepsilon}}\|_{L^1(I)} \leq c_{\varepsilon,\tilde{\varepsilon}}\mathfrak{C}(\|(\phi,\mu,\boldsymbol{u},\theta)\|_{\mathcal{W}})$. Choose $\tilde{\varepsilon} > \varepsilon > 0$ small enough so that the coefficients on the left hand side are positive. Thus, by applying the Gronwall Lemma, we deduce that $\{\psi_k\}_{k=1}^\infty$, $\{\xi_k\}_{k=1}^\infty$, $\{\boldsymbol{w}_k\}_{k=1}^\infty$, and $\{\zeta_k\}_{k=1}^\infty$ are bounded in $L^\infty(I;Y)\cap L^2(I;Y^3)$, $L^2(I;Y)$, $L^\infty(I;\boldsymbol{H})\cap L^2(I;\boldsymbol{V})$, and $L^\infty(I;X)\cap L^2(I;Y)$, respectively. In fact, we have

$$\begin{aligned}\|\psi_k\|_{L^\infty(Y)\cap L^2(Y^3)} + \|\xi_k\|_{L^2(Y)} + \|\boldsymbol{w}_k\|_{L^\infty(\boldsymbol{H})\cap L^2(\boldsymbol{V})} + \|\zeta_k - l_{\mathrm{h}}\psi_k\|_{L^\infty(X)} \\ + \|\nabla\zeta_k\|_{L^2(\boldsymbol{X})} \leq \mathfrak{C}(\|(\phi,\mu,\boldsymbol{u},\theta)\|_{\mathcal{W}})\|(s,\sigma,\boldsymbol{y},z,\phi_0,\boldsymbol{w}_0,\zeta_0)\|_{\mathcal{Q}^*\times Y\times\boldsymbol{H}\times X}\end{aligned} \tag{4.34}$$

and utilize $\|\zeta_k\|_{L^\infty(X)} \leq \|\zeta_k - l_{\mathrm{h}}\psi_k\|_{L^\infty(X)} + l_{\mathrm{h}}\|\psi_k\|_{L^\infty(X)}$ in order to bound $\|\zeta_k\|_{L^\infty(X)}$ by the right hand side of (4.34).

Following Step 4 in the proof of Theorem 3.2, and applying (3.2)-(3.6) together with the a priori estimate (4.34), the norms $\|\partial_t\psi_k\|_{L^2(Y^*)}$, $\|\partial_t\boldsymbol{w}_k\|_{L^2(\boldsymbol{V}^*)}$, and $\|\partial_t\zeta_k\|_{L^2(Y^*)}$ can be estimated from above by the right hand side of (4.34). This implies that (4.4) holds with $(\psi,\xi,\boldsymbol{w},\zeta)$ replaced by $(\psi_k,\xi_k,\boldsymbol{w}_k,\zeta_k)$. Thus, $\{(\psi_k,\xi_k,\boldsymbol{w}_k,\zeta_k)\}_{k=1}^\infty$ is bounded in $\mathcal{W}$. Extraction of a suitable subsequence leads to a solution of (4.3) satisfying (4.4). Finally, the uniqueness of solution follows from standard arguments, the a priori estimate (4.4), and the linearity of the system (4.3). □

For coefficients $(\phi,\mu,\boldsymbol{u},\theta)\in\mathcal{V}$, one can allow less regular source terms and initial data in the linearized system (4.3). For this, one has to consider *very weak solutions* to the linearized Cahn-Hilliard and convection-diffusion equations. By duality principles, this will lead to more regular solutions to the adjoint system. In this direction, we introduce the following weaker solution space and the predual space of less regular source terms

$$\begin{aligned}\mathcal{U} &:= W^2(I;Y^2)\times L^2(I;X)\times W^2(I;\boldsymbol{V})\times W^2(I;X,Y^{2*})\\ \mathcal{Y} &:= L^2(I;Y^2)\times L^2(I;X)\times L^2(I;\boldsymbol{V})\times L^2(I;Y^2).\end{aligned}$$

The inclusions $\mathcal{W}\subset\mathcal{U}$ and $\mathcal{Q}^*\subset\mathcal{Y}^*$ are continuous and dense. Notice that for these embeddings, the third components are left unchanged.

To facilitate the proof of the next theorem, let us consider the closed, positive and self-adjoint linear operator $C := I + A_N : Y^2 \to X$. Note that $C : Y^2 \to X$ is a unitary operator and it admits a unique extension $C : X \to Y^{2*}$ that is again unitary. Also, $C^{1/2} : Y \to X$ and $C^{1/2} : X \to Y^*$ are unitary operators, provided that $Y = H^1(\Omega)$ is equipped with its usual norm. Thus, $\|C^{-1}\theta\|_{Y^2} = \|\theta\|_X$ for every $\theta\in X$ and $\|C^{-1}\vartheta\|_Y = \|C^{-1/2}\vartheta\|_X = \|\vartheta\|_{Y^*}$ for every $\vartheta\in Y^*$. For a proof of these remarks, we refer to [58, Proposition 3.4.5]. The operator C will be also defined in the time-dependent case in the obvious way. Finally, we point out that the operator A_N has a unique extension as a mapping $A_N : L^2(I;X)\to L^2(I;Y^{2*})$ that is also linear and continuous.

Theorem 4.2. *Suppose that $(\phi,\mu,\boldsymbol{u},\theta)\in\mathcal{V}$, $(s,\sigma,\boldsymbol{y},z)\in\mathcal{Y}^*$, and $(\phi_0,\boldsymbol{w}_0,\zeta_0)\in X\times\boldsymbol{H}\times Y^*$. Then (4.3) has a unique solution $(\psi,\xi,\boldsymbol{w},\zeta)\in\mathcal{U}$ and there is a continuous function $\mathfrak{C}>0$ that is independent of $(\psi,\xi,\boldsymbol{w},\zeta)$ such that*

$$\|(\psi,\xi,\boldsymbol{w},\zeta)\|_{\mathcal{U}} \leq \mathfrak{C}(\|(\phi,\mu,\boldsymbol{u},\theta)\|_{\mathcal{V}})\|(s,\sigma,\boldsymbol{y},z,\phi_0,\boldsymbol{w}_0,\zeta_0)\|_{\mathcal{Y}^*\times X\times\boldsymbol{H}\times Y^*}. \tag{4.35}$$

In particular, $\mathcal{A}:\mathcal{V}\to\mathcal{L}(\mathcal{U},\mathcal{Y}^*\times X\times\boldsymbol{H}\times Y^*)$.

Proof. We shall proceed by a density argument. Choose sequences $\{(s_k, \sigma_k, z_k)\}_{k=1}^{\infty} \subset L^2(I;Y^*) \times L^2(I;Y) \times L^2(I;Y^*)$ and $\{(\phi_{k0}, \zeta_{k0})\}_{k=1}^{\infty} \subset Y \times X$ in such a way that $(s_k, \sigma_k, z_k) \to (s, \sigma, z)$ in $L^2(I;Y^{2*}) \times L^2(I;X) \times L^2(I;Y^{2*})$ and $(\phi_{k0}, \zeta_{k0}) \to (\phi_0, \zeta_0)$ in $X \times Y^*$. Then, by Theorem 4.1, there exists $(\psi_k, \xi_k, \boldsymbol{w}_k, \zeta_k) \in \mathcal{W}$ that satisfies the linear system $\mathcal{A}(\phi, \mu, \boldsymbol{u}, \theta)(\psi_k, \xi_k, \boldsymbol{w}_k, \zeta_k) = (s_k, \sigma_k, \boldsymbol{y}, z_k, \phi_{k0}, \boldsymbol{w}_0, \zeta_{k0})$ in $\mathcal{Q}^* \times Y \times \boldsymbol{H} \times X$.

We revisit some computations in the proof of the previous theorem. First, by replacing the estimate (4.7) by

$$|\langle s_k, \psi_k \rangle_{Y^{2*} \times Y^2}| \leq \varepsilon \|\Delta \psi_k\|_X^2 + c_\varepsilon (\|s_k\|_{Y^{2*}}^2 + \|\psi_k\|_X^2),$$

instead of (4.12) we get that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\psi_k\|_X^2 + \frac{m\alpha}{4}\|\Delta\psi_k\|_X^2 \\ \leq K_1(\|\psi_k\|_X^2 + \|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2) + c(\|\zeta_k - l_{\mathrm{h}}\psi_k\|_X^2 + \|s_k\|_{Y^{2*}}^2 + \|\sigma_k\|_X^2). \end{aligned} \tag{4.36}$$

On the other hand, the estimates for ξ_k and $\boldsymbol{w}_k$ given by (4.26) and (4.10), respectively, remain the same except that z must be replaced by z_k.

For the linearized convection-diffusion equation, we shall use the test function $C^{-1}(\zeta_k - l_{\mathrm{h}}\psi_k) \in L^2(I;Y^2)$ and write $A_N\zeta_k = C(\zeta_k - l_{\mathrm{h}}\psi_k) + l_{\mathrm{h}}C\psi_k - \zeta_k$ to the obtain equation

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|C^{-1/2}(\zeta_k - l_{\mathrm{h}}\psi_k)\|_X^2 + \kappa\|\zeta_k - l_{\mathrm{h}}\psi_k\|_X^2 = &- l_{\mathrm{h}}\kappa(\psi_k, \zeta_k - l_{\mathrm{h}}\psi_k)_X \\ &+ \kappa(\zeta_k, C^{-1}(\zeta_k - l_{\mathrm{h}}\psi_k))_X + r(\boldsymbol{w}_k, C^{-1}(\zeta_k - l_{\mathrm{h}}\psi_k), \theta - l_{\mathrm{h}}\phi) \\ &+ r(\boldsymbol{u}, C^{-1}(\zeta_k - l_{\mathrm{h}}\psi_k), \zeta_k - l_{\mathrm{h}}\psi_k) + \langle \boldsymbol{g}\cdot\boldsymbol{w}_k + z_k, C^{-1}(\zeta_k - l_{\mathrm{h}}\psi_k)\rangle_{Y^{2*}\times Y^2}. \end{aligned}$$

According to the discussion preceding the theorem, we have $\|C^{-1}(\zeta_k - l_{\mathrm{h}}\psi_k)\|_{Y^2} = \|\zeta_k - l_{\mathrm{h}}\psi_k\|_X$ and $\|C^{-1}(\zeta_k - l_{\mathrm{h}}\psi_k)\|_Y = \|C^{-1/2}(\zeta_k - l_{\mathrm{h}}\psi_k)\|_X = \|\zeta_k - l_{\mathrm{h}}\psi_k\|_{Y^*}$. Using the Cauchy-Schwarz inequality and $\|\boldsymbol{g}\cdot\boldsymbol{w}_k\|_{Y^{2*}} \leq c\|\boldsymbol{w}_k\|_{\boldsymbol{H}}$ it holds that

$$\begin{aligned} |l_{\mathrm{h}}\kappa(\psi_k, \zeta_k - l_{\mathrm{h}}\psi_k)_X| &\leq \frac{\kappa}{6}\|\zeta_k - l_{\mathrm{h}}\psi_k\|_X^2 + c\|\psi_k\|_X^2 \\ |\langle \boldsymbol{g}\cdot\boldsymbol{w}_k + z_k, C^{-1}(\zeta_k - l_{\mathrm{h}}\psi_k)\rangle_{Y^{2*}\times Y^2}| &\leq \frac{\kappa}{6}\|\zeta_k - l_{\mathrm{h}}\psi_k\|_X^2 + c(\|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2 + \|z_k\|_{Y^{2*}}^2). \end{aligned}$$

In the case of the trilinear terms, by applying the Hölder and Agmon inequalities, we get

$$\begin{aligned} |r(\boldsymbol{w}_k, C^{-1}(\zeta_k - l_{\mathrm{h}}\psi_k), \theta - l_{\mathrm{h}}\phi)| &\leq c(\|\zeta_k - l_{\mathrm{h}}\psi_k\|_{Y^*}^2 + c(\|\theta\|_{Y^2}^2 + \|\phi\|_{Y^2}^2)\|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2) \\ |r(\boldsymbol{u}, C^{-1}(\zeta_k - l_{\mathrm{h}}\psi_k), \zeta_k - l_{\mathrm{h}}\psi_k)| &\leq \frac{\kappa}{6}\|\zeta_k - l_{\mathrm{h}}\psi_k\|_X^2 + c\|\boldsymbol{u}\|_{\boldsymbol{V}^2}^2\|\zeta_k - l_{\mathrm{h}}\psi_k\|_{Y^*}^2. \end{aligned}$$

For the remaining term, let us write

$$(\zeta_k, C^{-1}(\zeta_k - l_{\mathrm{h}}\psi_k))_X = \|\zeta_k - l_{\mathrm{h}}\psi_k\|_{Y^*}^2 + l_{\mathrm{h}}(C^{-1/2}\psi_k, C^{-1/2}(\zeta_k - l_{\mathrm{h}}\psi_k))_X$$

and use the Cauchy-Schwarz inequality and $\|C^{-1/2}\psi_k\|_X \leq c\|\psi_k\|_X$, so that we have

$$|\kappa(\zeta_k, C^{-1}(\zeta_k - l_{\mathrm{h}}\psi_k))_X| \leq c(\|\zeta_k - l_{\mathrm{h}}\psi_k\|_{Y^*}^2 + \|\psi_k\|_X^2).$$

Taking the above estimates into consideration, putting $K_7 := c(1+\|\boldsymbol{u}\|_{\boldsymbol{V}^2}^2+\|\theta\|_{Y^2}^2+\|\phi\|_{Y^2}^2)$ and then invoking the Gronwall Lemma, we deduce that

$$\frac{1}{2}\frac{d}{dt}\|\zeta_k - l_{\rm h}\psi_k\|_{Y^*}^2 + \frac{\kappa}{2}\|\zeta_k - l_{\rm h}\psi_k\|_X^2 \leq K_7(\|\psi_k\|_X^2 + \|\zeta_k - l_{\rm h}\psi_k\|_{Y^*}^2 + \|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2) + c\|z\|_{Y^{2*}}. \tag{4.37}$$

Multiplying (4.26) and (4.36) by $\tilde\varepsilon > 0$, (4.10) by $\tilde\varepsilon^2$ and then taking the sum of the resulting estimates with (4.37) yields the inequality

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}E_{k,\tilde\varepsilon} + \left(\frac{m\alpha}{4} - c_\alpha\tilde\varepsilon - \varepsilon\right)\tilde\varepsilon\|\Delta\psi_k\|_X^2 \\ &\quad + (\tilde\varepsilon - \varepsilon)\tilde\varepsilon\|\xi_k\|_X^2 + \left(\frac{\kappa}{2} - c(2+\tilde\varepsilon)\tilde\varepsilon\right)\|\zeta_k - l_{\rm h}\psi_k\|_X^2 + \frac{\nu}{2}\|\boldsymbol{w}_k\|_{\boldsymbol{V}}^2 \\ &\quad \leq c_{\tilde\varepsilon}K_{\varepsilon,\tilde\varepsilon}E_{k,\tilde\varepsilon} + c_{\varepsilon,\tilde\varepsilon}(\|s_k\|_{Y^{2*}}^2 + \|\sigma_k\|_X^2 + \|\boldsymbol{y}\|_{\boldsymbol{V}^*}^2 + \|z_k\|_{Y^{2*}}^2),\end{aligned}$$

where $E_{k,\tilde\varepsilon} := \tilde\varepsilon\|\psi_k\|_X^2 + \|\zeta_k - \psi_k\|_{Y^*}^2 + \tilde\varepsilon\|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2$ and $K_{\varepsilon,\tilde\varepsilon} := \tilde\varepsilon^2K_0 + \tilde\varepsilon K_1 + \tilde\varepsilon K_{3\varepsilon} + K_7$. It is not difficult to see that $\|K_{\varepsilon,\tilde\varepsilon}\|_{L^1(I)} \leq c_{\varepsilon,\tilde\varepsilon}\mathfrak{C}(\|(\phi,\mu,\boldsymbol{u},\theta)\|_{\mathcal{V}})$. Taking $\tilde\varepsilon > \varepsilon > 0$ small enough, applying the Gronwall Lemma, and using the estimates $\|\psi_{k0}\|_{Y^*} \leq c\|\psi_{k0}\|_X$ and $\|\zeta_k\|_{L^\infty(Y^*)} \leq \|\zeta_k - l_{\rm h}\psi_k\|_{L^\infty(Y^*)} + c\|\psi_k\|_{L^\infty(X)}$, we obtain

$$\begin{aligned}&\|\psi_k\|_{L^\infty(X)\cap L^2(Y^2)} + \|\xi_k\|_{L^2(X)} + \|\boldsymbol{w}_k\|_{L^\infty(\boldsymbol{H})\cap L^2(\boldsymbol{V})} + \|\zeta_k\|_{L^\infty(Y^*)\cap L^2(X)} \\ &\quad \leq \mathfrak{C}(\|(\phi,\mu,\boldsymbol{u},\theta)\|_{\mathcal{V}})\|(s_k,\sigma_k,\boldsymbol{y},z_k,\phi_{k0},\boldsymbol{w}_0,\zeta_{k0})\|_{\mathcal{Y}^*\times X\times\boldsymbol{H}\times Y^*}.\end{aligned} \tag{4.38}$$

Let us turn into the estimates of the time derivatives. Applying the continuity of $L^2(I;Y^*) \subset L^2(I;Y^{2*})$ in the convection term and the boundedness of $A_N : L^2(I;X) \to L^2(I;Y^{2*})$, we have

$$\begin{aligned}\|\partial_t\psi_k\|_{L^2(Y^{2*})} &\leq c(\|\boldsymbol{u}\|_{L^\infty(\boldsymbol{H})}\|\psi_k\|_{L^2(Y)} + \|\psi_k\|_{L^\infty(X)}\|\boldsymbol{u}\|_{L^2(\boldsymbol{V})} + \|\xi_k\|_{L^2(X)} \\ &\quad + \|\phi\|_{L^\infty(X)}\|\boldsymbol{w}_k\|_{L^2(\boldsymbol{V})} + \|\boldsymbol{w}_k\|_{L^\infty(\boldsymbol{H})}\|\phi\|_{L^2(X)} + \|s_k\|_{L^2(Y^{2*})}).\end{aligned} \tag{4.39}$$

Similarly, using $\|B_1(\boldsymbol{u},\zeta_k - l_{\rm h}\psi_k)\|_{L^2(Y^{2*})} \leq c\|\boldsymbol{u}\|_{L^\infty(\boldsymbol{V})}(\|\zeta_k\|_{L^2(X)} + \|\psi_k\|_{L^2(X)})$ we obtain

$$\begin{aligned}\|\partial_t\zeta_k\|_{L^2(Y^{2*})} &\leq c(\|\partial_t\psi_k\|_{L^2(Y^{2*})} + \|\boldsymbol{u}\|_{L^\infty(\boldsymbol{V})}(\|\zeta_k\|_{L^2(X)} + \|\psi_k\|_{L^2(X)}) + \|\zeta_k\|_{L^2(X)} \\ &\quad + \|\boldsymbol{w}_k\|_{L^2(\boldsymbol{H})} + \|\boldsymbol{w}_k\|_{L^\infty(\boldsymbol{H})}\|\theta\|_{L^2(Y)} + \|\theta\|_{L^\infty(X)}\|\boldsymbol{w}_k\|_{L^2(\boldsymbol{V})} \\ &\quad + \|z_k\|_{L^2(Y^{2*})}).\end{aligned} \tag{4.40}$$

From $\|\boldsymbol{B}_2(\mu - l_{\rm c}\theta,\psi_k)\|_{L^2(\boldsymbol{V}^*)} \leq c(\|\mu\|_{L^\infty(X)} + \|\theta\|_{L^\infty(X)})\|\psi_k\|_{L^2(Y^2)}$ and $\|\boldsymbol{B}_2(\xi_k - l_{\rm c}\zeta_k,\phi)\|_{L^2(\boldsymbol{V}^*)} \leq c(\|\xi_k\|_{L^2(X)} + \|\zeta_k\|_{L^2(X)})\|\phi\|_{L^\infty(Y^2)}$, the time derivative of $\boldsymbol{w}_k$ can be estimated as follows:

$$\begin{aligned}\|\partial_t\boldsymbol{w}_k\|_{L^2(\boldsymbol{V}^*)} &\leq c(\|\boldsymbol{u}\|_{L^\infty(\boldsymbol{H})}\|\boldsymbol{w}_k\|_{L^2(\boldsymbol{V})} + \|\boldsymbol{w}_k\|_{L^\infty(\boldsymbol{H})}\|\boldsymbol{u}\|_{L^2(\boldsymbol{V})} + \|\boldsymbol{w}_k\|_{L^2(\boldsymbol{V})} \\ &\quad + (\|\mu\|_{L^\infty(X)} + \|\theta\|_{L^\infty(X)})\|\psi_k\|_{L^2(Y^2)} + (\|\xi_k\|_{L^2(X)} + \|\zeta_k\|_{L^2(X)})\|\phi\|_{L^\infty(Y^2)} \\ &\quad + \|\psi_k\|_{L^2(X)} + \|\zeta_k\|_{L^2(X)} + \|\boldsymbol{y}\|_{L^2(\boldsymbol{V}^*)}).\end{aligned} \tag{4.41}$$

Taking the sum of what we have obtained from (4.38) to (4.41), one can see that

$$\|(\psi_k,\xi_k,\boldsymbol{w}_k,\zeta_k)\|_{\mathcal{U}} \leq \mathfrak{C}(\|(\phi,\mu,\boldsymbol{u},\theta)\|_{\mathcal{V}})\|(s_k,\sigma_k,\boldsymbol{y},z_k,\phi_{k0},\boldsymbol{w}_0,\zeta_{k0})\|_{\mathcal{Y}^*\times X\times\boldsymbol{H}\times Y^*}.$$

When applied to the difference, this implies that $\{(\psi_k,\xi_k,\boldsymbol{w}_k,\zeta_k)\}_{k=1}^\infty$ is a Cauchy sequence in $\mathcal{U}$, so that $(\psi_k,\xi_k,\boldsymbol{w}_k,\zeta_k) \to (\psi,\xi,\boldsymbol{w},\zeta)$ in $\mathcal{U}$ and the limit is a solution of (4.3). For this solution, we have (4.35) due to strong convergence. Finally, the uniqueness of the solution follows from the previous theorem along with standard arguments. This completes the proof. □

In the above discussions, the main interest is when the initial conditions in the linearized system vanish. For this, we let $\mathcal{W}_0$ to be the space of all elements in $\mathcal{W}$ where the first, third, and last components vanish at $t = 0$. The function spaces $\mathcal{V}_0$ and $\mathcal{U}_0$ in relation to $\mathcal{V}$ and $\mathcal{U}$ are defined analogously.

Corollary 4.3. *The operator $A(\phi, \mu, \boldsymbol{u}, \theta) \in \mathcal{L}(\mathcal{W}_0, \mathcal{Q}^*)$ defined by (4.2) is an isomorphism for every $(\phi, \mu, \boldsymbol{u}, \theta) \in \mathcal{W}$. Also, $A(\phi, \mu, \boldsymbol{u}, \theta) \in \mathcal{L}(\mathcal{U}_0, \mathcal{Y}^*)$ is an isomorphism for all $(\phi, \mu, \boldsymbol{u}, \theta) \in \mathcal{V}$.*

Proof. This is a direct consequence of Theorem 4.1 and Theorem 4.2. □

The following theorem will be utilized in the proof of second order sufficient condition.

Theorem 4.4. *Suppose that $(\phi, \mu, \boldsymbol{u}, \theta) \in \mathcal{V}$, $\boldsymbol{y} \in L^2(I; \boldsymbol{X})$, and $z \in L^2(I; X)$. Then there is a unique solution $(\psi, \xi, \boldsymbol{w}, \zeta) \in \mathcal{V}_0$ to the equation*

$$A(\phi, \mu, \boldsymbol{u}, \theta)(\psi, \xi, \boldsymbol{w}, \zeta) = (0, 0, \boldsymbol{y}, z).$$

Moreover, there exists a continuous function $\mathfrak{C} > 0$ such that

$$\|(\psi, \xi, \boldsymbol{w}, \zeta)\|_{\mathcal{V}_0} \le \mathfrak{C}(\|(\phi, \mu, \boldsymbol{u}, \theta)\|_{\mathcal{V}})(\|\boldsymbol{y}\|_{L^2(\boldsymbol{X})} + \|z\|_{L^2(X)}). \tag{4.42}$$

Proof. We follow the proof provided in Theorem 3.4. The a priori estimates from Theorem 4.1 and the continuous embeddings $L^2(I; \boldsymbol{X}) \subset L^2(I; \boldsymbol{V}^*)$, $L^2(I; X) \subset L^2(I; Y^*)$, and $\mathcal{V} \subset \mathcal{W}$ imply

$$\begin{aligned}\|(\psi_k, \xi_k, \boldsymbol{w}_k, \zeta_k)\|_{\mathcal{W}_0} &\le \mathfrak{C}(\|(\phi, \mu, \boldsymbol{u}, \theta)\|_{\mathcal{V}})(\|\boldsymbol{y}\|_{L^2(\boldsymbol{X})} + \|z\|_{L^2(X)}) \\ &=: \mathfrak{C}(\phi, \mu, \boldsymbol{u}, \theta, \boldsymbol{y}, z).\end{aligned} \tag{4.43}$$

Choosing the test function $-(\Delta\zeta_k - l_{\mathrm{h}}\Delta\psi_k)$ in the equation satisfied by ζ_k and using the Gagliardo-Nirenberg and Agmon inequalities, we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\zeta_k - l_{\mathrm{h}}\nabla\psi_k\|_{\boldsymbol{X}}^2 + \kappa\|\Delta\zeta_k\|_X^2 = r(\boldsymbol{u}, \zeta_k - l_{\mathrm{h}}\psi_k, \Delta\zeta_k - l_{\mathrm{h}}\Delta\psi_k) \\ &\qquad + r(\boldsymbol{w}_k, \theta - l_{\mathrm{h}}\phi, \Delta\zeta_k - l_{\mathrm{h}}\Delta\psi_k) - (\boldsymbol{g}\cdot\boldsymbol{w}_k + z, \Delta\zeta_k - l_{\mathrm{h}}\Delta\psi_k)_X \\ &\le \frac{\kappa}{2}\|\Delta\zeta_k\|_X^2 + c(\|\boldsymbol{u}\|_{\boldsymbol{V}^2}^2(\|\nabla\zeta_k\|_{\boldsymbol{X}}^2 + \|\nabla\psi_k\|_{\boldsymbol{X}}^2) + \|\Delta\psi_k\|_X^2 + (\|\theta\|_{Y^2}^2 + \|\phi\|_{Y^2}^2)\|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2) \\ &\qquad + c((\|\theta\|_Y^2 + \|\phi\|_Y^2)\|\boldsymbol{w}_k\|_{\boldsymbol{V}}^2 + \|\boldsymbol{w}_k\|_{\boldsymbol{H}}^2 + \|z\|_X^2).\end{aligned}$$

Utilizing the Gronwall Lemma and applying (4.43) yields

$$\|\nabla\zeta_k\|_{L^\infty(\boldsymbol{X})} + \|\Delta\zeta_k\|_{L^2(X)} \le \mathfrak{C}(\phi, \mu, \boldsymbol{u}, \theta, \boldsymbol{y}, z). \tag{4.44}$$

Using the test function $\Delta^2\psi_k$ in the equation satisfied by ψ_k, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Delta\psi_k\|_X^2 - m(\Delta\xi_k, \Delta^2\psi_k)_X \\ &\qquad \le \frac{m\alpha}{16}\|\Delta^2\psi_k\|_X^2 + c(\|\boldsymbol{u}\|_{\boldsymbol{V}}^2\|\Delta\psi_k\|_X^2 + \|\boldsymbol{w}_k\|_{\boldsymbol{V}}^2\|\Delta\phi\|_X^2).\end{aligned} \tag{4.45}$$

Next, we shall estimate the norm of $\Delta(f'(\phi)\psi_k)$. By a simple calculation, we obtain from the chain rule that $\Delta(f'(\phi)\psi_k) = 6(\phi\Delta\phi + |\nabla\phi|^2)\psi_k + 12\phi\nabla\phi\cdot\nabla\psi_k + (3\phi^2 - 1)\Delta\psi_k$. Hence, by the Hölder and Gagliardo-Nirenberg inequalities as well as the Sobolev embedding

$$\|\Delta P_{X_k}(f'(\phi)\psi_k)\|_X^2 \leq c(\|\phi\|_Y^2\|\phi\|_{Y^3}^2 + \|\phi\|_{Y^2}^4)\|\psi_k\|_X^2 + c(\|\phi\|_Y^2\|\phi\|_{Y^2}^2 + \|\phi\|_Y^4 + 1)\|\Delta\psi_k\|_X^2.$$

This estimate and the one given in (4.43) imply that

$$\|\Delta P_{X_k}(f'(\phi)\psi_k)\|_{L^2(X)} \leq \mathfrak{C}(\phi,\mu,\boldsymbol{u},\theta,\boldsymbol{y},z) \tag{4.46}$$

since $\phi \in W^2(I;Y^4,X) \subset L^\infty(I;Y^2)$.

Note that for the second term on the left hand side of (4.45) it holds that

$$-m(\Delta\xi_k,\Delta^2\psi_k)_X \geq \frac{m\alpha}{4}\|\Delta^2\psi_k\|_X^2 - c(\|\Delta P_{X_k}(f'(\phi)\psi_k)\|_X^2 + \|\Delta\zeta_k\|_X^2). \tag{4.47}$$

In addition, we have $\|\Delta\xi_k\|_X^2 \leq c_\alpha\|\Delta^2\psi_k\|_X^2 + c(\|\Delta P_{X_k}(f'(\phi)\psi_k)\|_X^2 + \|\Delta\zeta_k\|_X^2)$. Upon substitution of (4.47) in (4.45) we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta\psi_k\|_X^2 + \frac{m\alpha}{8}\|\Delta^2\psi_k\|_X^2 + \frac{m}{16c_\alpha}\|\Delta\xi_k\|_X^2 \leq c(\|\boldsymbol{u}\|_{\boldsymbol{V}}^2\|\Delta\psi_k\|_X^2 + \|\boldsymbol{w}_k\|_{\boldsymbol{V}}^2\|\Delta\phi\|_X^2 + \|\Delta P_{X_k}(f'(\phi)\psi_k)\|_X^2 + \|\Delta\zeta_k\|_X^2).$$

Applying the Gronwall Lemma to this inequality and invoking the estimates (4.44) and (4.46), one has

$$\|\Delta\psi_k\|_{L^\infty(X)} + \|\Delta^2\psi_k\|_{L^2(X)} + \|\Delta\xi_k\|_{L^2(X)}^2 \leq \mathfrak{C}(\phi,\mu,\boldsymbol{u},\theta,\boldsymbol{y},z). \tag{4.48}$$

To derive an priori estimate for $\boldsymbol{w}$, let us note the following

$$\|(\mu - l_c\theta)\nabla\psi_k\|_{L^2(\boldsymbol{X})} + \|(\xi_k - l_c\zeta_k)\nabla\phi\|_{L^2(\boldsymbol{X})} \leq c((\|\mu\|_{L^2(Y)} + \|\theta\|_{L^2(Y)})\|\Delta\psi_k\|_{L^\infty(X)} + (\|\xi_k\|_{L^2(Y)} + \|\zeta_k\|_{L^2(Y)})\|\Delta\phi\|_{L^\infty(X)}).$$

From this, and by a similar argument as in the nonlinear state equation, it can be deduced that

$$\|\boldsymbol{w}_k\|_{L^\infty(\boldsymbol{V})} + \|\Delta\boldsymbol{w}_k\|_{L^2(\boldsymbol{H})} \leq \mathfrak{C}(\phi,\mu,\boldsymbol{u},\theta,\boldsymbol{y},z). \tag{4.49}$$

According to the differential equations satisfied by ψ_k, $\boldsymbol{w}_k$, and ζ_k, as well as the a priori estimates (4.44), (4.48) and (4.49), we have

$$\|\partial_t\psi_k\|_{L^2(X)} + \|\partial_t\boldsymbol{w}_k\|_{L^2(\boldsymbol{H})} + \|\partial_t\zeta_k\|_{L^2(X)} \leq \mathfrak{C}(\phi,\mu,\boldsymbol{u},\theta,\boldsymbol{y},z).$$

Finally, from $\partial_t\xi_k = -\alpha\Delta\partial_t\psi_k + P_{X_k}((3\phi^2-1)\partial_t\psi_k + 6\phi\psi_k\partial_t\phi) + \partial_t\zeta_k$, we obtain that

$$\|\partial_t\xi_k\|_{L^2(Y^{2*})} \leq c((1 + \|\phi\|_{L^\infty(Y^2)}^2)\|\partial_t\psi_k\|_{L^2(X)} + \|\phi\|_{L^\infty(Y^2)}\|\psi_k\|_{L^\infty(Y^2)}\|\partial_t\phi\|_{L^2(X)} + \|\partial_t\zeta_k\|_{L^2(X)}) \leq \mathfrak{C}(\phi,\mu,\boldsymbol{u},\theta,\boldsymbol{y},z).$$

Overall, we have established that $\{(\psi_k,\xi_k,\boldsymbol{w}_k,\zeta_k)\}_{k=1}^\infty$ is bounded in $\mathcal{V}_0$. Therefore the weak solution constructed from the Galerkin method satisfies $(\psi,\xi,\boldsymbol{w},\zeta) \in \mathcal{V}_0$. Finally, taking the limit inferior of the above a priori estimates for the Galerkin approximations yields (4.42). □

4.2. Differentiability of the control-to-state map. We shall now discuss the differentiability of the operator that maps a control to a solution of the state equation. All throughout, the dual of the ambient control space Q will be identified with itself. Define the nonlinear map

$$\mathcal{N} : \mathcal{W} \times Q \to \mathcal{Q}^* \times Y \times \boldsymbol{H} \times X$$

according to $\mathcal{N} = (N, N_0)$, where $N : \mathcal{W} \times Q \to \mathcal{Q}^*$ is given by

$$\mathcal{N}(\phi, \mu, \boldsymbol{u}, \theta, \boldsymbol{y}, z) := \begin{bmatrix} \partial_t\phi + B_1(\boldsymbol{u}, \phi) + mA_N\mu \\ \mu - \alpha A_N\phi - f(\phi) - l_c\theta \\ \partial_t\boldsymbol{u} + \boldsymbol{B}(\boldsymbol{u}) + \nu\boldsymbol{A}_S\boldsymbol{u} - \mathcal{K}\boldsymbol{B}_2(\mu - l_c\theta, \phi) - \ell(\phi,\theta)\boldsymbol{g} - \chi_{\omega_f}\boldsymbol{y} \\ \partial_t\theta - l_h\partial_t\psi + B_1(\boldsymbol{u}, \theta - l_h\phi) + \kappa A_N\theta - \boldsymbol{g}\cdot\boldsymbol{u} - \chi_{\omega_h}z \end{bmatrix}$$

and $N_0 : \mathcal{W} \times Q \to Y \times \boldsymbol{H} \times X$ is defined by $N_0(\phi, \mu, \boldsymbol{u}, \theta, \boldsymbol{y}, z) = (\phi(0) - \phi_0, \boldsymbol{u}(0) - \boldsymbol{u}_0, \theta(0) - \theta_0)$. Here, $(\phi_0, \boldsymbol{u}_0, \theta_0) \in Y \times \boldsymbol{H} \times X$ is a given fixed initial data.

According to the existence and uniqueness of weak solutions, cf. Theorem 3.2, for a given $(\boldsymbol{y}, z) \in Q$ there exists a unique $(\phi, \mu, \boldsymbol{u}, \theta) \in \mathcal{W}$ that satisfies the equation

$$\mathcal{N}(\phi, \mu, \boldsymbol{u}, \theta, \boldsymbol{y}, z) = 0. \tag{4.50}$$

Define the operator $S : Q \to \mathcal{W}$ by $S(\boldsymbol{y}, z) = (\phi, \mu, \boldsymbol{u}, \theta)$ if and only if (4.50) holds. We will prove that S is of class C^∞. To show this, it is enough to treat the nonlinear terms appearing in $\mathcal{N}$. For this, we need the following lemma.

Lemma 4.5. *The mappings $\boldsymbol{B} : W^2(I; \boldsymbol{V}) \to L^2(I; \boldsymbol{V}^*)$, $B_1 : W^2(I; \boldsymbol{V}) \times W^2(I; Y) \to L^2(I; Y^*)$, $\boldsymbol{B}_2 : L^2(I; Y) \times W^2(I; Y^3, Y^*) \to L^2(I; \boldsymbol{V}^*)$, and $f : W^2(I; Y^3, Y^*) \to L^2(I; Y)$ are of class C^∞. The derivatives of order at least 3 for B, B_1, and $\boldsymbol{B}_2$, and of order at least 4 for f all vanish.*

Proof. The differentiability of $\boldsymbol{B}$, B_1, and $\boldsymbol{B}_2$ follows from the bilinearity of these maps together with the estimates in (3.4)-(3.6). For future reference, we write the directional derivatives: For every $(\phi, \mu, \boldsymbol{u}, \theta)$, $(\psi, \xi, \boldsymbol{w}, \zeta) \in \mathcal{W}$ it holds that $\mathrm{D}\boldsymbol{B}(\boldsymbol{u})\boldsymbol{w} = \boldsymbol{B}(\boldsymbol{u}, \boldsymbol{w}) + \boldsymbol{B}(\boldsymbol{w}, \boldsymbol{u})$, $\mathrm{D}B_1(\boldsymbol{u}, \theta)(\boldsymbol{w}, \zeta) = B_1(\boldsymbol{u}, \zeta) + B_1(\boldsymbol{w}, \theta)$, and $\mathrm{D}\boldsymbol{B}_2(\mu, \phi)(\xi, \psi) = \boldsymbol{B}_2(\mu, \psi) + \boldsymbol{B}_2(\xi, \phi)$. Moreover, the action of the second derivatives are given by

$$\begin{aligned} \mathrm{D}^2\boldsymbol{B}(\boldsymbol{u})[\boldsymbol{w}_1, \boldsymbol{w}_2] &= \boldsymbol{B}(\boldsymbol{w}_1, \boldsymbol{w}_2) + \boldsymbol{B}(\boldsymbol{w}_2, \boldsymbol{w}_1), \\ \mathrm{D}^2B_1(\boldsymbol{u}, \theta)[(\boldsymbol{w}_1, \zeta_1), (\boldsymbol{w}_2, \zeta_2)] &= B_1(\boldsymbol{w}_1, \zeta_2) + B_1(\boldsymbol{w}_2, \zeta_1), \\ \mathrm{D}^2\boldsymbol{B}_2(\mu, \phi)[(\xi_1, \psi_1), (\xi_2, \psi_2)] &= \boldsymbol{B}_2(\xi_1, \psi_2) + \boldsymbol{B}_2(\xi_2, \psi_1), \end{aligned}$$

for every $(\psi_i, \xi_i, \boldsymbol{w}_i, \zeta_i) \in \mathcal{W}$ for $i = 1, 2$. Since these are independent on $(\phi, \mu, \boldsymbol{u}, \theta) \in \mathcal{W}$, it follows that the derivatives of order at least 3 for these operators vanish.

Let us establish the differentiability of f. Let $\phi, \psi \in W^2(I; Y^3, Y^*)$. By Taylor's expansion, $f(\phi + \psi) - f(\phi) - f'(\phi)\psi = 6(\phi\psi^2 + \psi^3)$. From the Hölder and Agmon inequalities we obtain

$$\|\phi\psi^2 + \psi^3\|_{L^2(X)} \le c(\|\phi\|_{L^2(Y)}\|\psi\|^2_{L^\infty(Y)} + \|\psi\|^3_{L^\infty(Y)}).$$

On the other hand, by computing the gradient of the right hand side, we get $\nabla(\phi\psi^2 + \psi^3) = \psi^2\nabla\phi + (2\phi\psi + 3\psi^2)\nabla\psi$. We then estimate from above according to

$$\|\psi^2\nabla\phi + (2\phi\psi + 3\psi^2)\nabla\psi\|_{L^2(\boldsymbol{X})}$$

$$\leq c(\|\phi\|_{L^2(Y^2)}\|\psi\|^2_{L^\infty(Y)} + \|\phi\|_{L^\infty(Y)}\|\psi\|_{L^\infty(Y)}\|\psi\|_{L^2(Y^2)} + \|\psi\|^2_{L^\infty(Y)}\|\psi\|_{L^2(Y^2)}).$$

Combining the previous estimates and recalling $W^2(I;Y^3,Y^*) \subset L^\infty(I;Y) \cap L^2(I;Y^2)$, there is a constant $c = c(\|\phi\|_{W^2(Y^3,Y^*)}) > 0$ such that

$$\|f(\phi+\psi) - f(\phi) - f'(\phi)\psi\|_{L^2(Y)} \leq c(1 + \|\psi\|_{W^2(Y^3,Y^*)})\|\psi\|^2_{W^2(Y^3,Y^*)}.$$

From this, we see that f is Frechét differentiable and $\mathrm{D}f(\phi) = f'(\phi)$, where the right hand side is to be understood as a multiplication operator.

For the second derivative, if $\phi, \psi_1, \psi_2 \in W^2(I;Y^3,Y^*)$ then we have $f'(\phi)\psi_1 - f'(\phi+\psi_2)\psi_1 - f''(\phi)\psi_1\psi_2 = 6\psi_1\psi_2^2$ and

$$\|\psi_1\psi_2^2\|_{L^2(Y)} \leq c\|\psi_1\|_{W^2(I;Y^3,Y^*)}\|\psi_2\|^2_{W^2(I;Y^3,Y^*)}.$$

Thus $\mathrm{D}^2 f(\phi) = f''(\phi)$. In addition, if $\psi_3 \in W^2(I;Y^3,Y^*)$ then $\mathrm{D}^3 f(\phi)\psi_1\psi_2\psi_3 = 6\psi_1\psi_2\psi_3$. Since this is independent on ϕ, the derivatives beyond order 3 of f vanish. This completes the proof of the lemma. □

Due to the fact that the controls are only present in the Oberbeck-Boussinesq system, it is advantageous to consider the operator $P : Q \to \mathcal{Q}^*$ defined by $P(\boldsymbol{y}, z) = (0, 0, \chi_{\omega_{\mathrm{f}}}\boldsymbol{y}, \chi_{\omega_{\mathrm{h}}}z)$. The adjoint operator $P^* : \mathcal{Q} \to Q$ is given by $P^*(\varphi, \eta, \boldsymbol{v}, \vartheta) = (\chi_{\omega_{\mathrm{f}}}\boldsymbol{v}, \chi_{\omega_{\mathrm{h}}}\vartheta)$. Note that one may also consider $P : Q \to \mathcal{Y}^*$ and $P^* : \mathcal{Y} \to Q$.

Theorem 4.6. *The map $S : Q \to \mathcal{W}$ is of class C^∞. For every $(\boldsymbol{y}, z), (\delta\boldsymbol{y}, \delta z) \in Q$ we have*

$$\mathrm{D}S(\boldsymbol{y}, z)(\delta\boldsymbol{y}, \delta z) = A(S(\boldsymbol{y}, z))^{-1}P(\delta\boldsymbol{y}, \delta z) \tag{4.51}$$

and for every $(\delta\boldsymbol{y}_1, \delta z_1), (\delta\boldsymbol{y}_2, \delta z_2) \in Q$ it holds that

$$\mathrm{D}^2 S(\boldsymbol{y}, z)((\delta\boldsymbol{y}_1, \delta z_1), (\delta\boldsymbol{y}_2, \delta z_2)) \tag{4.52}$$
$$= -A(S(\boldsymbol{y}, z))^{-1}\begin{bmatrix} B_1(\boldsymbol{w}_1, \psi_2) + B_1(\boldsymbol{w}_2, \psi_1) \\ 6\phi\psi_1\psi_2 \\ \mathrm{D}^2\boldsymbol{B}(\boldsymbol{u})[\boldsymbol{w}_1, \boldsymbol{w}_2] - \mathcal{K}\boldsymbol{B}_2(\xi_1 - l_{\mathrm{c}}\zeta_1, \psi_2) - \mathcal{K}\boldsymbol{B}_2(\xi_2 - l_{\mathrm{c}}\zeta_2, \psi_1) \\ B_1(\boldsymbol{w}_1, \zeta_2 - l_{\mathrm{h}}\psi_2) + B_1(\boldsymbol{w}_2, \zeta_1 - l_{\mathrm{h}}\psi_1) \end{bmatrix},$$

where $(\psi_i, \xi_i, \boldsymbol{w}_i, \zeta_i) = \mathrm{D}S(\boldsymbol{y}, z)(\delta\boldsymbol{y}_i, \delta z_i)$ for $i = 1, 2$ and ϕ is the first component of $S(\boldsymbol{y}, z)$.

Proof. From Lemma 4.5 it follows that $\mathcal{N} \in C^\infty(\mathcal{W} \times Q, \mathcal{Q}^* \times Y \times \boldsymbol{H} \times X)$. Let $(\bar{\boldsymbol{y}}, \bar{z}) \in Q$ so that there exists a unique $(\bar{\phi}, \bar{\mu}, \bar{\boldsymbol{u}}, \bar{\theta}) = S(\bar{\boldsymbol{y}}, \bar{z}) \in \mathcal{W}$ that satisfies $\mathcal{N}(\bar{\phi}, \bar{\mu}, \bar{\boldsymbol{u}}, \bar{\theta}, \bar{\boldsymbol{y}}, \bar{z}) = 0$. According to Theorem 4.1, the linear operator

$$\frac{\partial\mathcal{N}(\bar{\phi}, \bar{\mu}, \bar{\boldsymbol{u}}, \bar{\theta}, \bar{\boldsymbol{y}}, \bar{z})}{\partial(\phi, \mu, \boldsymbol{u}, \theta)} = \mathcal{A}(\bar{\phi}, \bar{\mu}, \bar{\boldsymbol{u}}, \bar{\theta}) \in \mathcal{L}(\mathcal{W}, \mathcal{Q}^* \times Y \times \boldsymbol{H} \times X)$$

is an isomorphism.

From the implicit function theorem, see [60, Section 4.7] for instance, there exist open sets $\mathcal{O}_{(\bar{\boldsymbol{y}},\bar{z})} \subset Q$ and $\mathcal{O}_{S(\bar{\boldsymbol{y}},\bar{z})} \subset \mathcal{W}$ containing $(\bar{\boldsymbol{y}}, \bar{z})$ and $S(\bar{\boldsymbol{y}}, \bar{z})$, respectively, and a map $\tilde{S} \in C^\infty(\mathcal{O}_{(\bar{\boldsymbol{y}},\bar{z})}, \mathcal{O}_{S(\bar{\boldsymbol{y}},\bar{z})})$ such that the equation $\mathcal{N}(\tilde{S}(\boldsymbol{y}, z), \boldsymbol{y}, z) = 0$ is satisfied for every $(\boldsymbol{y}, z) \in \mathcal{O}_{(\bar{\boldsymbol{y}},\bar{z})}$. However, this implies that $\tilde{S}(\boldsymbol{y}, z) = S(\boldsymbol{y}, z)$ by the definition of S. Since $(\bar{\boldsymbol{y}}, \bar{z})$ was an arbitrary element of Q, it follows that $S \in$

$C^\infty(Q, \mathcal{W})$. Furthermore, applying the chain rule to the identity $\mathcal{N}(S(\boldsymbol{y}, z), \boldsymbol{y}, z) = 0$, we have

$$\begin{aligned}\mathrm{D}S(\boldsymbol{y}, z)(\delta\boldsymbol{y}, \delta z) &= -\mathcal{A}(S(\boldsymbol{y}, z))^{-1}\frac{\partial}{\partial(\boldsymbol{y}, z)}\mathcal{N}(S(\boldsymbol{y}, z), \boldsymbol{y}, z)(\delta\boldsymbol{y}, \delta z)\\ &= A(S(\boldsymbol{y}, z))^{-1}P(\delta\boldsymbol{y}, \delta z)\end{aligned}$$

for every $(\boldsymbol{y}, z), (\delta\boldsymbol{y}, \delta z) \in Q$, and thus (4.51). In particular, $A(S(\boldsymbol{y}, z))\mathrm{D}S(\boldsymbol{y}, z)(\delta\boldsymbol{y}, \delta z) = P(\delta\boldsymbol{y}, \delta z)$. By applying the chain and product rules to the latter equation, we get

$$\begin{aligned}&\mathrm{D}^2S(\boldsymbol{y}, z)((\delta\boldsymbol{y}_1, \delta z_1), (\delta\boldsymbol{y}_2, \delta z_2))\\ &\quad= -A(S(\boldsymbol{y}, z))^{-1}\mathrm{D}A(S(\boldsymbol{y}, z))(\mathrm{D}S(\boldsymbol{y}, z)(\delta\boldsymbol{y}_1, \delta z_1), \mathrm{D}S(\boldsymbol{y}, z)(\delta\boldsymbol{y}_2, \delta z_2)).\end{aligned} \tag{4.53}$$

Here, note that $\mathrm{D}A : \mathcal{W} \to \mathcal{L}(\mathcal{W}, \mathcal{L}(\mathcal{W}_0, \mathcal{Q}^*))$, where the latter space is isometrically isomorphic to the Banach space $\mathcal{L}(\mathcal{W} \times \mathcal{W}_0, \mathcal{Q}^*)$. Using the second derivatives of the nonlinear operators presented in the proof of Lemma 4.5, we see that equation (4.53) implies (4.52). □

Remark 4.7. *In terms of PDEs,* $\mathrm{D}S(\boldsymbol{y}, z)(\delta\boldsymbol{y}, \delta z) = (\psi, \xi, \boldsymbol{w}, \zeta)$ *if and only if* $(\psi, \xi, \boldsymbol{w}, \zeta) \in \mathcal{W}_0$ *is the* weak solution *of the following linear system:*

$$\begin{cases}\partial_t\psi + \boldsymbol{u}\cdot\nabla\psi + \boldsymbol{w}\cdot\nabla\phi - m\Delta\xi = 0 & \text{in } I\times\Omega\\ \xi = -\alpha\Delta\psi + (3\phi^2 - 1)\psi + l_{\mathrm{c}}\zeta & \text{in } I\times\Omega,\\ \partial_t\boldsymbol{w} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{w} + \nabla\pi & \\ \qquad = \mathcal{K}(\mu - l_{\mathrm{c}}\theta)\nabla\psi + \mathcal{K}(\xi - l_{\mathrm{c}}\zeta)\nabla\phi + (\alpha_2\psi + \alpha_3\zeta)\boldsymbol{g} + \chi_{\omega_{\mathrm{f}}}\delta\boldsymbol{y} & \text{in } I\times\Omega,\\ \operatorname{div}\boldsymbol{w} = 0 & \text{in } I\times\Omega,\\ \partial_t\zeta - l_{\mathrm{h}}\partial_t\psi + \boldsymbol{u}\cdot\nabla(\zeta - l_{\mathrm{h}}\psi) + \boldsymbol{w}\cdot\nabla(\theta - l_{\mathrm{h}}\phi) - \kappa\Delta\zeta = \boldsymbol{g}\cdot\boldsymbol{w} + \chi_{\omega_{\mathrm{h}}}\delta z & \text{in } I\times\Omega,\end{cases}$$

satisfying the boundary conditions $\partial_{\boldsymbol{n}}\psi = \partial_{\boldsymbol{n}}\Delta\psi = 0$, $\boldsymbol{w} = 0$, *and* $\partial_{\boldsymbol{n}}\zeta = 0$ *on* $I \times \Gamma$, *and the initial conditions* $\psi(0) = 0$, $\boldsymbol{w}(0) = 0$, *and* $\zeta(0) = 0$ *in* Ω, *where* $(\phi, \mu, \boldsymbol{u}, \theta) = S(\boldsymbol{y}, z)$.

Similarly, $\mathrm{D}^2S(\boldsymbol{y}, z)[(\delta\boldsymbol{y}_1, \delta z_1), (\delta\boldsymbol{y}_2, \delta z_2)] = (\psi, \xi, \boldsymbol{w}, \zeta)$ *if and only if* $(\psi, \xi, \boldsymbol{w}, \zeta) \in \mathcal{W}_0$ *is the* weak solution *of the linear system*

$$\begin{cases}\partial_t\psi + \boldsymbol{u}\cdot\nabla\psi + \boldsymbol{w}\cdot\nabla\phi - m\Delta\xi = -\boldsymbol{w}_1\cdot\nabla\psi_2 - \boldsymbol{w}_2\cdot\nabla\psi_1 & \text{in } I\times\Omega,\\ \xi = -\alpha\Delta\psi + (3\phi^2 - 1)\psi + 6\phi\psi_1\psi_2 + l_{\mathrm{c}}\zeta & \text{in } I\times\Omega,\\ \partial_t\boldsymbol{w} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{w} + \nabla\pi = \mathcal{K}(\mu - l_{\mathrm{c}}\theta)\nabla\psi & \\ \qquad + \mathcal{K}(\xi - l_{\mathrm{c}}\zeta)\nabla\phi + \mathcal{K}(\xi_1 - l_{\mathrm{c}}\zeta_1)\nabla\psi_2 + \mathcal{K}(\xi_2 - l_{\mathrm{c}}\zeta_2)\nabla\psi_1 & \\ \qquad + (\alpha_2\psi + \alpha_3\zeta)\boldsymbol{g} - (\boldsymbol{w}_1\cdot\nabla)\boldsymbol{w}_2 - (\boldsymbol{w}_2\cdot\nabla)\boldsymbol{w}_1 & \text{in } I\times\Omega,\\ \operatorname{div}\boldsymbol{w} = 0 & \text{in } I\times\Omega,\\ \partial_t\zeta - l_{\mathrm{h}}\partial_t\psi + \boldsymbol{u}\cdot\nabla(\zeta - l_{\mathrm{h}}\psi) + \boldsymbol{w}\cdot\nabla(\theta - l_{\mathrm{h}}\phi) - \kappa\Delta\zeta & \\ \qquad = \boldsymbol{g}\cdot\boldsymbol{w} - \boldsymbol{w}_1\cdot\nabla(\zeta_2 - l_{\mathrm{h}}\psi_2) - \boldsymbol{w}_2\cdot\nabla(\zeta_1 - l_{\mathrm{h}}\psi_1) & \text{in } I\times\Omega,\end{cases}$$

with the boundary conditions $\partial_{\boldsymbol{n}}\psi = \partial_{\boldsymbol{n}}\Delta\psi = 0$, $\boldsymbol{w} = 0$, *and* $\partial_{\boldsymbol{n}}\zeta = 0$ *on* $I \times \Gamma$, *and the homogeneous initial conditions* $\psi(0) = 0$, $\boldsymbol{w}(0) = 0$, *and* $\zeta(0) = 0$ *in* Ω. *Here,* $(\psi_i, \xi_i, \boldsymbol{w}_i, \zeta_i) = \mathrm{D}S(\boldsymbol{y}, z)(\delta y_i, \delta z_i)$ *for* $i = 1, 2$ *and* $(\phi, \mu, \boldsymbol{u}, \theta) = S(\boldsymbol{y}, z)$.

Lemma 4.8. *The map $S : Q \to \mathcal{W}$ is weak-weak continuous, that is, if $(\boldsymbol{y}_k, z_k) \rightharpoonup (\boldsymbol{y}, z)$ in Q then $S(\boldsymbol{y}_k, z_k) \rightharpoonup S(\boldsymbol{y}, z)$ in $\mathcal{W}$.*

Proof. First we note that since the involved function spaces are both reflexive and separable, the notions of continuity and sequential continuity are equivalent with respect to the weak topologies, see [22, Theorem V.5.2]. Suppose $(\boldsymbol{y}_k, z_k) \rightharpoonup (\boldsymbol{y}, z)$ in Q and let $(\phi_k, \mu_k, \boldsymbol{u}_k, \theta_k) = S(\boldsymbol{y}_k, z_k)$. Then $\{(\boldsymbol{y}_k, z_k)\}_{k=1}^\infty$ is bounded in Q, and as consequence, $\{S(\boldsymbol{y}_k, z_k)\}_{k=1}^\infty$ is also bounded in $\mathcal{W}$ by (3.9). Then after taking a subsequence, denoted by the same indices for simplicity, we have $S(\boldsymbol{y}_k, z_k) \rightharpoonup (\phi, \mu, \boldsymbol{u}, \theta)$ in $\mathcal{W}$ for some $(\phi, \mu, \boldsymbol{u}, \theta) \in \mathcal{W}$. Recall from the Aubin-Lions-Simon Lemma that the embeddings $W^2(I; Y^3, Y^*) \subset L^2(I; Y^2)$, $W^2(I; \boldsymbol{V}) \subset L^2(I; \boldsymbol{H})$, and $W^2(I; Y) \subset L^2(I; X)$ are compact, and thus one can further extract a subsequence so that $\phi_k \to \phi$ in $L^2(I; Y^2)$, $\boldsymbol{u}_k \to \boldsymbol{u}$ in $L^2(I; \boldsymbol{H})$, and $\theta_k \to \theta$ in $L^2(I; X)$. By adapting the argument in Step 5 of the proof of Theorem 3.2, we have $N(\phi, \mu, \boldsymbol{u}, \theta, \boldsymbol{y}, z) = 0$.

Since the map $\varphi \mapsto \varphi(0)$ is continuous from $W^2(I; Y^3, Y^*)$ into Y, it follows that $\phi_k(0) \rightharpoonup \phi(0)$ in Y and thus $\phi(0) = \phi_0$. In a similar fashion, $\boldsymbol{u}(0) = \boldsymbol{u}_0$ in $\boldsymbol{H}$ and $\theta(0) = \theta_0$ in X. Thus, $N_0(\phi, \mu, \boldsymbol{u}, \theta, \boldsymbol{y}, z) = 0$ and hence $\mathcal{N}(\phi, \mu, \boldsymbol{u}, \theta, \boldsymbol{y}, z) = 0$. This implies that $(\phi, \mu, \boldsymbol{u}, \theta) = S(\boldsymbol{y}, z)$. In particular, $(\phi, \mu, \boldsymbol{u}, \theta)$ is uniquely determined, and as a result the whole sequence $\{S(\boldsymbol{y}_k, z_k)\}_{k=1}^\infty$ and not only the chosen subsequence must be weakly convergent. This means that $S(\boldsymbol{y}_k, z_k) \rightharpoonup S(\boldsymbol{y}, z)$ in $\mathcal{W}$. This establishes the weak-weak continuity of S. □

5. The Adjoint System

In this section, we shall analyze the adjoint system corresponding to the linearized state equation. Note from Theorem 4.6 that the adjoint operator $\mathrm{D}S(\boldsymbol{y}, z)^*$ of $\mathrm{D}S(\boldsymbol{y}, z)$ is given by

$$\mathrm{D}S(\boldsymbol{y}, z)^* = P^* A(S(\boldsymbol{y}, z))^{-*}, \tag{5.1}$$

where $A(S(\boldsymbol{y}, z))^{-*}$ denotes the inverse of the adjoint of $A(S(\boldsymbol{y}, z))$. If $S(\boldsymbol{y}, z) \in \mathcal{W}$, then $\mathrm{D}S(\boldsymbol{y}, z)^* \in \mathcal{L}(\mathcal{W}_0^*, Q)$, and if $S(\boldsymbol{y}, z) \in \mathcal{V}$, then $\mathrm{D}S(\boldsymbol{y}, z)^* \in \mathcal{L}(\mathcal{V}_0^*, Q)$, see Corollary 4.3.

Theorem 5.1. *Let $(\phi, \mu, \boldsymbol{u}, \theta) \in \mathcal{W}$. Given $(g_1, g_2, \boldsymbol{g}_3, g_4) \in \mathcal{W}_0^*$, there exists a unique solution $(\varphi, \eta, \boldsymbol{v}, \vartheta) \in \mathcal{Q}$ to the variational equation*

$$\begin{aligned}
&\langle \partial_t \psi + B_1(\boldsymbol{u}, \psi) + B_1(\boldsymbol{w}, \phi) + m A_N \xi, \varphi \rangle_{L^2(Y^*) \times L^2(Y)} \\
&\quad + \langle \eta, \xi - \alpha A_N \psi - f'(\phi)\psi - l_{\mathrm{c}}\zeta \rangle_{L^2(Y^*) \times L^2(Y)} \\
&\quad + \langle \partial_t \boldsymbol{w} + \mathrm{D}\boldsymbol{B}(\boldsymbol{u})\boldsymbol{w} + \nu \boldsymbol{A}_S \boldsymbol{w} - \mathcal{K}\boldsymbol{B}_2(\xi - l_{\mathrm{c}}\zeta, \phi), \boldsymbol{v} \rangle_{L^2(\boldsymbol{V}^*) \times L^2(\boldsymbol{V})} \\
&\quad + \ \langle -\mathcal{K}\boldsymbol{B}_2(\mu - l_{\mathrm{c}}\theta, \psi) - (\alpha_2 \psi + \alpha_3 \zeta)\boldsymbol{g}, \boldsymbol{v} \rangle_{L^2(\boldsymbol{V}^*) \times L^2(\boldsymbol{V})} \\
&\quad + \langle \partial_t \zeta - l_{\mathrm{h}} \partial_t \psi + B_1(\boldsymbol{u}, \zeta - l_{\mathrm{h}}\psi) + B_1(\boldsymbol{w}, \theta - l_{\mathrm{h}}\phi) + \kappa A_N \zeta - \boldsymbol{g} \cdot \boldsymbol{w}, \vartheta \rangle_{L^2(Y^*) \times L^2(Y)} \\
&\quad = \langle (g_1, g_2, \boldsymbol{g}_3, g_4), (\psi, \xi, \boldsymbol{w}, \zeta) \rangle_{\mathcal{W}_0^* \times \mathcal{W}_0} \qquad \forall (\psi, \xi, \boldsymbol{w}, \zeta) \in \mathcal{W}_0.
\end{aligned} \tag{5.2}$$

There exists a constant $c > 0$ depending on $\|(\phi, \mu, \boldsymbol{u}, \theta)\|_{\mathcal{W}}$ but not on $(\varphi, \eta, \boldsymbol{v}, \vartheta)$ such that

$$\|(\varphi, \eta, \boldsymbol{v}, \vartheta)\|_{\mathcal{Q}} \le c \|(g_1, g_2, \boldsymbol{g}_3, g_4)\|_{\mathcal{W}_0^*}. \tag{5.3}$$

Proof. Let us note that the variational equation (5.2) is equivalent to the equation $A(\phi,\mu,\boldsymbol{u},\theta)^*(\varphi,\eta,\boldsymbol{v},\vartheta) = (g_1,g_2,\boldsymbol{g}_3,g_4)$ in $\mathcal{W}_0^*$. Therefore, the existence, uniqueness, and stability of the solution to the variational equation is a direct consequence of the fact that $A(\phi,\mu,\boldsymbol{u},\theta)^*: \mathcal{Q} \to \mathcal{W}_0^*$ is an invertible operator having a bounded inverse. This remark follows directly from Theorem 4.1. Furthermore, one can take $c = \|A(\phi,\mu,\boldsymbol{u},\theta)^{-*}\|_{\mathcal{L}(\mathcal{W}_0^*,\mathcal{Q})}$ in (5.3). □

Remark 5.2. *Suppose that $(\phi,\mu,\boldsymbol{u},\theta) \in \mathcal{V}$ and $(g_1,g_2,\boldsymbol{g}_3,g_4) \in \mathcal{U}_0^* \subset \mathcal{W}_0^*$. Then for the unique solution of (5.2), it holds that $(\varphi,\eta,\boldsymbol{v},\vartheta) \in \mathcal{Y}$, and moreover, it satisfies the variational equation posed in the space $\mathcal{U}_0$ instead of $\mathcal{W}_0$. To be precise, in lieu of the duality pairings $\langle\cdot,\cdot\rangle_{L^2(Y^*)\times L^2(Y)}$, we have $\langle\cdot,\cdot\rangle_{L^2(Y^{2*})\times L^2(Y^2)}$ in the first and third terms and $(\cdot,\cdot)_{L^2(X)}$ in the second term, and $\langle\cdot,\cdot\rangle_{\mathcal{W}_0^*\times\mathcal{W}_0}$ is replaced by $\langle\cdot,\cdot\rangle_{\mathcal{U}_0^*\times\mathcal{U}_0}$. In this case, there exists a constant $c > 0$ depending on $\|(\phi,\mu,\boldsymbol{u},\theta)\|_{\mathcal{V}}$ but not on $(\varphi,\eta,\boldsymbol{v},\vartheta)$ such that*

$$\|(\varphi,\eta,\boldsymbol{v},\vartheta)\|_{\mathcal{Y}} \le c\|(g_1,g_2,\boldsymbol{g}_3,g_4)\|_{\mathcal{U}_0^*}. \tag{5.4}$$

These statements follow immediately from $A(\phi,\mu,\boldsymbol{u},\theta)^{-*} \in \mathcal{L}(\mathcal{U}_0^*,\mathcal{Y})$, see Corollary 4.3 above.

In the following theorem, we shall write the evolution system that governs the adjoint states under additional assumptions on $(g_1,g_2,\boldsymbol{g}_3,g_4)$. Before we proceed, we note that $A_N : L^2(I;Y^3) \to L^2(I;Y)$ and thus for the adjoint operator we have $A_N^* : L^2(I;Y^*) \to L^2(I;Y^{3*})$.

Theorem 5.3. *Let $(\phi,\mu,\boldsymbol{u},\theta) \in \mathcal{W}$. Suppose that the first, third, and fourth components of $(g_1,g_2,\boldsymbol{g}_3,g_4) \in \mathcal{W}_0^*$ have the following decompositions:*

$$g_1 = g_{1\Omega_T} + g_{1T}, \quad \boldsymbol{g}_3 = \boldsymbol{g}_{3\Omega_T} + \boldsymbol{g}_{3T}, \quad g_4 = g_{4\Omega_T} + g_{4T}, \tag{5.5}$$

where the first terms satisfy the regularity conditions

$$\begin{aligned} g_{1\Omega_T} &\in W_0^2(I;Y^3,Y^*)^* \cap L^{4/3}(I;Y^{3*}),\\ \boldsymbol{g}_{3\Omega_T} &\in W_0^2(I;\boldsymbol{V})^* \cap L^{4/3}(I;\boldsymbol{V}^*),\\ g_{4\Omega_T} &\in W_0^2(I;Y)^* \cap L^{4/3}(I;Y^*), \end{aligned}$$

and the second terms are defined by

$$\begin{aligned} \langle g_{1T},\psi\rangle_{W_0^2(Y^3,Y^*)^*\times W_0^2(Y^3,Y^*)} &= \langle\varphi_T,\psi(T)\rangle_{Y^*\times Y}\\ \langle \boldsymbol{g}_{3T},\boldsymbol{w}\rangle_{W_0^2(\boldsymbol{V})^*\times W_0^2(\boldsymbol{V})} &= (\boldsymbol{v}_T,\boldsymbol{w}(T))_{\boldsymbol{H}}\\ \langle g_{4T},\zeta\rangle_{W_0^2(Y)^*\times W_0^2(Y)} &= (\vartheta_T,\zeta(T))_X, \end{aligned}$$

with $\varphi_T \in Y^*$, $\boldsymbol{v}_T \in \boldsymbol{H}$, and $\vartheta_T \in X$. Then the solution $(\varphi, \eta, \boldsymbol{v}, \vartheta) \in \mathcal{Q}$ of (5.2) is given equivalently as the weak solution of the linear system

$$\begin{cases} -\partial_t\varphi + l_{\mathrm{h}}\partial_t\vartheta - B_1(\boldsymbol{u}, \varphi - l_{\mathrm{h}}\vartheta) - \alpha A_N^*\eta & \\ \qquad = \mathrm{D}f(\phi)^*\eta + \alpha_2 \boldsymbol{g}\cdot\boldsymbol{v} - \mathcal{K}B_1(\boldsymbol{v}, \mu - l_{\mathrm{c}}\theta) + g_{1\Omega_T} & \text{in } L^{4/3}(Y^{3*}), \\ -\eta = mA_N\varphi - \mathcal{K}B_1(\boldsymbol{v}, \phi) - g_2 & \text{in } L^2(I; Y^*), \\ -\partial_t\boldsymbol{v} + \mathrm{D}\boldsymbol{B}(\boldsymbol{u})^*\boldsymbol{v} + \nu\boldsymbol{A}_S\boldsymbol{v} & \\ \qquad = \vartheta\boldsymbol{g} - \boldsymbol{B}_2(\varphi, \phi) - \boldsymbol{B}_2(\vartheta, \theta - l_{\mathrm{h}}\phi) + \boldsymbol{g}_{3\Omega_T} & \text{in } L^{4/3}(I; \boldsymbol{V}^*), \\ -\partial_t\vartheta - B_1(\boldsymbol{u}, \vartheta) + \mathcal{K}l_{\mathrm{c}}B_1(\boldsymbol{v}, \phi) + \kappa A_N\vartheta & \\ \qquad = \alpha_3\boldsymbol{g}\cdot\boldsymbol{v} + l_{\mathrm{c}}\eta + g_{4\Omega_T} & \text{in } L^{4/3}(I; Y^*), \end{cases} \tag{5.6}$$

with the terminal conditions

$$\varphi(T) - l_{\mathrm{h}}\vartheta(T) = \varphi_T \text{ in } Y^*, \quad \boldsymbol{v}(T) = \boldsymbol{v}_T \text{ in } \boldsymbol{H}, \quad \vartheta(T) = \vartheta_T \text{ in } X. \tag{5.7}$$

Thus, $\varphi \in W^{4/3}(I; Y, Y^{3*})$, $\eta \in L^2(I; Y^*)$, $\boldsymbol{v} \in W^{4/3}(I; \boldsymbol{V})$, and $\vartheta \in W^{4/3}(I; Y)$. Furthermore, there exists $\mathfrak{C} = \mathfrak{C}(\|(\phi, \mu, \boldsymbol{u}, \theta)\|_{\mathcal{W}}) > 0$ such that

$$\begin{aligned} &\|\varphi\|_{W^{4/3}(Y,Y^{3*})} + \|\eta\|_{L^2(Y^*)} + \|\boldsymbol{v}\|_{W^{4/3}(\boldsymbol{V})} + \|\vartheta\|_{W^{4/3}(Y)} \\ &\quad \le \mathfrak{C}(\|\varphi_T\|_{Y^*} + \|\boldsymbol{v}_T\|_{\boldsymbol{H}} + \|\vartheta_T\|_X + \|g_{1\Omega_T}\|_{W_0^2(Y^3, Y^*)^* \cap L^{4/3}(Y^{3*})} \\ &\quad + \|g_2\|_{L^2(Y^*)} + \|\boldsymbol{g}_{3\Omega_T}\|_{W_0^2(\boldsymbol{V})^* \cap L^{4/3}(\boldsymbol{V}^*)} + \|g_{4\Omega_T}\|_{W_0^2(Y)^* \cap L^{4/3}(Y^*)}). \end{aligned} \tag{5.8}$$

Proof. We shall proceed through integration by parts and density arguments. The main idea is to take one arbitrary component of the product space $\mathcal{W}_0$ of test functions and the rest are set to zero. First, let us show that the solution of the variational equation (5.2) satisfies (5.6)-(5.8).

Step 1. *Time regularity of ϑ.* Taking $\varphi = 0$, $\xi = 0$, $\boldsymbol{w} = 0$, and $\zeta \in W_0^2(I; Y) \cap L^4(I; Y)$ in (5.2) yields the variational equation

$$\begin{aligned} &\langle \partial_t\zeta, \vartheta\rangle_{L^2(Y^*)\times L^2(Y)} + \langle B_1(\boldsymbol{u}, \zeta), \vartheta\rangle_{L^2(Y^*)\times L^2(Y)} + \kappa\langle A_N\zeta, \vartheta\rangle_{L^2(Y^*)\times L^2(Y)} \\ &\quad + \mathcal{K}l_{\mathrm{c}}\langle \boldsymbol{B}_2(\zeta, \phi), \boldsymbol{v}\rangle_{L^2(\boldsymbol{V}^*)\times L^2(\boldsymbol{V})} - (\alpha_3\boldsymbol{g}\cdot\boldsymbol{v}, \zeta)_{L^2(X)} - l_{\mathrm{c}}\langle \eta, \zeta\rangle_{L^2(Y^*)\times L^2(Y)} \\ &\quad = \langle g_{4\Omega_T}, \zeta\rangle_{L^{4/3}(Y^*)\times L^4(Y)} + (\vartheta_T, \zeta(T))_X. \end{aligned}$$

From the Hölder and Gagliardo-Nirenberg inequalities, we obtain that

$$|\langle B_1(\boldsymbol{u}, \zeta), \vartheta\rangle_{L^2(Y^*)\times L^2(Y)}| \le c\|\boldsymbol{u}\|_{L^\infty(\boldsymbol{H})}^{1/2}\|\boldsymbol{u}\|_{L^2(\boldsymbol{V})}^{1/2}\|\vartheta\|_{L^2(Y)}\|\zeta\|_{L^4(Y)}.$$

Therefore, by duality we have

$$\langle B_1(\boldsymbol{u}, \zeta), \vartheta\rangle_{L^2(Y^*)\times L^2(Y)} = -\langle B_1(\boldsymbol{u}, \vartheta), \zeta\rangle_{L^{4/3}(Y^*)\times L^4(Y)}$$

and $\|B_1(\boldsymbol{u}, \vartheta)\|_{L^{4/3}(Y^*)} \le c\|\boldsymbol{u}\|_{W^2(\boldsymbol{V})}\|\vartheta\|_{L^2(Y)}$. Similarly, we have

$$|\langle \boldsymbol{B}_2(\zeta, \phi), \boldsymbol{v}\rangle_{L^2(\boldsymbol{V}^*)\times L^2(\boldsymbol{V})}| \le c\|\phi\|_{W^2(Y^3, Y^*)}\|\boldsymbol{v}\|_{L^2(\boldsymbol{V})}\|\zeta\|_{L^4(Y)}.$$

By duality once again, this implies

$$\langle \boldsymbol{B}_2(\zeta, \phi), \boldsymbol{v}\rangle_{L^2(\boldsymbol{V}^*)\times L^2(\boldsymbol{V})} = \langle B_1(\boldsymbol{v}, \phi), \zeta\rangle_{L^{4/3}(Y^*)\times L^4(Y)}$$

and $\|B_1(\boldsymbol{v}, \phi)\|_{L^{4/3}(Y^*)} \le c\|\phi\|_{W^2(Y^3, Y^*)}\|\boldsymbol{v}\|_{L^2(\boldsymbol{V})}$. As a consequence, it holds that

$$\begin{aligned} &\langle \partial_t\zeta, \vartheta\rangle_{L^2(Y^*)\times L^2(Y)} = (\vartheta_T, \zeta(T))_X \\ &\quad + \langle B_1(\boldsymbol{u}, \vartheta) - \mathcal{K}l_{\mathrm{c}}B_1(\boldsymbol{v}, \phi) - \kappa A_N\vartheta + \alpha_3\boldsymbol{g}\cdot\boldsymbol{v} + l_{\mathrm{c}}\eta + g_{4\Omega_T}, \zeta\rangle_{L^{4/3}(Y^*)\times L^4(Y)}. \end{aligned} \tag{5.9}$$

Taking $\zeta \in C_0^\infty(I;Y)$ in (5.9) shows that the fourth equation in (5.6) is satisfied in the sense of vector-valued distributions and $\partial_t \vartheta \in L^{4/3}(I;Y^*)$. Thus, $\vartheta \in W^{4/3}(I;Y)$ and we deduce that

$$\begin{aligned}\|\partial_t\vartheta\|_{L^{4/3}(Y^*)} &\le c(\|B_1(\boldsymbol{u},\vartheta)\|_{L^{4/3}(Y^*)} + \|B_1(\boldsymbol{v},\phi)\|_{L^{4/3}(Y^*)}) \\ &\quad + c\|A_N\zeta\|_{L^2(Y^*)} + \|\boldsymbol{v}\|_{L^2(\boldsymbol{H})} + \|\eta\|_{L^2(Y^*)} + \|g_{4\Omega_T}\|_{L^{4/3}(Y^*)}) \\ &\le \mathfrak{C}(\|\vartheta\|_{L^2(Y)} + \|\zeta\|_{L^2(Y)} + \|\boldsymbol{v}\|_{L^2(\boldsymbol{V})} + \|\eta\|_{L^2(Y^*)} + \|g_{4\Omega_T}\|_{L^{4/3}(Y^*)}),\end{aligned} \tag{5.10}$$

where $\mathfrak{C} = \mathfrak{C}(\|\phi\|_{W^2(Y^3,Y^*)}, \|\boldsymbol{u}\|_{W^2(\boldsymbol{V})})$. To prove that the terminal condition for ϑ holds, let us note that $C^1(\bar{I};Y)$ is dense in $W^{4/3}(I;Y)$, see [49, Lemma 7.2] for instance, and therefore we can find a sequence $\{\vartheta_k\}_{k=1}^\infty \subset C^1(\bar{I};Y)$ such that $\vartheta_k \to \vartheta$ in $W^{4/3}(I;Y)$. Hence, for every $\zeta \in C^1(\bar{I};Y)$ such that $\zeta(0) = 0$, we have

$$\begin{aligned}\langle \partial_t\zeta, \vartheta\rangle_{L^2(Y^*)\times L^2(Y)} &= \lim_{k\to\infty}(\partial_t\zeta,\vartheta_k)_{L^2(X)} = \lim_{k\to\infty}[(\zeta(T),\vartheta_k(T))_X - (\zeta,\partial_t\vartheta_k)_{L^2(X)}] \\ &= \langle \vartheta(T),\zeta(T)\rangle_{Y^*\times Y} - \langle\partial_t\vartheta,\zeta\rangle_{L^{4/3}(Y^*)\times L^4(Y)},\end{aligned}$$

where in the last equation we used the continuous embedding $W^{4/3}(I;Y) \subset C(\bar{I};Y^*)$. Using this in (5.9) we obtain $\langle \vartheta(T) - \vartheta_T, \zeta(T)\rangle_{Y^*\times Y} = 0$, and since $\zeta(T)$ can assume any value in Y, it follows that $\vartheta(T) = \vartheta_T$ in X.

Step 2. *Time regularity of φ.* By taking $\psi \in W_0^2(I;Y^3,Y^*) \cap L^4(I;Y^3)$, $\xi = 0$, $\boldsymbol{w} = 0$, and $\zeta = 0$ in (5.2), we obtain the variational equation

$$\begin{aligned}&\langle\partial_t\psi, \varphi - l_{\mathrm{h}}\vartheta\rangle_{L^2(Y^*)\times L^2(Y)} + \langle B_1(\boldsymbol{u},\psi),\varphi - l_{\mathrm{h}}\vartheta\rangle_{L^2(Y^*)\times L^2(Y)} \\ &\quad - \langle\eta,\alpha A_N\psi\rangle_{L^2(Y^*)\times L^2(Y)} - \langle\eta, f'(\phi)\psi\rangle_{L^2(Y^*)\times L^2(Y)} \\ &\quad - \mathcal{K}\langle\boldsymbol{B}_2(\mu - l_{\mathrm{c}}\theta,\psi),\boldsymbol{v}\rangle_{L^2(\boldsymbol{V}^*)\times L^2(\boldsymbol{V})} - (\alpha_2\boldsymbol{g}\cdot\boldsymbol{v},\psi)_{L^2(X)} \\ &\quad = \langle g_{1\Omega_T},\psi\rangle_{L^{4/3}(Y^{3*})\times L^4(Y^3)} + \langle\varphi_T,\psi(T)\rangle_{Y^*\times Y}.\end{aligned} \tag{5.11}$$

Let us note that by assumption $\psi \in L^4(I;Y) \cap L^2(I;Y^3)$. By using a similar argument as in the previous step, we obtain that

$$\begin{aligned}\langle B_1(\boldsymbol{u},\psi),\varphi - l_{\mathrm{h}}\vartheta\rangle_{L^2(Y^*)\times L^2(Y)} &= -\langle B_1(\boldsymbol{u},\varphi - l_{\mathrm{h}}\vartheta),\psi\rangle_{L^{4/3}(Y^*)\times L^4(Y)} \\ \|B_1(\boldsymbol{u},\varphi - l_{\mathrm{h}}\vartheta)\|_{L^{4/3}(Y^*)} &\le c\|\boldsymbol{u}\|_{W^2(\boldsymbol{V})}(\|\varphi\|_{L^2(Y)} + \|\vartheta\|_{L^2(Y)}).\end{aligned}$$

For the third term in (5.11), we have $\|A_N^*\eta\|_{L^2(Y^{3*})} \le c\|\eta\|_{L^2(Y^*)}$ and

$$\langle\eta,\alpha A_N\psi\rangle_{L^2(Y^*)\times L^2(Y)} = \langle\alpha A_N^*\eta,\psi\rangle_{L^2(Y^{3*})\times L^2(Y^3)}.$$

To treat the fourth duality pairing in (5.11), observe that $\mathrm{D}f(\phi)\psi = f'(\phi)\psi \in L^2(I;Y)$ and

$$\begin{aligned}\|\mathrm{D}f(\phi)\psi\|_{L^2(Y)} &\le \|(3\phi^2-1)\psi\|_{L^2(X)} + \|(3\phi^2-1)\nabla\psi\|_{L^2(\boldsymbol{X})} + \|6\phi\psi\nabla\phi\|_{L^2(\boldsymbol{X})} \\ &\le c((\|\phi\|_{L^\infty(Y)}^2 + 1)\|\psi\|_{L^2(Y^2)} + \|\phi\|_{L^\infty(Y)}^{3/2}\|\psi\|_{L^4(Y)}\|\phi\|_{L^2(Y^2)}^{1/2}) \\ &\le \mathfrak{C}(\|\phi\|_{W^2(Y^3,Y^*)})\|\psi\|_{L^4(Y^2)}.\end{aligned}$$

This estimate implies that $\|\mathrm{D}f(\phi)^*\eta\|_{L^{4/3}(Y^{2*})} \le \mathfrak{C}(\|\phi\|_{W^2(Y^3,Y^*)})\|\eta\|_{L^2(Y^*)}$ and

$$\langle\eta, f'(\phi)\psi\rangle_{L^2(Y^*)\times L^2(Y)} = \langle\mathrm{D}f(\phi)^*\eta,\psi\rangle_{L^{4/3}(Y^{2*})\times L^4(Y^2)}.$$

On the other hand, substituting the equation $\mu - l_c\theta = -\alpha\Delta\phi + \phi^3 - \phi$ in the fifth term of (5.11) and then applying the Hölder inequality, Green identity, and the estimate (2.4), we have

$$\begin{aligned}
|\langle \boldsymbol{B}_2(\mu - l_c\theta, \psi), \boldsymbol{v}\rangle_{L^2(\boldsymbol{V}^*)\times L^2(\boldsymbol{V})}| &\le \int_0^T |(\boldsymbol{v}\cdot\nabla\psi, \mu - l_c\theta)_X|\,\mathrm{d}t \\
\le \int_0^T \alpha|((\nabla^2\psi)\boldsymbol{v}, \nabla\phi)_{\boldsymbol{X}} &+ \alpha|((\nabla\boldsymbol{v})\nabla\psi, \nabla\phi)_{\boldsymbol{X}}| + |(\boldsymbol{v}\cdot\nabla\psi, \phi^3 - \phi)_X|\,\mathrm{d}t \\
\le c\int_0^T \|\boldsymbol{v}\|_{\boldsymbol{V}}\|\phi\|_Y^{1/2}\|\phi\|_{Y^2}^{1/2}\|\psi\|_{Y^2} &+ (\|\phi\|_Y^3 + \|\phi\|_X)\|\boldsymbol{v}\|_{\boldsymbol{V}}\|\psi\|_{Y^2}\,\mathrm{d}t \\
\le c(1 + \|\phi\|_{L^\infty(Y)}^{1/2}\|\phi\|_{L^2(Y^2)}^{1/2} &+ \|\phi\|_{L^\infty(Y)}^3 + \|\phi\|_{L^\infty(X)})\|\boldsymbol{v}\|_{L^2(\boldsymbol{V})}\|\psi\|_{L^4(Y^2)}.
\end{aligned}$$

It follows that $\|B_1(\boldsymbol{v}, \mu - l_c\theta)\|_{L^{4/3}(Y^{2*})} \le \mathfrak{C}(\|\phi\|_{W^2(Y^3,Y^*)})\|\boldsymbol{v}\|_{L^2(\boldsymbol{V})}$ and by duality

$$\langle \boldsymbol{B}_2(\mu - l_c\theta, \psi), \boldsymbol{v}\rangle_{L^2(\boldsymbol{V}^*)\times L^2(\boldsymbol{V})} = -\langle B_1(\boldsymbol{v}, \mu - l_c\theta), \psi\rangle_{L^{4/3}(Y^{2*})\times L^4(Y^2)}.$$

Using the continuous embeddings $L^{4/3}(I;Y^*) \subset L^{4/3}(I;Y^{2*}) \subset L^{4/3}(I;Y^{3*})$ and $L^2(I;Y^{3*}) \subset L^{4/3}(I;Y^{3*})$, we find that

$$\begin{aligned}
\langle \partial_t\psi, \varphi - l_h\vartheta\rangle_{L^2(Y^*)\times L^2(Y)} &= \langle \varphi_T, \psi(T)\rangle_{Y^*\times Y} + \langle B_1(\boldsymbol{u}, \varphi - l_h\vartheta), \psi\rangle_{L^{4/3}(Y^{3*})\times L^4(Y^3)} \\
&+ \langle \alpha A_N^*\eta + \mathrm{D}f(\phi)^*\eta + \alpha_2\boldsymbol{g}\cdot\boldsymbol{v} - \mathcal{K}B_1(\boldsymbol{v}, \mu - l_c\theta) + g_{1\Omega_T}, \psi\rangle_{L^{4/3}(Y^{3*})\times L^4(Y^3)}.
\end{aligned}$$

For $\psi \in C_0^\infty(I;Y^3)$, this implies that the first equation in (5.6) holds true and $\partial_t\varphi - l_h\partial_t\vartheta \in L^{4/3}(I;Y^{3*})$. Since $\partial_t\vartheta \in L^{4/3}(I;Y^*) \subset L^{4/3}(I;Y^{3*})$ we have $\partial_t\varphi \in L^{4/3}(I;Y^{3*})$, and hence $\varphi \in W^{4/3}(I;Y,Y^{3*})$. Moreover, one has

$$\begin{aligned}
\|\partial_t\varphi\|_{L^{4/3}(I;Y^{3*})} &\le c(\|\partial_t\vartheta\|_{L^{4/3}(Y^*)} + \|B_1(\boldsymbol{u}, \varphi - l_h\vartheta)\|_{L^{4/3}(Y^*)} + \|A_N^*\eta\|_{L^2(Y^{3*})}) \\
&+ c(\|\mathrm{D}f(\phi)^*\eta\|_{L^{4/3}(Y^{2*})} + \|\boldsymbol{v}\|_{L^2(\boldsymbol{H})} + \|B_1(\boldsymbol{v}, \mu - l_c\theta)\|_{L^{4/3}(Y^{2*})}) \\
&+ c\|g_{1\Omega_T}\|_{L^{4/3}(Y^{3*})} \le \mathfrak{C}(\|\partial_t\vartheta\|_{L^{4/3}(Y^*)} + \|\varphi\|_{L^2(Y)} + \|\eta\|_{L^2(Y^*)} + \|\boldsymbol{v}\|_{L^2(\boldsymbol{V})}) \\
&+ \mathfrak{C}(\|\vartheta\|_{L^2(Y)} + \|g_{1\Omega_T}\|_{L^{4/3}(Y^{3*})}) \qquad (5.12)
\end{aligned}$$

where $\mathfrak{C} = \mathfrak{C}(\|\phi\|_{W^2(Y^3,Y^*)}, \|\boldsymbol{u}\|_{W^2(\boldsymbol{V})})$. Using a similar density argument as in Step 1, the terminal condition $\varphi(T) - l_h\vartheta(T) = \varphi_T$ is satisfied in Y^*.

Step 3. *Equation for η.* If $\xi \in L^2(I;Y)$, $\psi = 0$, $\boldsymbol{w} = 0$, and $\zeta = 0$ in (5.2) then we get

$$\langle \eta + mA_N\varphi - \mathcal{K}B_1(\boldsymbol{v}, \phi) - g_2, \xi\rangle_{L^2(Y^*)\times L^2(Y)} = 0.$$

Note that $\|B_1(\boldsymbol{v}, \phi)\|_{L^2(Y^*)} \le c\|\boldsymbol{v}\|_{L^2(\boldsymbol{V})}\|\phi\|_{L^\infty(Y)}$. These imply that the second equation in (5.6) holds and

$$\|\eta\|_{L^2(Y^*)} \le \mathfrak{C}(\|\phi\|_{W^2(Y^3,Y^*)})(\|\varphi\|_{L^2(Y)} + \|\boldsymbol{v}\|_{L^2(\boldsymbol{V})} + \|g_2\|_{L^2(Y^*)}). \qquad (5.13)$$

Step 4. *Time regularity of v.* To prove the regularity of $\partial_t\boldsymbol{v}$, we take $\boldsymbol{w} \in W_0^2(I;\boldsymbol{V}) \cap L^4(I;\boldsymbol{V})$, $\psi = 0$, $\xi = 0$, and $\zeta = 0$ in (5.2) so that

$$\begin{aligned}
\langle \partial_t\boldsymbol{w}, \boldsymbol{v}\rangle_{L^2(\boldsymbol{V}^*)\times L^2(\boldsymbol{V})} &+ \langle \mathrm{D}\boldsymbol{B}(\boldsymbol{u})\boldsymbol{w} + \nu\boldsymbol{A}_S\boldsymbol{w}, \boldsymbol{v}\rangle_{L^2(\boldsymbol{V}^*)\times L^2(\boldsymbol{V})} \\
&+ \langle B_1(\boldsymbol{w}, \phi), \varphi\rangle_{L^2(Y^*)\times L^2(Y)} + \langle B_1(\boldsymbol{w}, \theta - l_h\phi), \vartheta\rangle_{L^2(Y^*)\times L^2(Y)} \\
&- (\vartheta\boldsymbol{g}, \boldsymbol{w})_{L^2(\boldsymbol{X})} = \langle \boldsymbol{g}_{3\Omega_T}, \boldsymbol{w}\rangle_{L^{4/3}(\boldsymbol{V}^*)\times L^4(\boldsymbol{V})} + (\boldsymbol{v}_T, \boldsymbol{w}(T))_{\boldsymbol{H}}.
\end{aligned}$$

From the Hölder and Gagliardo-Nirenberg inequalities we obtain the following

$$\begin{aligned}
|\langle \mathrm{D}\boldsymbol{B}(\boldsymbol{u})\boldsymbol{w}, \boldsymbol{v}\rangle_{L^2(\boldsymbol{V}^*)\times L^2(\boldsymbol{V})}| &\le c\|\boldsymbol{u}\|_{L^\infty(\boldsymbol{H})}^{1/2}\|\boldsymbol{u}\|_{L^2(\boldsymbol{V})}^{1/2}\|\boldsymbol{v}\|_{L^2(\boldsymbol{V})}\|\boldsymbol{w}\|_{L^4(\boldsymbol{V})}\\
|\langle B_1(\boldsymbol{w},\phi),\varphi\rangle_{L^2(Y^*)\times L^2(Y)}| &\le c\|\phi\|_{L^\infty(Y)}\|\varphi\|_{L^2(Y)}\|\boldsymbol{w}\|_{L^2(\boldsymbol{V})}\\
|\langle B_1(\boldsymbol{w},\theta-l_{\mathrm{h}}\phi),\vartheta\rangle_{L^2(Y^*)\times L^2(Y)}| &\le c(\|\theta\|_{W^2(Y)}+\|\phi\|_{L^\infty(Y)})\|\vartheta\|_{L^2(Y)}\|\boldsymbol{w}\|_{L^4(\boldsymbol{V})}.
\end{aligned}$$

Applying duality argument once more, it follows from these estimates that

$$\begin{aligned}
\|\mathrm{D}\boldsymbol{B}(\boldsymbol{u})^*\boldsymbol{v}\|_{L^{4/3}(\boldsymbol{V}^*)} &\le c\|\boldsymbol{u}\|_{W^2(\boldsymbol{V})}\|\boldsymbol{v}\|_{L^2(\boldsymbol{V})}\\
\|\boldsymbol{B}_2(\phi,\varphi)\|_{L^2(\boldsymbol{V}^*)} &\le c\|\phi\|_{W^2(Y^3,Y^*)}\|\varphi\|_{L^2(Y)}\\
\|\boldsymbol{B}_2(\vartheta,\theta-l_{\mathrm{h}}\phi)\|_{L^{4/3}(\boldsymbol{V}^*)} &\le c(\|\theta\|_{W^2(Y)}+\|\phi\|_{W^2(Y^3,Y^*)})\|\vartheta\|_{L^2(Y)}
\end{aligned}$$

and moreover we have

$$\begin{aligned}
\langle \mathrm{D}\boldsymbol{B}(\boldsymbol{u})\boldsymbol{w},\boldsymbol{v}\rangle_{L^2(\boldsymbol{V}^*)\times L^2(\boldsymbol{V})} &= \langle \mathrm{D}\boldsymbol{B}(\boldsymbol{u})^*\boldsymbol{v},\boldsymbol{w}\rangle_{L^{4/3}(\boldsymbol{V}^*)\times L^4(\boldsymbol{V})}\\
\langle B_1(\boldsymbol{w},\phi),\varphi\rangle_{L^2(Y^*)\times L^2(Y)} &= \langle \boldsymbol{B}_2(\varphi,\phi),\boldsymbol{w}\rangle_{L^2(\boldsymbol{V}^*)\times L^2(\boldsymbol{V})}\\
\langle B_1(\boldsymbol{w},\theta-l_{\mathrm{h}}\phi),\vartheta\rangle_{L^2(Y^*)\times L^2(Y)} &= \langle \boldsymbol{B}_2(\vartheta,\theta-l_{\mathrm{h}}\phi),\boldsymbol{w}\rangle_{L^{4/3}(\boldsymbol{V}^*)\times L^4(\boldsymbol{V})}.
\end{aligned}$$

Taking these into account, we can now rewrite the variational equation for $\boldsymbol{v}$ according to

$$\begin{aligned}
\langle \partial_t\boldsymbol{w},\boldsymbol{v}\rangle_{L^2(\boldsymbol{V}^*)\times L^2(\boldsymbol{V})} &= (\boldsymbol{v}_T,\boldsymbol{w}(T))_{\boldsymbol{H}} + \langle \vartheta\boldsymbol{g} - \mathrm{D}\boldsymbol{B}(\boldsymbol{u})^*\boldsymbol{v},\boldsymbol{w}\rangle_{L^{4/3}(\boldsymbol{V}^*)\times L^4(\boldsymbol{V})}\\
&\quad + \langle -\nu\boldsymbol{A}_S\boldsymbol{v} - \boldsymbol{B}_2(\varphi,\phi) - \boldsymbol{B}_2(\vartheta,\theta-l_{\mathrm{h}}\phi) + \boldsymbol{g}_{3\Omega_T},\boldsymbol{w}\rangle_{L^{4/3}(\boldsymbol{V}^*)\times L^4(\boldsymbol{V})}.
\end{aligned}$$

This implies the third equation in (5.6), and as a consequence $\partial_t\boldsymbol{v}\in L^{4/3}(I;\boldsymbol{V}^*)$. Hence $\boldsymbol{v}\in W^{4/3}(I;\boldsymbol{V})$ and

$$\|\partial_t\boldsymbol{v}\|_{L^{4/3}(\boldsymbol{V}^*)} \le \mathfrak{C}(\|\varphi\|_{L^2(Y)}+\|\boldsymbol{v}\|_{L^2(\boldsymbol{V})}+\|\vartheta\|_{L^2(Y)}+\|\boldsymbol{g}_{3\Omega_T}\|_{L^{4/3}(\boldsymbol{V}^*)}) \tag{5.14}$$

where $\mathfrak{C}=\mathfrak{C}(\|\phi\|_{W^2(Y^3,Y^*)},\|\boldsymbol{u}\|_{W^2(\boldsymbol{V})},\|\theta\|_{W^2(Y)})$. Using a similar argument as in Step 1, one can deduce that $\boldsymbol{v}(T)=\boldsymbol{v}_T$ in $\boldsymbol{H}$.

It remains to show the estimate (5.8). From the decompositions of g_1, $\boldsymbol{g}_3$, g_4, and the continuity of $W_0^2(I;Y^3,Y^*)\subset C(\bar{I};Y)$, $W_0^2(I;Y)\subset C(\bar{I};X)$, and $W_0^2(I;\boldsymbol{V})\subset C(\bar{I};\boldsymbol{H})$, it follows that

$$\begin{aligned}
\|g_4\|_{W_0^2(Y)^*} &\le c(\|g_{4\Omega_T}\|_{W_0^2(Y)^*}+\|\vartheta_T\|_X)\\
\|\boldsymbol{g}_3\|_{W_0^2(\boldsymbol{V})^*} &\le c(\|\boldsymbol{g}_{3\Omega_T}\|_{W_0^2(\boldsymbol{V})^*}+\|\boldsymbol{v}_T\|_{\boldsymbol{H}})\\
\|g_1\|_{W_0^2(Y^3,Y^*)^*} &\le c(\|g_{1\Omega_T}\|_{W_0^2(Y^3,Y^*)^*}+\|\varphi_T\|_{Y^*}).
\end{aligned}$$

Using these estimates along with (5.3), (5.10), (5.12), (5.13), and (5.14), we obtain (5.8).

Finally, to show that every solution of (5.6) also satisfies (5.2), one can apply smooth test functions in (5.6), take the sum of the resulting equations and perform the above steps backward by passing all derivatives to the test functions. The proof of the theorem is now complete. □

Remark 5.4. *In terms of PDEs, the solution of the adjoint system* (5.6) *can be equivalently characterized as the* very weak solution *of the following linear system:*

$$\begin{cases} -\partial_t\varphi + l_{\rm h}\partial_t\vartheta - \boldsymbol{u}\cdot\nabla(\varphi - l_{\rm h}\vartheta) + \alpha\Delta\eta \\ \qquad = f'(\phi)\eta + \alpha_2\boldsymbol{g}\cdot\boldsymbol{v} - \mathcal{K}\boldsymbol{v}\cdot\nabla(\mu - l_{\rm c}\theta) + g_{1\Omega_T} & \text{in } I\times\Omega, \\ -\eta = -m\Delta\varphi - \mathcal{K}\boldsymbol{v}\cdot\nabla\phi - g_2 & \text{in } I\times\Omega, \\ -\partial_t\boldsymbol{v} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{v} + (\nabla\boldsymbol{u})^\top\boldsymbol{v} - \nu\Delta\boldsymbol{v} + \nabla\pi \\ \qquad = \vartheta\boldsymbol{g} - \varphi\nabla\phi - \vartheta\nabla(\theta - l_{\rm h}\phi) + \boldsymbol{g}_{3\Omega_T} & \text{in } I\times\Omega, \\ \operatorname{div}\boldsymbol{v} = 0 & \text{in } I\times\Omega, \\ -\partial_t\vartheta - \boldsymbol{u}\cdot\nabla\vartheta + \mathcal{K}l_{\rm c}\boldsymbol{v}\cdot\nabla\phi - \kappa\Delta\vartheta = \alpha_3\boldsymbol{g}\cdot\boldsymbol{v} + l_{\rm c}\eta + g_{4\Omega_T} & \text{in } I\times\Omega, \end{cases}$$

satisfying the terminal conditions $\varphi(T) = \varphi_T + l_{\rm h}\vartheta_T$, $\boldsymbol{v}(T) = \boldsymbol{v}_T$, $\vartheta(T) = \vartheta_T$ *in* Ω, *and the boundary conditions* $\partial_{\boldsymbol{n}}\varphi = \partial_{\boldsymbol{n}}\eta = 0$, $\boldsymbol{v} = 0$, *and* $\partial_{\boldsymbol{n}}\theta = 0$ *on* $I\times\Gamma$ *in the weak sense.*

Next, we shall prove the regularity of the adjoint states by gradually considering additional regularity on the desired data for the temperature, fluid velocity, order parameter, and chemical potential.

Corollary 5.5. *Let* $(\phi,\mu,\boldsymbol{u},\theta)\in\mathcal{V}$. *Suppose that* $(g_1, g_2, \boldsymbol{g}_3, g_4)\in\mathcal{U}_0^*$ *admits the decompositions* (5.5) *with* $g_{1\Omega_T}\in L^2(I;Y^{2*})$, $\boldsymbol{g}_{3\Omega_T}\in L^2(I;\boldsymbol{V}^*)$, $g_{4\Omega_T}\in L^2(I;X)$,

$$\langle g_{1T},\psi\rangle_{W_0^2(Y^2)^*\times W_0^2(Y^2)} = (\varphi_T, \psi(T))_X$$
$$\langle \boldsymbol{g}_{3T},\boldsymbol{w}\rangle_{W_0^2(\boldsymbol{V})^*\times W_0^2(\boldsymbol{V})} = (\boldsymbol{v}_T, \boldsymbol{w}(T))_{\boldsymbol{H}}$$
$$\langle g_{4T},\zeta\rangle_{W_0^2(X,Y^{2*})^*\times W_0^2(X,Y^{2*})} = \langle \zeta(T), \vartheta_T\rangle_{Y^*\times Y}$$

where $\varphi_T\in X$, $\boldsymbol{v}_T\in\boldsymbol{H}$, *and* $\vartheta_T\in Y$. *Then for the solution of* (5.6) *it holds that* $\varphi\in W^2(I;Y^2)$, $\eta\in L^2(I;X)$, $\boldsymbol{v}\in W^2(I;\boldsymbol{V})$, *and* $\vartheta\in W^2(I;Y^2,X)$. *Moreover, there exists* $\mathfrak{C} = \mathfrak{C}(\|(\phi,\mu,\boldsymbol{u},\theta)\|_{\mathcal{V}}) > 0$ *such that*

$$\begin{aligned} \|\varphi\|_{W^2(Y^2)} + \|\eta\|_{L^2(X)} + \|\boldsymbol{v}\|_{W^2(\boldsymbol{V})} + \|\vartheta\|_{W^2(Y^2,X)} \le \mathfrak{C}(\|\varphi_T\|_X + \|\boldsymbol{v}_T\|_{\boldsymbol{H}} \\ + \|\vartheta_T\|_Y + \|g_{1\Omega_T}\|_{L^2(Y^{2*})} + \|g_2\|_{L^2(X)} + \|\boldsymbol{g}_{3\Omega_T}\|_{L^2(\boldsymbol{V}^*)} + \|g_{4\Omega_T}\|_{L^2(X)}). \end{aligned} \tag{5.15}$$

Proof. Let us recall that $\mathcal{U}_0^* = W_0^2(I;Y^2)^*\times L^2(I;X)\times W_0^2(I;\boldsymbol{V})^*\times W_0^2(I;X,Y^{2*})^*$. Under the given assumptions, the solution to (5.2) satisfies $(\varphi,\eta,\boldsymbol{v},\vartheta)\in\mathcal{Y}$, see Remark 5.2. Furthermore, the solution satisfies the estimate

$$\|\varphi\|_{L^2(Y^2)} + \|\eta\|_{L^2(X)} + \|\boldsymbol{v}\|_{L^2(\boldsymbol{V})} + \|\vartheta\|_{L^2(Y^2)} \le c\|(g_1,g_2,\boldsymbol{g}_3,g_4)\|_{\mathcal{U}_0^*} \tag{5.16}$$

thanks to (5.4). Using the decompositions of g_1, $\boldsymbol{g}_3$, g_4, and the continuity of the embeddings $W_0^2(I;Y^2)\subset C(\bar I;X)$, $W_0^2(I;\boldsymbol{V})\subset C(\bar I;\boldsymbol{H})$, and $W_0^2(I;X,Y^{2*})\subset C(\bar I;Y^*)$, we have

$$\|g_1\|_{W_0^2(Y^2)^*} \le c(\|g_{1\Omega_T}\|_{L^2(Y^{2*})} + \|\varphi_T\|_X) \tag{5.17}$$
$$\|\boldsymbol{g}_3\|_{W_0^2(\boldsymbol{V})^*} \le c(\|\boldsymbol{g}_{3\Omega_T}\|_{L^2(V^*)} + \|\boldsymbol{v}_T\|_{\boldsymbol{H}}) \tag{5.18}$$
$$\|g_4\|_{W_0^2(X,Y^{2*})^*} \le c(\|g_{4\Omega_T}\|_{L^2(X)} + \|\vartheta_T\|_Y). \tag{5.19}$$

Observe that $g_{1\Omega_T}\in L^2(I;Y^{2*})\subset W_0^2(I;Y^3,Y^*)^*\cap L^{4/3}(I;Y^{3*})$, $g_2\in L^2(I;X)\subset L^2(I;Y^*)$, $\boldsymbol{g}_{3\Omega_T}\in L^2(I;\boldsymbol{V}^*)\subset W_0^2(I;\boldsymbol{V})^*\cap L^{4/3}(I;\boldsymbol{V}^*)$, and $g_{4\Omega_T}\in L^2(I;X)\subset W_0^2(I;Y)^*\cap L^{4/3}(I;Y^*)$. Hence $(\varphi,\eta,\boldsymbol{v},\vartheta)$ satisfies (5.6) according to Theorem 5.3.

Moreover, the term $\mathrm{D}f(\phi)^*\eta$ in the first equation equation of (5.6) can be replaced by $f'(\phi)\eta$.

Using the Hölder inequality, we obtain $\|B_1(\boldsymbol{u},\vartheta)\|_{L^2(X)} \le c\|\boldsymbol{u}\|_{L^\infty(\boldsymbol{V})}\|\vartheta\|_{L^2(Y^2)}$ and $\|B_1(\boldsymbol{v},\phi)\|_{L^2(X)} \le c\|\phi\|_{W^2(Y^4,X)}\|\boldsymbol{v}\|_{L^2(\boldsymbol{V})}$. Thus, $\partial_t\vartheta \in L^2(I;X)$ and

$$\|\partial_t\vartheta\|_{L^2(X)} \le \mathfrak{C}(\|\vartheta\|_{L^2(Y^2)} + \|\boldsymbol{v}\|_{L^2(\boldsymbol{V})} + \|\eta\|_{L^2(X)} + \|g_{4\Omega_T}\|_{L^2(X)}), \tag{5.20}$$

where $\mathfrak{C} = \mathfrak{C}(\|\phi\|_{W^2(Y^4,X)}, \|\boldsymbol{u}\|_{W^2(\boldsymbol{V}^2,\boldsymbol{H})})$. From the equation satisfied by η and $\|B_1(\boldsymbol{v},\phi)\|_{L^2(X)} \le c\|\boldsymbol{v}\|_{L^2(\boldsymbol{V})}\|\phi\|_{L^\infty(Y^2)}$ we obtain that

$$\|\eta\|_{L^2(X)} \le \mathfrak{C}(\|\phi\|_{W^2(Y^4,X)})(\|\varphi\|_{L^2(Y^2)} + \|\boldsymbol{v}\|_{L^2(\boldsymbol{V})} + \|g_2\|_{L^2(X)}). \tag{5.21}$$

Note that $\|B_1(\boldsymbol{u},\varphi - l_{\mathrm{h}}\vartheta)\|_{L^2(X)} \le c\|\boldsymbol{u}\|_{L^\infty(\boldsymbol{V})}(\|\varphi\|_{L^2(Y^2)} + \|\vartheta\|_{L^2(Y^2)})$ and $\|B_1(\boldsymbol{v},\mu - l_{\mathrm{c}}\theta)\|_{L^2(Y^{2*})} \le c\|\boldsymbol{v}\|_{L^2(\boldsymbol{V})}(\|\mu\|_{L^\infty(X)} + \|\theta\|_{L^\infty(X)})$. Also, using the continuity of the embedding $L^\infty(I;Y^2) \subset L^\infty(\Omega_T)$ we get $\|f'(\phi)\|_{L^\infty(\Omega_T)} \le c(\|\phi\|^2_{L^\infty(Y^2)} + 1)$. Thus, the time derivative of φ can be estimated as follows:

$$\begin{aligned}\|\partial_t\varphi\|_{L^2(Y^{2*})} &\le c(\|\partial_t\vartheta\|_{L^2(Y^*)} + \|B_1(\boldsymbol{u},\varphi - l_{\mathrm{h}}\vartheta)\|_{L^2(X)} + \|A_N^*\eta\|_{L^2(Y^{2*})})\\ &\quad + c(\|\mathrm{D}f(\phi)^*\eta\|_{L^2(X)} + \|\boldsymbol{v}\|_{L^2(\boldsymbol{H})} + \|B_1(\boldsymbol{v},\mu - l_{\mathrm{c}}\theta)\|_{L^2(Y^{2*})})\\ &\quad + c\|g_{1\Omega_T}\|_{L^2(Y^{2*})} \le \mathfrak{C}(\|\partial_t\vartheta\|_{L^2(X)} + \|\varphi\|_{L^2(Y^2)} + \|\eta\|_{L^2(X)})\\ &\quad + \mathfrak{C}(\|\boldsymbol{v}\|_{L^2(\boldsymbol{V})} + \|\vartheta\|_{L^2(Y^2)} + \|g_{1\Omega_T}\|_{L^2(Y^{2*})})\end{aligned} \tag{5.22}$$

where $\mathfrak{C} = \mathfrak{C}(\|\phi\|_{W^2(Y^4,X)}, \|\mu\|_{W^2(Y^2)}, \|\boldsymbol{u}\|_{W^2(\boldsymbol{V})}, \|\theta\|_{W^2(Y^2,X)})$.

Likewise, we also have the estimates $\|\mathrm{D}\boldsymbol{B}(\boldsymbol{u})^*\boldsymbol{v}\|_{L^2(\boldsymbol{V}^*)} \le c\|\boldsymbol{u}\|_{L^\infty(\boldsymbol{V})}\|\boldsymbol{v}\|_{L^2(\boldsymbol{V})}$, $\|\boldsymbol{B}_2(\vartheta,\theta - l_{\mathrm{h}}\phi)\|_{L^2(X)} \le c\|\vartheta\|_{L^2(Y^2)}(\|\theta\|_{L^\infty(Y)} + \|\phi\|_{L^\infty(Y)})$, and $\|\boldsymbol{B}_2(\varphi,\phi)\|_{L^2(X)} \le c\|\varphi\|_{L^2(Y)}\|\phi\|_{L^\infty(Y^2)}$. As a consequence, it holds that

$$\|\partial_t\boldsymbol{v}\|_{L^2(\boldsymbol{V}^*)} \le \mathfrak{C}(\|\varphi\|_{L^2(Y^2)} + \|\boldsymbol{v}\|_{L^2(\boldsymbol{V})} + \|\vartheta\|_{L^2(Y^2)} + \|\boldsymbol{g}_{3\Omega_T}\|_{L^2(\boldsymbol{V}^*)}) \tag{5.23}$$

where $\mathfrak{C} = \mathfrak{C}(\|\phi\|_{W^2(Y^4,X)}, \|\boldsymbol{u}\|_{W^2(\boldsymbol{V}^2,\boldsymbol{H})}, \|\theta\|_{W^2(Y^2,X)})$. Combining the above a priori estimates from (5.16) to (5.23) yields (5.15). □

Corollary 5.6. *Consider the assumptions of Corollary 5.5 and in addition $\boldsymbol{v}_T \in \boldsymbol{V}$ and $\boldsymbol{g}_{3\Omega_T} \in L^2(I;\boldsymbol{X})$. Then the components of the weak solution to (5.6) satisfy $\varphi \in W^2(I;Y^2)$, $\eta \in L^2(I;X)$, $\boldsymbol{v} \in W^2(I;\boldsymbol{V}^2,\boldsymbol{H})$, and $\vartheta \in W^2(I;Y^2,X)$. Furthermore, there exists $\mathfrak{C} = \mathfrak{C}(\|(\phi,\mu,\boldsymbol{u},\theta)\|_{\mathcal{V}}) > 0$ such that*

$$\begin{aligned}&\|\varphi\|_{W^2(Y^2)} + \|\eta\|_{L^2(X)} + \|\boldsymbol{v}\|_{W^2(\boldsymbol{V}^2,\boldsymbol{H})} + \|\vartheta\|_{W^2(Y^2,X)} \le \mathfrak{C}(\|\varphi_T\|_X + \|\boldsymbol{v}_T\|_{\boldsymbol{V}}\\ &\quad + \|\vartheta_T\|_Y + \|g_{1\Omega_T}\|_{L^2(Y^{2*})} + \|g_2\|_{L^2(X)} + \|\boldsymbol{g}_{3\Omega_T}\|_{L^2(\boldsymbol{X})} + \|g_{4\Omega_T}\|_{L^2(X)}).\end{aligned} \tag{5.24}$$

Proof. From the previous corollary, it remains to demonstrate the regularity of $\boldsymbol{v}$ with the given additional assumptions $\boldsymbol{v}_T \in \boldsymbol{V}$ and $\boldsymbol{g}_{3\Omega_T} \in L^2(I;\boldsymbol{X})$. Let $\boldsymbol{F}_3 := \vartheta\boldsymbol{g} - \varphi\nabla\phi - \vartheta\nabla(\theta - l_{\mathrm{h}}\phi) + \boldsymbol{g}_{3\Omega_T}$. Therefore, $\boldsymbol{v}$ is the weak solution to the following adjoint equation to the linearized Navier-Stokes equation:

$$\begin{cases} -\partial_t\boldsymbol{v} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{v} + (\nabla\boldsymbol{u})^\top\boldsymbol{v} + \nu\Delta\boldsymbol{v} + \nabla\pi = \boldsymbol{F}_3 \quad \text{in } I\times\Omega,\\ \operatorname{div}\boldsymbol{v} = 0 \quad \text{in } I\times\Omega, \qquad \boldsymbol{v} = 0 \quad \text{on } I\times\Gamma, \qquad \boldsymbol{v}(T) = \boldsymbol{v}_T \quad \text{in } \boldsymbol{V}.\end{cases}$$

From $\vartheta \in W^2(I;Y^2,X) \subset L^\infty(I;Y)$, $\theta \in L^2(I;Y^2)$, and $\phi \in W^2(I;Y^4,X) \subset L^\infty(I;Y^2)$, we have $\vartheta\nabla(\theta - l_{\mathrm{h}}\phi) \in L^2(I;\boldsymbol{X})$. Similarly, since $\varphi \in L^2(I;Y^2)$ and

$\phi \in L^\infty(I;Y^2)$, we deduce that $\varphi\nabla\phi \in L^2(I;\boldsymbol{X})$. Thus, $\boldsymbol{F}_3 \in L^2(I;\boldsymbol{X})$ and

$$\|\boldsymbol{F}_3\|_{L^2(\boldsymbol{X})} \le \mathfrak{C}(\|\vartheta\|_{W^2(Y^2,X)} + \|\varphi\|_{L^2(Y)} + \|\boldsymbol{g}_{3\Omega_T}\|_{L^2(\boldsymbol{X})}),$$

where $\mathfrak{C} = \mathfrak{C}(\|\phi\|_{W^2(Y^4,X)}, \|\theta\|_{W^2(Y^2,X)})$. Applying the regularity result in [31, Proposition 2.4] yields $\boldsymbol{v} \in W^2(I;\boldsymbol{V}^2, X)$ and

$$\|\boldsymbol{v}\|_{W^2(\boldsymbol{V}^2,\boldsymbol{H})} \le \mathfrak{C}(\|\boldsymbol{u}\|_{W^2(\boldsymbol{V}^2,\boldsymbol{H})})(\|\boldsymbol{v}_T\|_{\boldsymbol{V}} + \|\boldsymbol{F}_3\|_{L^2(\boldsymbol{X})}).$$

The previous two estimates and (5.15) imply the a priori estimate (5.24). □

Corollary 5.7. *Consider the assumptions of Corollary 5.6 and in addition $g_{1\Omega_T} \in L^2(I;Y^*)$, $g_2 \in L^2(I;Y)$ and $\varphi_T \in Y$. Then the weak solution to (5.6) satisfies $\varphi \in W^2(I;Y^3,Y^*)$, $\eta \in L^2(I;Y)$, $\boldsymbol{v} \in W^2(I;\boldsymbol{V}^2,\boldsymbol{H})$, and $\vartheta \in W^2(I;Y^2,X)$. Furthermore, there exists $\mathfrak{C} = \mathfrak{C}(\|(\phi,\mu,\boldsymbol{u},\theta)\|_{\mathcal{V}}) > 0$ such that*

$$\|\varphi\|_{W^2(Y^3,Y^*)} + \|\eta\|_{L^2(Y)} + \|\boldsymbol{v}\|_{W^2(\boldsymbol{V}^2,\boldsymbol{H})} + \|\vartheta\|_{W^2(Y^2,X)} \le \mathfrak{C}(\|\varphi_T\|_Y + \|\boldsymbol{v}_T\|_{\boldsymbol{V}} + \|\vartheta_T\|_Y + \|g_{1\Omega_T}\|_{L^2(Y^*)} + \|g_2\|_{L^2(Y)} + \|\boldsymbol{g}_{3\Omega_T}\|_{L^2(\boldsymbol{X})} + \|g_{4\Omega_T}\|_{L^2(X)}).$$

Proof. Let $F_1 := \boldsymbol{u}\cdot\nabla(\varphi - l_{\rm h}\vartheta) + f'(\phi)\eta + \alpha_2\boldsymbol{g}\cdot\boldsymbol{v} - \alpha\Delta(\mathcal{K}\boldsymbol{v}\cdot\nabla\phi + g_2) - \mathcal{K}\boldsymbol{v}\cdot\nabla(\mu - l_{\rm c}\theta) + g_{1\Omega_T} - l_{\rm h}\partial_t\vartheta$, so that φ is a weak solution of the following backward-in-time biharmonic heat equation

$$\begin{cases} -\partial_t\varphi + m\alpha\Delta^2\varphi = F_1 & \text{in } I\times\Omega,\\ \partial_{\boldsymbol{n}}\varphi = \partial_{\boldsymbol{n}}(m\Delta\varphi + \mathcal{K}\boldsymbol{v}\cdot\nabla\phi + g_2) = 0 & \text{on } I\times\Gamma,\\ \varphi(T) = \varphi_T + l_{\rm h}\vartheta_T & \text{in } Y. \end{cases}$$

We claim that $\boldsymbol{v}\cdot\nabla\phi \in L^2(I;Y)$. Indeed, since $\nabla(\boldsymbol{v}\cdot\nabla\phi) = (\nabla\boldsymbol{v})\nabla\phi + (\nabla^2\phi)\boldsymbol{v}$, we get from $\boldsymbol{v} \in L^\infty(I;\boldsymbol{V}) \cap L^2(I;\boldsymbol{V}^2)$ and $\phi \in L^\infty(I;Y^2)$ that

$$\|\boldsymbol{v}\cdot\nabla\phi\|_{L^2(Y)} \le c(\|\boldsymbol{v}\|_{L^\infty(\boldsymbol{V})}\|\phi\|_{L^2(Y^2)} + \|\boldsymbol{v}\|_{L^2(\boldsymbol{V}^2)}\|\phi\|_{L^\infty(Y^2)}).$$

It can be also shown that the other terms in F_1 lies in $L^2(I;Y^*)$. Hence, it holds that $F_1 \in L^2(I;Y^*)$ and

$$\|F_1\|_{L^2(Y^*)} \le \mathfrak{C}(\|\varphi\|_{L^2(Y^2)} + \|\eta\|_{L^2(X)} + \|\boldsymbol{v}\|_{W^2(\boldsymbol{V}^2,\boldsymbol{H})} + \|g_2\|_{L^2(Y)} + \|g_{1\Omega_T}\|_{L^2(Y^*)} + \|\vartheta\|_{W^2(Y^2,X)})$$

where $\mathfrak{C} = \mathfrak{C}(\|\phi\|_{W^2(Y^4,X)}, \|\boldsymbol{u}\|_{W^2(\boldsymbol{V}^2,\boldsymbol{H})}, \|\mu\|_{L^2(Y^2)}, \|\theta\|_{W^2(Y^2,X)})$. Therefore, we obtain that $\varphi \in W^2(I;Y^3,Y^*)$.

Following the methods given in the linearized system, cf. Theorem 4.1, one can deduce that

$$\|\varphi\|_{W^2(Y^3,Y^*)} \le \mathfrak{C}(\|\varphi_T\|_Y + \|\vartheta_T\|_Y + \|F_1\|_{L^2(Y^*)}).$$

From this we also get that $\eta \in L^2(I;Y)$ and

$$\|\eta\|_{L^2(Y)} \le \mathfrak{C}(\|\varphi\|_{L^2(Y^3)} + \|\boldsymbol{v}\|_{W^2(\boldsymbol{V}^2,\boldsymbol{H})} + \|g_2\|_{L^2(Y)}).$$

Combining the above a priori estimates and the one given in the previous corollary, we obtain the desired stability estimate as stated by the corollary. □

6. Analysis of Optimal Control Problem

In this section we analyze the existence of solutions to the optimal control problem (1.6) and characterize the necessary and sufficient conditions for optimality. Introducing the reduced cost functional $j : Q \to \mathbb{R}$ by

$$j(\boldsymbol{y}, z) := J(S(\boldsymbol{y}, z), \boldsymbol{y}, z) = G(S(\boldsymbol{y}, z)) + \frac{\gamma_{\mathrm{f}}}{2}\int_0^T \|\boldsymbol{y}\|^2_{L^2(\omega_{\mathrm{f}})^2}\,\mathrm{d}t + \frac{\gamma_{\mathrm{h}}}{2}\int_0^T \|z\|^2_{L^2(\omega_{\mathrm{h}})}\,\mathrm{d}t$$

we can equivalently write this problem as a constrained optimization on Q_{ad}:

$$\min_{(\boldsymbol{y},z)\in Q_{\mathrm{ad}}} j(\boldsymbol{y}, z). \tag{6.1}$$

We define the set of all feasible directions at $(\boldsymbol{y}, z) \in Q_{\mathrm{ad}}$ by

$$\mathcal{F}_{\mathrm{ad}}(\boldsymbol{y}, z) := \{(\delta\boldsymbol{y}, \delta z) \in Q : \exists\omega > 0 \text{ such that } (\boldsymbol{y} + \varepsilon\delta\boldsymbol{y}, z + \varepsilon\delta z) \in Q_{\mathrm{ad}}\ \forall\varepsilon \in [0, \omega]\}.$$

The minimum requirement for the initial data and target data in order for G to be well-defined is as follows:

(A) *It holds that* ϕ_d, μ_d, $\theta_d \in L^2(I; X)$, $\boldsymbol{\psi}_d$, $\boldsymbol{\xi}_d$, $\boldsymbol{\zeta}_d \in L^2(I; \boldsymbol{X})$, $\boldsymbol{u}_d \in L^2(I; \boldsymbol{X})$, $\psi_0 \in Y$, $\boldsymbol{u}_0$, $\boldsymbol{u}_T \in \boldsymbol{H}$, θ_0, θ_T, $\phi_T \in X$, *and* $\boldsymbol{\psi}_T \in \boldsymbol{X}$.

Theorem 6.1. *Suppose that* (A) *holds. The optimization problem* (6.1) *admits a global solution, that is, there exists* $(\boldsymbol{y}^\star, z^\star) \in Q_{\mathrm{ad}}$ *such that* $j(\boldsymbol{y}^\star, z^\star) \le j(\boldsymbol{y}, z)$ *for every* $(\boldsymbol{y}, z) \in Q_{\mathrm{ad}}$.

Proof. The proof is based on classical sequential compactness arguments in [42, 56], which we outline for the sake of the reader. Since j is bounded from below, j admits a minimizing sequence $\{(\boldsymbol{y}_k, z_k)\}_{k=1}^\infty \subset Q_{\mathrm{ad}}$, that is, $j(\boldsymbol{y}_k, z_k) \to \inf_{(\boldsymbol{y},z)\in Q_{\mathrm{ad}}} j(\boldsymbol{y}, z)$. Let $(\phi_k, \mu_k, \boldsymbol{u}_k, \theta_k) = S(\boldsymbol{y}_k, z_k)$. It follows that $\{(\boldsymbol{y}_k, z_k)\}_{k=1}^\infty$ is bounded in Q, and consequently $\{(\phi_k, \mu_k, \boldsymbol{u}_k, \theta_k)\}_{k=1}^\infty$ is bounded in $\mathcal{W}$ by (3.9). Since Q_{ad} is closed and convex, it is weakly closed, so that for a subsequence we have $(\boldsymbol{y}_k, z_k) \rightharpoonup (\boldsymbol{y}^\star, z^\star)$ in Q for some $(\boldsymbol{y}^\star, z^\star) \in Q_{\mathrm{ad}}$. According to the weak-weak continuity of S in Lemma 4.8, we get $S(\boldsymbol{y}_k, z_k) \rightharpoonup S(\boldsymbol{y}^\star, z^\star)$ in $\mathcal{W}$.

Let $(\phi^\star, \mu^\star, \boldsymbol{u}^\star, \theta^\star) = S(\boldsymbol{y}^\star, z^\star)$. Since the map $\varphi \mapsto \varphi(T)$ from $W^2(I; Y^3, Y^*)$ into Y is continuous, we have $\phi_k(T) \rightharpoonup \phi^\star(T)$ in X and $\nabla\phi_k(T) \rightharpoonup \nabla\phi^\star(T)$ in $\boldsymbol{X}$. Similarly, $\boldsymbol{u}_k(T) \rightharpoonup \boldsymbol{u}^\star(T)$ in $\boldsymbol{H}$ and $\theta_k(T) \rightharpoonup \theta^\star(T)$ in X. Passing to the limit inferior and using the weak lower semicontinuity of norms, we obtain

$$j(\boldsymbol{y}^\star, z^\star) = J(S(\boldsymbol{y}^\star, z^\star), \boldsymbol{y}^\star, z^\star) \le \liminf_{k\to\infty} J(S(\boldsymbol{y}_k, z_k), \boldsymbol{y}_k, z_k) = \inf_{(\boldsymbol{y},z)\in Q_{\mathrm{ad}}} j(\boldsymbol{y}, z).$$

Thus, $j(\boldsymbol{y}^\star, z^\star) \le j(\boldsymbol{y}, z)$ for every $(\boldsymbol{y}, z) \in Q_{\mathrm{ad}}$ and this proves the existence of a global solution to (6.1). □

6.1. First order optimality condition. A control $(\boldsymbol{y}^\star, z^\star) \in Q_{\mathrm{ad}}$ is said to be a *local solution* to (6.1) if there exists a constant $\varepsilon > 0$ such that $j(\boldsymbol{y}^\star, z^\star) \le j(\boldsymbol{y}, z)$ for every $(\boldsymbol{y}, z) \in Q_{\mathrm{ad}}$ with $\|(\boldsymbol{y} - \boldsymbol{y}^\star, z - z^\star)\|_Q < \varepsilon$. A local solution is said to be *strict* if there is a neighborhood for which it is only the local solution to the reduced problem. For the action of the second derivatives, we shall simply write $\mathrm{D}^2 j(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}, z)^2$ instead of $\mathrm{D}^2 j(\boldsymbol{y}^\star, z^\star)((\boldsymbol{y}, z), (\boldsymbol{y}, z))$ for instance.

Define the Y^*-distributional divergence operator $\mathrm{Div} : \boldsymbol{X} \to Y^*$ by

$$\langle \mathrm{Div}\, \boldsymbol{\phi}, \psi\rangle_{Y^*\times Y} := -(\boldsymbol{\phi}, \nabla\psi)_{\boldsymbol{X}}, \qquad \boldsymbol{\phi} \in \boldsymbol{X},\ \psi \in Y. \tag{6.2}$$

One can easily see that $A_N = -\mathrm{Div}\,\nabla$ as a map from Y into Y^*, and by the divergence theorem that $\mathrm{Div}\,\boldsymbol{\phi} = \mathrm{div}\,\boldsymbol{\phi}$ if $\boldsymbol{\phi} \in \boldsymbol{Y}$ satisfies $\boldsymbol{\phi}\cdot\boldsymbol{n} = 0$. Let us introduce $(g_1, g_2, \boldsymbol{g}_3, g_4) = (g_1(\phi), g_2(\mu), \boldsymbol{g}_3(\boldsymbol{u}), g_4(\theta)) \in \mathcal{W}_0^*$ with components defined as follows: Given $(\phi, \mu, \boldsymbol{u}, \theta) \in \mathcal{W}$ and under the hypothesis (A), let

$$g_2(\mu) := \alpha_{\mathrm{c}}(\mu - \mu_d) + \delta_{\mathrm{c}}(A_N\mu + \mathrm{Div}\,\boldsymbol{\xi}_d) \in L^2(I; Y^*)$$

and $g_1(\theta)$, $\boldsymbol{g}_3(\boldsymbol{u})$, and $g_4(\theta)$ have the decompositions (5.5) with

$$\begin{aligned}
&g_{1\Omega_T}(\phi) := \alpha_{\mathrm{o}}(\phi - \phi_d) + \delta_{\mathrm{o}}(A_N\phi + \mathrm{Div}\,\boldsymbol{\psi}_d) \in L^2(I; Y^*)\\
&\boldsymbol{g}_{3\Omega_T}(\boldsymbol{u}) := \alpha_{\mathrm{f}}(\boldsymbol{u} - \boldsymbol{u}_d) + \delta_{\mathrm{f}}\nabla\times(\nabla\times\boldsymbol{u}) \in L^2(I; \boldsymbol{V}^*)\\
&g_{4\Omega_T}(\theta) := \alpha_{\mathrm{h}}(\theta - \theta_d) + \delta_{\mathrm{h}}(A_N\theta + \mathrm{Div}\,\boldsymbol{\zeta}_d) \in L^2(I; Y^*)\\
&\langle g_{1T}(\phi), \varphi\rangle_{W_0^2(Y^3,Y^*)^*\times W_0^2(Y^3,Y^*)}\\
&\qquad := \beta_{\mathrm{o}}(\phi(T) - \phi_T, \varphi(T))_X + \omega_{\mathrm{o}}\langle A_N\phi(T) + \mathrm{Div}\,\boldsymbol{\psi}_T, \varphi(T)\rangle_{Y^*\times Y}\\
&\langle \boldsymbol{g}_{3T}(\boldsymbol{u}), \boldsymbol{w}\rangle_{W_0^2(\boldsymbol{V})^*\times W_0^2(\boldsymbol{V})} := \beta_{\mathrm{f}}(\boldsymbol{u}(T) - \boldsymbol{u}_T, \boldsymbol{w}(T))_{\boldsymbol{H}}\\
&\langle g_{4T}(\theta), \zeta\rangle_{W_0^2(Y)^*\times W_0^2(Y)} := \beta_{\mathrm{h}}(\theta(T) - \theta_T, \zeta(T))_X.
\end{aligned}$$

With regards to the tracking part of J, let us note that $G \in C^\infty(\mathcal{W}, \mathbb{R})$, and for $(\phi, \mu, \boldsymbol{u}, \theta) \in \mathcal{W}$ and $(\psi, \xi, \boldsymbol{w}, \zeta) \in \mathcal{W}_0$ we have

$$\mathrm{D}G(\phi, \mu, \boldsymbol{u}, \theta)(\psi, \xi, \boldsymbol{w}, \zeta) = \langle (g_1(\phi), g_2(\mu), \boldsymbol{g}_3(\boldsymbol{u}), g_4(\theta)), (\psi, \xi, \boldsymbol{w}, \zeta)\rangle_{\mathcal{W}_0^*\times\mathcal{W}_0}. \tag{6.3}$$

The action of the second derivative is given by

$$\begin{aligned}
\mathrm{D}^2G(\phi, \mu, \boldsymbol{u}, \theta)(\psi, \xi, \boldsymbol{w}, \zeta)^2 &= \beta_{\mathrm{o}}\|\psi(T)\|_X^2 + \omega_{\mathrm{o}}\|\nabla\psi(T)\|_{\boldsymbol{X}}^2 + \beta_{\mathrm{f}}\|\boldsymbol{w}(T)\|_{\boldsymbol{H}}^2 + \beta_{\mathrm{h}}\|\zeta(T)\|_X^2\\
&\quad + \int_0^T \alpha_{\mathrm{o}}\|\psi\|_X^2 + \delta_{\mathrm{o}}\|\nabla\psi\|_{\boldsymbol{X}}^2 + \alpha_{\mathrm{c}}\|\xi\|_X^2 + \delta_{\mathrm{c}}\|\nabla\xi\|_{\boldsymbol{X}}^2\,\mathrm{d}t\\
&\quad + \int_0^T \alpha_{\mathrm{f}}\|\boldsymbol{w}\|_{\boldsymbol{H}}^2 + \delta_{\mathrm{f}}\|\nabla\times\boldsymbol{w}\|_X^2 + \alpha_{\mathrm{h}}\|\zeta\|_X^2 + \delta_{\mathrm{h}}\|\nabla\zeta\|_{\boldsymbol{X}}^2\,\mathrm{d}t.
\end{aligned}$$

Notice that the right hand side is independent on the argument $(\phi, \mu, \boldsymbol{u}, \theta)$, and therefore we simply write $\mathrm{D}^2G(\psi, \xi, \boldsymbol{w}, \zeta)^2$ for the left hand side.

Lemma 6.2. *Assume that* (A) *is satisfied. Then the map $j : Q \to \mathbb{R}$ is of class C^∞. Given $(\boldsymbol{y}, z), (\delta\boldsymbol{y}, \delta z) \in Q$, denote the respective solutions of the state, linearized state, and adjoint systems by $(\phi, \mu, \boldsymbol{u}, \theta) = S(\boldsymbol{y}, z) \in \mathcal{W}$, $(\psi, \xi, \boldsymbol{w}, \zeta) = \mathrm{D}S(\boldsymbol{y}, z)(\delta\boldsymbol{y}, \delta z) \in \mathcal{W}_0$, and*

$$(\varphi, \eta, \boldsymbol{v}, \vartheta) = A(S(\boldsymbol{y}, z))^{-*}(g_1(\phi), g_2(\mu), \boldsymbol{g}_3(\boldsymbol{u}), g_4(\theta)) \in \mathcal{Q}.$$

The first and second order derivatives of j at $(\boldsymbol{y}, z)$ in the direction $(\delta\boldsymbol{y}, \delta z)$ are given by

$$\mathrm{D}j(\boldsymbol{y}, z)(\delta\boldsymbol{y}, \delta z) = \int_0^T (\boldsymbol{v} + \gamma_{\mathrm{f}}\boldsymbol{y}, \delta\boldsymbol{y})_{L^2(\omega_{\mathrm{f}})^2}\,\mathrm{d}t + \int_0^T (\vartheta + \gamma_{\mathrm{h}}z, \delta z)_{L^2(\omega_{\mathrm{h}})}\,\mathrm{d}t \tag{6.4}$$

$$\mathrm{D}^2j(\boldsymbol{y}, z)(\delta\boldsymbol{y}, \delta z)^2 = \mathrm{D}^2G(\psi, \xi, \boldsymbol{w}, \zeta)^2 - \int_0^T 2(\boldsymbol{w}\cdot\nabla\psi, \varphi)_X\,\mathrm{d}t$$

$$- \int_0^T 6\langle \eta, \phi\psi^2\rangle_{Y^*\times Y}\,\mathrm{d}t - \int_0^T 2[((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\boldsymbol{v})_{\boldsymbol{X}} - \mathcal{K}(\boldsymbol{v},(\xi - l_{\mathrm{c}}\zeta)\nabla\psi)_{\boldsymbol{X}}]\,\mathrm{d}t$$

$$- \int_0^T 2(\boldsymbol{w}\cdot\nabla(\zeta - l_{\mathrm{h}}\psi),\vartheta)_X\,\mathrm{d}t + \int_0^T \gamma_{\mathrm{f}}\|\delta\boldsymbol{y}\|^2_{L^2(\omega_{\mathrm{f}})^2}\,\mathrm{d}t + \int_0^T \gamma_{\mathrm{h}}\|\delta z\|^2_{L^2(\omega_{\mathrm{h}})}\,\mathrm{d}t. \quad (6.5)$$

Proof. Since $G \in C^\infty(\mathcal{W},\mathbb{R})$ and $S \in C^\infty(Q,\mathcal{W})$, it follows that $j \in C^\infty(Q,\mathbb{R})$. Let $(\boldsymbol{y},z),(\delta\boldsymbol{y},\delta z) \in Q$. According to the chain rule

$$\mathrm{D}j(\boldsymbol{y},z)(\delta\boldsymbol{y},\delta z) = \mathrm{D}G(S(\boldsymbol{y},z))\mathrm{D}S(\boldsymbol{y},z)(\delta\boldsymbol{y},\delta z) + \int_0^T \gamma_{\mathrm{f}}(\boldsymbol{y},\delta\boldsymbol{y})_{L^2(\omega_{\mathrm{f}})^2}\,\mathrm{d}t + \int_0^T \gamma_{\mathrm{h}}(z,\delta z)_{L^2(\omega_{\mathrm{h}})}\,\mathrm{d}t.$$

From Theorem 5.1 and (6.3), the first term on the right hand side can be written as

$$\mathrm{D}G(S(\boldsymbol{y},z))\mathrm{D}S(\boldsymbol{y},z)(\delta\boldsymbol{y},\delta z) = (P^*A(S(\boldsymbol{y},z))^{-*}(g_1(\phi),g_2(\mu),\boldsymbol{g}_3(\boldsymbol{u}),g_4(\theta)),(\delta\boldsymbol{y},\delta z))_Q = \int_0^T (\boldsymbol{v},\delta\boldsymbol{y})_{L^2(\omega_{\mathrm{f}})^2}\,\mathrm{d}t + \int_0^T (\vartheta,\delta z)_{L^2(\omega_{\mathrm{h}})}\,\mathrm{d}t.$$

Thus (6.4) is verified. On the other hand, applying the chain rule once more we obtain

$$\mathrm{D}^2 j(\boldsymbol{y},z)(\delta\boldsymbol{y},\delta z)^2 = \mathrm{D}^2 G(\mathrm{D}S(\boldsymbol{y},z)(\delta\boldsymbol{y},\delta z))^2 + \mathrm{D}G(S(\boldsymbol{y},z))\mathrm{D}^2 S(\boldsymbol{y},z)(\delta\boldsymbol{y},\delta z)^2 + \int_0^T \gamma_{\mathrm{f}}\|\delta\boldsymbol{y}\|^2_{L^2(\omega_{\mathrm{f}})^2}\,\mathrm{d}t + \int_0^T \gamma_{\mathrm{h}}\|\delta z\|^2_{L^2(\omega_{\mathrm{h}})}\,\mathrm{d}t. \quad (6.6)$$

The first term on the right hand side of this equation is precisely the term $\mathrm{D}^2G(\psi,\xi,\boldsymbol{w},\zeta)^2$ in (6.5) since $(\psi,\xi,\boldsymbol{w},\zeta) = \mathrm{D}S(\boldsymbol{y},z)(\delta\boldsymbol{y},\delta z)$. For the second term, we apply (4.53) and (6.3) to get

$$\mathrm{D}G(S(\boldsymbol{y},z))\mathrm{D}^2S(\boldsymbol{y},z)(\delta\boldsymbol{y},\delta z)^2$$
$$= -\langle (g_1(\phi),g_2(\mu),\boldsymbol{g}_3(\boldsymbol{u}),g_4(\theta)), A(S(\boldsymbol{y},z))^{-1}\mathrm{D}A(S(\boldsymbol{y},z))(\mathrm{D}S(\boldsymbol{y},z)(\delta\boldsymbol{y},\delta z))^2\rangle_{\mathcal{W}_0^*\times\mathcal{W}_0}$$
$$= -\langle \mathrm{D}A(S(\boldsymbol{y},z))(\psi,\xi,\boldsymbol{w},\zeta)^2,(\varphi,\eta,\boldsymbol{v},\vartheta)\rangle_{\mathcal{Q}^*\times\mathcal{Q}}. \quad (6.7)$$

Comparing (4.52) and (4.53), we see that the right hand side of this equation corresponds to the first four integrals in (6.5). □

With the help of the previous lemma, one can now establish the following first order necessary condition for local optimality.

Theorem 6.3. *Suppose that* (A) *is satisfied and* $(\boldsymbol{y}^\star,z^\star) \in Q_{\mathrm{ad}}$ *is a local solution to the optimization problem* (6.1). *Then*

$$\int_0^T (\boldsymbol{v}^\star + \gamma_{\mathrm{f}}\boldsymbol{y}^\star,\boldsymbol{y}-\boldsymbol{y}^\star)_{L^2(\omega_{\mathrm{f}})^2}\,\mathrm{d}t + \int_0^T (\vartheta^\star + \gamma_{\mathrm{h}}z^\star, z - z^\star)_{L^2(\omega_{\mathrm{h}})}\,\mathrm{d}t \geq 0 \quad (6.8)$$

for all $(\boldsymbol{y},z) \in Q_{\mathrm{ad}}$, *where* $(\boldsymbol{v}^\star,\vartheta^\star) \in W^{4/3}(I;\boldsymbol{V}) \times W^{4/3}(I;Y)$ *are the last two components of the solution for the adjoint system* (5.6) *corresponding to the source term* $(g_1(\phi^\star),g_2(\mu^\star),\boldsymbol{g}_3(\boldsymbol{u}^\star),g_4(\theta^\star)) \in \mathcal{W}_0^*$.

Proof. If $(\boldsymbol{y}^\star,z^\star) \in Q_{\mathrm{ad}}$ is a local solution to (6.1) then $\mathrm{D}j(\boldsymbol{y}^\star,z^\star)(\boldsymbol{y}-\boldsymbol{y}^\star,z-z^\star) \geq 0$ for every $(\boldsymbol{y},z) \in Q_{\mathrm{ad}}$, see [42]. The variational inequality (6.8) now follows from

(6.4), while the regularity of $(\boldsymbol{v}^\star, \vartheta^\star)$ is a consequence of Theorem 5.3. □

From the above theorem, a local optimal solution $(\boldsymbol{y}^\star, z^\star)$ is given equivalently as

$$\boldsymbol{y}^\star(t,x) = \mathrm{Proj}_{[\boldsymbol{a}_\mathrm{f},\boldsymbol{b}_\mathrm{f}]}(-\gamma_\mathrm{f}^{-1}\boldsymbol{v}^\star(t,x)) \quad \text{a.e. } (t,x) \in I \times \omega_\mathrm{f},$$
$$z^\star(t,x) = \mathrm{Proj}_{[a_\mathrm{h},b_\mathrm{h}]}(-\gamma_\mathrm{h}^{-1}\vartheta^\star(t,x)) \quad \text{a.e. } (t,x) \in I \times \omega_\mathrm{h},$$

where $\mathrm{Proj}_{[\boldsymbol{a}_\mathrm{f},\boldsymbol{b}_\mathrm{f}]}$ and $\mathrm{Proj}_{[a_\mathrm{h},b_\mathrm{h}]}$ are the projections onto the rectangle $[\boldsymbol{a}_\mathrm{f}, \boldsymbol{b}_\mathrm{f}]$ and the interval $[a_\mathrm{h}, b_\mathrm{h}]$, respectively. This can be seen by taking $(\boldsymbol{y}, z)$ in (6.8) to be either $(y_1, y_2^\star, z^\star)$, $(y_1^\star, y_2, z^\star)$, and $(y_1^\star, y_2^\star, z)$ with $(y_1, y_2, z) \in Q_\mathrm{ad}$ and using classical arguments to pass from the variational inequalities to pointwise inequalities. Notice that if $\boldsymbol{y}^\star(t,x) \in (\boldsymbol{a}_\mathrm{f}, \boldsymbol{b}_\mathrm{f})$ then we have $\boldsymbol{y}^\star(t,x) = -\gamma_\mathrm{f}^{-1}\boldsymbol{v}^\star(t,x)$, hence $\boldsymbol{v}^\star(t,x) + \gamma_\mathrm{f}\boldsymbol{y}^\star(t,x) = 0$. In a similar way, $z^\star(t,z) \in (a_\mathrm{h}, b_\mathrm{h})$ implies that $\vartheta^\star(t,x) + \gamma_\mathrm{h} z^\star(t,x) = 0$.

In the unconstrained case $Q_\mathrm{ad} = Q$ and $\omega_\mathrm{f} = \omega_\mathrm{h} = \Omega$, we have $\boldsymbol{y}^\star = \gamma_\mathrm{f}^{-1}\boldsymbol{v}^\star$ and $z^\star = \gamma_\mathrm{h}^{-1}\vartheta^\star$. In this case, the regularity of $(\boldsymbol{v}^\star, \vartheta^\star)$ and the optimal control $(\boldsymbol{y}^\star, z^\star)$ coincide. Next, let us discuss the regularity of the adjoint states beyond the assumption (A):

(A′) *It holds that* ϕ_d, μ_d, $\theta_d \in L^2(I; X)$, $\boldsymbol{\psi}_d$, $\boldsymbol{\xi}_d$, $\boldsymbol{\zeta}_d \in L^2(I; \boldsymbol{Y})$, $\boldsymbol{u}_d \in L^2(I; \boldsymbol{X})$, $\boldsymbol{u}_0 \in \boldsymbol{V}$, $\boldsymbol{u}_T \in \boldsymbol{H}$, θ_0, $\theta_T \in Y$, $\boldsymbol{\psi}_T \in \boldsymbol{Y}$, $\phi_T \in X$, $\phi_0 \in Y^2$, $\boldsymbol{\psi}_d \cdot \boldsymbol{n} = \boldsymbol{\xi}_d \cdot \boldsymbol{n} = \boldsymbol{\zeta}_d \cdot \boldsymbol{n} = 0$ *on* $I \times \Gamma$, *and* $\boldsymbol{\psi}_T \cdot \boldsymbol{n} = 0$ *on* Γ.

Suppose that (A′) holds and consider a local solution $(\boldsymbol{y}^\star, z^\star)$ to (6.1). Denote the corresponding optimal states and adjoint states by $(\phi^\star, \mu^\star, \boldsymbol{u}^\star, \theta^\star) = S(\boldsymbol{y}^\star, z^\star)$ and

$$(\varphi^\star, \eta^\star, \boldsymbol{v}^\star, \vartheta^\star) = A(S(\boldsymbol{y}^\star, z^\star))^{-*}(g_1(\phi^\star), g_2(\mu^\star), \boldsymbol{g}_3(\boldsymbol{u}^\star), g_4(\theta^\star)). \tag{6.9}$$

Then $(\phi^\star, \mu^\star, \boldsymbol{u}^\star, \theta^\star) \in \mathcal{V}$ by Theorem 3.4 and

$$(\varphi^\star, \eta^\star, \boldsymbol{v}^\star, \vartheta^\star) \in W^2(I; Y^2) \times L^2(I; X) \times W^2(I; \boldsymbol{V}) \times W^2(I; Y^2, X)$$

by Corollary 5.5. Thus, one can replace the duality pairing $\langle \cdot, \cdot \rangle_{Y^* \times Y}$ by the inner product $(\cdot, \cdot)_X$ in (6.5). If in addition, $\boldsymbol{u}_T \in \boldsymbol{V}$ then $\boldsymbol{v}^\star \in W^2(I; \boldsymbol{V}^2, \boldsymbol{H})$ in virtue of Corollary 5.6.

Assume that (A′) is satisfied and in addition $\mu_d \in L^2(I; Y)$, $\boldsymbol{u}_T \in \boldsymbol{V}$, $\phi_T \in Y$, and $\omega_\mathrm{o} = \delta_\mathrm{c} = 0$. Then we can apply Corollary 5.7 so that

$$(\varphi^\star, \eta^\star, \boldsymbol{v}^\star, \vartheta^\star) \in W^2(I; Y^3, Y^*) \cap L^2(I; Y) \cap W^2(I; \boldsymbol{V}^2, \boldsymbol{H}) \cap W^2(I; Y^2, X).$$

In particular, $(\boldsymbol{y}^\star, z^\star) \in W^2(I; \boldsymbol{V}^2, \boldsymbol{H}) \cap W^2(I; Y^2, X)$ in the unconstrained case and $\omega_\mathrm{f} = \omega_\mathrm{h} = \Omega$. Note that this is not true anymore in the constrained scenario due to the projections. The assumption $\omega_\mathrm{o} = \delta_\mathrm{c} = 0$ is imposed since we only know that $\mu \in L^2(I; Y^2)$ and $\phi(T) \in Y^2$. More precisely, $g_2(\mu) = \alpha_\mathrm{c}(\mu - \mu_d) \in L^2(I; Y)$ and $g_{1T}(\phi) = \beta_\mathrm{o}(\phi(T) - \phi_T) \in Y$.

Finally, under additional assumptions on the data, it is possible to establish further regularity of the controls in the unconstrained case, cf. Theorems 3.5 and 3.6. However, we do not pursue the details here and leave the precise formulations to the reader.

6.2. Second order optimality conditions. To formulate the second order conditions, we follow [14] and consider the following directions corresponding to the set of points where the constraints are active

$$\begin{aligned}\mathcal{A}_s(\boldsymbol{y}^\star) &:= \{\boldsymbol{y} \in L^2(I;L^2(\omega_{\mathrm{f}})^2) : y_i(t,x) \geq 0 \text{ if } y_i^\star(t,x) = a_{\mathrm{f}i},\\ &\qquad y_i(t,x) \leq 0 \text{ if } y_i^\star(t,x) = b_{\mathrm{f}i} \text{ for a.e. } (t,x) \in I\times\omega_{\mathrm{f}},\ i=1,2\}\\ \mathcal{A}_s(z^\star) &:= \{z \in L^2(I;L^2(\omega_{\mathrm{h}})) : z(t,x) \geq 0 \text{ if } z^\star(t,x) = a_{\mathrm{h}},\\ &\qquad z(t,x) \leq 0 \text{ if } z^\star(t,x) = b_{\mathrm{h}} \text{ for a.e. } (t,x) \in I\times\omega_{\mathrm{h}}\}.\end{aligned}$$

From these, we define the cone of critical directions $\mathcal{C}(\boldsymbol{y}^\star, z^\star) := \mathcal{C}(\boldsymbol{y}^\star)\times\mathcal{C}(z^\star)$ with

$$\begin{aligned}\mathcal{C}(\boldsymbol{y}^\star) &:= \{y \in \mathcal{A}_s(\boldsymbol{y}^\star) : y_i(t,x) = 0 \text{ if } (v_i^\star + \gamma_{\mathrm{f}} y_i^\star)(t,x) \neq 0\\ &\qquad\qquad \text{for a.e. } (t,x) \in I\times\omega_{\mathrm{f}},\ i=1,2\}\\ \mathcal{C}(z^\star) &:= \{z \in \mathcal{A}_s(z^\star) : z(t,x) = 0 \text{ if } (\zeta^\star + \gamma_{\mathrm{h}} z^\star)(t,x) \neq 0 \text{ for a.e. } (t,x) \in I\times\omega_{\mathrm{h}}\}.\end{aligned}$$

Notice that $\mathcal{C}(\boldsymbol{y}^\star, z^\star)$ and $\mathcal{A}_s(\boldsymbol{y}^\star, z^\star) := \mathcal{A}_s(\boldsymbol{y}^\star)\times\mathcal{A}_s(z^\star)$ are closed and convex subsets of Q. From the definition of the critical cones, it follows that $\mathrm{D}j(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}, z) = 0$ for every $(\boldsymbol{y}, z) \in \mathcal{C}(\boldsymbol{y}^\star, z^\star)$.

Lemma 6.4. *Given $(\boldsymbol{y}, z) \in \mathcal{C}(\boldsymbol{y}^\star, z^\star)$, there is a sequence $\{(\boldsymbol{y}_k, z_k)\}_{k=1}^\infty \subset \mathcal{F}_{Q_{\mathrm{ad}}}(\boldsymbol{y}^\star, z^\star) \cap \mathcal{C}(\boldsymbol{y}^\star, z^\star)$ such that $(\boldsymbol{y}_k, z_k) \to (\boldsymbol{y}, z)$ in Q.*

Proof. We shall present the proof in the case $\boldsymbol{a}_{\mathrm{f}}, \boldsymbol{b}_{\mathrm{f}} \in \mathbb{R}^2$, $a_{\mathrm{h}}, b_{\mathrm{h}} \in \mathbb{R}$, $\boldsymbol{a}_{\mathrm{f}} < \boldsymbol{b}_{\mathrm{f}}$ and $a_{\mathrm{h}} < b_{\mathrm{h}}$. The other cases where at least one of the endpoints take an infinite value can be dealt with analogously. Following [14], let $(\boldsymbol{y}, z) \in \mathcal{C}(\boldsymbol{y}^\star, z^\star)$ and take $k_0 > 0$ large enough so that $\boldsymbol{a}_{\mathrm{f}} + (k_0+k)^{-1} < \boldsymbol{b}_{\mathrm{f}} - (k_0+k)^{-1}$ and $a_{\mathrm{h}} + (k_0+k)^{-1} < b_{\mathrm{h}} - (k_0+k)^{-1}$ for every positive integer k. Set

$$\begin{aligned}K_{\mathrm{f}}^k &:= \{(t,x) \in I\times\omega_{\mathrm{f}} : \boldsymbol{y}^\star(t,x) \in (\boldsymbol{a}_{\mathrm{f}}, \boldsymbol{a}_{\mathrm{f}} + (k_0+k)^{-1}) \cup (\boldsymbol{b}_{\mathrm{f}} - (k_0+k)^{-1}, \boldsymbol{b}_{\mathrm{f}})\}\\ K_{\mathrm{h}}^k &:= \{(t,x) \in I\times\omega_{\mathrm{h}} : z^\star(t,x) \in (a_{\mathrm{h}}, a_{\mathrm{h}} + (k_0+k)^{-1}) \cup (b_{\mathrm{h}} - (k_0+k)^{-1}, b_{\mathrm{h}})\}.\end{aligned}$$

Let us define the projected functions $\boldsymbol{y}_k := \mathrm{Proj}_{[-(k_0+k),k_0+k]^2}(1-\chi_{K_{\mathrm{f}}^k})\boldsymbol{y}$ and $z_k := \mathrm{Proj}_{[-(k_0+k),k_0+k]}(1-\chi_{K_{\mathrm{h}}^k})z$. By construction, one can see that $(\boldsymbol{y}_k, z_k) \in \mathcal{C}(\boldsymbol{y}^\star, z^\star)$. Also, $|\boldsymbol{y}_k| \leq |\boldsymbol{y}|$, $\boldsymbol{y}_k \to y$ a.e. in $I\times\omega_{\mathrm{f}}$ and $|z_k| \leq |z|$, $z_k \to z$ a.e. in $I\times\omega_{\mathrm{h}}$. From the Lebesgue Theorem, $(\boldsymbol{y}_k, z_k) \to (\boldsymbol{y}, z)$ in Q. By adapting the arguments in [14, Theorem 3.6], it can be shown that $(\boldsymbol{y}^\star + \rho\boldsymbol{y}_k, z^\star + \rho z_k) \in Q_{\mathrm{ad}}$ for every $0 < \rho < (k_0+k)^{-2}$, and therefore $(\boldsymbol{y}_k, z_k) \in \mathcal{F}_{Q_{\mathrm{ad}}}$ for every k. □

From this lemma, one can now establish a second order necessary optimality condition.

Theorem 6.5. *Under the assumption* (A), *if $(\boldsymbol{y}^\star, z^\star) \in Q_{\mathrm{ad}}$ is a local solution of* (6.1) *then $\mathrm{D}^2 j(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}, z)^2 \geq 0$ for every $(\boldsymbol{y}, z) \in \mathcal{C}(\boldsymbol{y}^\star, z^\star)$.*

Proof. Let $(\boldsymbol{y}, z) \in \mathcal{C}(\boldsymbol{y}^\star, z^\star)$. According to Lemma 6.4, there is a sequence $\{(\boldsymbol{y}_k, z_k)\}_{k=1}^\infty \subset \mathcal{F}_{Q_{\mathrm{ad}}}(\boldsymbol{y}^\star, z^\star) \cap \mathcal{C}(\boldsymbol{y}^\star, z^\star)$ with $(\boldsymbol{y}_k, z_k) \to (\boldsymbol{y}, z)$ in Q. For each k, there exists $\delta_k > 0$ such that $(\boldsymbol{y}^\star + \varepsilon\boldsymbol{y}_k, z^\star + \varepsilon z_k) \in Q_{\mathrm{ad}}$ for every $0 < \varepsilon < \delta_k$ by feasibility of $(\boldsymbol{y}_k, z_k)$. Therefore, by Taylor's Theorem and the fact that $\mathrm{D}j(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}_k, z_k) = 0$, we have

$$0 \leq j(\boldsymbol{y}^\star + \varepsilon\boldsymbol{y}_k, z^\star + \varepsilon z_k) - j(\boldsymbol{y}^\star, z^\star) \leq \frac{\varepsilon^2}{2}\mathrm{D}^2 j(\boldsymbol{y}^\star + \sigma_\varepsilon\varepsilon\boldsymbol{y}_k, z^\star + \sigma_\varepsilon\varepsilon z_k)(\boldsymbol{y}_k, z_k)^2$$

for some $0 < \sigma_\varepsilon < 1$. Dividing by $\varepsilon^2/2$ and passing $\varepsilon \to 0$ yield $\mathrm{D}^2 j(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}_k, z_k)^2 \geq 0$. Consequently, by letting $k \to \infty$ and using the fact that $\mathrm{D}^2 j(\boldsymbol{y}^\star, z^\star) \in \mathcal{L}(Q \times Q, \mathbb{R})$ we obtain that $\mathrm{D}^2 j(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}, z)^2 \geq 0$. □

We now discuss a second order sufficient condition under additional assumptions on the initial data. Similar to the case of finite-dimensional problems with box constraints, the non-negativity of the Hessian on $\mathcal{C}(\boldsymbol{y}^\star, z^\star)$ is a necessary optimality condition, while the positive-definiteness of the Hessian on $\mathcal{C}(\boldsymbol{y}^\star, z^\star)$ is a sufficient condition for optimality.

Theorem 6.6. *Consider the assumption* (A) *and in addition that either* $(\phi_0, \boldsymbol{u}_0, \theta_0) \in Y^2 \times \boldsymbol{V} \times Y$ *or* $\omega_{\mathrm{o}} = \delta_{\mathrm{c}} = \delta_{\mathrm{f}} = \delta_{\mathrm{h}} = \beta_{\mathrm{f}} = \beta_{\mathrm{h}} = 0$. *Let* $(\boldsymbol{y}^\star, z^\star) \in Q_{\mathrm{ad}}$ *satisfy* (6.8) *and suppose that there exists* $\delta > 0$ *such that*

$$\mathrm{D}^2 j(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}, z)^2 \geq \delta \|(\boldsymbol{y}, z)\|_Q^2 \quad \text{for all } (\boldsymbol{y}, z) \in \mathcal{C}(\boldsymbol{y}^\star, z^\star). \tag{6.10}$$

Then there exist $\varepsilon > 0$ *and* $\sigma > 0$ *such that*

$$j(\boldsymbol{y}^\star, z^\star) + \frac{\sigma}{2}\|(\boldsymbol{y} - \boldsymbol{y}^\star, z - z^\star)\|_Q^2 \leq j(\boldsymbol{y}, z) \tag{6.11}$$

holds for every $(\boldsymbol{y}, z) \in Q_{\mathrm{ad}}$ *with* $\|(\boldsymbol{y} - \boldsymbol{y}^\star, z - z^\star)\|_Q < \varepsilon$. *In particular,* $(\boldsymbol{y}^\star, z^\star)$ *is a strict local solution to* (6.1).

Proof. We shall only prove the case where $(\phi_0, \boldsymbol{u}_0, \theta_0) \in Y^2 \times \boldsymbol{V} \times Y$, while the other alternative can be shown in a similar way. Suppose on the contrary that for every $\varepsilon > 0$ and $\sigma > 0$ there exists $(\boldsymbol{y}_{\varepsilon,\sigma}, z_{\varepsilon,\sigma}) \in Q_{\mathrm{ad}}$ such that $\|(\boldsymbol{y}_{\varepsilon,\sigma} - \boldsymbol{y}^\star, z_{\varepsilon,\sigma} - z^\star)\|_Q < \varepsilon$ and $j(\boldsymbol{y}^\star, z^\star) + \frac{\sigma}{2}\|(\boldsymbol{y}, z)\|_Q^2 > j(\boldsymbol{y}_{\varepsilon,\sigma}, z_{\varepsilon,\sigma})$. In particular, taking $\sigma = \frac{2}{k}$ and $\varepsilon = \frac{1}{k}$ for every positive integer k, there is $(\tilde{\boldsymbol{y}}_k, \tilde{z}_k) \in Q_{\mathrm{ad}}$ such that $\|(\tilde{\boldsymbol{y}}_k - \boldsymbol{y}^\star, \tilde{z}_k - z^\star)\|_Q < \frac{1}{k}$ and

$$j(\tilde{\boldsymbol{y}}_k, \tilde{z}_k) < j(\boldsymbol{y}^\star, z^\star) + \frac{1}{k}\|(\tilde{\boldsymbol{y}}_k - \boldsymbol{y}^\star, \tilde{z}_k - z^\star)\|_Q^2. \tag{6.12}$$

Let $\rho_k = \|(\tilde{\boldsymbol{y}}_k - \boldsymbol{y}^\star, \tilde{z}_k - z^\star)\|_Q$ and $(\boldsymbol{y}_k, z_k) = (\tilde{\boldsymbol{y}}_k - \boldsymbol{y}^\star, \tilde{z}_k - z^\star)/\rho_k$ so that $\|(\boldsymbol{y}_k, z_k)\|_Q = 1$. Then there is a subsequence, still denoted by $(\boldsymbol{y}_k, z_k)$, such that $(\boldsymbol{y}_k, z_k) \rightharpoonup (\boldsymbol{y}, z)$ in Q. We claim that $(\boldsymbol{y}, z) \in \mathcal{C}(\boldsymbol{y}^\star, z^\star)$. Since $(\tilde{\boldsymbol{y}}_k, \tilde{z}_k) \in Q_{\mathrm{ad}}$, we have $(\boldsymbol{y}_k, z_k) \in \mathcal{A}_s(\boldsymbol{y}^\star, z^\star)$. The set $\mathcal{A}_s(\boldsymbol{y}^\star, z^\star)$ is closed and convex in Q, hence weakly closed, and we have $(\boldsymbol{y}, z) \in \mathcal{A}_s(\boldsymbol{y}^\star, z^\star)$. We will prove that in fact $(\boldsymbol{y}, z) \in \mathcal{C}(\boldsymbol{y}^\star, z^\star)$.

By Taylor's expansion, we obtain

$$\begin{aligned} j(\tilde{\boldsymbol{y}}_k, \tilde{z}_k) &= j(\boldsymbol{y}^\star + \rho_k \boldsymbol{y}_k, z^\star + \rho_k z_k) \\ &= j(\boldsymbol{y}^\star, z^\star) + \rho_k \mathrm{D}j(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}_k, z_k) + \frac{\rho_k^2}{2}\mathrm{D}^2 j(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}_k, z_k)^2 + o(\rho_k^2) \end{aligned} \tag{6.13}$$

where $o(\rho_k^2)/\rho_k^2 \to 0$ as $\rho_k \to 0$. Dividing by ρ_k and applying (6.12) we get

$$\begin{aligned} \mathrm{D}j(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}_k, z_k) &\leq \frac{\rho_k}{k} - \frac{\rho_k}{2}\mathrm{D}^2 j(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}_k, z_k)^2 - \frac{o(\rho_k^2)}{\rho_k} \\ &\leq \frac{1}{k^2} + \frac{1}{2k}\|\mathrm{D}^2 j(\boldsymbol{y}^\star, z^\star)\|_{\mathcal{L}(Q \times Q, \mathbb{R})} - \frac{o(\rho_k^2)}{\rho_k}. \end{aligned} \tag{6.14}$$

From (6.4) we see that $(\boldsymbol{y}_k, z_k) \rightharpoonup (\boldsymbol{y}, z)$ implies $\mathrm{D}j(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}_k, z_k) \to \mathrm{D}j(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}, z)$. Passing to the limit $k \to \infty$ in the inequality (6.14) yields $\mathrm{D}j(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}, z) \le 0$.

Let $(\phi^\star, \mu^\star, \boldsymbol{u}^\star, \theta^\star) = S(\boldsymbol{y}^\star, z^\star)$ and $(\varphi^\star, \eta^\star, \boldsymbol{v}^\star, \vartheta^\star)$ be given by (6.9). From the admissibility of $(\tilde{\boldsymbol{y}}_k, \tilde{z}_k)$, it holds that $\mathrm{D}j(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}_k, z_k) = \mathrm{D}j(\boldsymbol{y}^\star, z^\star)(\tilde{\boldsymbol{y}}_k - \boldsymbol{y}^\star, \tilde{z}_k - z^\star)/\rho_k \ge 0$, so that $\mathrm{D}j(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}, z) \ge 0$ after letting $k \to \infty$. Consequently, from (6.4) we have

$$\int_0^T (\boldsymbol{v}^\star + \gamma_{\mathrm{f}} \boldsymbol{y}^\star, y)_{L^2(\omega_{\mathrm{f}})^2} \,\mathrm{d}t + \int_0^T (\vartheta^\star + \gamma_{\mathrm{h}} z^\star, z)_{L^2(\omega_{\mathrm{h}})} \,\mathrm{d}t = 0. \tag{6.15}$$

The condition (6.8) implies that $(v_i^\star + \gamma_{\mathrm{f}} y_i^\star) y_i \ge 0$ for a.e. in $I \times \omega_{\mathrm{f}}$ for $i = 1, 2$ and $(\vartheta^\star + \gamma_{\mathrm{h}} z^\star) z \ge 0$ for a.e. in $I \times \omega_{\mathrm{h}}$. Hence, the equation (6.15) is equivalent to

$$\int_0^T \int_{\omega_{\mathrm{f}}} |(v_i^\star + \gamma_{\mathrm{f}} y_i^\star) y_i| \,\mathrm{d}x \,\mathrm{d}t = \int_0^T \int_{\omega_{\mathrm{h}}} |(\vartheta^\star + \gamma_{\mathrm{h}} z^\star) z| \,\mathrm{d}x \,\mathrm{d}t = 0, \quad i = 1, 2.$$

Thus, if $v_i^\star(t, x) + \gamma_{\mathrm{f}} y_i^\star(t, x) \ne 0$ then $y_i(t, x) = 0$ for a.e. $(t, x) \in I \times \omega_{\mathrm{f}}$ and $i = 1, 2$. Similarly, if $\vartheta^\star(t, x) + \gamma_{\mathrm{h}} z^\star(t, x) \ne 0$ then $z(t, x) = 0$ for a.e. $(t, x) \in I \times \omega_{\mathrm{h}}$. Together with $(\boldsymbol{y}, z) \in \mathcal{A}_s(\boldsymbol{y}^\star, z^\star)$, we have verified the claim that $(\boldsymbol{y}, z) \in \mathcal{C}(\boldsymbol{y}^\star, z^\star)$.

From (6.12), (6.13), and the fact that $\mathrm{D}j(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}_k, z_k) \ge 0$, we have

$$\mathrm{D}^2 j(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}_k, z_k)^2 < 2\left(\frac{1}{k} - \frac{o(\rho_k^2)}{\rho_k^2}\right). \tag{6.16}$$

Let $(\psi_k, \xi_k, \boldsymbol{w}_k, \zeta_k) := \mathrm{D}S(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}_k, z_k)$. According to Theorem 3.4, $(\phi^\star, \mu^\star, \boldsymbol{u}^\star, \theta^\star) \in \mathcal{V}$. Then it follows from Theorem 4.4 that $\{(\psi_k, \xi_k, \boldsymbol{w}_k, \zeta_k)\}_{k=1}^\infty$ is bounded in $\mathcal{V}_0$, so that in particular, $\{(\psi_k(T), \boldsymbol{w}_k(T), \zeta_k(T))\}_{k=1}$ is bounded in $Y^2 \times \boldsymbol{V} \times Y$. By further extracting a subsequence, we obtain that $(\psi_k, \xi_k, \boldsymbol{w}_k, \zeta_k) \rightharpoonup (\psi, \xi, \boldsymbol{w}, \zeta) = \mathrm{D}S(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}, z)$ in $\mathcal{V}_0$. Invoking the compact embeddings $Y^2 \times \boldsymbol{V} \times Y \subset Y \times \boldsymbol{H} \times X$ and $\mathcal{V}_0 \subset L^2(I; Y^2) \times L^2(I; Y) \times L^2(I; \boldsymbol{V}) \times L^2(I; Y)$, by extraction of another subsequence, the following strong convergences hold:

$$(\psi_k, \xi_k, \boldsymbol{w}_k, \zeta_k) \to (\psi, \xi, \boldsymbol{w}, \zeta) \text{ in } L^2(I; Y^2) \times L^2(I; Y) \times L^2(I; \boldsymbol{V}) \times L^2(I; Y) \tag{6.17}$$

and $(\psi_k(T), \boldsymbol{w}_k(T), \zeta_k(T)) \to (\psi(T), \boldsymbol{w}(T), \zeta(T))$ in $Y \times \boldsymbol{H} \times X$. These convergences imply that

$$\mathrm{D}^2 G(\psi_k, \xi_k, \boldsymbol{w}_k, \zeta_k)^2 \to \mathrm{D}^2 G(\psi, \xi, \boldsymbol{w}, \zeta)^2. \tag{6.18}$$

By adapting the argument presented in Step 5 in the proof of Theorem 3.2, we can deduce from the limit (6.17) that

$$\begin{aligned}
&\int_0^T 2(\boldsymbol{w}_k \cdot \nabla \psi_k, \varphi^\star)_X \,\mathrm{d}t + \int_0^T 2((\boldsymbol{w}_k \cdot \nabla)\boldsymbol{w}_k, \boldsymbol{v}^\star)_{\boldsymbol{X}} \,\mathrm{d}t \\
&\quad - \int_0^T 2\mathcal{K}(\boldsymbol{v}^\star, (\xi_k - l_{\mathrm{c}}\zeta_k)\nabla\psi_k)_{\boldsymbol{X}} \,\mathrm{d}t + \int_0^T 2(\boldsymbol{w}_k \cdot \nabla(\zeta_k - l_{\mathrm{h}}\psi_k), \vartheta^\star)_X \,\mathrm{d}t \\
&\to \int_0^T 2(\boldsymbol{w} \cdot \nabla \psi, \varphi^\star)_X \,\mathrm{d}t + \int_0^T 2((\boldsymbol{w} \cdot \nabla)\boldsymbol{w}, \boldsymbol{v}^\star)_{\boldsymbol{X}} \,\mathrm{d}t \\
&\quad - \int_0^T 2\mathcal{K}(\boldsymbol{v}^\star, (\xi - l_{\mathrm{c}}\zeta)\nabla\psi)_{\boldsymbol{X}} \,\mathrm{d}t + \int_0^T 2(\boldsymbol{w} \cdot \nabla(\zeta - l_{\mathrm{h}}\psi), \vartheta^\star)_X \,\mathrm{d}t.
\end{aligned} \tag{6.19}$$

On the other hand, we have $\phi^\star\psi_k^2 \to \phi^\star\psi^2$ in $L^2(I;Y)$ thanks to the estimate

$$\|\phi^\star\psi_k^2 - \phi^\star\psi^2\|_{L^2(Y)} \leq \mathfrak{C}(\|\phi^\star\|_{L^\infty(Y^2)}, \|\psi_k\|_{L^\infty(Y^2)}, \|\psi\|_{L^\infty(Y^2)})\|\psi_k - \psi\|_{L^2(Y^2)}$$

for some positive continuous function $\mathfrak{C}$. Thus, we have

$$\int_0^T \langle \eta^\star, \phi^\star\psi_k^2\rangle_{Y^*\times Y}\,\mathrm{d}t \to \int_0^T \langle \eta^\star, \phi^\star\psi^2\rangle_{Y^*\times Y}\,\mathrm{d}t. \tag{6.20}$$

From (6.7), (6.19), and (6.20), one obtains

$$\mathrm{D}G(S(\boldsymbol{y}^\star, z^\star))\mathrm{D}^2S(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}_k, z_k)^2 \to \mathrm{D}G(S(\boldsymbol{y}^\star, z^\star))\mathrm{D}^2S(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}, z)^2. \tag{6.21}$$

Passing to the limit inferior as $k \to \infty$ in (6.16) and recalling (6.10) lead us to $\delta\|(\boldsymbol{y}, z)\|_Q^2 \leq \mathrm{D}^2 j(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}, z)^2 \leq 0$. This is possible only if $(\boldsymbol{y}, z) = (\mathbf{0}, 0)$ in Q since $\delta > 0$. According to (6.6), (6.16), (6.18), and (6.21) we have

$$\begin{aligned}
&\limsup_{k\to\infty}\, (\gamma_{\mathrm{f}}\|\boldsymbol{y}_k\|^2_{L^2(L^2(\omega_{\mathrm{f}})^2)} + \gamma_{\mathrm{h}}\|z_k\|^2_{L^2(L^2(\omega_{\mathrm{h}}))})\\
&\leq \limsup_{k\to\infty}\left[2\left(\frac{1}{k} - \frac{o(\rho_k^2)}{\rho_k^2}\right) - \mathrm{D}^2G(\psi_k, \xi_k, \boldsymbol{w}_k, \zeta_k)^2\right.\\
&\qquad\qquad\qquad\left. - \mathrm{D}G(S(\boldsymbol{y}^\star, z^\star))\mathrm{D}^2S(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}_k, z_k)^2\right]\\
&= -\mathrm{D}^2G(\mathrm{D}S(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}, z))^2 - \mathrm{D}G(S(\boldsymbol{y}^\star, z^\star))\mathrm{D}^2S(\boldsymbol{y}^\star, z^\star)(\boldsymbol{y}, z)^2 = 0.
\end{aligned}$$

Since $\gamma_{\mathrm{f}} > 0$ and $\gamma_{\mathrm{h}} > 0$ we have $\|(\boldsymbol{y}_k, z_k)\|_Q \to 0$. Combined with $(\boldsymbol{y}_k, z_k) \rightharpoonup (\mathbf{0}, 0)$ in Q, we have $(\boldsymbol{y}_k, z_k) \to (\mathbf{0}, 0)$ in Q. However, this is a contradiction to the fact that $\|(\boldsymbol{y}_k, z_k)\|_Q = 1$ for every k. Therefore, (6.11) must be true and this completes the proof of the theorem. □

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