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Well-Posedness and Regularity of Linear Hyperbolic Systems with Dynamic Boundary Conditions

WELL-POSEDNESS AND REGULARITY OF LINEAR HYPERBOLIC SYSTEMS WITH DYNAMIC BOUNDARY CONDITIONS

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Abstract.
We consider first order hyperbolic systems on an interval with dynamic boundary conditions. These systems occur when the ODE dynamics on the boundary interact with the waves in the interior. The well-posedness for linear systems is established using an abstract Friedrichs Theorem. Due to the limited regularity of the coefficients we need to introduce the appropriate space of test functions for the weak formulation. It is shown that the weak solutions exhibit a hidden regularity at the boundary as well as at interior points. As a consequence, the dynamics of the boundary components satisfy an additional regularity, and both can not be achieved from standard semigroup methods. Nevertheless, we show that our weak solutions and the semigroup solutions coincide. For illustration, we give three particular physical examples that fit into our framework.

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1. Introduction

Hyperbolic partial differential equations are recognized mathematical models in areas such as fluid dynamics, acoustics, electromagnetics, scattering theory and the general theory of relativity. Because information travels along characteristic curves, discontinuities and oscillations propagate through time and space. Therefore, in general, one might expect the same regularity for the initial data and the solution. But what happens when a hyperbolic system has a dynamic boundary condition? There is an emerging interest in coupled hyperbolic systems with dynamic boundary conditions due to their applications in multiscale blood flow modelling and valveless pumping, see [4, 5, 6, 11, 21, 27, 29, 30] and the references therein.

In this paper, we consider general linear hyperbolic systems with variable coefficients coupled with linear ordinary differential equations at the boundary

$$\begin{cases} [\partial_t + A(v(t,x))\partial_x + R(t,x)]u(t,x) = f(t,x), \quad 0 < t < T,\ 0 < x < 1, \\ B_0 u(t,0) = g_0(t) + Q_0(t)h(t), \quad 0 < t < T, \\ B_1 u(t,1) = g_1(t) + Q_1(t)h(t), \quad 0 < t < T, \\ h'(t) = H(t)h(t) + G_0(t)u(t,0) + G_1(t)u(t,1) + S(t), \quad 0 < t < T, \\ u(0,x) = u_0(x), \quad 0 < x < 1, \\ h(0) = h_0 \end{cases} \tag{1.1}$$

for some appropriate matrices A, R, B_i, Q_i, H, G_i and S_i. Here, v is a Lipschitz function and it can be thought of as a frozen coefficient in an otherwise nonlinear system, see [23]. The present article is the first work (to the best of our knowledge) to deal with the well-posedness of general hyperbolic PDE-ODE systems, although specific cases have been studied separately, e.g. the wave equation with acoustic boundary conditions [2, 14] and flow in an elastic tube connected to tanks [24]. Here, our goal is to unify and improve these results.

The L^2-well-posedness of (1.1) is based on energy estimates. It is well-known that hyperbolic systems admit hidden boundary trace regularity. This is due to the fact that information travels along characteristics, and thus the boundary regularity of solutions is influenced by the regularity of the boundary and initial data. We would like to extend this phenomenon to the coupled system (1.1). It will be shown that u satisfies a hidden regularity property, i.e., it has L^2-trace at the boundary. This property implies that the ODE component h does not lie only in L^2 but in H^1. Thanks to this boundary trace regularity, we can also deduce an *interior-point trace regularity* for solutions using the multiplier method. Thus, the ODE have a smoothing effect not only at the boundary. We would like to point out that trace regularity plays an important role in the boundary controllability of hyperbolic systems. If one computes the optimal control via the HUM (Hilbert Uniqueness Method) then the cost functional contains traces of solutions of the adjoint problem.

One difficulty in deriving the weak form of (1.1) is to eliminate the traces $u_{|x=0}$ and $u_{|x=1}$ in the ODE part. If there are some structural conditions on G_i and B_i for $i = 0, 1$ then this would be an easier task. However, we will not impose any relationship between these matrices.

The weak solutions in L^2 satisfiy a variational equation that takes the form

$$(u, \Lambda w)_X = (f, w)_X + (g, \Psi w)_Z \quad \text{for all } w \in W, \tag{1.2}$$

for suitable function spaces X, W, Z and operators Λ, Ψ. This equation is obtained by multiplying the differential equation by appropriate test functions, integrating by parts and using the boundary and initial conditions. Due to the limited regularity of the coefficients, particularly on G_0 and G_1 which we assumed to be L^∞ only, we need to introduce a non-standard space of test functions for the weak formulation. In fact, they will be chosen to lie on a graph space. With an abstract a priori estimate, the variational equation (1.2) has a solution $u \in X$ (Section 2). Its proof is based on the Hahn-Banach and Riesz Representation Theorems. The idea of the proof can be traced back to the work of Friedrichs [12] for symmetric systems. Therefore, proving an a priori estimate is the first step in proving the existence of weak solutions. Our method is to consider the ODE part (Section 3) and PDE part (Section 5) separately.

How does the weak solution satisfy the initial-boundary value problem? To answer this, we need to consider the space of functions $u \in L^2(Q_T)$ with $Lu := \partial_t u + A\partial_x u \in L^2(Q_T)$, where A is at least Lipschitz and $Q_T = (0, T) \times (0, 1)$. This space is similar to the space of L^2-functions with L^2-distributional divergence which is used in studying the Navier-Stokes equation and the wave equation. These spaces are called *graph spaces*. The usual trace operator in H^1 can be extended to define a generalized trace operator for the graph space $\{u \in L^2(Q_T) : Lu \in L^2(Q_T)\}$, but the traces are now in $H^{-\frac{1}{2}}(\partial Q_T)$. To treat initial-boundary value problems, we will also restrict the trace to the edges of the time-space domain (Section 4). With these considerations, it will be seen that weak solutions satisfy the partial differential equation in the sense of distributions and the boundary conditions and initial condition are satisfied in the sense of (generalized) traces.

In the constant coefficient case, our well-posedness result implies that the weak solution generates a $\mathcal{C}_0$-semigroup (Section 7). As a reassuring result, the weak solution is the same as the solution given by the semigroup approach.

Notation. $L^p(O)$ and $W^{s,p}(O)$ denote the usual Lebesgue and Sobolev spaces on a nonempty open set $O \subset \mathbb{R}^d$ and we set $H^k(O) := W^{k,2}(O)$. The usual notation for the space of continuous functions $\mathscr{C}^k(O)$, $k \in \mathbb{N}_0 \cup \{\infty\}$, will be used. The space of smooth functions with compact support in O is denoted by $\mathscr{D}(O)$. For each nonnegative integer k we let $CH^k(Q_T) := \bigcap_{j=0}^k C^j([0,T], H^{k-j}(0,1))$.

If X is a Hilbert space consisting of functions depending on the variable t, we define the weighted space $e^{\gamma t}X = \{e^{\gamma t}u : u \in X\}$, where $\gamma \in \mathbb{R}$, equipped with the inner product $(u,v)_{e^{\gamma t}X} := (e^{-\gamma t}u, e^{-\gamma t}v)_X$. Given $n \in \mathbb{N}$, X^n denotes the product of n copies of X. However, if the context is clear we shall simply write X for X^n.

2. A Generalized Friedrichs Theorem

In this section we prove the existence and uniqueness of solutions of a variational problem. This general framework will be used in Section 6 to a coupled PDE-ODE systems with variable coefficients. Let X and Z be real Hilbert spaces and Y be a subspace of X. Suppose that $\Lambda : Y \to X$, $\Psi : Y \to Z$ and $\Phi : Y \to Z$ are linear operators. Let $W = \ker \Phi$. We assume that W and $\Lambda(W)$ are both nontrivial. Given $F \in X$ and $G \in Z$ we consider the variational problem:

Find $u \in X$ such that

$$(u, \Lambda w)_X = (F, w)_X + (G, \Psi w)_Z, \qquad \forall\, w \in W. \tag{2.1}$$

For the differential equations we consider, Ψ is a trace operator while Λ and Φ are the differential and trace operators associated with the adjoint problem. We note that the space of test functions W need not be dense with respect to the topology of the space X. For the examples in the succeeding sections, X will be the dual of the solution space.

Theorem 2.1. *Suppose that there exist $\gamma > 0$ and $C > 0$ such that*

$$\gamma\|w\|_X^2 + \|\Psi w\|_Z^2 \le C\left(\frac{1}{\gamma}\|\Lambda w\|_X^2 + \|\Phi w\|_Z^2\right), \qquad \forall\, w \in Y. \tag{2.2}$$

Then the variational equation (2.1) has a solution $u \in X$ satisfying

$$\gamma\|u\|_X^2 \le C\left(\frac{1}{\gamma}\|F\|_X^2 + \|G\|_Z^2\right). \tag{2.3}$$

In addition, the solution is unique if and only if $\Lambda(W)$ is dense in X.

Proof. By assumption, the restriction $\Lambda : W \to X$ of Λ to W is injective, and therefore it has a left inverse $\Lambda^{-1} : \Lambda(W) \subset X \to W$. According to (2.2)

$$\gamma\|\Lambda^{-1}\varphi\|_X^2 + \|\Psi\Lambda^{-1}\varphi\|_Z^2 \le \frac{C}{\gamma}\|\varphi\|_X^2, \qquad \forall\, \varphi \in \Lambda(W). \tag{2.4}$$

Define the linear map $\ell : \Lambda(W) \to \mathbb{R}$ by

$$\ell\varphi = (F, \Lambda^{-1}\varphi)_X + (G, \Psi\Lambda^{-1}\varphi)_Z,$$

for $\varphi \in \Lambda(W)$. We equipped $\Lambda(W)$ with the norm $\|\cdot\|_X$. The Cauchy-Schwarz inequality and (2.4) imply that

$$\begin{aligned} |\ell\varphi|^2 &\leq 2\|F\|_X^2\|\Lambda^{-1}\varphi\|_X^2 + 2\|G\|_Z^2\|\Psi\Lambda^{-1}\varphi\|_Z^2 \\ &\leq 2\left(\frac{1}{\gamma}\|F\|_X^2 + \|G\|_Z^2\right)(\gamma\|\Lambda^{-1}\varphi\|_X^2 + \|\Psi\Lambda^{-1}\varphi\|_Z^2) \\ &\leq \frac{C}{\gamma}\left(\frac{1}{\gamma}\|F\|_X^2 + \|G\|_Z^2\right)\|\varphi\|_X^2 \end{aligned}$$

for all $\varphi \in \Lambda(W)$. Thus $\ell \in [\Lambda(W)]'$ and

$$\gamma\|\ell\|_{[\Lambda(W)]'}^2 \leq C\left(\frac{1}{\gamma}\|F\|_X^2 + \|G\|_Z^2\right).$$

According to the Hahn-Banach Theorem, ℓ admits an extension $\tilde{\ell} \in X'$ such that $\|\tilde{\ell}\|_{X'} = \|\ell\|_{[\Lambda(W)]'}$. From the Riesz Representation Theorem there is a unique $u \in X$ such that $\|u\|_X = \|\tilde{\ell}\|_{X'}$ and $(u, v)_X = \tilde{\ell}v$ for all $v \in X$. In particular, for every $w \in W$

$$(u, \Lambda w)_X = \tilde{\ell}\Lambda w = \ell\Lambda w = (F, w)_X + (G, \Psi w)_Z.$$

Thus u is a solution of the variational equation (2.1) and it satisfies the estimate (2.3). Suppose that u_1 and u_2 solve (2.1). Then $(u_1 - u_2, \Lambda w) = 0$ for every $w \in W$. If $\Lambda(W)$ is dense in X then $u_1 - u_2 = 0$ and thus the solution of (2.1) is unique.

Conversely, suppose that $(v, \Lambda w)_X = 0$ for some $v \in X \setminus \{0\}$ and for all $w \in W$. If u is a solution of (2.1) then $u + v$ is also a solution and hence the solution is not unique. □

The idea of the proof of Theorem 2.1 can be traced back to the work of Friedrichs [12]. The same idea has been used in [3, 7, 15]. The constant γ is introduced because the a priori estimates will be derived in weighted Lebesgue spaces. This parameter is useful as well for the absorption arguments.

In the context of differential equations, the variational equation (2.1) can be derived by multiplying the differential equation by appropriate test functions and formally integrate by parts. To prove the existence of solutions of the variational equation (2.1), one has to prove the *abstract a priori estimate* (2.2). For hyperbolic systems, the a priori estimates can be obtained with the help of symmetrizers, see [3, 7, 8, 17, 20]. Before dealing with partial differential equations, we will first illustrate how Theorem 2.1 can be used to prove well-posedness of a system of ordinary differential equations. This will be done in the succeeding section.

To prove uniqueness, a sufficient condition is to show that for each $v \in X$ there exists $w \in Y$ with $\Lambda w = v$ and $\Phi w = 0$. This corresponds to a homogeneous dual problem. In most cases, the well-posedness of the dual problem follows from the primal problem after time reversal. However, the criterion that the solution lies in the space Y is not known a priori. In the context of PDEs a different approach in proving uniqueness is taken, namely the *weak equals strong* argument.

3. Linear Ordinary Differential Equations

Consider the ordinary differential equation

$$\begin{cases} h'(t) = H(t)h(t) + f(t), & t \in (0,T), \\ h(0) = h_0 \end{cases} \tag{3.1}$$

where $T > 0$, $h : (0,T) \to \mathbb{R}^m$, $h_0 \in \mathbb{R}^m$, $H \in L^\infty((0,T);\mathbb{R}^{m\times m})$ and $f \in L^2((0,T);\mathbb{R}^m)$. A function $h \in L^2(0,T)$ is called a *weak solution* of (3.1) if the variational equation

$$(h, \eta' + H^\top \eta)_{L^2(0,T)} = -\, h_0 \cdot \eta(0) - (f,\eta)_{L^2(0,T)} \tag{3.2}$$

holds for every $\eta \in H^1(0,T)$ such that $\eta(T) = 0$. If h is a weak solution of (3.1) then necessarily $h \in H^1(0,T)$ and $h' = Hh + f$ in the weak sense. This can be seen immediately from (3.2) by taking $\eta \in \mathscr{D}(0,T)$. In addition, integrating by parts we obtain $h(0) = h_0$. As a result, the variational equation (3.2) is equivalent to the ordinary differential equation (3.1).

The existence and uniqueness of weak solutions to (3.1) is well-known and established. However, we would like to apply Theorem 2.1 to prove its well-posedness and to use the corresponding results in studying the coupled system (1.1). The application of Theorem 2.1 to (3.1) relies on an a priori estimate that will be derived using the following proposition. For the proof we refer to [3, p. 283].

Proposition 3.1. *For each $\eta \in e^{\gamma t}H^1(-\infty,T)$ and $\gamma \geq 1$ we have*

$$\int_{-\infty}^T e^{-2\gamma t}|\eta(t)|^2 \,\mathrm{d}t \leq \frac{1}{\gamma^2}\int_{-\infty}^T e^{-2\gamma t}|\eta'(t)|^2 \,\mathrm{d}t.$$

As a consequence we have the following estimate.

Corollary 3.2. *For each $\gamma \geq 1$ and $\eta \in H^1(0,T)$ such that $\eta(T) = 0$ we have*

$$\int_0^T e^{2\gamma t}|\eta(t)|^2 \,\mathrm{d}t \leq \frac{1}{\gamma^2}\int_0^T e^{2\gamma t}|\eta'(t)|^2 \,\mathrm{d}t. \tag{3.3}$$

Proof. Extending η by zero for $t > T$ we have $\eta \in H^1(0,\infty)$. Define the function $\zeta \in e^{\gamma t}H^1(-\infty,T)$ by $\zeta(t) = \eta(T-t)$. Proposition 3.1 and the change of variable $s = T - t$ imply

$$\begin{aligned} \int_0^T e^{2\gamma t}|\eta(t)|^2 \,\mathrm{d}t &= \int_{-\infty}^T e^{-2\gamma(s-T)}|\zeta(s)|^2 \,\mathrm{d}s \\ &\leq \frac{1}{\gamma^2}\int_{-\infty}^T e^{-2\gamma(s-T)}|\zeta'(s)|^2 \,\mathrm{d}s. \end{aligned} \tag{3.4}$$

Using $\zeta'(s) = -\eta'(T-s)$ and the change of variable $t = T - s$ we have

$$\begin{aligned} \int_{-\infty}^T e^{-2\gamma(s-T)}|\zeta'(s)|^2 \,\mathrm{d}s &= \int_{-\infty}^T e^{-2\gamma(s-T)}|\eta'(T-s)|^2 \,\mathrm{d}s \\ &= \int_0^T e^{2\gamma t}|\eta'(t)|^2 \,\mathrm{d}t. \end{aligned} \tag{3.5}$$

The estimate (3.3) now follows from (3.4) and (3.5). □

With the estimate (3.3), it is now possible to derive an a priori estimate needed in the well-posedness of (3.2). This a priori estimate, which can be thought of a *Poincaré-type inequality*, will be also used in the PDE-ODE systems of Section 6.

Theorem 3.3. *Let $A \in L^\infty((0,T);\mathbb{R}^{m\times m})$. There exist constants $C > 0$ and $\gamma_0 \geq 1$ depending only on $\|A\|_{L^\infty(0,T)}$ such that for all $\eta \in H^1(0,T)$ and for all $\gamma \geq \gamma_0$ we have*

$$|\eta(0)|^2 + \gamma\|e^{\gamma t}\eta\|^2_{L^2(0,T)} \leq \frac{C}{\gamma}\|e^{\gamma t}(\eta' + A\eta)\|^2_{L^2(0,T)} + Ce^{2\gamma T}|\eta(T)|^2. \tag{3.6}$$

Proof. First, suppose that $\eta \in H^1(0,T)$ satisfies $\eta(T) = 0$. According to Corollary 3.2 and the triangle inequality we have

$$\gamma\|e^{\gamma t}\eta\|^2_{L^2(0,T)} \leq \frac{2}{\gamma}\|e^{\gamma t}(\eta' + A\eta)\|^2_{L^2(0,T)} + \frac{2}{\gamma}\|A\|^2_{L^\infty(0,T)}\|e^{\gamma t}\eta\|^2_{L^2(0,T)}. \tag{3.7}$$

For sufficiently large γ, the second term on the right hand side of (3.7) can be absorbed by the term on the left hand side. Thus there are constants $C > 0$ and $\gamma_0 \geq 1$ both depending only on the L^∞-norm of A such that for all $\gamma \geq \gamma_0$

$$\gamma\|e^{\gamma t}\eta\|^2_{L^2(0,T)} \leq \frac{C}{\gamma}\|e^{\gamma t}(\eta' + A\eta)\|^2_{L^2(0,T)}. \tag{3.8}$$

Define $\eta(t) = 0$ for $t > T$ and $w(t) = e^{\gamma(T-t)}\eta(T-t)$ for $-\infty < t < T$. Then $w \in H^1(-\infty,T)$ and therefore it satisfies the weighted Sobolev estimate

$$\|w\|^2_{L^\infty(-\infty,T)} \leq \gamma\|w\|^2_{L^2(-\infty,T)} + \frac{1}{\gamma}\|w'\|^2_{L^2(-\infty,T)} \tag{3.9}$$

for all $\gamma > 0$. Since $w'(t) = -\gamma e^{\gamma(T-t)}\eta(T-t) - e^{\gamma(T-t)}\eta'(T-t)$ the above estimate implies that for some $C > 0$ there holds

$$e^{2\gamma(T-t)}|\eta(T-t)|^2 \leq C\left(\gamma\|e^{\gamma t}\eta\|^2_{L^2(0,T)} + \frac{1}{\gamma}\|e^{\gamma t}\eta'\|^2_{L^2(0,T)}\right) \tag{3.10}$$

for all $t \in [0,T]$. Choosing $t = T$ in (3.10), writing $\eta' = (\eta' + A\eta) - A\eta$ and using the same argument as before we obtain, by increasing γ_0 if necessary, that for all $\gamma \geq \gamma_0$

$$|\eta(0)|^2 \leq C\left(\gamma\|e^{\gamma t}\eta\|^2_{L^2(0,T)} + \frac{1}{\gamma}\|e^{\gamma t}(\eta' + A\eta)\|^2_{L^2(0,T)}\right) \tag{3.11}$$

for some $C > 0$. The estimate

$$|\eta(0)|^2 + \gamma\|e^{\gamma t}\eta\|^2_{L^2(0,T)} \leq \frac{C}{\gamma}\|e^{\gamma t}(\eta' + A\eta)\|^2_{L^2(0,T)} \tag{3.12}$$

follows from (3.8) and (3.11).

Now suppose that $\eta \in H^1(0,T)$. Define $\zeta \in H^1(0,T)$ by $\zeta(t) = \eta(t) - \eta(T)$ for $0 < t < T$. Applying (3.12) to ζ, using the triangle inequality and the fact that $2\gamma\|e^{\gamma t}\|^2_{L^2(0,T)} = e^{2\gamma T} - 1$ we obtain (3.6). □

We are now in a position to use Theorem 2.1 in proving that (3.2) is well-posed. We take $X = e^{-\gamma t}L^2(0,T)$, $Y = H^1(0,T)$ and $Z = \mathbb{R}^m$. The operators Λ, Ψ and Φ

are given by $\Lambda\eta = \eta' + H^\top\eta$, $\Psi\eta = \eta(0)$ and $\Phi\eta = \eta(T)$ for all $\eta \in Y$, respectively. Thus the variational equation (3.2) can be written in the form

$$(e^{-2\gamma t}h, \Lambda\eta)_X = (-e^{-2\gamma t}f, \eta)_X + (-h_0, \Psi\eta)_Z, \qquad \forall\, \eta \in W \tag{3.13}$$

where $W = \{\eta \in Y : \eta(T) = 0\}$. Note that the set X coincides with $L^2(0,T)$.

Theorem 3.4. *Let $h_0 \in \mathbb{R}^m$, $H \in L^\infty(0,T)$ and $f \in L^2(0,T)$. Then (3.1) has a unique weak solution $h \in L^2(0,T)$. Furthermore, $h \in H^1(0,T)$ and it satisfies the energy estimates*

$$\gamma\|e^{-\gamma t}h\|^2_{L^2(0,T)} \le C\left(\frac{1}{\gamma}\|e^{-\gamma t}f\|^2_{L^2(0,T)} + |h_0|^2\right) \tag{3.14}$$

and

$$\|e^{-\gamma t}h'\|^2_{L^2(0,T)} \le C(\|e^{-\gamma t}f\|^2_{L^2(0,T)} + |h_0|^2) \tag{3.15}$$

for all $\gamma \ge \gamma_0$ for some $C > 0$ and $\gamma_0 \ge 1$ both depending only on $\|H\|_{L^\infty(0,T)}$.

Proof. Using the notations of the paragraph preceding the theorem, the a priori estimate (2.3) follows directly from Theorem 3.3. Hence Theorem 2.1 implies the existence of $g \in X$ such that

$$(g, \Lambda\eta)_X = (-e^{-2\gamma t}f, \eta)_X + (-h_0, \Psi\eta)_Z, \qquad \forall\, \eta \in W,$$

and it satisfies

$$\gamma\|g\|^2_X \le C\left(\frac{1}{\gamma}\|e^{-2\gamma t}f\|^2_X + |h_0|^2\right). \tag{3.16}$$

Then $h = e^{2\gamma t}g \in L^2(0,T)$ is a weak solution of (3.1) and it satisfies (3.14) due to (3.16). From the discussion at the beginning of this section, we already know that the weak solution h lies in $H^1(0,T)$ and it satisfies $h' = Hh + f$ in $L^2(0,T)$. The estimate (3.15) follows from the differential equation $h' = Hh + f$ and (3.14). Given $f \in X$, the dual problem $\eta' + H^\top\eta = f$, $\eta(T) = 0$ admits a solution $\eta \in H^1(0,T)$, which was just shown for the forward problem. Hence $\Lambda(W) = X$ and therefore the weak solution is unique by Theorem 2.1. □

4. GRAPH SPACES AND THEIR TRACES

Let $\mathcal{O}$ be a non-empty open subset of $\mathbb{R}^2$, $A \in W^{1,\infty}(\mathcal{O})$ and $R \in L^\infty(\mathcal{O})$. Consider the linear operator $L : H^1(\mathcal{O}) \to L^2(\mathcal{O})$ defined by

$$Lu = \partial_t u + A\partial_x u + Ru.$$

By duality, we can extend the definition of L for $u \in L^1_{\mathrm{loc}}(\mathcal{O})$ in the sense of distributions. Define $L : L^1_{\mathrm{loc}}(\mathcal{O}) \to \mathscr{D}(\mathcal{O})'$ by

$$Lu(\varphi) = (Lu, \varphi)_{\mathscr{D}(\mathcal{O})' \times \mathscr{D}(\mathcal{O})} = \int_{\mathcal{O}} u \cdot L^*\varphi \,\mathrm{d}x\,\mathrm{d}t, \quad \forall\, \varphi \in \mathscr{D}(\mathcal{O})$$

where L^* denotes the formal adjoint of L given by

$$L^*\varphi = -\partial_t\varphi - A^\top\partial_x\varphi - (\partial_x A)^\top\varphi + R^\top\varphi. \tag{4.1}$$

By the definition of distributional derivatives, it can be seen that

$$Lu = \partial_t u + \partial_x(Au) - (\partial_x A)u + Ru$$

for all $u \in L^1_{\mathrm{loc}}(\mathcal{O})$ in the sense of distributions. It is clear from the definition that $L \in \mathcal{L}(L^2(\mathcal{O}); H^{-1}(\mathcal{O}))$.

Given $u \in L^2(\mathcal{O})$, suppose that there exists $C > 0$ such that

$$|Lu(\varphi)| \leq C\|\varphi\|_{L^2(\mathcal{O})}, \qquad \forall\, \varphi \in \mathscr{D}(\mathcal{O}). \tag{4.2}$$

From the Riesz Representation Theorem, there exists a unique $f \in L^2(\mathcal{O})$ such that $Lu(\varphi) = (f, \varphi)_{L^2(\mathcal{O})}$ for all $\varphi \in L^2(\mathcal{O})$ whenever (4.2) holds. Identifying $L^2(\mathcal{O})$ with its dual, we write $Lu = f$. Thus, $Lu = f$ for some $f \in L^2(\mathcal{O})$, with $u \in L^2(\mathcal{O})$, is equivalent to

$$(u, L^*\varphi)_{L^2(\mathcal{O})} = (f, \varphi)_{L^2(\mathcal{O})}, \qquad \forall\, \varphi \in \mathscr{D}(\mathcal{O}).$$

If $u \in H^1(\mathcal{O})$ then $Lu = \partial_t u + A\partial_x u + Ru$ in the weak sense. In other words, the operator L defined in the sense of distributions and the differential operator $\partial_t + A\partial_x + R$ coincide in $H^1(\mathcal{O})$.

For $\theta \in \mathscr{C}^\infty(\overline{\mathcal{O}}; \mathbb{R})$ the distribution $\theta Lu \in \mathscr{D}(\mathcal{O})'$ is defined by

$$\theta Lu(\varphi) = Lu(\theta\varphi) = (u, L^*(\theta\varphi))_{L^2(\mathcal{O})}, \quad \forall \varphi \in \mathscr{D}(\mathcal{O}).$$

The product rule for smooth functions implies that $\theta Lu = L(\theta u) - (\partial_t\theta + (\partial_x\theta)A)u$ in the sense of distributions.

Consider the following subspace of $L^2(\mathcal{O})$

$$E(\mathcal{O}) = \{u \in L^2(\mathcal{O}) : Lu \in L^2(\mathcal{O})\}.$$

Induced by the graph norm

$$\|u\|_{E(\mathcal{O})} = (\|u\|^2_{L^2(\mathcal{O})} + \|Lu\|^2_{L^2(\mathcal{O})})^{\frac{1}{2}}$$

$E(\mathcal{O})$ becomes a Hilbert space, called a *graph space*. Furthermore, the zero order terms of L are immaterial in the definition of $E(\mathcal{O})$, that is,

$$E(\mathcal{O}) = \{u \in L^2(\mathcal{O}) : \partial_t u + \partial_x(Au) \in L^2(\mathcal{O})\}.$$

The space $E(\mathcal{O})$ is closed under multiplication with functions in $\mathscr{C}^\infty_b(\overline{\mathcal{O}}; \mathbb{R})$ and if $u_j \to u$ in $E(\mathcal{O})$ then $\theta u_j \to \theta u$ in $E(\mathcal{O})$ for every $\theta \in \mathscr{C}^\infty_b(\overline{\mathcal{O}}; \mathbb{R})$.

We need traces of functions in $E(Q_T)$, where $Q_T = (0, T) \times (0, 1)$, which will be used for initial-boundary value problems. This has been done in [1] for general Lipschitz domains and in [15] for general graph spaces. It is shown in [1] that $\mathscr{D}(\overline{Q}_T)$ is dense in $E(Q_T)$. This information allows us to extend the trace operator $\Gamma : H^1(Q_T) \to H^{\frac{1}{2}}(\partial Q_T)$ to functions in $E(Q_T)$. Given $u \in E(Q_T)$ define $\Gamma_g u : H^{\frac{1}{2}}(\partial Q_T) \to \mathbb{R}$ by

$$\Gamma_g u(\varphi) = \lim_{j\to\infty} (\Gamma u_j, A_\partial \varphi)_{L^2(\partial Q_T)}, \qquad \varphi \in H^{\frac{1}{2}}(Q_T),$$

where

$$A_\partial = -\mathbf{1}_{\{x=0\}} + \mathbf{1}_{\{x=1\}} - A^{-\top}\mathbf{1}_{\{t=0\}} + A^{-\top}\mathbf{1}_{\{t=T\}}, \qquad \text{in } \partial Q_T$$

and $(u_j)_j \subset H^1(Q_T)$ and $u_j \to u$ in $E(Q_T)$. Here, $\mathbf{1}_S$ denotes the indicator function of a set S. Using the same arguments as in [1] we have $\Gamma_g u \in H^{-\frac{1}{2}}(\partial Q_T)$ and $\Gamma_g \in \mathcal{L}(E(Q_T); H^{-\frac{1}{2}}(\partial Q_T))$. Moreover, if $u \in H^1(Q)$ then $\Gamma_g u = A_\partial^\top \Gamma u$ and $\Gamma_g(\theta u) = \theta_{|\partial Q_T}\Gamma_g u$ for every $\theta \in \mathscr{C}^\infty(\overline{Q}_T; \mathbb{R})$ and $u \in E(Q_T)$.

The next step is to localize the trace defined in the previous discussion. Given a nonempty $\Sigma \subset \partial Q_T$ we define

$$\mathcal{V}(\Sigma) = \{\varphi \in H^{\frac{1}{2}}(\partial Q_T) : \operatorname{supp} \varphi \subset \Sigma\}. \tag{4.3}$$

It is known that $\mathcal{V}(\Sigma)$ is dense in $L^2(\Sigma)$, see [31, Theorem 13.6.10]. Denote by $V(\Sigma)$ the completion of $\mathcal{V}(\Sigma)$ with respect to the norm of $H^{\frac{1}{2}}(\partial Q_T)$. Thus we have the Gelfand triple

$$V(\Sigma) \subset L^2(\Sigma) \subset V(\Sigma)'. \tag{4.4}$$

If $\varphi \in V(\Sigma)$ then there exists a sequence $(\varphi_j)_j \subset \mathcal{V}(\Sigma)$ such that $\|\varphi_j - \varphi\|_{H^{\frac{1}{2}}(\partial Q_T)} \to 0$. If $a \in W^{1,\infty}(\Sigma)$ then $a^\top \varphi_j \in \mathcal{V}(\Sigma)$ and $\|a^\top \varphi_j - a^\top \varphi\|_{H^{\frac{1}{2}}(\partial Q_T)} \to 0$. Hence $a^\top \varphi \in V(\Sigma)$. As a result, we can define the product $au \in V(\Sigma)'$ where $u \in V(\Sigma)'$ and $a \in W^{1,\infty}(\Sigma)$ by

$$\langle au, \varphi \rangle_{V(\Sigma)' \times V(\Sigma)} = \langle u, a^\top \varphi \rangle_{V(\Sigma)' \times V(\Sigma)}, \qquad \varphi \in V(\Sigma).$$

Let us denote $\Sigma_0 = \{0\} \times (0,1)$, $\Sigma_1 = (0,T) \times \{0\}$, $\Sigma_2 = (0,T) \times \{1\}$ and $\Sigma_3 = \{T\} \times (0,1)$. Given $u \in E(Q_T)$ we define the generalized trace $u_{|\Sigma_1} : V(\Sigma_1) \to \mathbb{R}$ of u on Σ_1 by

$$u_{|\Sigma_1}(\varphi) = -\lim_{j\to\infty} \langle \Gamma_g u, \varphi_j \rangle_{H^{-\frac{1}{2}}(\partial Q_T) \times H^{\frac{1}{2}}(\partial Q_T)}, \qquad \varphi \in V(\Sigma_1), \tag{4.5}$$

where $(\varphi_j)_j \subset \mathcal{V}(\Sigma_1)$ and $\|\varphi_j - \varphi\|_{H^{\frac{1}{2}}(\partial Q_T)} \to 0$. By definition, we have

$$|u_{|\Sigma_1}(\varphi)| \le \|\Gamma_g u\|_{H^{-\frac{1}{2}}(\partial Q_T)} \|\varphi\|_{H^{\frac{1}{2}}(\partial Q_T)}.$$

Thus $u_{|\Sigma_1} \in V(\Sigma_1)'$ and $\|u_{|\Sigma_1}\|_{V(\Sigma_1)'} \le \|\Gamma_g u\|_{H^{-\frac{1}{2}}(\partial Q_T)}$. In particular, $u \mapsto u_{|\Sigma_1} \in \mathcal{L}(E(Q_T); V(\Sigma_1)')$ because Γ_g is bounded. It follows from the definition that

$$\langle u_{|\Sigma_1}, \varphi \rangle_{V(\Sigma_1)' \times V(\Sigma_1)} = -\langle \Gamma_g u, \varphi \rangle_{H^{-\frac{1}{2}}(\partial Q_T) \times H^{\frac{1}{2}}(\partial Q_T)} \tag{4.6}$$

for all $u \in E(Q_T)$ and $\varphi \in \mathcal{V}(\Sigma_1)$. Also,

$$u_{|\Sigma_1} = (\Gamma u)_{|\Sigma_1}, \qquad \forall\, u \in H^1(Q_T). \tag{4.7}$$

The other trace operators are defined as follows

$$\begin{aligned}
\langle u_{|\Sigma_2}, \varphi_2 \rangle_{V(\Sigma_2)' \times V(\Sigma_2)} &= \lim_{j\to\infty} \langle \Gamma_g u, \varphi_{2j} \rangle_{H^{-\frac{1}{2}}(\partial Q_T) \times H^{\frac{1}{2}}(\partial Q_T)} \\
\langle u_{|\Sigma_0}, \varphi_0 \rangle_{V(\Sigma_0)' \times V(\Sigma_0)} &= -\lim_{j\to\infty} \langle \Gamma_g u, A(0,\cdot)^\top \varphi_{0j} \rangle_{H^{-\frac{1}{2}}(\partial Q_T) \times H^{\frac{1}{2}}(\partial Q_T)} \\
\langle u_{|\Sigma_3}, \varphi_3 \rangle_{V(\Sigma_3)' \times V(\Sigma_3)} &= \lim_{j\to\infty} \langle \Gamma_g u, A(T,\cdot)^\top \varphi_{3j} \rangle_{H^{-\frac{1}{2}}(\partial Q_T) \times H^{\frac{1}{2}}(\partial Q_T)}
\end{aligned}$$

where $\varphi_i \in V(\Sigma_i)$, $\varphi_{ij} \in \mathcal{V}(\Sigma_i)$ and $\|\varphi_{ij} - \varphi_i\|_{H^{\frac{1}{2}}(\partial Q_T)} \to 0$ for $i = 0, 2, 3$. The properties of the trace $u_{|\Sigma_1}$ are carried by these traces as well. We note that the localization process we introduced above is different from the one mentioned in [7].

Using a standard density argument, we can show that

$$\int_0^T \int_0^1 Lu \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_0^1 u \cdot L^* \varphi \, \mathrm{d}x \, \mathrm{d}t + \langle A \Gamma_g u, \Gamma \varphi \rangle_{V(\Sigma_1)' \times V(\Sigma_1)} \tag{4.8}$$

for every $u \in E(Q_T)$ and $\varphi \in H^1(Q_T)$ such that $\Gamma\varphi \in \mathcal{V}(\Sigma_1)$. Similarly, we have

$$\int_0^T \int_0^1 Lu \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_0^1 u \cdot L^* \varphi \, \mathrm{d}x \, \mathrm{d}t - \langle \Gamma_g u, \Gamma\varphi \rangle_{V(\Sigma_0)' \times V(\Sigma_0)} \tag{4.9}$$

for every $u \in E(Q_T)$ and $\varphi \in H^1(Q_T)$ satisfying $\Gamma\varphi \in \mathcal{V}(\Sigma_0)$.

Let us simplify the notation for the traces we have introduced in this section. For functions $u \in E(Q_T)$ we shall also use the notations $u_{|x=0}$, $u_{|x=1}$, $u_{|t=0}$ and $u_{|t=T}$ for $u_{|\Sigma_1}$, $u_{|\Sigma_2}$, $u_{|\Sigma_0}$, and $u_{|\Sigma_3}$, respectively.

5. Weak and Strong Solutions for Linear Hyperbolic Systems

The present section is devoted to hyperbolic systems on an interval in the absence of ODE boundary conditions. We shall recall the notion of weak and strong solutions for such systems. Most of the results stated here are without proofs. We refer to [3, Chapter 9] for more details on the multidimensional case and to [25, Chapter 4] in the case of one-space dimension. For the sake of completeness and clarity, we review these results and in a form (e.g. Theorem 5.7) which will be used later. All throughout this section, we assume the following hypotheses, similar to those given in [3], see also [23].

(FS) *Friedrichs Symmetrizability.* Let $\mathcal{U} \subset \mathbb{R}^n$ be open and convex. The differential operator

$$L_w = \partial_t + A(w)\partial_x$$

is Friedrichs symmetrizable for all $w \in \mathcal{U}$, i.e., there exists a symmetric positive-definite matrix-valued function $S \in \mathscr{C}^\infty(\mathcal{U}; \mathbb{R}^{n\times n})$, called the *Friedrichs symmetrizer*, that is bounded as well as its derivatives, $S(w)A(w)$ is symmetric for all $w \in \mathcal{U}$, and there exists $\alpha > 0$ such that $S(w) \geq \alpha I_n$ for all $w \in \mathcal{U}$.

(D) *Diagonalizability.* It holds that $A \in \mathscr{C}^\infty(\mathcal{U}; \mathbb{R}^{n\times n})$ and for each $w \in \mathcal{U}$, $A(w)$ is diagonalizable with p positive eigenvalues and $n-p$ negative eigenvalues. In particular, $A(w)$ is invertible and has n independent eigenvectors.

(UKL) *Uniform Kreiss-Lopatinskiĭ Condition.* The matrices $B_0 \in \mathscr{C}^\infty(\mathcal{U}; \mathbb{R}^{p\times n})$ and $B_1 \in \mathscr{C}^\infty(\mathcal{U}; \mathbb{R}^{(n-p)\times n})$ are of full rank and there exists $C > 0$ such that for all $w \in \mathcal{U}$

$$|V| \leq C|B_0(w)V|, \qquad \text{for all } V \in E^u(A(w))$$

and

$$|V| \leq C|B_1(w)V|, \qquad \text{for all } V \in E^s(A(w))$$

where $E^u(A)$ and $E^s(A)$ denote the unstable and stable subspaces of a matrix A, respectively.

Using the full-rank assumptions on B_0 and B_1, one can prove the following decomposition of the flux matrix in terms of the boundary matrices B_0 and B_1. A proof can be found in [3, Lemma 9.4]. This decomposition is important in deriving the weak form of (1.1).

Lemma 5.1. *Assume that (D) holds and suppose that the boundary matrices* $B_0 \in \mathscr{C}^\infty(\mathcal{U};\mathbb{R}^{p\times n})$ *and* $B_1 \in \mathscr{C}^\infty(\mathcal{U};\mathbb{R}^{(n-p)\times n})$ *have full ranks at each point of* $\mathcal{U}$*. Then there exist matrix-valued maps* $N_0, C_0, M_1 \in \mathscr{C}^\infty(\mathcal{U};\mathbb{R}^{(n-p)\times n})$ *and* $N_1, C_1, M_0 \in \mathscr{C}^\infty(\mathcal{U};\mathbb{R}^{p\times n})$ *such that*

$$A(w) = M_x(w)^\top B_x(w) + C_x(w)^\top N_x(w), \qquad \forall\ (w,x) \in \mathcal{U}\times\{0,1\}. \tag{5.1}$$

In fact, N_0 is chosen so that $\binom{B_0}{N_0} \in \mathscr{C}^\infty(\mathcal{U};\mathbb{R}^{n\times n})$ is invertible with inverse $(Y_0\ D_0)$ where $Y_0 \in \mathscr{C}^\infty(\mathcal{U};\mathbb{R}^{n\times p})$ and $D_0 \in \mathscr{C}^\infty(\mathcal{U};\mathbb{R}^{n\times(n-p)})$. Thus we can take

$$M_0 = (AY_0)^\top \qquad \text{and} \qquad C_0 = (AD_0)^\top. \tag{5.2}$$

Consider the initial-boundary value problem (IBVP)

$$\begin{cases} \partial_t u + A\partial_x u + Ru = f, & 0<t<T,\ 0<x<1\\ B_0 u_{|x=0} = g_0, & 0<t<T,\\ B_1 u_{|x=1} = g_1, & 0<t<T,\\ u_{|t=0} = u_0, & 0<x<1, \end{cases} \tag{5.3}$$

where $A = A(v)$, $B_0 = B_0(v)$, $B_1 = B_1(v)$, $v \in W^{1,\infty}(Q_T)$ and $R \in L^\infty(Q_T;\mathbb{R}^{n\times n})$. All throughout this paper, we suppose that the range of v lies in a compact subset $\mathcal{K}$ of $\mathcal{U}$, $\|v\|_{W^{1,\infty}(Q_T)} \le K$ and $\|R\|_{L^\infty(Q_T)} \le \varrho$. Here, $K>0$ and $\varrho>0$ are fixed.

Definition 5.2. Let $f \in L^2(Q_T)$, $g_0, g_1 \in L^2(0,T)$ and $u_0 \in L^2(0,1)$. A function $u \in L^2(Q_T)$ is called a *weak solution* of the initial-boundary value problem (5.3) if

$$\begin{aligned}\int_0^T\!\!\int_0^1 u\cdot L^*\varphi\,\mathrm{d}x\,\mathrm{d}t \ &=\ \int_0^T\!\!\int_0^1 f\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T g_1\cdot M_1\varphi_{|x=1}\,\mathrm{d}t\\ &\quad + \int_0^T g_0\cdot M_0\varphi_{|x=0}\,\mathrm{d}t + \int_0^1 u_0\cdot\varphi_{|t=0}\,\mathrm{d}x \end{aligned} \tag{5.4}$$

holds for all $\varphi \in H^1(Q_T)$ such that $C_0\varphi_{|x=0} = 0$, $C_1\varphi_{|x=1} = 0$ and $\varphi_{|t=T} = 0$.

It is clear that the space of test functions in Definition 5.2 is dense in the solution space $L^2(Q_T)$. The following theorem states how the weak solution satisfies the IBVP (5.3) in some sense.

Theorem 5.3. *If* $u \in L^2(Q_T)$ *is a weak solution of* (5.3) *then* $u \in E(Q_T)$*. The equation* $Lu = f$ *holds in* $L^2(Q_T)$ *in the sense of distributions and the boundary and initial conditions are satisfied in the following sense*

$$B_0 u_{|x=0} = g_0 \quad \textit{in } V(\Sigma_1)', \tag{5.5}$$

$$B_1 u_{|x=1} = g_1 \quad \textit{in } V(\Sigma_2)', \tag{5.6}$$

$$u_{|t=0} = u_0 \quad \textit{in } V(\Sigma_0)'. \tag{5.7}$$

Proof. By taking $\varphi \in \mathscr{D}(Q_T)$ in the definition, the equation $Lu = f$ holds in the sense of distributions and hence $u \in E(Q_T)$. By Green's identity (4.8), (5.1) and (5.4) we have

$$\langle B_0 u_{|\Sigma_1}, M_0\varphi_{|x=0}\rangle_{V(\Sigma_1)'\times V(\Sigma_1)} = \int_0^T g_0\cdot M_0\varphi_{|x=0}\,\mathrm{d}t \tag{5.8}$$

for every $\varphi \in H^1(Q_T)$ such that $\Gamma\varphi \in \mathcal{V}(\Sigma_1)$ and $C_0\varphi_{|x=0} = 0$. Given $\psi \in \mathcal{V}(\Sigma_1)$, let $\phi \in H^1(Q_T)^p$ be such that $\Gamma\phi = \psi$ and define $\varphi \in H^1(Q_T)$ by

$$\varphi(t,x) = A(t,x)^{-\top} \begin{pmatrix} Y_0(t,x)^\top \\ D_0(t,x)^\top \end{pmatrix}^{-1} \begin{pmatrix} \phi(t,x) \\ O_{(n-p)\times 1} \end{pmatrix}.$$

It is clear that $\Gamma\varphi \in \mathcal{V}(\Sigma_1)$ and $C_0\varphi_{|x=0} = D_0^\top A^\top \varphi_{|x=0} = 0$. Also, $M_0\varphi_{|x=0} = Y_0^\top A^\top \varphi_{|x=0} = \phi_{|x=0} = \psi$. With this φ in (5.8) we have

$$\langle B_0 u_{|\Sigma_1}, \psi\rangle_{V(\Sigma_1)'\times V(\Sigma_1)} = \int_0^T g_0 \cdot \psi \,\mathrm{d}t.$$

By the density of $\mathcal{V}(\Sigma_1)$ in $V(\Sigma_1)$ this means that (5.5) holds. A similar argument shows that (5.6) holds as well.

Let us prove (5.7). For $\psi \in \mathcal{V}(\Sigma_0)$ we let $\varphi \in H^1(Q_T)$ be such that $\Gamma\varphi = \psi$. Then $C_0\varphi_{|x=0} = 0$, $C_1\varphi_{|x=1} = 0$, $\varphi_{|t=T} = 0$ and so

$$\langle u_{|\Sigma_0}, \psi\rangle_{V(\Sigma_0)'\times V(\Sigma_0)} \quad = \quad \int_0^1 u_0 \cdot \psi \,\mathrm{d}x$$

from (4.9) and (5.4). Thus $u_{|\Sigma_0} = u_0$ in $V(\Sigma_0)'$. □

We can also introduce a stronger notion of solution for the IBVP (5.3).

Definition 5.4. A function $u \in L^2(Q_T)$ is called a *strong solution* of (5.3) if there exist sequences $(u_j)_j \subset H^1(Q_T)$, $(f_j)_j \subset L^2(Q_T)$, $(g_{0j})_j \subset H^{\frac{1}{2}}(0,T)$, $(g_{1j})_j \subset H^{\frac{1}{2}}(0,T)$ and $(u_{0j})_j \subset H^{\frac{1}{2}}(0,1)$ such that

$$\begin{cases} Lu_j = f_j, \quad 0<t<T,\ 0<x<1, \\ B_0 u_{j|x=0} = g_{0j}, \quad 0<t<T, \\ B_1 u_{j|x=1} = g_{1j}, \quad 0<t<T, \\ u_{j|t=0} = u_{0j}, \quad 0<x<1, \end{cases}$$

with $u_j \to u$ and $f_j \to f$ in $L^2(Q_T)$, $g_{0j} \to g_0$ in $L^2(0,T)$, $g_{1j} \to g_1$ in $L^2(0,T)$ and $u_{0j} \to u_0$ in $L^2(0,1)$.

It can be easily seen that every strong solution of (5.3) is also a weak solution. The convergence of the sequence approximating a strong solution can be improved to $E(Q_T)$. The proof of the following theorem can be deduced immediately from the definition of strong solutions and the continuity of the trace operators.

Theorem 5.5. *If u is a strong solution of (5.3) and $(u_j)_j \subset H^1(Q_T)$ is a corresponding approximating sequence of u then $u_j \to u$ in $E(Q_T)$. In particular, $u_{j|\Sigma_i} \to u_{|\Sigma_i}$ in $V(\Sigma_i)'$ for $i = 1,2,3,4$.*

We let $\mathcal{E}(Q_T)$ be the space of all functions $\varphi \in E(Q_T)$ such that $\varphi_{|\partial Q_T} \in L^2(\partial Q_T)$ and there exists a sequence $(\varphi_j)_j \subset H^1(Q_T)$ with the property that

$$\lim_{j\to\infty} \|u_j - u\|_{E(Q_T)} + \|u_{j|\partial Q_T} - u_{|\partial Q_T}\|_{L^2(\partial Q_T)} = 0. \tag{5.9}$$

Obviously, we have $H^1(Q_T) \subset \mathcal{E}(Q_T)$. One can check that $\mathcal{E}(Q_T)$ is the completion of $H^1(Q_T)$ with respect to the norm

$$\|u\|_{\mathcal{E}(Q_T)} := (\|u\|^2_{E(Q_T)} + \|u_{|\partial Q_T}\|^2_{L^2(\partial Q_T)})^{\frac{1}{2}}. \tag{5.10}$$

The space $\mathcal{E}^*(Q_T)$ is also defined in a similar manner where L is replaced by L^*.

We can extend Green's identity to functions in $\mathcal{E}(Q_T)$ and $\mathcal{E}^*(Q_T)$.

Theorem 5.6. *For every $u \in \mathcal{E}(Q_T)$ and $\varphi \in \mathcal{E}^*(Q_T)$ we have*

$$\begin{aligned}\int_0^T\int_0^1 u \cdot L^*\varphi \,\mathrm{d}x\,\mathrm{d}t \;=\; & \int_0^T\int_0^1 Lu \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T A(t,1)u(t,1)\cdot\varphi(t,1)\,\mathrm{d}t \\ & + \int_0^T A(t,0)u(t,0)\cdot\varphi(t,0)\,\mathrm{d}t - \int_0^1 u(T,x)\cdot\varphi(T,x)\,\mathrm{d}x \\ & + \int_0^1 u(0,x)\cdot\varphi(0,x)\,\mathrm{d}x. \end{aligned} \tag{5.11}$$

Proof. Using integration by parts, (5.11) holds for all $u, v \in \mathscr{D}(\overline{Q}_T)$ and hence for all $u, v \in H^1(Q_T)$. The conclusion now follows from the density of $H^1(Q_T)$ in $\mathcal{E}(Q_T)$ and $\mathcal{E}^*(Q_T)$. □

Theorem 5.7. *Suppose that (FS), (D) and (UKL) hold. Then there exist $C = C(\varrho, K, \mathcal{K}) > 0$ and $\gamma_0 = \gamma_0(\varrho, K, \mathcal{K}) \geq 1$ such that the a priori estimate*

$$\begin{aligned}&\|u_{|t=0}\|^2_{L^2(0,1)} + \gamma\|e^{\gamma t}u\|^2_{L^2(Q_T)} + \|e^{\gamma t}u_{|x=0}\|^2_{L^2(0,T)} + \|e^{\gamma t}u_{|x=1}\|^2_{L^2(0,T)} \\ &\quad\leq C\Bigg(e^{2\gamma T}\|u_{|t=T}\|^2_{L^2(0,1)} + \frac{1}{\gamma}\|e^{\gamma t}L^*_v u\|^2_{L^2(Q_T)} \\ &\qquad + \|e^{\gamma t}C_0(v)u_{|x=0}\|^2_{L^2(0,T)} + \|e^{\gamma t}C_1(v)u_{|x=1}\|^2_{L^2(0,T)}\Bigg)\end{aligned} \tag{5.12}$$

holds for all $u \in \mathcal{E}^*(Q_T)$ and $\gamma \geq \gamma_0$.

The proof of this theorem can be found in [3, Chapter 9] in the case where $u \in H^1(Q_T)$. The fact that it holds for all $u \in \mathcal{E}^*(Q_T)$ follows immediately from the definition of the space $\mathcal{E}^*(Q_T)$. The proof of (5.12) is obtained by successively deriving various a priori estimates. These are the a priori estimates for (i) pure boundary value problems using symmetrizers, (ii) initial-boundary value problems with homogeneous initial data with the help of a causality principle and (iii) general initial-boundary value problems using duality.

Now with the help of the a priori estimate (5.12), the well-posedness of (5.3) can be obtained from Theorem 2.1, see [3, Chapter 9] and [25, Chapter 4] for the details.

Theorem 5.8. *In the situation of Theorem 5.7, the hyperbolic system (5.3) has a unique weak solution u such that $u \in C([0,T], L^2(0,1)) \cap \mathcal{E}(Q_T)$. The weak solution u is strong and there exists a sequence $(u_j)_j \subset H^1(Q_T)$ such that $u_j \to u$ in $C([0,T], L^2(0,1)) \cap E(Q_T)$ and $u_{j|x=y} \to u_{|x=y}$ in $L^2(0,T)$ for $y = 0, 1$. Furthermore, there exist $\gamma_0 = \gamma_0(\varrho, K, \mathcal{K}) \geq 1$ and $C = C(\varrho, K, \mathcal{K}) > 0$ such that u satisfies the energy estimate*

$$e^{-2\gamma T}\|u\|^2_{C([0,T],L^2(0,1))} + \gamma\|e^{-\gamma t}u\|^2_{L^2(Q_T)} + \|e^{-\gamma t}u_{|x=0}\|^2_{L^2(0,T)}$$

$$+ \|e^{-\gamma t} u_{|x=1}\|^2_{L^2(0,T)} \leq C\bigg(\|u_0\|^2_{L^2(0,1)} + \frac{1}{\gamma}\|e^{-\gamma t} f\|^2_{L^2(Q_T)} + \|e^{-\gamma t} g_0\|^2_{L^2(0,T)} + \|e^{-\gamma t} g_1\|^2_{L^2(0,T)}\bigg) \tag{5.13}$$

for every $\gamma \geq \gamma_0$.

Remark 5.9. According to Green's identity (5.11) and Theorem 5.8, the weak solution u of the IBVP (5.3) satisfies

$$\begin{aligned} \int_0^T\int_0^1 u \cdot L_v^* \varphi \, \mathrm{d}x\, \mathrm{d}t \;=\; & \int_0^T\int_0^1 f \cdot \varphi \, \mathrm{d}x\, \mathrm{d}t - \int_0^T A(v(t,1))u(t,1)\cdot \varphi(t,1)\, \mathrm{d}t \\ & + \int_0^T A(v(t,0))u(t,0)\cdot\varphi(t,0)\,\mathrm{d}t \\ & - \int_0^1 u(T,x)\cdot\varphi(T,x)\,\mathrm{d}x + \int_0^1 u_0(x)\cdot\varphi(0,x)\,\mathrm{d}x. \end{aligned}$$

for every $\varphi \in \mathcal{E}^*(Q_T)$. In particular, (5.4) holds for every $\varphi \in \mathcal{E}^*(Q_T)$ with the properties

$$C_0\varphi_{|x=0} = 0, \qquad C_1\varphi_{|x=1} = 0, \qquad \varphi_{|t=T} = 0. \tag{5.14}$$

On the other hand, if u satisfies (5.4) for every $\varphi \in \mathcal{E}^*(Q_T)$ such that (5.14) hold then u must be the unique weak solution of (5.4).

To close this section, we state the following regularity result which will be needed in Section 7. In this theorem, we limit ourselves to the case where A, B_0, B_1 and R are constant matrices.

Theorem 5.10. *Let $k \in \mathbb{N}$. If $f \in H^k(Q_T)$, $g_0, g_1 \in H^k(0,T)$ and $u_0 \in H^k(0,1)$ satisfy an appropriate compatibility condition up to order $k-1$ (e.g. (7.4) below) then the weak solution of*

$$Lu = f, \qquad B_0 u_{|x=0} = g_0, \qquad B_1 u_{|x=1} = g_1, \qquad u_{|t=0} = u_0 \tag{5.15}$$

satisfies $u \in CH^k(Q_T)$ and $u_{|x=0}, u_{|x=1} \in H^k(0,T)$. There is a sequence $(u_j)_j \subset H^{k+1}(Q_T)$ with the properties $u_j \to u$ in $CH^k(Q_T)$, $Lu_j \to Lu$ in $H^k(Q_T)$ and $u_{j|x=y} \to u_{|x=y}$ in $H^k(0,T)$ for $y = 0,1$. Moreoever, u satisfies the energy estimate

$$\begin{aligned} & e^{-\gamma T} \sum_{|\alpha|\leq k} \gamma^{2(k-|\alpha|)} \sup_{\tau\in[0,T]} \|\partial^\alpha u(\tau)\|^2_{L^2(0,1)} + \gamma\|e^{-\gamma t}u\|^2_{H^k_\gamma(Q_T)} \\ & + \|e^{-\gamma t}u_{|x=0}\|^2_{H^k_\gamma(0,T)} + \|e^{-\gamma t}u_{|x=1}\|^2_{H^k_\gamma(0,T)} \leq C_k\bigg(\sum_{j=0}^k \|u_j\|^2_{H^{k-j}(0,1)} \\ & + \frac{1}{\gamma}\|e^{-\gamma t} f\|^2_{H^k_\gamma(Q_T)} + \|e^{-\gamma t} g_0\|^2_{H^k_\gamma(0,T)} + \|e^{-\gamma t} g_1\|^2_{H^k_\gamma(0,T)}\bigg) \end{aligned} \tag{5.16}$$

for all $\gamma \geq \gamma_k$ and for some $C_k > 0$ and $\gamma_k \geq 1$.

Proof. See [25, 28] for example. □

6. Linear Hyperbolic PDE-ODE Systems

In this section we prove the existence, uniqueness and regularity of weak solutions to a linear hyperbolic system of partial differential equations coupled with a differential equation at the boundary. We are interested in the L^2-well-posedness of the following system

$$\begin{cases} L_v u(t,x) = f(t,x), \quad 0<t<T,\ 0<x<1, \\ B_0 u(t,0) = g_0(t) + Q_0(t)h(t), \quad 0<t<T, \\ B_1 u(t,1) = g_1(t) + Q_1(t)h(t), \quad 0<t<T, \\ h'(t) = H(t)h(t) + G_0(t)u(t,0) + G_1(t)u(t,1) + S(t), \quad 0<t<T, \\ u(0,x) = u_0(x), \quad 0<x<1, \\ h(0) = h_0 \end{cases} \tag{6.1}$$

where

$$L_v u(t,x) = \partial_t u(t,x) + A(v(t,x))\partial_x u(x) + R(t,x)u(t,x)$$

and $v \in W^{1,\infty}(Q_T;\mathbb{R}^n)$ satisfies the conditions stated in the previous section. All throughout this section we assume that $B_0 \in \mathbb{R}^{p\times n}$ and $B_1 \in \mathbb{R}^{(n-p)\times p}$ have full ranks, $R \in L^\infty(Q_T;\mathbb{R}^{n\times n})$, $Q_0 \in L^\infty((0,T);\mathbb{R}^{p\times m})$, $Q_1 \in L^\infty((0,T);\mathbb{R}^{(n-p)\times m})$, $H \in L^\infty((0,T);\mathbb{R}^{m\times m})$, $G_0, G_1 \in L^\infty((0,T);\mathbb{R}^{m\times n})$, $S \in L^2((0,T);\mathbb{R}^m)$. Furthermore, we suppose that (FS), (D), and (UKL) hold.

Definition 6.1. Given $f \in L^2(Q_T)$, $g_0 \in L^2(0,T)$, $g_1 \in L^2(0,T)$, $S \in L^2(0,T)$, $u_0 \in L^2(0,1)$ and $h_0 \in \mathbb{R}^m$, a pair of functions $(u,h) \in L^2(Q_T) \times L^2(0,T)$ is called a *weak solution* of the system (6.1) if the variational equality

$$\begin{aligned} &\int_0^T\int_0^1 u(t,x)\cdot L_v^*\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &+\int_0^T h(t)\cdot(\eta'(t) + \tilde{H}(t)\eta(t) + Q_1(t)^\top M_1(t)\varphi(t,1) - Q_0(t)^\top M_0(t)\varphi(t,0))\,\mathrm{d}t \\ &= \int_0^T\int_0^1 f(t,x)\cdot\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T g_1(t)\cdot(M_1(t)\varphi(t,1) + (G_1(t)Y_1)^\top\eta(t))\,\mathrm{d}t \\ &+\int_0^T g_0(t)\cdot(M_0(t)\varphi(t,0) - (G_0(t)Y_0)^\top\eta(t))\,\mathrm{d}t - \int_0^T S(t)\cdot\eta(t)\,\mathrm{d}t \\ &+\int_0^1 u_0(x)\cdot\varphi(0,x)\,\mathrm{d}x - h_0\cdot\eta(0) \end{aligned} \tag{6.2}$$

where

$$\tilde{H} = (H + G_1Y_1Q_1 + G_0Y_0Q_0)^\top,$$

holds for all $\varphi \in \mathcal{E}^*(Q_T)$ and for all $\eta \in H^1(0,T)$ such that $\varphi(T,\cdot) = 0$, $\eta(T) = 0$, $C_1\varphi_{|x=1} = -(G_1D_1)^\top\eta$ and $C_0\varphi_{|x=0} = (G_0D_0)^\top\eta$.

In Definition 6.1, the matrices M_i, Y_i and D_i are those given in Lemma 5.1. The definition of a weak solution is obtained by multiplying the system (6.1) with appropriate test functions and integrating by parts. The space of test functions in the above definition is denoted by

$$W = \{(\varphi,\eta) \in \mathcal{E}^*(Q_T) \times H^1(0,T) : \eta_{|t=T} = 0,\ \varphi_{|t=T} = 0,$$

$$C_1\varphi_{|x=1} = -(G_1D_1)^\top\eta,\ C_0\varphi_{|x=0} = (G_0D_0)^\top\eta\}.$$

Because G_0 and G_1 are in L^∞, the functions $(G_1D_1)^\top\eta$ and $(G_0D_0)^\top\eta$ may be only in $L^2(0,T)$ even for $\eta \in H^1(0,T)$. In order for the compatibility conditions $C_1\varphi_{|x=1} = -(G_1D_1)^\top\eta$ and $C_0\varphi_{|x=0} = (G_0D_0)^\top\eta$ to be meaningful, we take the space $\mathcal{E}^*(Q_T)$ to be the space for the first component instead of the space $H^1(Q_T)$ which was used in Definition 5.2.

Theorem 6.2. *The space W is dense in $L^2(Q_T) \times L^2(0,T)$.*

Proof. Take $(u,h) \in L^2(Q_T) \times L^2(0,T)$ and $\epsilon > 0$. Let $\eta \in H^1(0,T)$ be such that $\eta(T) = 0$ and $\|\eta - h\|_{L^2(0,T)} < \epsilon$. Take $w \in H_0^1(Q_T)$ satisfying $\|u - w\|_{L^2(Q_T)} < \epsilon$. Consider the IBVP

$$L_v^*\psi = 0, \quad C_0\psi_{|x=0} = (G_0D_0)^\top\eta, \quad C_1\psi_{|x=1} = -(G_1D_1)^\top\eta, \quad \psi_{|t=T} = 0. \tag{6.3}$$

This IBVP has a unique solution $\psi \in L^2(Q_T)$ and furthermore $\psi \in \mathcal{E}^*(Q_T)$ according to the dual version of Theorem 5.8.

By the absolute continuity of the Lebesgue integral, there exists $\delta = \delta(\epsilon) > 0$ such that if $\mathcal{O} \subset Q_T$ has Lebesgue measure less than or equal to δ then $\|u - \psi\|_{L^2(\mathcal{O})} < \epsilon$. Without loss of generality, we can assume that $\delta < 4T$. Let $\theta \in \mathscr{D}[0,1]$ be such that $0 \le \theta \le 1$ on $[0,1]$, $\theta = 1$ on $(0,\delta/4T) \cup (1-\delta/4T,1)$ and $\theta = 0$ on $(\delta/2T, 1-\delta/2T)$. Define $\varphi = \theta\psi + (1-\theta)w$. Since $\mathcal{E}^*(Q_T)$ is closed under addition and multiplication with smooth functions it holds that $\varphi \in \mathcal{E}^*(Q_T)$. From (6.3) and the definition of θ we have $(\varphi,\eta) \in W$. Furthermore,

$$\|u-\varphi\|_{L^2(Q_T)} \le \|\theta\|_{L^\infty(Q_T)}\|u-\psi\|_{L^2(R_{\delta,T})} + \|1-\theta\|_{L^\infty(Q_T)}\|u-w\|_{L^2(Q_T)} < 2\epsilon$$

where $R_{\delta,T} = (0,T) \times ((0,\delta/2T) \cup (1-\delta/2T,1))$. Therefore

$$\|(u,h) - (\varphi,\eta)\|_{L^2(Q_T)\times L^2(0,T)} < \sqrt{5}\epsilon$$

and consequently W is dense in $L^2(Q_T) \times L^2(0,T)$. □

We would like to apply Theorem 2.1 to prove the well-posedness of (6.1). Therefore the crucial step is to prove an a priori estimate. But first we need to rewrite (6.2) in the form (2.1). For this purpose, we set $X = e^{-\gamma t}L^2(Q_T) \times e^{-\gamma t}L^2(0,T)$, $Y = \mathcal{E}^*(Q_T) \times H^1(0,T)$ and $Z = e^{-\gamma t}L^2(0,T) \times e^{-\gamma t}L^2(0,T) \times L^2(0,1) \times \mathbb{R}^m$. Define $\Lambda : Y \to X$, $\Psi : Y \to Z$ and $\Phi : Y \to Z$ as follows

$$\begin{aligned}
\Lambda\begin{pmatrix}\varphi\\ \eta\end{pmatrix} &= \begin{pmatrix} L_v^*\varphi \\ \eta' + \tilde{H}\eta + Q_1^\top M_1\varphi_{|x=1} - Q_0^\top M_0\varphi_{|x=0}\end{pmatrix}\\
\Phi\begin{pmatrix}\varphi\\ \eta\end{pmatrix} &= \begin{pmatrix} C_0\varphi_{|x=0} - (G_0D_0)^\top\eta \\ C_1\varphi_{|x=1} + (G_1D_1)^\top\eta \\ \varphi_{|t=T} \\ \eta(T)\end{pmatrix}\\
\Psi\begin{pmatrix}\varphi\\ \eta\end{pmatrix} &= \begin{pmatrix} M_0\varphi_{|x=0} - (G_0Y_0)^\top\eta \\ -(M_1\varphi_{|x=1} + (G_1Y_1)^\top\eta) \\ \varphi_{|t=0} \\ -\eta(0).\end{pmatrix}
\end{aligned}$$

for every $(\varphi, \eta) \in Y$. With these notations, the variational equation (6.2) can be rewritten as

$$\begin{aligned}\left(e^{-2\gamma t}\begin{pmatrix}u\\h\end{pmatrix}, \Lambda\begin{pmatrix}\varphi\\\eta\end{pmatrix}\right)_X &= \left(e^{-2\gamma t}\begin{pmatrix}f\\-S\end{pmatrix}, \begin{pmatrix}\varphi\\\eta\end{pmatrix}\right)_X \\ &\quad + \left((e^{-2\gamma t}g_0, e^{-2\gamma t}g_1, u_0, h_0)^\top, \Psi\begin{pmatrix}\varphi\\\eta\end{pmatrix}\right)_Z \end{aligned} \tag{6.4}$$

for all $(\varphi, \eta) \in W = \ker \Phi$.

Theorem 6.3. *In the notation of the previous paragraph, there exist $\gamma_0 \geq 1$ and $C > 0$ such that*

$$\gamma\|(\varphi,\eta)\|_X^2 + \|\Psi(\varphi,\eta)\|_Z^2 \leq C\left(\frac{1}{\gamma}\|\Lambda(\varphi,\eta)\|_X^2 + \|\Phi(\varphi,\eta)\|_Z^2\right)$$

holds for all $(\varphi, \eta) \in Y$ and $\gamma \geq \gamma_0$.

Proof. Let $(\varphi, \eta) \in Y$. From the priori estimate (5.12) and the triangle inequality it follows that there is a constant $C > 0$ such that

$$\begin{aligned}&\|\varphi_{|t=0}\|_{L^2(0,1)}^2 + \gamma\|e^{\gamma t}\varphi\|_{L^2(Q_T)}^2 + \|e^{\gamma t}\varphi_{|x=0}\|_{L^2(0,T)}^2 + \|e^{\gamma t}\varphi_{|x=1}\|_{L^2(0,T)}^2 \\ &+ \|e^{\gamma t}(M_0\varphi_{|x=0} - (G_0Y_0)^\top\eta)\|_{L^2(0,T)}^2 + \|e^{\gamma t}(M_1\varphi_{|x=1} + (G_1Y_1)^\top\eta)\|_{L^2(0,T)}^2 \\ &\leq C\Bigg(\frac{1}{\gamma}\|e^{\gamma t}L_v^*\varphi\|_{L^2(Q_T)}^2 + \|e^{\gamma t}(C_0\varphi_{|x=0} - (G_0D_0)^\top\eta)\|_{L^2(0,T)}^2 \\ &+ \|e^{\gamma t}(C_1\varphi_{|x=1} + (G_1D_1)^\top\eta)\|_{L^2(0,T)}^2 + \|e^{\gamma t}\eta\|_{L^2(0,T)}^2 + e^{2\gamma T}\|\varphi_{|t=T}\|_{L^2(0,1)}^2\Bigg)\end{aligned} \tag{6.5}$$

for all $\gamma \geq \gamma_0$ where γ_0 is the constant in Theorem 5.7. From the a priori estimate (3.6) in Theorem 3.3 and the triangle inequality we obtain

$$\begin{aligned}&|\eta(0)|^2 + \gamma\|e^{\gamma t}\eta\|_{L^2(0,T)}^2 \\ &\leq \frac{C}{\gamma}\|e^{\gamma t}(\eta' + \tilde{H}\eta + Q_1^\top M_1\varphi_{|x=1} - Q_0^\top M_0\varphi_{|x=0})\|_{L^2(0,T)}^2 \\ &+ \frac{C}{\gamma}\|e^{\gamma t}\varphi_{|x=0}\|_{L^2(0,T)}^2 + \frac{C}{\gamma}\|e^{\gamma t}\varphi_{|x=1}\|_{L^2(0,T)}^2 + Ce^{2\gamma T}|\eta(T)|^2.\end{aligned} \tag{6.6}$$

From (6.5) and (6.6) and upon choosing γ_0 large enough, the estimate in the theorem follows after absorbing the terms $\|e^{\gamma t}\varphi_{|x=0}\|_{L^2(0,T)}^2$ and $\|e^{\gamma t}\varphi_{|x=1}\|_{L^2(0,T)}^2$. □

It is now possible to prove the existence and uniqueness of weak solutions of the system (6.1).

Theorem 6.4. *Let $f \in L^2(Q_T)$, $g_0 \in L^2(0,T)$, $g_1 \in L^2(0,T)$, $S \in L^2(0,T)$, $u_0 \in L^2(0,1)$ and $h_0 \in \mathbb{R}^m$. With the assumptions in the beginning of this section, the system (6.1) has a unique weak solution $(u,h) \in L^2(Q_T) \times L^2(0,T)$. Furthermore, $(u,h) \in [C([0,T], L^2(0,1)) \cap \mathcal{E}(Q_T)] \times H^1(0,T)$ and in particular $u_{|x=0}, u_{|x=1} \in$*

$L^2(0,T)$. *The function u is the weak solution of the IBVP*

$$\begin{cases} L_v u(t,x) = f(t,x), \quad 0 < t < T,\ 0 < x < 1, \\ B_0 u(t,0) = g_0(t) + Q_0(t)h(t), \quad 0 < t < T, \\ B_1 u(t,1) = g_1(t) + Q_1(t)h(t), \quad 0 < t < T, \\ u(0,x) = u_0(x), \quad 0 < x < 1, \end{cases} \tag{6.7}$$

and h is the solution of the ODE

$$\begin{cases} h'(t) = H(t)h(t) + G_0(t)u(t,0) + G_1(t)u(t,1) + S(t), \quad 0 < t < T, \\ h(0) = h_0. \end{cases} \tag{6.8}$$

The weak solution (u,h) satisfies the energy estimate

$$\begin{aligned} & e^{-2\gamma T}\|u\|^2_{C([0,T],L^2(0,1))} + \gamma\|e^{-\gamma t}u\|^2_{L^2(Q_T)} + \|e^{-\gamma t}u_{|x=0}\|^2_{L^2(0,T)} \\ & + \|e^{-\gamma t}u_{|x=1}\|^2_{L^2(0,T)} + \gamma\|e^{-\gamma t}h\|^2_{L^2(0,T)} \le C\Big(\|u_0\|^2_{L^2(0,1)} + |h_0|^2 \\ & + \frac{1}{\gamma}\|e^{-\gamma t}f\|^2_{L^2(Q_T)} + \|e^{-\gamma t}g_0\|^2_{L^2(0,T)} + \|e^{-\gamma t}g_1\|^2_{L^2(0,T)} + \frac{1}{\gamma}\|e^{-\gamma t}S\|^2_{L^2(0,T)}\Big) \end{aligned}$$

for all $\gamma \ge \gamma_0$ for some $C > 0$ and $\gamma_0 \ge 1$.

Proof. The existence of a weak solution is a direct consequence of Theorem 2.1 and Theorem 6.3. The next step is to show that if (u,h) is any weak solution of (6.1) then u is the weak solution of (6.7) and h is the solution of (6.8). Suppose that (u,h) is a weak solution of (6.1). Taking $\eta = 0$ and $\varphi \in H^1(Q_T)$ satisfying (5.14) we have $(\varphi,\eta) \in W$. With this (φ,η) in (6.2) we can see that u is the weak solution of (6.7). Therefore from Theorem 5.8, $u \in C([0,T],L^2(0,1)) \cap \mathcal{E}(Q_T)$ and in particular $u_{|x=0}, u_{|x=1} \in L^2(0,T)$. Moreover, from Remark 5.9 and Lemma 5.1 u satisfies the variational equation

$$\begin{aligned} & \int_0^T\int_0^1 u(t,x)\cdot L_v^*\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ & = \int_0^T\int_0^1 f(t,x)\cdot\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T (g_1(t) + Q_1(t)h(t))\cdot M_1(t)\varphi(t,1)\,\mathrm{d}t \\ & + \int_0^T (g_0(t) + Q_0(t)h_0(t))\cdot M_0(t)\varphi(t,0)\,\mathrm{d}t - \int_0^T N_1u(t,1)\cdot C_1(t)\varphi(t,1)\,\mathrm{d}t \\ & + \int_0^T N_0u(t,0)\cdot C_0(t)\varphi(t,0)\,\mathrm{d}t - \int_0^1 u(T,x)\cdot\varphi(T,x)\,\mathrm{d}x \\ & + \int_0^1 u_0(x)\cdot\varphi(0,x)\,\mathrm{d}x \end{aligned} \tag{6.9}$$

for all $\varphi \in \mathcal{E}^*(Q_T)$.

Given $\eta \in H^1(0,T)$ with $\eta(T) = 0$ consider the IBVP

$$L_v^*\varphi = 0, \quad C_0\varphi_{|x=0} = (G_0D_0)^\top\eta, \quad C_1\varphi_{|x=1} = -(G_1D_1)^\top\eta, \quad \varphi_{|t=T} = 0. \tag{6.10}$$

The dual version of Theorem 5.8 implies that (6.10) has a unique weak solution $\varphi \in L^2(Q_T)$ such that $\varphi \in \mathcal{E}^*(Q_T)$. Thus $(\varphi,\eta) \in W$. From the identity (see the remark following Lemma 5.1)

$$Y_y B_y + D_y N_y = I_n, \qquad y = 0, 1,$$

(5.1), (6.2) and (6.9) we can see that

$$\begin{aligned}&\int_0^T h(t) \cdot (\eta'(t) + H(t)^\top \eta(t))\,\mathrm{d}t \\ &= -\,h_0 \cdot \eta(0) - \int_0^T (G_0(t)u(t,0) + G_1(t)u(t,1) + S(t)) \cdot \eta(t)\,\mathrm{d}t. \qquad (6.11)\end{aligned}$$

According to (6.11) and Theorem 3.4, h is the solution of the ordinary differential equation (6.8) and $h \in H^1(0,T)$.

The energy estimate in the statement of the theorem follows from the energy estimate (5.13) for u, the energy estimate (3.14) for h and an absorption argument. Thus, any weak solution of (6.1) satisfies the energy estimate. Consequently, (6.1) has a unique weak solution. □

In particular, if (u,h) is the weak solution of (6.1) then Theorem 5.8 and Theorem 6.4 imply that the PDE is satisfied in the sense of distributions, the boundary conditions and the ODE are satisfied in $L^2(0,T)$ and the initial conditions are satisfied in $L^2(0,1) \times \mathbb{R}^m$. Due to the L^2-trace boundary regularity we have the following *interior-point trace regularity.*

Theorem 6.5. *If (u,h) is the unique weak solution of (6.1) then $u_{|x=\xi} \in L^2(0,T)$ for every $\xi \in (0,1)$.*

Proof. From the diagonalizability assumption (D), there exists an invertible matrix $T \in \mathscr{C}^\infty(\mathcal{U}; \mathbb{R}^{n\times n})$ such that $T^{-1}AT = \Lambda$ where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ consists of the eigenvalues of A. Introduce the new variables $\tilde{u} = T^{-1}u$. Because $T(\tilde{u}), T(\tilde{u})^{-1} \in W^{1,\infty}(Q_T)$ we have $\tilde{u}_{|x=\xi} \in L^2(0,T)$ if and only if $u_{|x=\xi} \in L^2(0,T)$.

Given $w \in H^1((0,T)\times(0,\xi))$, $\lambda \in W^{1,\infty}((0,T)\times(0,\xi))$ and $m \in W^{1,\infty}(0,\xi)$ we have the identity

$$\begin{aligned}&\frac{1}{2}\int_0^T \lambda(t,\xi)m(\xi)|w(t,\xi)|^2\,\mathrm{d}t = \frac{1}{2}\int_0^T \lambda(t,0)m(\xi)|w(t,0)|^2\,\mathrm{d}t \\ &- \frac{1}{2}\int_0^\xi m(x)|w(t,x)|^2\,\mathrm{d}x\Big|_{t=0}^{t=T} + \frac{1}{2}\int_0^T\int_0^\xi (\lambda(t,x)m(x))_x |w(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T\int_0^\xi (w_t(t,x) + \lambda(t,x)w_x(t,x))m(x)w(t,x)\,\mathrm{d}x\,\mathrm{d}t. \qquad (6.12)\end{aligned}$$

This can be obtained by multiplying the expression $w_t + \lambda w_x$ by mw, integrating by parts and rearranging the terms. Suppose that λ is uniformly bounded away from zero. Choose m such that $\lambda(t,\xi)m(\xi) > 0$ for every $t \in [0,T]$. From (6.12) we get the estimate, by choosing appropriate multipliers for each eigenvalue of A and taking the sum of the components

$$\begin{aligned}\|\tilde{u}_{|x=\xi}\|^2_{L^2(0,T)} \;\leq\; & C(\|\tilde{u}\|^2_{C([0,T],L^2(0,1))} + \|\tilde{u}_t + \Lambda\tilde{u}_x\|^2_{L^2(Q_T)} \\ & + \|\tilde{u}\|^2_{L^2(Q_T)} + \|\tilde{u}_{|x=0}\|^2_{L^2(0,1)}) \qquad (6.13)\end{aligned}$$

for some $C = C(\|\Lambda\|_{W^{1,\infty}}, \|m\|_{W^{1,\infty}}) > 0$ independent of $\tilde{u}$ and ξ, whenever $\tilde{u} \in H^1(Q_T)$.

According to Theorem 5.8 and Theorem 6.4 the solution $\tilde{u}$ can be approximated by a sequence of functions $(\tilde{u}^j)_j \subset H^1(Q_T)$. We can apply the estimate (6.13) to each $\tilde{u}^j$ and then pass to the limit thanks to convergence $\tilde{u}^j \to \tilde{u}$ in $C([0,T], L^2(0,1))$, $\tilde{u}^j_t + \Lambda\tilde{u}^j_x \to \tilde{u}_t + \Lambda\tilde{u}_x$ in $L^2(Q_T)$ and $\tilde{u}^j_{|x=0} \to \tilde{u}_{|x=0}$ in $L^2(0,T)$ due to Theorem 5.8. Thus $\tilde{u}_{|x=\xi} \in L^2(0,T)$ and consequently $u_{|x=\xi} \in L^2(0,T)$. □

7. Constant Coefficient Hyperbolic PDE-ODE Systems

The goal of the present section is to show that in the case where the coefficients in (6.1) are constant, the weak solution defined in the previous section coincides with the one given by semigroup theory. Consider the weak solution $(u,h) \in C([0,\infty); L^2(0,1) \times \mathbb{R}^m)$ of the system

$$\begin{cases} \partial_t u(t,x) + A\partial_x u(t,x) + Ru(t,x) = 0, \quad t > 0,\ 0 < x < 1, \\ B_0 u(t,0) = Q_0 h(t), \quad t > 0, \\ B_1 u(t,1) = Q_1 h(t), \quad t > 0, \\ h'(t) = Hh(t) + G_0 u(t,0) + G_1 u(t,1), \quad t > 0, \\ u(0,x) = u_0(x), \quad 0 < x < 1, \\ h(0) = h_0. \end{cases} \tag{7.1}$$

The boundary conditions for u and the ODE for h can be viewed as a nonlocal boundary condition for u

$$B_x u(t,x) = Q_x e^{tH} h_0 + \int_0^t Q_x e^{(t-s)H}(G_0 u(s,0) + G_1 u(s,1))\,\mathrm{d}s, \quad x = 0,1.$$

This can be derived by using the variation of parameters formula for the differential equation for h and substituting it to the boundary conditions for u. However, we will not treat the boundary conditions in this way.

Let k be a positive integer. For each $u_0 \in H^k(0,1)$ we define

$$u_i = -A\partial_x u_{i-1} - Ru_{i-1}, \qquad i = 1,\dots,k. \tag{7.2}$$

The data $(u_0,h_0) \in H^k(0,1) \times \mathbb{R}^m$ is said to be *compatible* up to order $k-1$ if

$$B_y u_i(y) = Q_y h_i, \qquad i = 0,\dots,k-1 \text{ and } y = 0,1, \tag{7.3}$$

where

$$h_i = Hh_{i-1} + G_0 u_{i-1}(0) + G_1 u_{i-1}(1), \qquad i = 1,\dots,k. \tag{7.4}$$

Theorem 7.1. *Let $k \in \mathbb{N}$. If the data $(u_0,h_0) \in H^k(0,1) \times \mathbb{R}^m$ is compatible up to order $k-1$ then the weak solution (u,h) of (7.1) satisfies $(u,h) \in CH^k(Q_T) \times H^{k+1}(0,T)$ and $u_{|x=0}, u_{|x=1} \in H^k(0,T)$.*

Proof. From Theorem 6.4, $h \in H^1(0,T)$ and u is the weak solution of the system

$$\begin{cases} \partial_t u(t,x) + A\partial_x u(t,x) + Ru(t,x) = 0, & t>0,\ 0<x<1, \\ B_0 u(t,0) = Q_0 h(t), & t>0, \\ B_1 u(t,1) = Q_1 h(t), & t>0, \\ u(0,x) = u_0(x), & 0<x<1. \end{cases} \tag{7.5}$$

From (7.3) it can be seen that the data $(u_0, 0, Q_0 h, Q_1 h)$ is compatible up to order 0 for the system (7.5). Thus Theorem 5.10 implies that $u \in CH^1(Q_T)$ and $u_{|x=0}, u_{|x=1} \in H^1(0,T)$. On the other hand, h satisfies the ODE

$$\begin{cases} h'(t) = Hh(t) + G_0 u(t,0) + G_1 u(t,1), & t>0, \\ h(0) = h_0 \end{cases} \tag{7.6}$$

still from Theorem 6.4. Since $u_{|x=0}, u_{|x=1} \in H^1(0,T)$, it follows from (7.6) that $h \in H^2(0,T)$. Consequently, Theorem 5.10 and (7.3) imply that $u \in CH^2(Q_T)$ and $u_{|x=0}, u_{|x=1} \in H^2(0,T)$. Repeating this process, one eventually arrives at $u \in CH^k(Q_T)$, $u_{|x=0}, u_{|x=1} \in H^k(0,T)$ and $h \in H^{k+1}(0,T)$. □

Next we present the following theorem stating that compatible data can be approximated by a sequence of smoother data that are still compatible. This theorem can be viewed as a generalization of Theorem 6.2. A proof is given in the Appendix.

Theorem 7.2. *Let $k \in \mathbb{N}$. If $(u_0, h_0) \in H^k(0,1) \times \mathbb{R}^m$ is compatible up to order $k-1$, then there exists a sequence $(u_0^\nu)_\nu \subset H^{k+1}(0,1)$ such that (u_0^ν, h_0) is compatible up to order k for each ν and $\|u_0^\nu - u_0\|_{H^k(0,1)} \to 0$.*

Using a diagonalization argument, the following result can be shown.

Corollary 7.3. *For every $(u_0, h_0) \in L^2(0,1) \times \mathbb{R}^m$ and $k \in \mathbb{N}$, there exists a sequence $(u_0^\nu)_\nu \subset H^k(0,1)$ such that (u_0^ν, h_0) is compatible up to order $k-1$ and $\|u_0^\nu - u_0\|_{L^2(0,1)} \to 0$.*

For each $t \geq 0$, define the operator $\mathcal{T}(t) : L^2(0,1) \times \mathbb{R}^m \to L^2(0,1) \times \mathbb{R}^m$ by

$$\mathcal{T}(t)(u_0, h_0) = (u(t,\cdot), h(t)), \qquad t \geq 0,\ (u_0, h_0) \in L^2(0,1) \times \mathbb{R}^m,$$

where (u,h) is the unique weak solution of the system (7.1). The linearity of $\mathcal{T}(t)$ follows from the linearity of the system (7.1) and the uniqueness of weak solutions. The boundedness follows from the energy estimate in Theorem 6.4. Also, $\mathcal{T}(0) = I$ and $(\mathcal{T}(t))_{t\geq 0}$ is strongly continuous since $(u,h) \in C([0,T]; L^2(0,1) \times \mathbb{R}^m)$ for any $T > 0$. Finally, since the system (7.1) is autonomous, $(\mathcal{T}(t))_{t\geq 0}$ satisfies the semigroup property.

The goal is to determine the generator of the $\mathcal{C}_0$-semigroup $(\mathcal{T}(t))_{t\geq 0}$, which we denote by $\mathcal{A}$. A candidate generator is given by the linear operator $\tilde{\mathcal{A}} : D(\tilde{\mathcal{A}}) \to L^2(0,1) \times \mathbb{R}^m$ defined by

$$\tilde{\mathcal{A}} \begin{pmatrix} u \\ h \end{pmatrix} = \begin{pmatrix} -Au_x - Ru \\ Hh + G_0 u(0) + G_1 u(1) \end{pmatrix} \tag{7.7}$$

where

$$D(\tilde{\mathcal{A}}) = \{(u,h) \in H^1(0,1) \times \mathbb{R}^m : B_0 u(0) = Q_0 h,\ B_1 u(1) = Q_1 h\}.$$

To prove that $\mathcal{A} = \tilde{\mathcal{A}}$ we proceed using the method in [9] applied to delay equations. This requires the following three steps: (1) characterize the resolvent $R(\lambda, \mathcal{A})$, (2) show that $\lambda I - \tilde{\mathcal{A}}$ is injective and (3) the resolvent of $\mathcal{A}$ and $\tilde{\mathcal{A}}$ at λ coincide. It is sufficient to prove these three steps for large enough λ.

Step 1. Suppose that $(u_0, h_0) \in H^1(0,1) \times \mathbb{R}^m$ satisfies the compatibility condition up to order 0, in other words, $(u_0, h_0) \in \mathcal{D}(\tilde{A})$. Then $u \in CH^1(Q_T)$ and $h \in H^2(0,T)$ from Theorem 7.1. For $\lambda > \omega_0$, where ω_0 is the growth bound of $\mathcal{T}(t)$, the resolvent of $\mathcal{A}$ at λ is given by the Laplace transform of the semigroup $\mathcal{T}(t)$, i.e.,

$$R(\lambda, \mathcal{A})(u_0, h_0) = \int_0^\infty e^{-\lambda t} \mathcal{T}(t)(u_0, h_0)\,\mathrm{d}t = \int_0^\infty e^{-\lambda t}(u(t,\cdot), h(t))\,\mathrm{d}t,$$

see [22] for example.

Define $w : (0,1) \to \mathbb{R}^n$ and $g \in \mathbb{R}^m$ by

$$\begin{aligned} w(x) &= \int_0^\infty e^{-\lambda t} u(t,x)\,\mathrm{d}t \\ g &= \int_0^\infty e^{-\lambda t} h(t)\,\mathrm{d}t \end{aligned}$$

so that $R(\lambda, \mathcal{A})(u_0, h_0) = (w, g)$.

Because $\partial_x : H^1(0,1) \to L^2(0,1)$ is a closed operator, $u \in C([0,T]; H^1(0,1))$ and $t \mapsto e^{-\lambda t} u_x(t,\cdot)$ is integrable for $\lambda > \gamma_1$ according to (5.16), (3.14) and (3.15), we can interchange differentiation and integration to obtain

$$w'(x) = \int_0^\infty e^{-\lambda t} u_x(t,x)\,\mathrm{d}t,$$

see [13, Theorem 3.7.12] and [10, Chap. II, Theorem 6]. Thus we take $\lambda > \max(\omega_0, \gamma_0, \gamma_1)$. Integrating by parts

$$\begin{aligned} \lambda w(x) &= -\, e^{-\lambda t} u(t,x)\Big|_{t=0}^{t=\infty} + \int_0^\infty e^{-\lambda t} u_t(t,x)\,\mathrm{d}t \\ &= u_0(x) - \int_0^\infty e^{-\lambda t}(A u_x(t,x) + R u(t,x))\,\mathrm{d}t \\ &= u_0(x) - A w'(x) - R w(x). \end{aligned} \tag{7.8}$$

Because we already know that $w \in L^2(0,1)$, (7.8) implies that $w \in H^1(0,1)$. Furthermore, for $y = 0, 1$ we have

$$B_y w(y) = \int_0^\infty e^{-\lambda t} B_y u(t,y)\,\mathrm{d}t = \int_0^\infty e^{-\lambda t} Q_y h(t)\,\mathrm{d}t = Q_y g.$$

Similarly,

$$\lambda g = H g + h_0 + G_0 w(0) + G_1 w(1).$$

Therefore the resolvent of $\mathcal{A}$ at $\lambda > \max(\omega_0, \gamma_0, \gamma_1)$ is given by $R(\lambda)(u_0, h_0) = (w, g)$, for $(u_0, h_0) \in \mathcal{D}(\tilde{A})$, where w and g satisfy the system

$$\begin{cases} Aw'(x) + (\lambda I_n + R)w(x) = u_0(x) \\ B_0 w(0) = Q_0 g \\ B_1 w(1) = Q_1 g \\ (\lambda I_m - H)g = h_0 + G_0 w(0) + G_1 w(1) \end{cases} \tag{7.9}$$

and in particular $(w, g) \in D(\tilde{\mathcal{A}})$.

Step 2. In this step we wish to show that $\lambda I - \tilde{A}$ is injective for sufficiently large λ. However, we only consider the case where $R = 0$ and $H = 0$ in this step. Let us denote the operator $\tilde{\mathcal{A}}$ by $\mathcal{A}_0$ when $R = 0$ and $H = 0$. We even prove the stronger property that $\lambda I - \mathcal{A}_0$ is bijective for λ large enough. Given $(u_0, h_0) \in L^2(0, 1) \times \mathbb{R}^m$ we show that there exists a unique $(w, g) \in D(\mathcal{A}_0)$ such that $(\lambda I - \mathcal{A}_0)(w, g) = (u_0, h_0)$. This is equivalent to the system

$$\begin{cases} Aw'(x) + \lambda w(x) = u_0(x) \\ B_0 w(0) = Q_0 g \\ B_1 w(1) = Q_1 g \\ \lambda g = h_0 + G_0 w(0) + G_1 w(1). \end{cases} \tag{7.10}$$

Thus w satisfies the two-point boundary value problem

$$\begin{cases} Aw'(x) + \lambda w(x) = u_0(x) \\ \lambda B_0 w(0) = Q_0(h_0 + G_0 w(0) + G_1 w(1)) \\ \lambda B_1 w(1) = Q_1(h_0 + G_0 w(0) + G_1 w(1)). \end{cases} \tag{7.11}$$

Therefore to show that there exists a unique (w, g) satisfying (7.10) it is enough to prove that the two-point boundary value problem (7.11) has a unique solution.

Due to the assumption on the matrix A, there exists an invertible matrix T such that $T^{-1}AT = \Lambda$ where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. By rearranging the columns of T we can assume without loss of generality that $\lambda_1 \leq \cdots \leq \lambda_{n-p} < 0 < \lambda_{n-p+1} \leq \cdots \lambda_n$. Let $v = T^{-1}w$, $v_0 = T^{-1}u_0$ and $\tilde{B}_y = B_y T$ for $y = 0, 1$. Then (7.11) is equivalent to

$$\begin{cases} \lambda v + \Lambda v_x = v_0 \\ \lambda \tilde{B}_0 v(0) = Q_0 h_0 + Q_0 G_0 T v(0) + Q_0 G_1 T v(1) \\ \lambda \tilde{B}_1 v(1) = Q_1 h_0 + Q_0 G_0 T v(0) + Q_1 G_1 T v(1). \end{cases} \tag{7.12}$$

Note that $(\Lambda, \tilde{B}_0, \tilde{B}_1)$ still satisfies the uniform Lopatinskii condition. Thus $\tilde{B}_0$ is injective on the unstable subspace of Λ which is $\{0\}^{n-p} \oplus \mathbb{R}^p$, while $\tilde{B}_1$ is injective on the stable subspace of Λ which is $\mathbb{R}^{n-p} \oplus \{0\}^p$. We will decompose a vector v in $\mathbb{R}^n$ by $v = \binom{v^-}{v^+}$ where $v^- \in \mathbb{R}^{n-p}$ and $v^+ \in \mathbb{R}^p$. Partitioning $\tilde{B}_0 = (\tilde{B}_0^- \ \tilde{B}_0^+)$ we have

$$\tilde{B}_0 v(0) = \tilde{B}_0^- v^-(0) + \tilde{B}_0^+ v^+(0). \tag{7.13}$$

where $\tilde{B}_0^+ \in \mathbb{R}^{p \times p}$ and $\tilde{B}_0^- \in \mathbb{R}^{p \times (n-p)}$. The matrix $\tilde{B}_0^+$ is invertible and so from (7.13) the boundary condition at $x = 0$ in (7.12) can be written as

$$(\lambda I_p + R_1)v^+(0) = (\lambda R_2 + R_3)v^-(0) + R_4 v^-(1) + R_5 v^+(1) + R_6 h_0 \tag{7.14}$$

for some matrices R_i. Similarly, the boundary condition at $x = 1$ is equivalent to

$$(\lambda I_{n-p} + S_1)v^-(1) = (\lambda S_2 + S_3)v^+(1) + S_4 v^-(0) + S_5 v^+(0) + S_6 h_0 \tag{7.15}$$

for some matrices S_i.

By the variation of parameters formula, the function v in (7.12) is given by

$$v(x) = e^{-x\lambda\Lambda^{-1}} \begin{pmatrix} c^- \\ c^+ \end{pmatrix} + \int_0^x e^{-(x-y)\lambda\Lambda^{-1}} \Lambda^{-1} v_0(y)\, \mathrm{d}y \tag{7.16}$$

and from (7.14) and (7.15) the vectors c^- and c^+ satisfy the equations

$$\begin{cases} (\lambda I_p + R_1)c^+ = (\lambda R_2 + R_3)c^- + R_4(e^{-\lambda(\Lambda^-)^{-1}} c^- + d^-) \\ \qquad\qquad + R_5(e^{-\lambda(\Lambda^+)^{-1}} c^+ + d^+) + R_6 h_0 \\ (\lambda I_{n-p} + S_1)(e^{-\lambda(\Lambda^-)^{-1}} c^- + d^-) = (\lambda S_2 + S_3)(e^{-\lambda(\Lambda^+)^{-1}} c^+ + d^+) \\ \qquad\qquad + S_4 c^- + S_5 c^+ + S_6 h_0 \end{cases} \tag{7.17}$$

where $\Lambda^- = \mathrm{diag}(\lambda_1, \dots, \lambda_{n-p})$, $\Lambda^+ = \mathrm{diag}(\lambda_{n-p+1}, \dots, \lambda_n)$ and

$$d = \int_0^1 e^{-(1-y)\lambda\Lambda^{-1}} \Lambda^{-1} v_0(y)\, \mathrm{d}y. \tag{7.18}$$

The system (7.17) can be written in matrix form as

$$\begin{pmatrix} R_5 e^{-\lambda(\Lambda^+)^{-1}} - R_1 - \lambda I_p & \lambda R_2 + R_3 + R_4 e^{-\lambda(\Lambda^-)^{-1}} \\ (\lambda S_2 + S_3)e^{-\lambda(\Lambda^+)^{-1}} + S_5 & S_4 - (\lambda I_{n-p} + S_1)e^{-\lambda(\Lambda^-)^{-1}} \end{pmatrix} \begin{pmatrix} c^+ \\ c^- \end{pmatrix}$$
$$= \begin{pmatrix} -R_6 h_0 + R_7 d \\ -S_6 h_0 + S_7(\lambda) d \end{pmatrix}. \tag{7.19}$$

Therefore to show that (7.12) has a unique solution, we must prove that the 2×2 matrix on the left hand side of (7.19) is invertible. To prove this, we need the following result in linear algebra.

Lemma 7.4. *Let A, B, C and D be $m \times m$, $m \times (n-m)$, $(n-m) \times m$ and $(n-m) \times (n-m)$ matrices, respectively. If A and $D - CA^{-1}B$ are invertible then the block matrix*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{7.20}$$

is invertible.

For sufficiently large $\lambda > 0$, the matrix

$$\Xi_\lambda := \lambda^{-1}(R_5 e^{-\lambda(\Lambda^+)^{-1}} - R_1) - I_p$$

is invertible and so $\lambda\Xi_\lambda$ is invertible. Consider the matrix

$$\begin{aligned} \Sigma_\lambda \ := \ & S_4 - (\lambda I_{n-p} + S_1)e^{-\lambda(\Lambda^-)^{-1}} \\ & - [(\lambda S_2 + S_3)e^{-\lambda(\Lambda^+)^{-1}} + S_5]\lambda^{-1}\Xi_\lambda^{-1}[\lambda R_2 + R_3 + R_4 e^{-\lambda(\Lambda^-)^{-1}}]. \end{aligned}$$

It can be seen that the matrix

$$\lambda^{-1}\Sigma_\lambda e^{\lambda(\Lambda^-)^{-1}} = \lambda^{-1}(S_4 e^{\lambda(\Lambda^-)^{-1}} - S_1) - I_{n-p}$$
$$- [(S_2 + \lambda^{-1}S_3)e^{-\lambda(\Lambda^+)^{-1}} + \lambda^{-1}S_5]\Xi_\lambda^{-1}[R_2 e^{\lambda(\Lambda^-)^{-1}} + \lambda^{-1}R_3 e^{\lambda(\Lambda^-)^{-1}} + \lambda^{-1}R_4]$$

is invertible for large $\lambda > 0$. Consequently, Σ_λ is invertible for sufficiently large $\lambda > 0$. Therefore from Lemma 7.4, the system (7.19) has a unique solution $(c^+ \ c^-)$ and so (7.12) has a unique solution v. As a result, (7.9) has a unique solution (w, g).

From (7.16), (7.18) and (7.19) there exists a constant $C_\lambda > 0$ such that

$$\|w\|_{L^2(0,1)} = \|Tv\|_{L^2(0,1)} \leq C_\lambda(\|u_0\|_{L^2(0,1)} + |h_0|).$$

The last equation in (7.10) together with (7.16), (7.18) and (7.19) imply that

$$|g| \leq C_\lambda(\|u_0\|_{L^2(0,1)} + |h_0|)$$

for some $C_\lambda > 0$. Therefore $R(\lambda, \mathcal{A}_0) \in \mathcal{L}(L^2(0,1) \times \mathbb{R}^m)$ so that $\mathcal{A}_0$ has a nonempty resolvent. Hence $\mathcal{A}_0$ is closed.

Step 3. In this step we show that the resolvents of $\mathcal{A}$ (with $R = 0$ and $H = 0$) and $\mathcal{A}_0$ at λ are the same for sufficiently large λ. Let $(u_0, h_0) \in D(\mathcal{A}_0)$. From (7.9) and (7.10) we have

$$(\lambda I - \mathcal{A}_0)R(\lambda, \mathcal{A})(u_0, h_0) = (\lambda I - \mathcal{A}_0)(w, g) = (u_0, h_0).$$

Thus $(\lambda I - \mathcal{A}_0)R(\lambda, \mathcal{A}) = I$ in $D(\mathcal{A}_0)$. Since $R(\lambda, \mathcal{A}) \in \mathcal{L}(L^2(0,1) \times \mathbb{R}^m)$, $\mathcal{A}_0$ is closed and $D(\mathcal{A}_0)$ is dense in $L^2(0,1) \times \mathbb{R}^m$ according to Corollary 7.3, we have $(\lambda I - \mathcal{A}_0)R(\lambda, \mathcal{A}) = I$ in $L^2(0,1) \times \mathbb{R}^m$.

Let $z \in D(\mathcal{A}_0)$ and $y = R(\lambda, \mathcal{A})(\lambda I - \mathcal{A}_0)z$. Then $(\lambda I - \mathcal{A}_0)y = (\lambda I - \mathcal{A}_0)z$. Since $\lambda I - \mathcal{A}_0$ is injective for sufficiently large $\lambda > 0$ it follows that $y = z$ and hence $R(\lambda, \mathcal{A})(\lambda I - \mathcal{A}_0)z = z$ for all $z \in D(\mathcal{A}_0)$. Therefore $R(\lambda, \mathcal{A}_0) = R(\lambda, \mathcal{A})$ and also the domain of $\mathcal{A}$ is $D(\mathcal{A}_0)$. Since

$$\begin{aligned} \lambda I - \mathcal{A} &= (\lambda I - \mathcal{A}_0)R(\lambda, \mathcal{A}_0)(\lambda I - \mathcal{A}) \\ &= (\lambda I - \mathcal{A}_0)R(\lambda, \mathcal{A})(\lambda I - \mathcal{A}) \ = \ \lambda I - \mathcal{A}_0 \end{aligned}$$

we conclude that $\mathcal{A} = \mathcal{A}_0$.

Now let us turn to the general case where R and H are not necessarily zero. We can write the operator $\tilde{\mathcal{A}}$ defined by (7.7) as $\tilde{\mathcal{A}} = \mathcal{A}_0 + \mathcal{B}$ where $\mathcal{A}_0 : D(\tilde{\mathcal{A}}) \to L^2(0,1) \times \mathbb{R}^m$ and $\mathcal{B} : L^2(0,1) \times \mathbb{R}^m \to L^2(0,1) \times \mathbb{R}^m$ are given by

$$\begin{aligned} \mathcal{A}_0\begin{pmatrix} u \\ h \end{pmatrix} &= \begin{pmatrix} -Au_x \\ G_0u(0) + G_1u(1) \end{pmatrix} \\ \mathcal{B}\begin{pmatrix} u \\ h \end{pmatrix} &= \begin{pmatrix} -Ru \\ Hh \end{pmatrix}. \end{aligned}$$

Since $\mathcal{A}_0$ is closed and $\mathcal{B}$ is bounded, $\tilde{\mathcal{A}}$ is closed. We know from above that $\mathcal{A}_0$ generates a $\mathcal{C}_0$-semigroup on $L^2(0,1) \times \mathbb{R}^m$. It follows from the bounded perturbation theorem of semigroups that $\tilde{\mathcal{A}}$ generates a $\mathcal{C}_0$-semigroup on $L^2(0,1) \times \mathbb{R}^m$. Therefore $\lambda I - \tilde{\mathcal{A}}$ is invertible for sufficiently large $\lambda > 0$. Similar arguments as in Step 3 show that $\mathcal{A} = \tilde{\mathcal{A}}$.

Therefore, the solution of the system (7.1) given by semigroup theory coincides with the weak solution given in Definition 6.1. An alternative way of proving that the weak and semigroup solutions are the same is to prove that the operator $\tilde{A}$ generates a $\mathcal{C}_0$-semigroup. For initial data in $\mathcal{D}(\tilde{A}^2)$ we have a classical solution and so we can multiply the system with the appropriate test functions and use integration by parts to show that the semigroup solution is the weak solution. By the density of $\mathcal{D}(\tilde{A}^2)$ in $L^2(0,1) \times \mathbb{R}^m$, this is also true for every initial data in

$L^2(0,1)\times\mathbb{R}^m$. However, proving that $\tilde{\mathcal{A}}$ is a generator is a difficult task, specifically it is hard to show that $\tilde{\mathcal{A}}-\lambda I$ is dissipative for some $\lambda\in\mathbb{R}$.

If (u,h) is the weak solution of (7.1) then $u_{|x=0}, u_{|x=1}\in L^2(0,T)$ and $h\in H^1(0,T)$ for every $T>0$ according to Theorem 6.4. These properties are called *hidden regularity*. Note that these cannot be obtained directly from standard semigroup methods because in general the solution given by semigroup theory only satisfies $(u,h)\in C([0,\infty);L^2(0,1)\times\mathbb{R}^m)$. In the literature, hidden regularity properties for weak solutions of partial differential equations were established using Fourier analysis and multiplier methods, see [16, 18, 19].

As an application, we provide a class of admissible observation operators for the semigroup $(\mathcal{T}(t))_{t\geq 0}$.

Example 7.5. If we define the operator $\mathcal{C}: D(\mathcal{A})\to\mathbb{R}^s$ by

$$\mathcal{C}\begin{pmatrix}u_0\\h_0\end{pmatrix}=\sum_{i=1}^N J_i u_0(\xi_i),\qquad \xi_i\in[0,1],$$

where $D(\mathcal{A})$ is the domain of the generator $\mathcal{A}$ of the semigroup $(\mathcal{T}(t))_{t\geq 0}$ defined above and $J_i\in\mathbb{R}^{s\times n}$ for $1\leq i\leq N$, then $\mathcal{C}$ is an admissible observation operator for $(\mathcal{T}(t))_{t\geq 0}$, see [31]. Indeed, the direct inequality

$$\int_0^T\left|\mathcal{C}\mathcal{T}(t)\begin{pmatrix}u_0\\h_0\end{pmatrix}\right|^2\mathrm{d}t\leq M_T\left\|\begin{pmatrix}u_0\\h_0\end{pmatrix}\right\|^2_{L^2(0,1)^n\times\mathbb{R}^m}\qquad\forall\,(u_0,h_0)\in D(\mathcal{A})$$

follows immediately from the energy estimate in Theorem 6.4 and the estimate (6.13).

8. EXAMPLES

Example 8.1. *(Linearized Flow in an Elastic Tube* [21, 27]*)* Consider an elastic tube of length ℓ filled with an incompressible fluid whose ends are attached to a tank with cross section A_T. Looking at the dynamics near the steady state, the following linear model can be derived

$$\begin{cases}\partial_t A(t,x)+A_e\partial_x u(t,x)=0, & t>0,\ 0<x<\ell,\\ \partial_t u(t,x)+\alpha\partial_x A(t,x)+\beta u(t,x)=0, & t>0,\ 0<x<\ell,\\ A(t,0)=\gamma h_0(t), & t>0,\\ A(t,\ell)=\gamma h_\ell(t), & t>0,\\ A_T h_0'(t)=-A_e u(t,0), & t>0,\\ A_T h_\ell'(t)=A_e u(t,\ell), & t>0,\\ A(0,x)=A^0(x), & 0<x<\ell,\\ u(0,x)=u^0(x), & 0<x<\ell,\\ h_0(0)=h_0^0, & h_\ell(0)=h_\ell^0.\end{cases}\tag{8.1}$$

Here (A,u,h_0,h_ℓ) are the deviations of the cross-sectional area of the tube, the fluid velocity and the level heights from the equilibrium $(A_e,0,h_{0e},h_{\ell e})$. Also, $\alpha,\gamma>0$ and $\beta\geq 0$ are parameters based on either the physical properties of the fluid or the material properties of the tube or both.

It follows from Theorem 6.4 that (8.1) admits a unique weak solution $A, u \in C([0,T], L^2(0,\ell))$, $h_0, h_\ell \in H^1(0,T)$ with boundary traces $A(\cdot,0), A(\cdot,\ell), u(\cdot,0)$, $u(\cdot,\ell) \in L^2(0,T)$. The boundary conditions further imply that $A(\cdot,0), A(\cdot,\ell) \in H^1(0,T)$. Furthermore, the previous section shows that this solution coincides with the one given by semigroup theory. In an earlier work [24], it is shown that the velocity admits L^2-traces at the boundary using tools from control theory and Fourier analysis.

Example 8.2. *(Wave Equations with Oscillator Boundary Conditions* [2, 14]*)* Consider the one-dimensional undamped wave equation with oscillator boundary conditions

$$\begin{cases} \partial_{tt}\psi(t,x) - \partial_{xx}\psi(t,x) = 0, \quad t>0,\ 0<x<\ell, \\ \partial_x\psi(t,0) = -\delta_0'(t), \quad t>0, \\ \partial_x\psi(t,\ell) = \delta_\ell'(t), \quad t>0, \\ m_0\delta_0''(t) + d_0\delta_0'(t) + k_0\delta_0(t) = -\rho\partial_t\psi(t,0), \quad t>0, \\ m_\ell\delta_\ell''(t) + d_\ell\delta_\ell'(t) + k_\ell\delta_\ell(t) = -\rho\partial_t\psi(t,\ell), \quad t>0, \\ \psi(0,x) = \psi_0(x), \quad 0<x<\ell, \\ \partial_t\psi(0,x) = \psi_1(x), \quad 0<x<\ell, \\ \delta_i(0) = \delta_i^0, \quad i=0,\ell, \\ \delta_i'(0) = v_i^0, \quad i=0,\ell. \end{cases} \tag{8.2}$$

The system (8.2) models the velocity potential ψ of the acoustics in a homogeneous fluid with nominal density ρ contained in a wave guide of length ℓ and terminated by oscillators. In this model it is assumed that the fluid does not penetrate the surface of the oscillators.

As in Ito and Propst [14], we introduce the variables $\phi^- = \frac{1}{2}(\partial_t\psi + \partial_x\psi)$, $\phi^+ = \frac{1}{2}(\partial_t\psi - \partial_x\psi)$, $v_0 = \delta_0'$ and $v_\ell = \delta_\ell'$. The system (8.2) can be put in the form (7.1) as

$$\begin{cases} \partial_t\phi^-(t,x) - \partial_x\phi^-(t,x) = 0, \quad t>0,\ 0<x<\ell, \\ \partial_t\phi^+(t,x) + \partial_x\phi^+(t,x) = 0, \quad t>0,\ 0<x<\ell, \\ \phi^-(t,0) - \phi^+(t,0) = -v_0(t), \quad t>0, \\ \phi^-(t,\ell) - \phi^+(t,\ell) = v_\ell(t), \quad t>0, \\ \delta_0'(t) = v_0(t), \quad t>0, \\ \delta_\ell'(t) = v_\ell(t), \quad t>0, \\ v_0'(t) = -\dfrac{d_0}{m_0}v_0(t) - \dfrac{k_0}{m_0}\delta_0(t) - \dfrac{\rho}{m_0}(\phi^-(t,0) + \phi^+(t,0)), \quad t>0, \\ v_\ell'(t) = -\dfrac{d_\ell}{m_\ell}v_\ell(t) - \dfrac{k_\ell}{m_\ell}\delta_\ell(t) - \dfrac{\rho}{m_\ell}(\phi^-(t,\ell) + \phi^+(t,\ell)), \quad t>0, \\ \phi^-(0,x) = \phi_0^-(x), \quad 0<x<\ell, \\ \phi^+(0,x) = \phi_0^+(x), \quad 0<x<\ell, \\ \delta_i(0) = \delta_i^0, \quad i=0,\ell, \\ v_i(0) = v_i^0, \quad i=0,\ell, \end{cases} \tag{8.3}$$

where $\phi_0^- = \frac{1}{2}(\psi_1 + \psi_0')$ and $\phi_0^+ = \frac{1}{2}(\psi_1 - \psi_0')$. It can be checked that all the requirements in Theorem 6.4 are satisfied by the system (8.3). Therefore for every

$(\phi_0^-, \phi_0^+, \delta_0, \delta_\ell, v_0, v_\ell) \in L^2(0,\ell)^2 \times \mathbb{R}^4$ the system (8.3) has a unique weak solution $(\phi^-, \phi^+, \delta_0, \delta_\ell, v_0, v_\ell) \in C([0,\infty); L^2(0,\ell)^2 \times \mathbb{R}^4)$ and it satisfies $\phi^\pm(\cdot,0), \phi^\pm(\cdot,\ell) \in L^2(0,T)$ and $\delta_0, \delta_\ell, v_0, v_\ell \in H^1(0,T)$ for every $T > 0$. Consequently, $\delta_0, \delta_\ell \in H^2(0,T)$ and $\phi^-(\cdot,0) - \phi^+(\cdot,0), \phi^-(\cdot,\ell) - \phi^+(\cdot,\ell) \in H^1(0,T)$. The well-posedness of (8.3) was established in [14] using semigroup methods. Here, we improved this result by showing that the solutions admit traces in L^2 and that the oscillator components are more regular.

Example 8.3. *(Wave Equations with Exponential Memory Kernel* [26]*)* The next example is the normalized damped wave equation with memory boundary conditions

$$\begin{cases} \partial_{tt}\phi(t,x) - \partial_{xx}\phi(t,x) + \partial_t\phi(t,x) = 0, \quad t > 0,\ 0 < x < 1, \\ \displaystyle\int_0^t a_0(t-s)\partial_t\phi(s,0)\,\mathrm{d}s - \partial_x\phi(t,0) = 0, \quad t > 0, \\ \displaystyle\int_0^t a_1(t-s)\partial_t\phi(s,1)\,\mathrm{d}s + \partial_x\phi(t,1) = 0, \quad t > 0, \\ \phi(0,x) = \phi_0(x), \quad 0 < x < 1, \\ \partial_t\phi(0,x) = \phi_1(x), \quad 0 < x < 1. \end{cases} \tag{8.4}$$

Suppose that the kernels a_0 and a_1 take the form $a_0(t) = \kappa_0 e^{\alpha_0 t}$ and $a_1(t) = \kappa_1 e^{\alpha_1 t}$ for some nonzero real numbers $\kappa_0, \kappa_1, \alpha_0, \alpha_1$. Introducing the state vector

$$(u,v,h,g)(t) = \left(\phi_t(t,\cdot), \phi_x(t,\cdot), \int_0^t e^{\alpha_0(t-s)}\phi_t(s,0)\,\mathrm{d}s, \int_0^t e^{\alpha_1(t-s)}\phi_t(s,1)\,\mathrm{d}s\right)$$

at time t, the system (8.4) can be written in the form of (7.1) as

$$\begin{cases} \partial_t u(t,x) - \partial_x v(t,x) + u(t,x) = 0, \quad t > 0,\ 0 < x < 1, \\ \partial_t v(t,x) - \partial_x u(t,x) = 0, \quad t > 0,\ 0 < x < 1, \\ v(t,0) = \kappa_0 h(t), \quad t > 0, \\ v(t,1) = -\kappa_1 g(t), \quad t > 0, \\ h'(t) = \alpha_0 h(t) + u(t,0), \quad t > 0, \\ g'(t) = \alpha_1 g(t) + u(t,1), \quad t > 0, \\ u(0,x) = u_0(x), \quad 0 < x < 1, \\ v(0,x) = v_0(x), \quad 0 < x < 1, \\ h(0) = h_0, \\ g(0) = g_0, \end{cases} \tag{8.5}$$

where $u_0 = \phi_1$, $v_0 = \phi_0'$ and $h_0 = g_0 = 0$. The conditions of Theorem 6.4 are satisfied by the system (8.5). Thus, for each initial data $(u_0, v_0, h_0, g_0) \in L^2(0,1)^2 \times \mathbb{R}^2$ the system (8.5) admits a unique weak solution $(u,v,h,g) \in C([0,\infty); L^2(0,1)^2 \times \mathbb{R}^2)$, and moreover, $u(\cdot,0), v(\cdot,0), u(\cdot,1), v(\cdot,1) \in L^2(0,T)$ and $h, g \in H^1(0,T)$ for every $T > 0$. As a consequence, $v(\cdot,0), v(\cdot,1) \in H^1(0,T)$.

9. APPENDIX

We give the proof of Theorem 7.2. The proof follows the ideas presented in [28] for hyperbolic systems. Pick a sequence $(v_\nu)_\nu \subset H^{k+1}(0,1)$ satisfying $v_\nu \to u_0$ in

$H^k(0,1)$. Define $u_0^\nu = v_\nu - w_\nu$ where $w_\nu \in H^{k+1}(0,1)$ satisfies $w_\nu \to 0$ in $H^k(0,1)$ and to be constructed below. The compatibility conditions for u_0^ν are given by

$$B_y w_{\nu,i}(y) = B_y v_{\nu,i}(y) - Q_y h_{\nu,i}, \qquad 0 \le i \le k,\ y = 0,1, \tag{9.1}$$

where

$$\begin{aligned} w_{\nu,0} &= w_\nu, \qquad v_{\nu,0} = v_\nu, \qquad h_{\nu,0} = h_0,\\ w_{\nu,i} &= -A\partial_x w_{\nu,i-1} - Rw_{\nu,i-1}, \quad 1 \le i \le k+1\\ v_{\nu,i} &= -A\partial_x v_{\nu,i-1} - Rv_{\nu,i-1}, \quad 1 \le i \le k+1\\ h_{\nu,i} &= Hh_{\nu,i-1} + G_0(v_{\nu,i-1}(0) - w_{\nu,i-1}(0))\\ &\quad + G_1(v_{\nu,i-1}(1) - w_{\nu,i-1}(1)), \qquad 1 \le i \le k. \end{aligned}$$

The compatibility conditions (9.1) can be rewritten as

$$B_y w_\nu(y) = B_y v_\nu(y) - Q_y h_0 \tag{9.2}$$

$$\begin{aligned} B_y A^i \partial_x^i w_\nu(y) = B_y A^i \partial_x^i v_\nu(y) + \ell_{y,i}(&h_0, v_\nu - w_\nu, \ldots, \partial_x^{i-1} v_\nu - \partial_x^{i-1} w_\nu,\\ &v_\nu(0) - w_\nu(0), v_\nu(1) - w_\nu(1), \ldots, \partial_x^{i-1} v_\nu(0) - \partial_x^{i-1} w_\nu(0),\\ &\partial_x^{i-1} v_\nu(1) - \partial_x^{i-1} w_\nu(1)) \end{aligned} \tag{9.3}$$

for $y = 0,1$ and $i = 1,\ldots,k$, where $\ell_{y,i}$ is linear in all its arguments.

Recall that there exits a matrix Y_y such that $B_y Y_y = I$ where I is the identity matrix I_p if $y = 0$ and I_{n-p} if $y = 1$. Consider the following equations

$$w_\nu(y) = Y_y(B_y v_\nu(y) - Q_y h_0) \tag{9.4}$$

$$\begin{aligned} \partial_x^i w_\nu(y) = A^{-i} Y_y(B_y A^i \partial_x^i v_\nu(y) + \ell_{y,i}(&h_0, v_\nu - w_\nu, \ldots, \partial_x^{i-1} v_\nu - \partial_x^{i-1} w_\nu,\\ &v_\nu(0) - w_\nu(0), v_\nu(1) - w_\nu(1), \ldots, \partial_x^{i-1} v_\nu(0) - \partial_x^{i-1} w_\nu(0),\\ &\partial_x^{i-1} v_\nu(1) - \partial_x^{i-1} w_\nu(1))) \end{aligned} \tag{9.5}$$

for $y = 0,1$ and $i = 1,\ldots,k$. By multiplying B_y and $B_y A^i$ to both sides of (9.4) and (9.5), respectively, we obtain (9.2) and (9.3), respectively. For this reason we construct w_ν that satisfies (9.4) and (9.5) in addition to the property $w_\nu \to 0$ in $H^k(0,1)$.

For $i = 0,\ldots k$ and $\nu \in \mathbb{N}$, let $\sigma_{\nu,i}(y)$ denote the right hand side of (9.4) and (9.5). Since $v_\nu \to u_0$ and $w_\nu \to 0$ both in $H^k(0,1)$, we have $\partial_x^i v_\nu(y) \to \partial_x^i u_0(y)$ and $\partial_x^i w_\nu(y) \to 0$ for all $0 \le i \le k-1$ by the Sobolev embedding. Thus, by the compatibility conditions for (u_0, h) we have $\sigma_{\nu,i}(y) \to 0$ for $0 \le i \le k-1$ and $y = 0,1$. Now given $(\sigma_{\nu,0}(0), \sigma_{\nu,0}(1), \ldots, \sigma_{\nu,k-1}(0), \sigma_{\nu,k-1}(1), 0, 0) \in \mathbb{R}^{2n\times(k+1)}$ there exists $\tilde{v}_\nu \in H^{k+1}(0,1)$ such that $\partial_x^i \tilde{v}_\nu(y) = \sigma_{\nu,i}(y)$ for $0 \le i \le k-1$, $\partial_x^k \tilde{v}_\nu(y) = 0$ and

$$\|\tilde{v}_\nu\|_{H^{k+1}(0,1)} \le C \sum_{i=0}^{k-1} (|\sigma_{\nu,i}(0)| + |\sigma_{\nu,i}(1)|) \to 0 \tag{9.6}$$

for some $C > 0$ independent of ν. Define $w_\nu = \tilde{v}_\nu + \tilde{w}_\nu$ where $\tilde{w}_\nu \in H^{k+1}(0,1)$ satisfies $\partial_x^i \tilde{w}_\nu(y) = 0$ for $0 \le i \le k-1$, $\partial_x^k \tilde{w}_\nu(y) = \sigma_{\nu,k}(y)$, and $\|\tilde{w}_\nu\|_{H^k(0,1)} \to 0$. Then w_ν satisfies the desired properties $w_\nu \to 0$ in $H^k(0,1)$ and $\partial_x^i w_\nu(y) = \sigma_{\nu,i}(y)$ for $0 \le i \le k$ and $y = 0,1$.

Thus the last step is to construct the function $\tilde{w}_\nu$. Set $c_\nu = \sigma_{\nu,k}(0)$. Because it is enough to consider each component of c_ν separately, we may assume without loss of generality that c_ν is scalar. Let us consider the two cases $|c_\nu| \le 1$ and $|c_\nu| > 1$ separately. Suppose that $|c_\nu| \le 1$. Let $\phi \in \mathscr{D}(\mathbb{R})$ be such that $\phi(x) = 1$ for $|x| \le \epsilon$ for some $\epsilon > 0$ small enough and supp $\phi \subset [-1, 1]$. Define

$$\psi_\nu(x) = \frac{x^k}{k!}\phi(\nu x)c_\nu.$$

Then by Leibniz' formula we have for $1 \le j \le k$

$$\partial_x^j \psi_\nu(x) = \sum_{i=0}^{j} \binom{j}{i} \frac{x^{k-i}}{(k-i)!} \nu^{j-i} \partial_x^{j-i}\phi(\nu x)c_\nu. \tag{9.7}$$

It can be seen from (9.7) that $\partial_x^j \psi_\nu(0) = 0$ for $1 \le j \le k-1$ and $\partial_x^k \psi_\nu(0) = c_\nu$. Moreover, using the change of variable $y = \nu x$ we obtain

$$\begin{aligned}
\|\partial_x^j \psi_\nu\|_{L^2(\mathbb{R})}^2 &\le C(k) \sum_{i=0}^{j} \int_{\mathbb{R}} |x|^{2(k-i)} \nu^{2(j-i)} |\partial_x^{j-i}\phi(\nu x)|^2 |c_\nu|^2 \, \mathrm{d}x \\
&= C(k) \sum_{i=0}^{j} \int_{\mathbb{R}} |y|^{2(k-i)} \nu^{2(j-k)} |\partial_x^{j-i}\phi(y)|^2 \, \frac{\mathrm{d}y}{\nu} \\
&\le \frac{C(k)}{\nu} \sum_{i=0}^{j} \int_{\mathbb{R}} |y|^{2(k-i)} |\partial_x^{j-i}\phi(y)|^2 \, \mathrm{d}y \le \frac{C(k,\phi)}{\nu}
\end{aligned}$$

for $0 \le j \le k$.

If $|c_\nu| > 1$ then we take

$$\psi_\nu(x) = \frac{x^k}{k!}\phi(|c_\nu|^2 \nu x)c_\nu.$$

For $1 \le j \le k$, applying Leibniz' rule yields

$$\partial_x^j \psi_\nu(x) = \sum_{i=0}^{j} \binom{j}{i} \frac{x^{k-i}}{(k-i)!} (|c_\nu|^2\nu)^{j-i} \partial_x^{j-i}\phi(|c_\nu|^2 \nu x)c_\nu. \tag{9.8}$$

From (9.8) we obtain $\partial_x^j \psi_\nu(0) = 0$ for $1 \le j \le k-1$, $\partial_x^k \psi^\nu(0) = c_\nu$ and

$$\begin{aligned}
\|\partial_x^j \psi_\nu\|_{L^2(\mathbb{R})}^2 &\le C(k) \sum_{i=0}^{j} \int_{\mathbb{R}} |x|^{2(k-i)} (|c_\nu|^2\nu)^{2(j-i)} |\partial_x^{j-i}\phi(|c_\nu|^2 \nu x)|^2 |c_\nu|^2 \, \mathrm{d}x \\
&= C(k) \sum_{i=0}^{j} \int_{\mathbb{R}} |y|^{2(k-i)} (|c_\nu|^2\nu)^{2(j-k)} |\partial_x^{j-i}\phi(y)|^2 \, \frac{\mathrm{d}y}{\nu} \\
&\le \frac{C(k)}{\nu} \sum_{i=0}^{j} \int_{\mathbb{R}} |y|^{2(k-i)} |\partial_x^{j-i}\phi(y)|^2 \, \mathrm{d}y \le \frac{C(k,\phi)}{\nu}
\end{aligned}$$

since $j - k \le 0$ and $|c_\nu|^2\nu > 1$. Therefore in any case we have $\|\psi_\nu\|_{H^k(\mathbb{R})} \le C(k,\phi)\nu^{-1/2}$.

For $\sigma_{\nu,k}(1)$ we can also do the same construction by replacing ϕ by a smooth function that is equal to 1 in an ϵ-neigborhood of $x = 1$. By taking the sum of the functions ψ_ν constructed for $x = 0$ and $x = 1$ and choosing ϵ small enough so that their supports do not intersect we obtain an appropriate $\tilde{w}_\nu$ satisfying all the required properties.

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