Department of Mathematics
and Computer Science

College of Science

University of the Philippines Baguio

Existence of Local-in-Time Classical Solutions of a Model of Flow in a Bounded Elastic Tube

EXISTENCE OF LOCAL-IN-TIME CLASSICAL SOLUTIONS OF A MODEL OF FLOW IN A BOUNDED ELASTIC TUBE

GILBERT PERALTA AND GEORG PROPST*

Abstract.
This paper studies the local-in-time existence of classical solutions to a hyperbolic system with differential boundary conditions modelling a flow in an elastic tube. The well-known Lax transformations used for obtaining a priori estimates for conservation laws are difficult to apply here due to the inhomogeneity of the partial differential equations. Rather, our method relies on a suitable splitting of the original system into the PDE part and the ODE part, the characteristics for the PDE part, appropriate modulus of continuity estimates and a compactness argument.

2010 Mathematics Subject Classification.
35L45, 35Q35

Keywords.
Flow in elastic tube, hyperbolic PDEs, differential boundary conditions, fixed-point technique, method of characteristics.

Citation.
G. Peralta and G. Propst, *Existence of local-in-time classical solutions of a model of flow in a bounded elastic tube*, Mathematical Methods in the Applied Sciences 39 (18), pp. 5315-5329, 2016.
DOI: https://doi.org/10.1002/mma.3917

The first author is supported by the grant *Technologiestipendien Südostasien* in the frame of ASEA-Uninet granted by the Austrian Agency for International Cooperation in Education and Research (OeAD-GmbH) and financed by the Austrian Federal Ministry for Science and Research (BMWF).

*Institut für Mathematik und Wissenschaftliches Rechnen, NAWI Graz, Karl-Franzens-Universität Graz, Heinrichstrasse 36, A-8010 Graz, Austria

Department of Mathematics and Computer Science, University of the Philippines Baguio, Governor Pack Road, Baguio, 2600 Philippines. Email: grperalta@up.edu.ph.

Table of Contents

1. The Model

A consequence of the results of this study is the existence and uniqueness of classical solutions to the following hyperbolic PDE-ODE system in [11], see also [1] and [9],

$$\begin{cases} A_t(t,x) + u(t,x)A_x(t,x) + A(t,x)u_x(t,x) = 0, \\ u_t(t,x) + u(t,x)u_x(t,x) + \dfrac{sE}{2\rho r_0\sqrt{A_0 A(t,x)}}A_x(t,x) + \dfrac{8\pi\mu_0}{\rho A_0}u(t,x) = 0, \\ A_T h_0'(t) = -A(t,0)u(t,0), \\ A_T h'(t) = A(t,\ell)u(t,\ell), \\ A(t,0) = A_0\left(1 + \dfrac{r_0}{sE}(\rho g h_0(t) + p_f(t))\right)^2, \\ A(t,\ell) = A_0\left(1 + \dfrac{r_0}{sE}(\rho g h(t))\right)^2, \\ A(0,x) = A^0(x), \qquad u(0,x) = u^0(x), \qquad h_0(0) = h_0^0, \qquad h(0) = h^0, \end{cases} \tag{1.1}$$

for $t \geq 0$ and $0 \leq x \leq \ell$; the unknown functions are A, u, h_0 and h. By classical solutions we mean that they are at least continuously differentiable functions. This system describes the flow of an incompressible fluid in an elastic tube whose ends are attached to cylindrical tanks with horizontal cross section A_T. The state variables A and u represent the cross-sectional area of the tube, which is assumed to be circular, and the velocity of the fluid inside the tube, while h_0 and h are the level heights of the fluid in the left and right tanks, respectively. The constants ρ and μ_0 are the density and viscosity of the fluid, s, E, r_0 and A_0 are the thickness, Young's modulus, inner rest radius and rest cross-sectional area of the tube material, and g is the gravitational constant. The function p_f represents an external pressure that is applied above the left tank.

Systems of type (1.1) occur in models of cardiovascular blood flow [6, 12, 13, 14] and investigations of valveless pumping [9, 1]. They are derived from conservation of mass, balance of momentum and an equation of state that relates the cross section of the tube and the pressure in the fluid. (1.1) is a two component system of balance laws that is coupled to ODEs via boundary conditions. The well-posedness of (1.1) in terms of weak solutions in Sobolev spaces has been studied in [10, 11].

Using an appropriate change of the unknown variables, (1.1) can be transformed into diagonal form. To do so, let us first note that the eigenvalues of the system (1.1) are given by $\lambda = u - \kappa A^{1/4}$ and $\mu = u + \kappa A^{1/4}$, where $\kappa = (sE/2\rho r_0\sqrt{A_0})^{1/2}$. With the characteristic variables $w(t,x) = -u(t,x) + 4\kappa A^{1/4}(t,x)$ and $z(t,x) = u(t,x) + 4\kappa A^{1/4}(t,x)$, the PDEs in (1.1) can be diagonalized as

$$\begin{aligned} w_t + \lambda(w,z)w_x &= \frac{c_0}{2}(z-w) \\ z_t + \mu(w,z)z_x &= \frac{c_0}{2}(w-z), \end{aligned}$$

where $c_0 = 8\pi\mu_0/A_0$.

The state variables can be written in terms of the characteristic variables as $u = (z-w)/2$ and $A = ((w+z)/8\kappa)^4$. To transform the boundary conditions in terms of the characteristic variables (in the form of mixed boundary data), we note that

$$\left(\frac{w(t,0)+z(t,0)}{8\kappa}\right)^4 = A_0\left[1 + \frac{r_0}{sE}(\rho g h_0(t) + p_f(t))\right]^2.$$

Assuming that $w(t,0) + z(t,0)$ remains positive for all $t \in [0,T]$, for some $T > 0$ (this will follow from (H4) with a proper choice of $\mathcal{O}$ and the continuity of solutions), we can solve for $z(t,0)$ and obtain

$$z(t,0) = 8\kappa A_0^{1/4}\left[1 + \frac{r_0}{sE}(\rho g h_0(t) + p_f(t))\right]^{1/2} - w(t,0).$$

We explain the reason why we solve $z(t,0)$ in terms of $w(t,0)$. As we can see from the diagonal form of (1.1), the characteristic curves corresponding to w are left-propagating, and hence the boundary values of w at $x = 0$ can be determined from the forcing function and the initial data w^0 up to a certain positive time. In this way, the values of z on the boundary $x = 0$ can be determined from the above equation for $z(t,0)$. A similar procedure yields the following correct form for the boundary condition at the right tank

$$w(t,\ell) = 8\kappa A_0^{1/4}\left[1 + \frac{r_0}{sE}\rho g h(t)\right]^{1/2} - z(t,\ell).$$

The state components h_0 and h in terms of the characteristic variables are as follows

$$\begin{aligned} 2^{13}\kappa A_T h_0'(t) &= -(w(t,0)+z(t,0))^4(z(t,0)-w(t,0)) \\ 2^{13}\kappa A_T h'(t) &= (w(t,\ell)+z(t,\ell))^4(z(t,\ell)-w(t,\ell)). \end{aligned}$$

The system (1.1) is a special case of the abstract system (compare with [6])

$$\begin{cases} w_t + \lambda(w,z)w_x = f(t,x,w,z), & 0<t<T,\ 0<x<\ell, \\ z_t + \mu(w,z)z_x = g(t,x,w,z), & 0<t<T,\ 0<x<\ell, \\ z(t,0) = G_0(t,h_0(t),w(t,0)), & 0<t<T, \\ w(t,\ell) = G(t,h(t),z(t,\ell)), & 0<t<T, \\ h_0'(t) = H_0(w(t,0),z(t,0)), & 0<t<T, \\ h'(t) = H(w(t,\ell),z(t,\ell)), & 0<t<T, \\ w(0,x) = w^0(x), \qquad z(0,x) = z^0(x), & 0<x<\ell, \\ h_0(0) = h_0^0, \qquad h(0) = h^0. \end{cases} \tag{1.2}$$

The initial conditions in (1.1) and (1.2) are related by $w^0 = -u^0 + 4\kappa(A^0)^{1/4}$ and $z^0 = u^0 + 4\kappa(A^0)^{1/4}$. We would like to point out that the methods presented here can be extended to differential boundary conditions

$$h_0'(t) = H_0(t,h_0(t),w(t,0),z(t,0)), \qquad h'(t) = H(t,h(t),w(t,0),z(t,0)),$$

where $H_0, H \in C^1(\mathbb{R}^4)$.

In what follows, we will analyze the coupled system (1.2), where $T>0$ is a generic time horizon. To guarantee the existence and uniqueness of a classical solution of this coupled system, the following hypotheses are sufficient.

- (H1) There exists an open set $\mathcal{O} \subset \mathbb{R}^2$ such that $\lambda, \mu \in C^1(\mathcal{O})$ and $\lambda(w,z) < \mu(w,z)$ for all $(w,z) \in \mathcal{O}$.
- (H2) $H, H_0 \in C^1(\mathbb{R}^2)$ and $f, g \in C^1([0,T]\times[0,\ell]\times\mathcal{O})$
- (H3) There exist constants $M_2 > 0$ and $T > 0$ such that $G_0 \in C^1([0,T]\times[h_0^0 - M_2, h_0^0 + M_2]\times\mathbb{R})$ and $G \in C^1([0,T]\times[h^0 - M_2, h^0 + M_2]\times\mathbb{R})$.
- (H4) The initial data satisfy $w^0, z^0 \in C^1[0,\ell]$, $(w^0(x), z^0(x)) \in \mathcal{O}$ for all $x \in [0,\ell]$, and $h_0^0, h^0 > 0$.
- (H5) It holds that $\lambda(w^0(x), z^0(x)) < 0 < \mu(w^0(x), z^0(x))$ for $x = 0, \ell$.
- (H6) The initial data at the left and right endpoints satisfy the following compatibility conditions

$$\begin{aligned} z^0(0) &= G_0(0,h_0^0,w^0(0)) \\ w^0(\ell) &= G(0,h^0,z^0(\ell)) \\ -\mu(w^0(0),z^0(0))(z^0)'(0) &= \nabla G_0(0,h_0^0,w^0(0)) \cdot \big(1, H_0(w^0(0),z^0(0)), \\ &\quad - \lambda(w^0(0),z^0(0))(w^0)'(0) + f(0,0,w^0(0),z^0(0))\big) \\ &\quad - g(0,0,w^0(0),z^0(0)) \\ -\lambda(w^0(\ell),z^0(\ell))(w^0)'(\ell) &= \nabla G(0,h^0,z^0(\ell)) \cdot \big(1, H(w^0(\ell),z^0(\ell)), \\ &\quad - \mu(w^0(\ell),z^0(\ell))(z^0)'(\ell) + g(0,\ell,w^0(\ell),z^0(\ell))\big) \\ &\quad - f(0,\ell,w^0(\ell),z^0(\ell)). \end{aligned}$$

Let us explain what these assumptions mean. The first hypothesis (H1) simply states that the quasilinear PDEs must be strictly hyperbolic. The smoothness requirement for the boundary data are given by (H2) and (H3), while (H4) imposes the smoothness requirement for the initial data and a range condition. We can view (H5) and (H6) as additional constraints on the initial data w^0 and z^0. These compatibility conditions imply the continuity of the state components and their derivatives. The assumption (H5) guarantees that the left and right boundaries are non-characteristic.

Our assumption (H1) is weaker than (H1)(i) in Fernandez et al. [6]. Notice that (H1)(ii) in [6] is used in [7] for global existence and uniqueness. However, we are only interested in local existence. (H2) and (H3) is stronger than (H6) and (H3), respectively, in [6]. (H4) is similar to (H2) in [6] but we have no assumptions on the derivatives of the initial data. Our hypothesis (H5) is similar to (H4) in [6], where the half line $x \in \mathbb{R}^+$ is considered with boundary and compatibility conditions at $x = 0$. Indeed, if the time horizon is small enough so that the two characteristic curves (x_0, x_ℓ in Figure 1) emanating from the two boundaries do not intersect, the bounded domain can be replaced by two half lines as in [6, Section 2.3]. However, some of the methods in [6] can not be used for our system that includes friction which leads to inhomogeneous right hand sides of the PDEs.

Theorem 1.1. *If the hypotheses* (H1)–(H6) *hold, then there exists a positive time* $\breve{T} \in (0, T]$ *such that the coupled system* (1.2) *has a unique classical solution* $(w, z, h_0, h) \in C^1([0, \breve{T}] \times [0, \ell])^2 \times C^2[0, \breve{T}]^2$.

Now we will apply the abstract result of Theorem 1.1 to obtain the local existence and uniqueness of a classical solution to system (1.1). It suffices to verify that all of (H1)–(H6) are satisfied. (H1) The open set can be chosen to be $\mathcal{O} = \{(w, z) \in \mathbb{R}^2 : w + z > 0\}$ in $\mathbb{R}^2$. (H2) Note that H and H_0 are polynomial functions. (H3) Let $M_2 = \min(h_0^0, h^0)$ and so $[h_0^0 - M_2, h_0^0 + M_2], [h^0 - M_2, h^0 + M_2] \subset [0, h_0^0 + h^0]$. The condition follows once we assume that $p_f \in C^1[0, T]$ and $p_f(t) \geq -\frac{sE}{r_0} - \frac{1}{2}\rho g h_0^0$ for all $t \in [0, T]$. For (H4), the conditions are $u_0, A_0 \in C^1[0, \ell]$, $A_0(x) > 0$ for all $x \in [0, \ell]$ and $h_0^0, h^0 > 0$. For the boundary conditions, (H5) translates into $|u^0(x)| \leq \kappa(A^0(x))^{1/4}$ for $x = 0, \ell$. The condition (H6) should be translated in terms of u^0 and A^0.

Corollary 1.2. *Assume that* $h_0^0, h^0 > 0$ *and* $u^0, A^0 \in C^1[0, \ell]$ *satisfy* $A^0(x) > 0$, $|u^0(x)| \leq \kappa(A^0(x))^{1/4}$ *for* $x = 0, \ell$*, and the compatibilty conditions. If the forcing* $p_f \in C^1[0, T]$ *satisfies* $p_f(t) \geq -\frac{sE}{r_0} - \frac{1}{2}\rho g h_0^0$ *for all* $t \geq 0$*, then the system* (1.1) *has a unique classical solution* $(u, A, h_0, h) \in C^1([0, \breve{T}] \times [0, \ell])^2 \times C^2[0, \breve{T}]^2$ *for some* $\breve{T} > 0$.

The method presented in this paper is a combination of the splitting method in [6] and iteration methods as in [4, 5, 8]. We divide the system into two parts, namely, the PDE part and the ODE part. This splitting method has been also used in [3] to prove the existence and uniqueness of solutions to the coupling of quasilinear hyperbolic and parabolic PDEs that describes the flow of a fluid in a porous medium that is connected by a pipe. The process is to define two mappings associated with these two problems in such a way that a fixed point of the composition corresponds to a solution of the system, and hence continuity properties of these mappings are required. Hence, existence and uniqueness will be established using a fixed-point argument, specifically the contraction principle. Similar problems have been considered in the series of papers [12, 13, 14]. The authors analyzed multiscale blood flow models, a coupled system of ODEs and hyperbolic PDEs, and prove the well-posedness of such systems. We note that our method is direct and does not use the approximation argument as in [7] and [8].

Now let us set the basic notations and assumptions. For each nonnegative integer n, positive integer m and positive T, we denote by $C^n([0,T],\mathbb{R}^m)$ the space of functions on $[0,T]$ whose derivatives up to the order n are continuous and it is equipped with the usual norm. For $r>0$, denote the closed ball in $C^n([0,T],\mathbb{R}^m)$ centered at the origin with radius r by $B^{n,m}[T,r]$.

First, we split the coupled system into two parts, an ODE part and a PDE part. Let M_1 be a positive constant which will be specified later. For fixed h_0^0 and h^0, define $\mathfrak{S}_1 : B^{0,4}[T,M_1] \to C^1([0,T],\mathbb{R}^2)$ by $\mathfrak{S}_1(\varphi_0,\theta_0,\varphi,\theta) = (h_0,h)$ where h_0 and h satisfy the ODEs

$$\begin{cases} h_0'(t) = H_0(\varphi_0(t),\theta_0(t)), & h_0(0) = h_0^0, \\ h'(t) = H(\varphi(t),\theta(t)), & h(0) = h^0. \end{cases} \tag{1.3}$$

This is the ODE part. The regularity of H_0, H implies that $\mathfrak{S}_1$ is well-defined.

The PDE part is posed in the following way. Given $M_3 > M_2 + |(h_0^0,h^0)|$ and for fixed w^0 and z^0, define $\mathfrak{S}_2 : B^{1,2}[T,M_3] \to C([0,T],\mathbb{R}^4)$ by

$$\mathfrak{S}_2(h_0,h) = (w(\cdot,0), z(\cdot,0), w(\cdot,\ell), z(\cdot,\ell))$$

where (w,z) is the classical solution on the rectangle $[0,T]\times[0,\ell]$ to the PDE

$$\begin{cases} w_t + \lambda(w,z)w_x = f(t,x,w,z) \\ z_t + \mu(w,z)z_x = g(t,x,w,z) \\ z(t,0) = G_0(t,h_0(t),w(t,0)) \\ w(t,\ell) = G(t,h(t),z(t,\ell)) \\ w(0,x) = w^0(x), \qquad z(0,x) = z^0(x). \end{cases} \tag{1.4}$$

The well-definedness of $\mathfrak{S}_2$ is not clear for the moment. Although the (local) existence and uniqueness of a classical solution of the initial-boundary value problem (1.4) for a given (h_0,h) has been already established [7], it is not obvious that the time of existence is independent on the choice of the boundary data (h_0,h). This problem has been solved in [6] by providing a positive existence time that does not depend on the particular choice of the boundary data but only on the bounds of their derivatives. To obtain such results, the authors used the well-known Lax transformations to obtain bounds for the derivatives of the solution of the quasilinear system. This method works for conservation laws but not on balance laws, which is the case in the present paper.

To obtain such desired time of existence, we shall proceed in the classical way. First we consider a linear system of PDEs associated with the quasilinear system (1.4) and provide estimates on the solutions of such linear systems. These estimates together with an iteration scheme will then prove the existence and uniqueness of continuously differentiable functions w and z satisfying (1.4) on a rectangular domain $[0,T]\times[0,\ell]$ with T being independent of (h_0,h), at least in $B^{1,2}[T,M_3]$.

If we can show that $\operatorname{ran}\mathfrak{S}_1 \subset \operatorname{dom}\mathfrak{S}_2$, then it follows that the map $\mathfrak{S} : B^{0,4}[T,M_1] \to C([0,T],\mathbb{R}^4)$, for appropriate T and M_1, given by the composition $\mathfrak{S} = \mathfrak{S}_2 \circ \mathfrak{S}_1$ is well-defined. Furthermore, every fixed point of $\mathfrak{S}$ corresponds to a solution to the coupled system (1.2). Indeed, assume that $(\varphi_0, \theta_0, \varphi, \theta)$ is a fixed point of $\mathfrak{S}$. Using $(h_0, h) = \mathfrak{S}_1(\varphi_0, \theta_0, \varphi, \theta)$ in (1.4), gives us a classical solution (w, z) of (1.4). Now $(\varphi_0, \theta_0, \varphi, \theta)$ being a fixed point gives us the property $(\varphi_0, \theta_0, \varphi, \theta) = (w(\cdot, 0), z(\cdot, 0), w(\cdot, \ell), z(\cdot, \ell))$ and plugging these in (1.3), we can see that (w, z) is a classical solution of the coupled system (1.2).

2. The ODE Part

The aim of the present section is to prove the claim that the range of the mapping $\mathfrak{S}_1$ is contained in the domain of the mapping $\mathfrak{S}_2$. In the following, $M_1 > 0$ is given.

Theorem 2.1. *There exists a solution $(h_0, h) \in C^1[0, T]^2$ of (1.3) such that for some $\hat{T} = \hat{T}(M_1, M_2) \in (0, T]$ we have $(h_0, h) \in B^{1,2}[\hat{T}, M_3]$, where M_3 depends only on T, M_1, M_2, (h_0^0, h^0) and not on the particular choice of the data $(\varphi_0, \theta_0, \varphi, \theta) \in B^{0,4}[T, M_1]$. In other words,* ran $\mathfrak{S}_1 \subset$ dom $\mathfrak{S}_2$.

Proof. The solution of (1.3) is

$$h_0(t) = h_0^0 + \int_0^t H_0(\varphi_0(t), \theta_0(t))\, dt, \qquad h(t) = h^0 + \int_0^t H(\varphi(t), \theta(t))\, dt.$$

Since $H_0, H \in C^1([-M_1, M_1]^2)$, there exists a constant $C = C(M_1) > 0$ such that we have $|H_0(a^1, b^1)| + |H(a^2, b^2)| \leq C$ for every $a^1, a^2, b^1, b^2 \in [-M_1, M_1]$. Thus $|H_0(\varphi_0(t), \theta_0(t))| + |H(\varphi(t), \theta(t))| \leq C$ for every $(\varphi_0, \theta_0, \varphi, \theta) \in B^{0,4}[T, M_1]$ and $t \in [0, T]$. Choose $\hat{T} > 0$ such that $\hat{T}C \leq M_2$. In this case, $\|(h_0, h) - (h_0^0, h^0)\|_{C[0,\hat{T}]^2} \leq \hat{T}C \leq M_2$. Also, $\|(h_0', h')\|_{C[0,\hat{T}]^2} \leq \|H_0(\varphi_0, \theta_0)\|_{C[0,\hat{T}]} + \|H(\varphi, \theta)\|_{C[0,\hat{T}]} \leq C$. Taking $M_3 = M_2 + |(h_0^0, h^0)| + C$ shows that $(h_0, h) \in B^{1,2}[\hat{T}, M_3]$. □

The following theorem states the continuity of the mapping $\mathfrak{S}_1$.

Theorem 2.2. *Let (h_0^1, h^1) and (h_0^2, h^2) be solutions of (1.3) with respective data $\mathbf{v}^1 = (\varphi_0^1, \theta_0^1, \varphi^1, \theta^1)$ and $\mathbf{v}^2 = (\varphi_0^2, \theta_0^2, \varphi^2, \theta^2)$. Then for any $T \in (0, \hat{T}]$ it holds that*

$$\|(h_0^1, h^1) - (h_0^2, h^2)\|_{C[0,T]^2} \leq LT\|\mathbf{v}^1 - \mathbf{v}^2\|_{C[0,T]^4},$$

where $L = \max(\|H_0\|_{C^1([-M_1,M_1]^2)}, \|H\|_{C^1([-M_1,M_1]^2)})$.

Proof. This follows immediately from the fact that

$$\|h_0^1 - h_0^2\|_{C[0,T]} \leq \|H_0\|_{C^1([-M_1,M_1]^2)} T \|(\varphi_0^1, \theta_0^1) - (\varphi_0^2, \theta_0^2)\|_{C[0,T]^2}$$

and a similar estimate for $\|h^1 - h^2\|_{C[0,T]}$. □

3. The PDE Part 1 : Linear System

In this section, we prove the existence and uniqueness result for the linear system corresponding to (1.4). More precisely, we consider the linear system with nonlinear boundary data

$$\begin{cases} w_t + \lambda(t,x)w_x = f(t,x) \\ z_t + \mu(t,x)z_x = g(t,x) \\ z(t,0) = G_0(t,h_0(t),w(t,0)) \\ w(t,\ell) = G(t,h(t),z(t,\ell)) \\ w(0,x) = w^0(x), \qquad z(0,x) = z^0(x) \end{cases} \tag{3.1}$$

where (h_0, h) is a fixed element of $B^{1,2}[T, M_3]$. Let $\Omega_T = [0,T] \times [0,\ell]$. In this section, we assume that

(L1) $\lambda, \mu, f, g \in C^1(\Omega_T)$
(L2) $w^0, z^0 \in C^1[0,\ell]$
(L3) $G_0 \in C^1([0,T] \times [h_0^0 - M_2, h_0^0 + M_2] \times \mathbb{R})$ and $G \in C^1([0,T] \times [h^0 - M_2, h^0 + M_2] \times \mathbb{R})$
(L4) $\lambda(t,x) < \mu(t,x)$ for all $(t,x) \in \Omega_T$
(L5) $\lambda(t,x) < 0 < \mu(t,x)$ for all $(t,x) \in [0,T] \times \{0,\ell\}$
(L6) The boundary and initial data satisfy C^1-compatibility conditions

$$\begin{aligned} z^0(0) &= G_0(0,h_0^0,w^0(0)) \\ w^0(\ell) &= G(0,h^0,z^0(\ell)) \\ -\mu(0,0)(z^0)'(0) &= \nabla G_0(0,h_0^0,w^0(0)) \cdot (1, H_0(w^0(0),z^0(0)), -\lambda(0,0)(w^0)'(0) \\ &\quad + f(0,0)) - g(0,0) \\ -\lambda(0,\ell)(w^0)'(\ell) &= \nabla G(0,h^0,z^0(\ell)) \cdot (1, H(w^0(0),z^0(0)), -\mu(0,\ell)(z^0)'(\ell) \\ &\quad + g(0,\ell)) - f(0,\ell). \end{aligned}$$

Here, the functions stated in (L1)-(L3) are given. Also, (L1) and (L5) imply that there exists a constant $d > 0$ such that $\lambda(t,x) \le -d < 0 < d \le \mu(t,x)$ for every $(t,x) \in [0,T] \times \{0,\ell\}$. Without loss of generality we may take $d \in (0,1)$.

Remark 3.1. *In (L3) we assumed that the second argument of G_0 and G lies in the intervals centered at the initial level heights h_0^0 and h^0, with radius M_2 as in (H3). However, in (L3) a larger radius is admissible. Moreover, a more general case where the right hand sides of the first two equations of* (3.1) *include multiples of z and w could be treated. However, for our purpose the above setting is sufficient. Because we will utilize the linear theory to prove the local existence of solution for the quasilinear case, it is also sufficient to prove local existence in the linear case.*

3.1. CHARACTERISTIC CURVES. For each $(t,x) \in \Omega_T$ we have the λ-characteristic curve $x_\lambda = x_\lambda(\tau; t, x)$ at (t,x), where

$$x'_\lambda(\tau;t,x) = \lambda(\tau, x_\lambda(\tau;t,x)), \qquad x_\lambda(t;t,x) = x. \tag{3.2}$$

Since $\lambda \in C^1(\Omega_T)$, it follows from the Picard-Lindelöf Theorem that such curve exists and it is unique. Furthermore, two distinct λ-characteristic curves will never intersect. Similarly, we have the μ-characteristic curve passing through (t,x), $x_\mu = x_\mu(\tau;t,x)$, where

$$x'_\mu(\tau;t,x) = \mu(\tau, x_\mu(\tau;t,x)), \qquad x_\mu(t;t,x) = x.$$

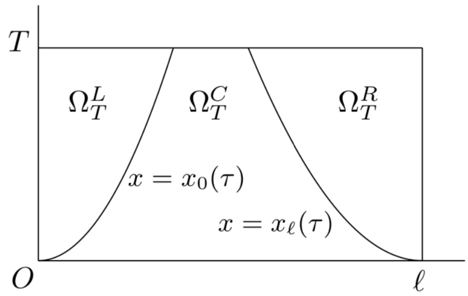

Figure 1. The regions determined by the left most and right most characteristic curves.

Let $x_0 = x_\mu(\tau; 0, 0)$ and $x_\ell = x_\lambda(\tau; 0, \ell)$ be the μ-characteristic curve and λ-characteristic curve passing through $(0, 0)$ and $(0, \ell)$, respectively. Temporarily, we denote by $T' > 0$ the time of intersection of the characteristic curves $x_0(\tau)$ and $x_\ell(\tau)$ and set $T = \min\{\hat{T}, T'\}$ and define

$$\begin{aligned} \Omega_T^L &= \{(t, x) \in \Omega_T : 0 \leq x \leq x_0(t)\}, \\ \Omega_T^C &= \{(t, x) \in \Omega_T : x_0(t) \leq x \leq x_\ell(t)\}, \\ \Omega_T^R &= \{(t, x) \in \Omega_T : x_\ell(t) \leq x \leq \ell\}. \end{aligned}$$

There are two possible scenarios. The characteristic curve x_λ intersects the x-axis at a unique point $(0, \tilde{x})$ and so $\tilde{x} = x_\lambda(0; t, x)$. This is the case if and only if $(t, x) \in \Omega_T^L \cup \Omega_T^C$. Define $\alpha : \Omega_T^L \cup \Omega_T^C \to [0, \ell]$ by $\alpha(t, x) = x_\lambda(0; t, x)$. On the other hand, the characteristic curve x_λ will intersect the line $x = \ell$ at a unique point $(\tilde{t}, \ell)$ and so $\ell = x_\lambda(\tilde{t}; t, x)$. This is true if and only if $(t, x) \in \Omega_T^R$ and we define $\sigma : \Omega_T^R \to [0, T]$ such that $x_\lambda(\sigma(t, x); t, x) = \ell$.

With the same procedure as above, we notice that the curve x_μ either intersects the x-axis at the unique point $(0, \beta(t, x))$, where $\beta : \Omega_T^C \cup \Omega_T^R \to [0, \ell]$ is given by $\beta(t, x) = x_\mu(0; t, x)$ or it will intersect the line $x = 0$ at the unique point $(\zeta(t, x), 0)$ where $\zeta : \Omega_T^L \to [0, T]$ satisfies $x_\mu(\zeta(t, x); t, x) = 0$.

Define the following sets

$$\begin{aligned} \Theta_{T,\lambda}^1 &= [0, T] \times (\Omega_T^L \cup \Omega_T^C), \\ \Theta_{T,\lambda}^2 &= \{(\tau, t, x) : (t, x) \in \Omega_T^R \text{ and } \sigma(t, x) \leq \tau \leq T\}, \\ \Theta_{T,\mu}^1 &= [0, T] \times (\Omega_T^C \cup \Omega_T^R), \\ \Theta_{T,\mu}^2 &= \{(\tau, t, x) : (t, x) \in \Omega_T^L \text{ and } \zeta(t, x) \leq \tau \leq T\}. \end{aligned}$$

In the following, we prove some properties of the characteristic curves and estimates of their derivatives.

Theorem 3.2. *It holds that $x_\lambda \in C^1(\Theta_{T,\lambda}^i)$ and $x_\mu \in C^1(\Theta_{T,\mu}^i)$ for $i = 1, 2$. Furthermore, we have*

$$\begin{aligned} \|x_\lambda\|_{C^1(\Theta_{T,\lambda}^i)} &\leq \|\lambda\|_{C(\Omega_T)} + (1 + \|\lambda\|_{C(\Omega_T)}) \exp(T\|\lambda\|_{C^1(\Omega_T)}) \\ \|x_\mu\|_{C^1(\Theta_{T,\mu}^i)} &\leq \|\mu\|_{C(\Omega_T)} + (1 + \|\mu\|_{C(\Omega_T)}) \exp(T\|\mu\|_{C^1(\Omega_T)}) \end{aligned}$$

for $i = 1, 2$. In particular, $\alpha \in C^1(\Omega_T^L \cup \Omega_T^C)$ and $\beta \in C^1(\Omega_T^C \cup \Omega_T^R)$.

Proof. Suppose that $(\tau; t, x) \in \Theta^i_{T,\lambda}$. Let

$$\eta_h(\tau) = h^{-1}[x_\lambda(\tau; t, x+h) - x_\lambda(\tau; t, x)]$$

for sufficiently small h such that $(t, x+h) \in \Omega^L_T \cup \Omega^C_T$ if $i = 1$ or $(t, x+h) \in \Omega^R_T$ if $i = 2$. Taking the derivative

$$\eta'_h(\tau) = h^{-1}[\lambda(\tau, x_\lambda(\tau; t, x+h)) - \lambda(\tau, x_\lambda(\tau; t, x))].$$

Since $\lambda \in C^1(\Omega_T)$, the mean value theorem implies the existence of a number $\xi_h(\tau)$ between $x_\lambda(\tau; t, x)$ and $x_\lambda(\tau; t, x+h)$ such that

$$\eta'_h(\tau) = h^{-1}\lambda_x(\tau, \xi_h(\tau))[x_\lambda(\tau; t, x+h) - x_\lambda(\tau; t, x)].$$

Therefore we have the ODE

$$\begin{cases} \eta'_h(\tau) &= \lambda_x(\tau, \xi_h(\tau))\eta_h(\tau), \qquad 0 \le \tau \le t, \\ \eta_h(t) &= h^{-1}[x_\lambda(t; t, x+h) - x] =: \eta^0_h. \end{cases}$$

The solution of this ODE is given by

$$\eta_h(\tau) = \eta^0_h \exp\left(\int_t^\tau \lambda_x(\vartheta, \xi_h(\vartheta))\, d\vartheta\right).$$

As $h \to 0$ we have, using $x_\lambda(t; t, x) = x$, that $\eta^0_h \to 1$ and $\xi_h(\vartheta) \to x_\lambda(\vartheta; t, x)$. Hence, taking the limit $h \to 0$ we get

$$(x_\lambda)_x(\tau; t, x) = \exp\left(\int_t^\tau \lambda_x(\vartheta, x_\lambda(\vartheta; t, x))\, d\vartheta\right). \tag{3.3}$$

From the definition of the characteristic curve, we have

$$x_\lambda(\tau; t+h, x) = x + \int_{t+h}^\tau \lambda(\vartheta, x_\lambda(\vartheta; t+h, x))\, d\vartheta.$$

Using the Lipschitz property of x_λ and λ, for every $\epsilon > 0$, it follows that

$$\left| \frac{1}{h} \int_{t+h}^t \lambda(\vartheta, x_\lambda(\vartheta; t, x)) - \lambda(\vartheta, x_\lambda(\vartheta; t+h, x))\, d\vartheta \right| < \epsilon \tag{3.4}$$

for sufficiently small values of h. Furthermore, we have

$$\begin{aligned} \left| \frac{x_\lambda(t; t+h, x) - x}{h} + \lambda(t, x) \right| \ &\le \ \left| \lambda(t, x) - \frac{1}{h} \int_{t+h}^t \lambda(\vartheta, x_\lambda(\vartheta; t, x))\, d\vartheta \right| \\ &+ \left| \frac{1}{h} \int_{t+h}^t \lambda(\vartheta, x_\lambda(\vartheta; t, x)) - \lambda(\vartheta, x_\lambda(\vartheta; t+h, x))\, d\vartheta \right|. \end{aligned}$$

From the continuity of λ and x_λ, the first term of the right hand side of the above inequality can be made arbitrarily small as long as $|h|$ is also small. Combining this with (3.4), we have $(x_\lambda)_t(t; t, x) = -\lambda(t, x)$. A similar procedure as above proves

$$(x_\lambda)_t(\tau; t, x) = -\lambda(t, x) \exp\left(\int_t^\tau \lambda_x(\vartheta, x_\lambda(\vartheta; t, x))\, d\vartheta\right).$$

Hence $x_\lambda \in C^1(\Theta^i_{T,\lambda})$. Similarly, $x_\mu \in C^1(\Theta^i_{T,\mu})$. The estimates for the derivative follows immediately. □

In the above proof, one can see that the λ-characteristic curves satisfy

$$(x_\lambda)_t(\tau; t, x) + \lambda(t, x)(x_\lambda)_x(\tau; t, x) = 0. \tag{3.5}$$

An analogous identity holds for the μ-characteristic curves.

Theorem 3.3. *It holds that $\sigma \in C^1(\Omega_T^R)$ and $\zeta \in C^1(\Omega_T^L)$ and*

$$\begin{aligned} \|\sigma_x\|_{C(\Omega_T^R)} &\leq (1/d)\exp(T\|\lambda\|_{C^1(\Omega_T)}) \\ \|\zeta_x\|_{C(\Omega_T^L)} &\leq (1/d)\exp(T\|\mu\|_{C^1(\Omega_T)}). \end{aligned}$$

Proof. The regularity of σ and ζ follows from the implicit function theorem. Differentiating $x_\mu(\zeta(t,x); t, x) = 0$ with respect to x gives us

$$(x_\mu)_x(\zeta(t, x); t, x) + x_\mu'(\zeta(t, x); t, x)\zeta_x(t, x) = 0.$$

Since $x_\mu'(\zeta(t, x); t, x) = \mu(\zeta(t, x), 0)$, we have

$$\zeta_x(t, x) = -\frac{1}{\mu(\zeta(t, x), 0)}(x_\mu)_x(\zeta(t, x); t, x) \tag{3.6}$$

and the first estimate follows from (3.3). The other one can be shown similarly. □

Our method is to divide (3.1) into four problems, namely, the decoupled initial-value problems

$$w_t + \lambda w_x = f, \qquad w(0, x) = w^0(x), \qquad \text{on } \Omega_T^L \cup \Omega_T^C, \tag{3.7}$$

$$z_t + \mu z_x = g, \qquad z(0, x) = z^0(x), \qquad \text{on } \Omega_T^C \cup \Omega_T^R, \tag{3.8}$$

and the boundary-value problems

$$z_t + \mu z_x = g, \qquad z(t, 0) = G_0(t, h_0(t), w(t, 0)), \qquad \text{on } \Omega_T^L, \tag{3.9}$$

$$w_t + \lambda w_x = f, \qquad w(t, \ell) = G(t, h(t), z(t, \ell)), \qquad \text{on } \Omega_T^R. \tag{3.10}$$

The existence of w on the region $\Omega_T^L \cup \Omega_T^C$ will then be used to solve (3.9), while the data for z on the region $\Omega_T^C \cup \Omega_T^R$ will be used to prove the existence of w on Ω_T^R.

We will deal with constants that depend on some functions, and so we shall make the following notations. For every positive $R > 0$, let

$$\begin{aligned} Q_0[R] &= [0, T] \times [h_0^0 - M_2, h_0^0 + M_2] \times [-R, R] \\ Q[R] &= [0, T] \times [h^0 - M_2, h^0 + M_2] \times [-R, R], \end{aligned}$$

which are the sets to which G_0 and G are to be restricted. Suppose for the moment that the solution of (3.1) satisfies the bounds $\|w(\cdot, 0)\|_{C[0,T]} \leq M_1$ and $\|z(\cdot, \ell)\|_{C[0,T]} \leq M_1$. Let Λ_1 denote the set of C^1-norms of w^0 and z^0 on $[0, \ell]$, G_0 on $Q_0[M_1]$, G on $Q[M_1]$, the supremum norms of f, g, λ and μ on Ω_T, and the constants M_2 and M_3. Let Λ_2 be the set of the supremum norms of the derivatives of f, g, λ and μ on Ω_T. Set $\Lambda = \Lambda_1 \cup \Lambda_2$. In the following, C_1 will denote constants, which may have a different value at different instances, that depend on a subset of Λ_1, and analogously for C_2 with Λ_2.

3.2. Existence of Solutions for the IVPs (3.7) and (3.8). First, let us consider the IVP (3.7). If w is a C^1-solution of (3.7) and $(t,x) \in \Omega_T^L \cup \Omega_T^C$ then integrating the first equation in (3.1) along the λ-characteristic at (t,x), we have

$$w(t,x) = w^0(\alpha(t,x)) + \int_0^t f(\tau, x_\lambda(\tau;t,x))\,d\tau. \tag{3.11}$$

We show that (3.11) is indeed the C^1-solution of (3.7). Differentiating (3.11) with respect to x and t gives us, using the Leibniz rule,

$$\begin{aligned} w_t(t,x) &= (w^0)'(\alpha(t,x))\alpha_t(t,x) + f(t,x) \\ &\quad + \int_0^t f_x(\tau, x_\lambda(\tau;t,x))(x_\lambda)_t(\tau;t,x)\,d\tau, \end{aligned} \tag{3.12}$$

$$w_x(t,x) = (w^0)'(\alpha(t,x))\alpha_x(t,x) + \int_0^t f_x(\tau, x_\lambda(\tau;t,x))(x_\lambda)_x(\tau;t,x))\,d\tau. \tag{3.13}$$

Since $\alpha \in C^1(\Omega_T^L \cup \Omega_T^C)$, $f \in C^1(\Omega_T)$ and $x_\lambda \in C^1(\Theta^1_{T,\lambda})$ it follows from (3.12) and (3.13) that $w \in C^1(\Omega_T^L \cup \Omega_T^C)$. Furthermore, these equations together with (3.5) imply that w satisfies (3.7). Its uniqueness can be shown in a standard manner.

Theorem 3.4. *The initial-value problem* (3.7) *has a unique solution in* $C^1(\Omega_T^L \cup \Omega_T^C)$. *Moreover,* $\|w - w^0\|_{C(\Omega_T^L\cup\Omega_T^C)} \le C(\Lambda)T$ *and*

$$\|w_x\|_{C(\Omega_T^L\cup\Omega_T^C)} + \|w_t\|_{C(\Omega_T^L\cup\Omega_T^C)} \le (C_1 + TC(\Lambda))e^{TC(\Lambda)}.$$

Proof. From the definition of α, we have $|\alpha(t,x) - x| = |x_\lambda(0;t,x) - x_\lambda(t;t,x)| \le \|\lambda\|_{C(\Omega_T)}t$ and so $|w^0(\alpha(t,x)) - w^0(x)| \le \|w^0\|_{C^1[0,\ell]}\|\lambda\|_{C(\Omega_T)}t$, and the estimate $\|w - w^0\|_{C(\Omega_T^L\cup\Omega_T^C)} \le C(\Lambda)T$ follows from this inequality and (3.11).

The estimate for the derivative with respect to x follows from (3.13). Indeed, using the said equation and Theorem 3.2 we have

$$\begin{aligned} |w_x(t,x)| \le \|w^0\|_{C^1[0,\ell]}[\|\lambda\|_{C(\Omega_T)} + (1 + \|\lambda\|_{C(\Omega_T)}) \exp(T\|\lambda\|_{C^1(\Omega_T)})] \\ + (\|f\|_{C^1(\Omega_T)}[\|\lambda\|_{C(\Omega_T)} + (1 + \|\lambda\|_{C(\Omega_T)}) \exp(T\|\lambda\|_{C^1(\Omega_T)})])T \end{aligned}$$

whenever $(t,x) \in \Omega_T^L \cup \Omega_T^C$. It can be easily seen that the above estimate is of the form given in the theorem. We can also use (3.12) to prove the estimate for the derivative with respect to t. Alternatively, we can use the PDE and then apply the bound for the derivative with respect to x. □

In an analogous manner, we have the following result for the IVP (3.8).

Theorem 3.5. *The initial-value problem* (3.8) *has a unique solution in* $C^1(\Omega_T^C \cup \Omega_T^R)$. *Moreover,* $\|z - z^0\|_{C(\Omega_T^C\cup\Omega_T^R)} \le C(\Lambda)T$ *and*

$$\|z_x\|_{C(\Omega_T^C\cup\Omega_T^R)} + \|z_t\|_{C(\Omega_T^C\cup\Omega_T^R)} \le (C_1 + TC(\Lambda))e^{TC(\Lambda)}.$$

3.3. Existence of Solutions for the BVPs (3.9) and (3.10). Integrating along the μ-characteristic, we obtain the integral equation

$$z(t,x) = G_0(\zeta(t,x), h_0(\zeta(t,x)), w(\zeta(t,x),0)) + \int_{\zeta(t,x)}^t g(\tau, x_\mu(\tau;t,x))\,d\tau,$$

where w at $x=0$ is from Theorem 3.4.

Using the same procedure as before, we can show that this is the unique solution of the BVP (3.9) whose derivatives are given by

$$\begin{aligned} z_t(t,x) &= P(t,x)\zeta_t(t,x) + g(t,x) - g(\zeta(t,x),0)\zeta_t(t,x) \\ &\quad + \int_{\zeta(t,x)}^t g_x(\tau, x_\mu(\tau;t,x))(x_\mu)_t(\tau;t,x)\,d\tau \qquad (3.14) \\ z_x(t,x) &= P(t,x)\zeta_x(t,x) - g(\zeta(t,x),0)\zeta_x(t,x) \\ &\quad + \int_{\zeta(t,x)}^t g_x(\tau, x_\mu(\tau;t,x))(x_\mu)_x(\tau;t,x))\,d\tau \qquad (3.15) \end{aligned}$$

where

$$P(t,x) = \nabla G_0(\zeta(t,x), h_0(\zeta(t,x)), w(\zeta(t,x),0)) \cdot (1, h_0'(\zeta(t,x)), w_t(\zeta(t,x),0)). \qquad (3.16)$$

Theorem 3.6. *Let $M_1 > 0$ be such that $\|w(\cdot,0)\|_{C[0,T]} \le M_1$. Then (3.9) has a unique solution $z \in C^1(\Omega_T^L)$ such that $\|z - z^0(0)\|_{C(\Omega_T^L)} \le C(\Lambda)T$ and*

$$\|z_x\|_{C(\Omega_T^L)} + \|z_t\|_{C(\Omega_T^L)} \le (1/d)(C_1 + (T+T^2)C(\Lambda))e^{TC(\Lambda)}.$$

Proof. The compatibility conditions in (L6) and the fact that $\zeta(t,x) \in [0,T]$ imply

$$\begin{aligned} |z(t,x) - z^0(0)| &\le |G_0(\zeta(t,x), h_0(\zeta(t,x)), w(\zeta(t,x),0)) - G_0(0,h(0),w^0(0))| \\ &\quad + T\|g\|_{C(\Omega_T)} \\ &\le \|\nabla G_0\|_{C(Q_0[M_1])}(1 + M_3 + \|w_t(\cdot,0)\|_{C([0,T])})T + T\|g\|_{C(\Omega_T)} \end{aligned}$$

for all $(t,x) \in \Omega_T^L$. Using the estimate for w_t in Theorem 3.4 in the above inequality, we obtain the desired bound. From the equation (3.15) and Theorem 3.3,

$$\begin{aligned} \|z_x\|_{C(\Omega_T^L)} &\le \frac{1}{d}\|\nabla G_0\|_{C(Q_0[M_1])}(1 + M_3 + \|w_t(\cdot,0)\|_{C[0,T]})\exp(T\|\mu\|_{C^1(\Omega_T)}) \\ &\quad + (\|g\|_{C^1(\Omega_T)}[\|\mu\|_{C(\Omega_T)} + (1 + \|\mu\|_{C(\Omega_T)})\exp(T\|\mu\|_{C^1(\Omega_T)})])T \\ &\quad + \frac{1}{d}\exp(T\|\mu\|_{C^1(\Omega_T)})\|g\|_{C(\Omega_T)} \end{aligned}$$

which has the form given by the theorem. Again, the bound for the time derivative of z can be obtained from the PDE. This completes the proof of the theorem. □

Similar to the previous theorem, we have the following.

Theorem 3.7. *Let $M_1 > 0$ be such that $\|z(\cdot,\ell)\|_{C[0,T]} \le M_1$. Then (3.10) has a unique solution $w \in C^1(\Omega_T^R)$ satisfying $\|w - w^0(\ell)\|_{C(\Omega_T^R)} \le C(\Lambda)T$ and*

$$\|w_x\|_{C(\Omega_T^R)} + \|w_t\|_{C(\Omega_T^R)} \le (1/d)(C_1 + (T+T^2)C(\Lambda))e^{TC(\Lambda)}.$$

It can be easily verified using the compatibility conditions in (L6) that the functions z and w are continuously differentiable on the whole rectangle Ω_T. Combining Theorem 3.4 through Theorem 3.7, we obtain the following.

Theorem 3.8. *Assume that* (L1)–(L6) *hold. Then for each* $(h_0, h) \in B^{1,2}[T, M_3]$ *the system* (3.1) *has a unique solution* $(w, z) \in C^1([0, T] \times [0, \ell])^2$. *Furthermore,*

$$\|(w_x, z_x, w_t, z_t)\|_{C(\Omega_T)^4} \le (1/d)(C(\Lambda_1) + (T + T^2)C(\Lambda))e^{TC(\Lambda)}. \tag{3.17}$$

4. Modulus of Continuity Estimates

Because the space where we look for a local solution is not a closed subset of $C(\Omega_T)^2$, the Banach Fixed Point Theorem cannot be applied. However, we can still find a continuously differentiable solution with the help of the notion of equicontinuity. We define equicontinuity in this paper through the modulus of continuity, precisely speaking as follows.

Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. We define the *modulus of continuity* of f to be the extended-real valued function $\omega(f, \cdot) : [0, \infty) \to [0, \infty]$ by $\omega(f, \delta) = \sup\{|f(x) - f(x')| : x, x' \in \Omega,\ |x - x'| \le \delta\}$. If $\mathcal{F} = (f_i)_{i \in I}$, where I is some nonempty index set, is a family of functions $f_i : \Omega_i \to \mathbb{R}$ we define $\omega(\mathcal{F}, \delta) = \sup_{i \in I} \omega(f_i, \delta)$. A family $\mathcal{F}$ of functions defined on the same set is called *equicontinuous* if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\omega(\mathcal{F}, \delta) < \epsilon$.

Let $\Omega \subset \mathbb{R}^2$, $a : \Omega \to \mathbb{R}$, $b : \Omega \to \mathbb{R}$, and $f : \{(\tau, t, x) : (t, x) \in \Omega,\ a(t, x) \le \tau \le b(t, x)\} \to \mathbb{R}$. If $F : \Omega \to \mathbb{R}$ is defined by

$$F(t, x) = \int_{a(t,x)}^{b(t,x)} f(\tau, t, x)\, d\tau$$

and f is bounded, then

$$\begin{aligned} |F(t, x) - F(t', x')| \ &\le\ \|f\|_\infty(|a(t, x) - a(t', x')| + |b(t, x) - b(t', x')|) \\ &\quad + \int_{a(t',x')}^{b(t,x)} |f(\tau, t, x) - f(\tau, t', x')|\, d\tau. \end{aligned}$$

In the sequel, we shall use this inequality frequently.

Let $\mathcal{F}_1$ be the set which consists of $(w^0)', (z^0)', h_0', h', \nabla G_0$ and ∇G, and $\mathcal{F}_2$ be the set containing the functions λ_x, μ_x, f_x and g_x.

Theorem 4.1. *Let* $M > 0$ *and* (w, z) *be the solution of the system* (3.1) *and suppose that* $\|w\|_{C^1(\Omega_T)} \le M$ *and* $\|z\|_{C^1(\Omega_T)} \le M$. *Then*

$$\omega(w_x, \delta) + \omega(z_x, \delta) \le (1/d^2)C(\Lambda)(\delta + \omega(\mathcal{F}_1, \delta) + T\omega(\mathcal{F}_2, \delta)).$$

Proof.

The proof is established in several steps.

Step 1. If $(\tau, t, x), (\tau, t', x') \in \Theta^i_{T,\lambda}$, $i = 1, 2$, *satisfy* $|(t, x) - (t', x')| \le \delta$ *then*

$$|(x_\lambda)_x(\tau; t, x) - (x_\lambda)_x(\tau; t', x')| \le C(\Lambda)\left(\delta + T\omega(\lambda_x, \delta)\right).$$

An analogous statement involving x_μ *is also true.* From Theorem 3.2 we have

$$|x_\lambda(\tau; t, x) - x_\lambda(\tau; t', x')| \ \le\ (1 + \|\lambda\|_{C^1(\Omega_T)})e^{T\|\lambda\|_{C^1(\Omega_T)}}\delta \ \le\ C(\Lambda)\delta. \tag{4.1}$$

Let $M' = \max(\|\lambda\|_{C^1(\Omega_T)}, \|\mu\|_{C^1(\Omega_T)})$. From the formula (3.3) of $(x_\lambda)_x$ we obtain

$$|(x_\lambda)_x(\tau; t, x) - (x_\lambda)_x(\tau; t', x')| \ \le\ e^{TM'}\|\lambda\|_{C^1(\Omega_T)}\delta$$

$$+\, e^{TM'} \int_{t'}^{\tau} |\lambda_x(\vartheta, x_\lambda(\vartheta; t, x)) - \lambda_x(\vartheta, x_\lambda(\vartheta; t', x'))|\, d\vartheta. \tag{4.2}$$

However, from (4.1) we have

$$|\lambda_x(\vartheta, x_\lambda(\vartheta; t, x)) - \lambda_x(\vartheta, x_\lambda(\vartheta; t', x'))| \;\leq\; \omega(\lambda_x, C(\Lambda)\delta) \;\leq\; C(\Lambda)\omega(\lambda_x, \delta)\,.$$

Using this in (4.2) and noting that $|\tau - t'| \leq T$ we obtain the required estimate.

Step 2. We have

$$\omega(w_x|_{\Omega_T^L \cup \Omega_T^C}, \delta) + \omega(w_t|_{\Omega_T^L \cup \Omega_T^C}, \delta) \leq C(\Lambda)(\delta + \omega(\mathcal{F}_1, \delta) + T\omega(\mathcal{F}_2, \delta)).$$

Also, $\omega(z_x|_{\Omega_T^C \cup \Omega_T^R}, \delta) + \omega(z_t|_{\Omega_T^C \cup \Omega_T^R}, \delta)$ has an upper bound of the same form. Define $F : \Omega_T^L \cup \Omega_T^C \to \mathbb{R}$ by

$$F(t, x) = \int_0^t f_x(\tau, x_\lambda(\tau; t, x))(x_\lambda)_t(\tau; t, x)\, d\tau.$$

This is the integral given in (3.12). Then

$$\begin{aligned}
|F(t, x) - F(t', x')| \;&\leq\; \|f\|_{C^1(\Omega_T)} \|x_\lambda\|_{C^1(\Theta^1_{T,\lambda})} \delta \\
&+ \int_0^t |f_x(\tau, x_\lambda(\tau; t, x))(x_\lambda)_x(\tau; t, x) - f_x(\tau, x_\lambda(\tau; t', x'))(x_\lambda)_x(\tau; t', x')|\, d\tau \\
&\leq\; C(\Lambda)\left(\delta + T\omega(f_x, \delta) + T\omega(\lambda_x, \delta)\right)
\end{aligned}$$

whenever $|(t, x) - (t', x')| \leq \delta$. Furthermore,

$$\begin{aligned}
\omega\left(((w^0)' \circ \alpha)\alpha_x, \delta\right) \;&\leq\; \|w^0\|_{C^1[0,\ell]}\, \omega(\alpha_x, \delta) + \omega((w^0)' \circ \alpha, \delta) \|\alpha_x\|_{C(\Omega_T^L \cup \Omega_T^C)} \\
&\leq\; C(\Lambda)\left(\delta + \omega((w^0)', \delta) + T\omega(\lambda_x, \delta)\right).
\end{aligned}$$

Adding these estimates and using (3.13) proves the first half. The second half follows from the PDE and the first half since

$$\begin{aligned}
\omega(w_t|_{\Omega_T^L \cup \Omega_T^C}, \delta) &\leq \|\lambda\|_{C(\Omega_T)} \omega(w_x|_{\Omega_T^L \cup \Omega_T^C}, \delta) + \omega(\lambda, \delta)\|w_x\|_{C(\Omega_T)} + \omega(f, \delta) \\
&\leq \|\lambda\|_{C(\Omega_T)} \omega(w_x|_{\Omega_T^L \cup \Omega_T^C}, \delta) + (\|\lambda\|_{C^1(\Omega_T)} \|w_x\|_{C(\Omega_T)} + \|f\|_{C^1(\Omega_T)})\delta.
\end{aligned}$$

Step 3. We have $\omega((\zeta_x, \sigma_x), \delta) \leq (1/d^2) C(\Lambda)\left(\delta + T\omega((\mu_x, \lambda_x), \delta)\right)$. Similar arguments as in the proof of Step 1 give us

$$|(x_\mu)_x(\zeta(t, x); t, x) - (x_\mu)_x(\zeta(t', x'); t', x')| \leq C(\Lambda)\left(\delta + T\omega(\mu_x, \delta)\right).$$

This inequality together with (3.6) implies

$$\begin{aligned}
\omega(\zeta_x, \delta) \;&\leq\; \frac{\|x_\mu\|_{C^1(\Theta^2_{T,\mu})}}{d^2} \omega(\mu(\zeta, 0), \delta) + \frac{1}{d} C(\Lambda)\left(\delta + T\omega(\mu_x, \delta)\right) \\
&\leq\; \frac{1}{d^2} C(\Lambda)\left(\delta + T\omega(\mu_x, \delta)\right).
\end{aligned}$$

The second inequality is similar.

Step 4. It holds that $\omega(P, \delta) \leq (1/d^2) C(\Lambda)(\delta + \omega(\mathcal{F}_1, \delta) + T\omega(\mathcal{F}_2, \delta))$, where P is given by (3.16). If $|(t, x) - (t', x')| \leq \delta$ then

$$\begin{aligned}
|h_0(\zeta(t, x)) - h_0(\zeta(t', x'))| \;&\leq\; M_3 \omega(\zeta, \delta) \\
|w(\zeta(t, x), 0) - w(\zeta(t, x), 0)| \;&\leq\; M\omega(\zeta, \delta).
\end{aligned}$$

These properties imply that

$$\omega(G_{0t}(\zeta, h\circ\zeta, w(\zeta,0)),\delta) \le \omega(G_{0t},(1+M_3+M)\omega(\zeta,\delta)) \le C(\Lambda)\omega(G_{0t},\delta).$$

A similar procedure for the other terms appearing in (3.16) shows that

$$\begin{aligned}\omega(P,\delta) \quad \le \quad & \frac{1}{d^2}C(\Lambda)\big[\omega(G_{0t},\delta)+M_3\omega(G_{0h_0},\delta)+\|G_{0h_0}\|_{C(Q_0[M])}\omega(h_0',\delta)\\ &+M\omega(G_{0w},\delta)+\|G_{0w}\|_{C(Q_0[M])}\omega(w_t|_{\Omega_T^L\cup\Omega_T^C},\delta)\big]\end{aligned}$$

and upon using the result of Step 2, we obtain the desired estimate.

Step 5. It holds that $\omega(z_x|_{\Omega_T^L},\delta) \le (1/d^2)C(\Lambda)(\delta+\omega(\mathcal{F}_1,\delta)+T\omega(\mathcal{F}_2,\delta))$ *and* $\omega(w_x|_{\Omega_T^R},\delta)$ *has also the same type of bound.* Utilize Step 3, Step 4 and a similar argument as in proving Step 2.

The proof of Theorem 4.1 follows directly from Step 2 and Step 5. □

5. The PDE Part 2: Quasilinear System

If $S\subset\mathbb{R}^2$ and $\epsilon>0$, we let $S_\epsilon=\{(x,y) : \mathrm{dist}((x,y),S)\le\epsilon\}$. Consider the curve $\Sigma := \{(w^0(x),z^0(x)) : x\in[0,\ell]\}$ in $\mathcal{O}$. Define $\delta : [0,\ell]\to\mathbb{R}$ by $\delta(x) = \mathrm{dist}((w^0(x),z^0(x)),\partial\mathcal{O})$. If the boundary of $\mathcal{O}$ is empty, then we can replace $\mathcal{O}$ by an open set with a nonempty boundary that contains Σ and is contained in $\mathcal{O}$. Then δ is continuous and has a positive minimum. Let $\epsilon_1=\frac{1}{2}\min_{x\in[0,\ell]}\delta(x)>0$. By construction, Σ_{ϵ_1} is compactly contained in $\mathcal{O}$. Furthermore, the continuity of λ and μ implies the existence of $\epsilon_2>0$ and a positive constant $d>0$ such that for $(w,z)\in\mathbb{R}^2$, if $\mathrm{dist}((w,z),(w^0(0),z^0(0)))\le\epsilon_2$ then $\lambda(w,z)\le -d<0<d\le\mu(w,z)$, and if $\mathrm{dist}((w,z),(w^0(\ell),z^0(\ell)))\le\epsilon_2$ then $\lambda(w,z)\le -d<0<d\le\mu(w,z)$.

Let $\epsilon=\min(\epsilon_1,\epsilon_2)>0$ and let $\mathcal{R}_T$ denote the set of all functions $(v,y)\in C(\Omega_T)^2$ such that

(1) ran $(v,y)\subset\Sigma_\epsilon$
(2) $\mathrm{dist}((v(t,x),y(t,x)),(w^0(x),z^0(x)))\le\epsilon$ for $(t,x)\in[0,T]\times\{0,\ell\}$
(3) $(v(0,x),y(0,x))=(w^0(x),z^0(x))$ for $x\in[0,\ell]$.

In the iteration scheme it is important that the resulting linear system must be strictly hyperbolic and that the boundaries are non-characteristic. The first and second criteria in $\mathcal{R}_T$ preserve these properties, respectively.

Let $N>0$ be sufficiently large, which will be made precise later, and

$$\mathcal{D}_T=\{(v,y)\in C^1(\Omega_T)^2 : \|(v_t,y_t)\|_{C(\Omega_T)^2}\le N,\ \|(v_x,y_x)\|_{C(\Omega_T)^2}\le N\}.$$

Notice that if $v(t,x)=w^0(x)$ and $y(t,x)=z^0(x)$ for all $(t,x)\in\Omega_T$ then $(v,y)\in\mathcal{R}_T\cap\mathcal{D}_T$ if $\|((w^0)',(z^0)')\|_{C[0,\ell]^2}\le N$, that is, $\mathcal{R}_T\cap\mathcal{D}_T$ is nonempty.

In this section we prove the well-posedness of the system

$$\begin{cases} w_t+\lambda(w,z)w_x=f(t,x,w,z)\\ z_t+\mu(w,z)z_x=g(t,x,w,z)\\ z(t,0)=G_0(t,h_0(t),w(t,0))\\ w(t,\ell)=G(t,h(t),z(t,\ell))\\ w(0,x)=w^0(x), \qquad z(0,x)=z^0(x)\end{cases} \tag{5.1}$$

where (h_0, h) is a fixed element of $B^{1,2}[T, M_3]$.

Theorem 5.1. *There exists a time $T^* > 0$ such that $x_0(\tau) \neq x_\ell(\tau)$ for $0 \leq \tau \leq T^*$, for all $(v, y) \in \mathcal{R}_{T^*}$, where*

$$\begin{aligned} x_0'(\tau) &= \mu(v(\tau, x_0(\tau)), y(\tau, x_0(\tau))), \qquad x_0(0) = 0, \\ x_\ell'(\tau) &= \lambda(v(\tau, x_\ell(\tau)), y(\tau, x_\ell(\tau))), \qquad x_\ell(0) = \ell. \end{aligned}$$

Proof. Suppose in contrary that there exists a sequence $(T_n)_n$ of positive numbers converging to 0 and a sequence $(v_n, y_n)_n \in \mathcal{R}_T$ satisfying $x_0(T_n; v_n, y_n) = x_\ell(T_n; v_n, y_n)$, and denote this common value by x_n, for all $n \in \mathbb{N}$. Since $(x_n)_n$ is a bounded sequence, there is convergent subsequence, which we still denote by $(x_n)_n$. Then there are two possible cases, either $x_n \geq c > 0$ for all positive integers n (this is the case where x_n does not converge to 0) or for each $\epsilon > 0$ there exists a positive integer n such that $x_n < \epsilon$ (this is the case where the limit is 0).

First let us consider the former case. Since $x_0(0) = 0$ and $x_0(T_n) = x_n$, by the mean-value theorem, there exists $\tau_n \in (0, T_n)$ such that $x_0'(\tau_n) = x_n/T_n$. Hence, it follows that we have $\mu(v_n(\tau_n, \xi_n), y_n(\tau_n, \xi_n)) \geq c/T_n$ for all n, where we put $\xi_n = x_0(\tau_n)$. For each $S > 0$, there exists $R = R(S)$ such that $\mu(\tilde{v}_R, \tilde{y}_R) \geq S$ and $(\tilde{v}_N, \tilde{y}_N) \in \Sigma_\epsilon$, a contradiction to the fact that μ is bounded on Σ_ϵ.

For the latter case, without loss of generality, we may take that $\epsilon < \ell/2$. Because $x_\ell(0) = \ell$ and $x_\ell(T_n) = x_n$, there exists $(\tau_n, \xi_n) \in \Omega_T$ such that $x_\ell'(\tau_n) = \lambda(v_n(\tau_n, \xi_n), y_n(\tau_n, \xi_n)) = (x_n - \ell)/T_n < (\epsilon - \ell)/T_n < -\ell/(2T_n)$. For each $m < 0$ there exists a positive integer $n' = n'(m)$ such that the inequality $\lambda(\tilde{v}_{n'}, \tilde{y}_{n'}) \leq m$ holds for some $(\tilde{v}_{n'}, \tilde{y}_{n'}) \in \Sigma_\epsilon$, which contradicts the boundedness of λ on Σ_ϵ. □

Now, we are ready to state and prove the local existence and uniqueness of solutions to the quasilinear system (1.4) whose life span is independent on the particular data (h_0, h) in $B^{1,2}[T, M_3]$. As mentioned in the earlier sections, this would imply that the mapping $\mathfrak{S}_2$ is well-defined. Before we state the result, we note the following elementary estimate.

Lemma 5.2. *Let $a \geq 0$, $b > 0$ and $(s_n)_{n\geq 0}$ be a sequence of nonnegative real numbers such that $s_n \leq a + bs_{n-1}$ for all $n \geq 1$. Then $s_n \leq a\sum_{k=0}^{n-1} b^k + b^n s_0$, $n \geq 1$.*

Theorem 5.3. *Let $(h_0, h) \in B^{1,2}[T, M_3]$ and assume that (H1)–(H6) holds. Then there exists a time $\tilde{T} = \tilde{T}(M_2, M_3) \in (0, T]$ independent of (h_0, h) such that the quasilinear system (5.1) has a unique solution (w, z) in $C^1(\Omega_{\tilde{T}})^2$. Moreover we have $(w(t, x), z(t, x)) \in \Sigma_\epsilon$ for every $(t, x) \in \Omega_{\tilde{T}}$ and it holds that $\|(w_x, z_x)\|_{C(\Omega_{T_1})^2} \leq N$ and $\|(w_t, z_t)\|_{C(\Omega_{\tilde{T}})^2} \leq N$.*

Proof. We divide the proof into several steps.

Step 1. Definition of the iteration map. Let $(v,y) \in \mathcal{R}_{T^*} \cap \mathcal{D}_{T^*}$ be given and consider the linear system

$$\begin{cases} w_t + \hat{\lambda}(t,x) w_x = \hat{f}(t,x), \\ z_t + \hat{\mu}(t,x) z_x = \hat{g}(t,x), \\ z(t,0) = G_0(t, h_0(t), w(t,0)), \\ w(t,\ell) = G(t, h(t), z(t,\ell)), \\ w(0,x) = w^0(x), \qquad z(0,x) = z^0(x), \end{cases} \tag{5.2}$$

where $\hat{\lambda}(t,x) = \lambda(v(t,x), y(t,x))$, $\hat{\mu}(t,x) = \mu(v(t,x), y(t,x))$, $\hat{f}(t,x) = f(t,x,v(t,x), y(t,x))$, $\hat{g}(t,x) = g(t,x,v(t,x),y(t,x))$. One can easily see that the above system satisfies (L1)–(L6). Therefore, by Theorem 3.8, there exists a unique solution $(w,z) \in C^1(\Omega_{T^*})^2$ of (5.2). This defines a mapping $\mathfrak{F} : \mathcal{R}_{T^*} \cap \mathcal{D}_{T^*} \to C^1(\Omega_{T^*})^2$ given by $\mathfrak{F}(v,y) = (w,z)$.

Step 2. Invariance property. We will show that there exists $T > 0$ such that $\mathfrak{F}(\mathcal{R}_\tau \cap \mathcal{D}_\tau) \subset \mathcal{R}_\tau \cap \mathcal{D}_\tau$ for all $\tau \in (0,T]$. The functions $\hat{\lambda}, \hat{\mu}, \hat{f}, \hat{g}$ and their derivatives with respect to x have uniform bounds independent of $(v,y) \in \mathcal{R}_{T^*} \cap \mathcal{D}_{T^*}$. More precisely, we have the estimates

$$\begin{aligned} \|\hat{f}\|_{C(\Omega_{T^*})} &\leq \|f\|_{C(\Omega_T \times \Sigma_\epsilon)}, \qquad \|\hat{\lambda}\|_{C(\Omega_{T^*})} \leq \|\lambda\|_{C(\Sigma_\epsilon)}, \\ \|\hat{f}_x\|_{C(\Omega_{T^*})} &\leq (1+2N)\|\nabla f\|_{C(\Omega_T \times \Sigma_\epsilon)}, \qquad \|\hat{\lambda}_x\|_{C(\Omega_{T^*})} \leq 2N \|\nabla \lambda\|_{C(\Sigma_\epsilon)}, \end{aligned}$$

and similar estimates for $\hat{\mu}$ and $\hat{g}$. Let $\hat{\Lambda}_1$ be the set Λ_1 in the statement of Theorem 3.8 where the constants $\|\hat{f}\|_{C(\Omega_{T^*})}$, $\|\hat{g}\|_{C(\Omega_{T^*})}$, $\|\hat{\lambda}\|_{C(\Omega_{T^*})}$, and $\|\hat{\mu}\|_{C(\Omega_{T^*})}$ are replaced by the constants $\|f\|_{C(\Omega_T \times \Sigma_\epsilon)}$, $\|g\|_{C(\Omega_T \times \Sigma_\epsilon)}$, $\|\lambda\|_{C(\Sigma_\epsilon)}$, and $\|\mu\|_{C(\Sigma_\epsilon)}$, respectively. Now we take $N > \frac{1}{d} C(\hat{\Lambda}_1)$.

Using this observation in Theorems 3.4 to 3.7, we can see that there exists $T^{(1)} \in (0, T^*]$ such that we have $\|w - w^0\|_{C(\Omega_T^L \cup \Omega_T^C)} \leq \epsilon/2$, $\|z - z^0\|_{C(\Omega_\tau^L \cup \Omega_\tau^C)} \leq \epsilon/2$, $\|z - z^0(0)\|_{C(\Omega_\tau^L)} \leq \epsilon/2$ and $\|w - w^0(\ell)\|_{C(\Omega_\tau^R)} \leq \epsilon/2$ for all $\tau \in (0, T^{(1)}]$. These estimates prove that $\mathrm{ran}(w,z) \in \Sigma_\epsilon$, and the last two also prove that $|(w(t,x), z(t,x)) - (w^0(x), z^0(x))| \leq \epsilon$ for $(t,x) \in [0, T^{(1)}] \times \{0, \ell\}$. The last criterion in $\mathcal{R}_{T^*}$ is obvious. From the choice of N and the estimate (3.17) in Theorem 3.8, we can deduce that there exists $T^{(2)} \in (0, T^*]$ such that $(w,z) \in \mathcal{D}_\tau$ for all $\tau \in (0, T^{(2)}]$. Taking $T^{(3)} = \min(T^{(1)}, T^{(2)})$ shows that $(w,z) \in \mathcal{R}_\tau \cap \mathcal{D}_\tau$ and so $\mathcal{R}_\tau \cap \mathcal{D}_\tau$ is invariant under $\mathfrak{F}$ for all $\tau \in (0, T^{(3)}]$.

Step 3. Contraction property. Let $(v_1, y_1), (v_2, y_2) \in \mathcal{R}_{T^{(3)}} \cap \mathcal{D}_{T^{(3)}}$ and $\mathfrak{F}(v_i, y_i) = (w_i, z_i)$ for $i = 1, 2$. Define $\tilde{w} = w_1 - w_2$ and $\tilde{z} = z_1 - z_2$. It follows that

$$\begin{cases} w_t + \lambda(v_1,y_1) w_x = f(t,x,v_1,y_1) - f(t,x,v_2,y_2) + (\lambda(v_1,y_1) - \lambda(v_2,y_2)) w_{2x} \\ z_t + \mu(v_1,y_1) z_x = g(t,x,v_1,y_1) - g(t,x,v_2,y_2) + (\mu(v_1,y_1) - \mu(v_2,y_2)) z_{2x} \\ \tilde{z}(t,0) = G_0(t, h_0(t), w_1(t,0)) - G_0(t, h_0(t), w_2(t,0)) \\ \tilde{w}(t,\ell) = G(t, h_0(t), w_1(t,\ell)) - G(t, h_0(t), w_2(t,\ell)) \\ \tilde{w}(0,x) = 0, \qquad \tilde{z}(0,x) = 0. \end{cases}$$

From Theorem 3.4 we have

$$\|\tilde{w}\|_{C(\Omega^L_{\tilde{T}}\cup\Omega^C_{\tilde{T}})} \leq \tilde{T}(\|f\|_{C^1(\Omega_T\times\Sigma_\epsilon)} + N\|\lambda\|_{C^1(\Sigma_\epsilon)})\|(v_1,y_1)-(v_2,y_2)\|_{C(\Omega_{\tilde{T}})^2}.$$

for each $\tilde{T} \in (0, T^{(3)}]$. Here, the regions are determined by $\lambda(v_1, y_1)$ and $\mu(v_1, y_1)$. Similarly, we have the estimate

$$\|\tilde{z}\|_{C(\Omega^C_{\tilde{T}}\cup\Omega^R_{\tilde{T}})} \leq \tilde{T}(\|g\|_{C^1(\Omega_T\times\Sigma_\epsilon)} + N\|\mu\|_{C^1(\Sigma_\epsilon)})\|(v_1,y_1)-(v_2,y_2)\|_{C(\Omega_{\tilde{T}})^2},$$

from Theorem 3.5. A procedure similar to the proofs of Theorems 3.6 and 3.8 gives

$$\|\tilde{w}\|_{C(\Omega^R_{\tilde{T}})} + \|\tilde{z}\|_{C(\Omega^L_{\tilde{T}})} \leq C\tilde{T}\|(v_1,y_1)-(v_2,y_2)\|_{C(\Omega_{\tilde{T}})^2},$$

where C is a positive constant independent of (v_1, y_1) and (v_2, y_2). Combining these, one can see that $\|(\tilde{w},\tilde{z})\|_{C(\Omega_{\tilde{T}})^2} \leq C\tilde{T}\|(v_1,y_1)-(v_2,y_2)\|_{C(\Omega_{\tilde{T}})^2}$ for some positive constant C independent of (v_1, y_1) and (v_2, y_2). Hence, $\mathfrak{F}$ is a contraction provided that $C\tilde{T} < 1$.

Step 4. Iteration scheme and compactness argument. One can easily see that $\mathcal{R}_{\tilde{T}} \cap \mathcal{D}_{\tilde{T}}$ is not closed. However, if we have a sequence $((v_n, y_n))_n$ in $\mathcal{R}_{\tilde{T}} \cap \mathcal{D}_{\tilde{T}}$ where (v_0, y_0) is fixed and we have recursively $(v_n, y_n) = \mathfrak{F}(v_{n-1}, y_{n-1})$ for all $n \in \mathbb{N}$, that is,

$$\begin{cases} v_{nt} + \lambda(v_{n-1}, y_{n-1})v_{nx} = f(t, x, v_{n-1}, y_{n-1}) \\ y_{nt} + \mu(v_{n-1}, y_{n-1})y_{nx} = g(t, x, v_{n-1}, y_{n-1}) \\ y_n(t, 0) = G_0(t, h_0(t), v_n(t, 0)), \\ v_n(t, \ell) = G(t, h(t), y_n(t, \ell)), \\ v_n(0, x) = w^0(x), \qquad y_n(0, x) = z^0(x), \end{cases} \tag{5.3}$$

then according to the contractive property of $\mathfrak{F}$, the sequence $((v_n, y_n))_n$ is a Cauchy sequence in $C(\Omega_{\tilde{T}})^2$, and hence converges to some element in $C(\Omega_{\tilde{T}})^2$, say (w, z). From the definition of $\mathcal{R}_{\tilde{T}} \cap \mathcal{D}_{\tilde{T}}$, the sequence $(v_{nx}, y_{nx})_n$ is equibounded with respect to the C-norm, indeed, $\|(v_{nx}, y_{nx})\|_{C(\Omega_{\tilde{T}})^2} \leq N$ for all n.

If $M = \max(\|(w^0, z^0)\|_{C[0,\ell]^2} + N + \epsilon, M_1)$ then $\|v_n\|_{C^1(\Omega_T)} \leq M$ and $\|y_n\|_{C^1(\Omega_T)} \leq M$ for all n. Hence, we take this value of M in the statement of Theorem 4.1. Let $\hat{\Lambda}_2$ be the set of supremum norms of f, g, λ, and μ, $\hat{\Lambda} = \hat{\Lambda}_1 \cup \hat{\Lambda}_2$, $\hat{\mathcal{F}}_2 = \{\nabla f, \nabla g, \nabla\lambda, \nabla\mu\}$ and $\hat{\mathcal{F}}_{2,n}$ be the set consisting of the derivatives with respect to x of the functions $\lambda(v_n, y_n)$, $\mu(v_n, y_n)$, $f(\cdot, \cdot, v_n, y_n)$, and $g(\cdot, \cdot, v_n, y_n)$.

With these in hand, Theorem 4.1 gives us the inequality

$$\omega(v_{nx}, \delta) + \omega(y_{nx}, \delta) \leq C(\hat{\Lambda})(\delta + \omega(\mathcal{F}_1, \delta) + \tilde{T}\omega(\hat{\mathcal{F}}_{2,n-1}, \delta))$$

One can check that $\omega(f_x(\cdot, \cdot, v_n, y_n), \delta) \leq C(\hat{\Lambda})(\omega(\nabla f, \delta) + \omega(v_{nx}, \delta) + \omega(y_{nx}, \delta))$. Using similar estimates for the other elements of $\hat{\mathcal{F}}_{2,n}$, we obtain that

$$\omega(\hat{\mathcal{F}}_{2,n}, \delta) \leq C(\hat{\Lambda})(\omega(\hat{\mathcal{F}}_2, \delta) + \omega(v_{nx}, \delta) + \omega(y_{nx}, \delta)).$$

Consequently,

$$\omega(v_{nx}, \delta) + \omega(y_{nx}, \delta) \leq C(\hat{\Lambda})(\delta + \omega(\mathcal{F}_1 \cup \hat{\mathcal{F}}_2, \delta) + \tilde{T}\omega((v_{n-1})_x, \delta) + \tilde{T}\omega((y_{n-1})_x, \delta)).$$

Choose $\tilde{T}$ such that $C(\hat{\Lambda})\tilde{T} < 1$. With this choice it follows from Lemma 5.2 that

$$\omega(v_{nx},\delta) + \omega(y_{nx},\delta) \le \frac{C(\hat{\Lambda})}{1 - C(\hat{\Lambda})\tilde{T}}[\delta + \omega(\mathcal{F}_1 \cup \hat{\mathcal{F}}_2, \delta) + \omega(v_{1x},\delta) + \omega(y_{1x},\delta)].$$

and hence $(v_{nx}, y_{nx})_n$ is equicontinuous.

It follows from the Arzela-Ascoli Theorem that there exists a convergent subsequence $((v_{n'})_x, (y_{n'})_x)_{n'}$ of $(v_{nx}, y_{nx})_n$. Let us denote the limit of this subsequence by $(W, Z) \in C(\Omega_{\tilde{T}})^2$. From the integral representation

$$w_{n'}(x,t) = w_{n'}(0,t) + \int_0^x (w_{n'})_x(t,\xi)\, d\xi$$

and from the uniform convergence we obtain, by passing through the limit, that $w_x = W \in C(\Omega_{\tilde{T}})$. Similarly, $z_x = Z \in C(\Omega_{\tilde{T}})$.

From the PDE and the equiboundedness of the derivatives of v_n and y_n with respect to x, it can be shown that the subsequence $((v_{n'})_t, (y_{n'})_t)_{n'}$ of $(v_{nt}, y_{nt})_n$ is equicontinuous and so it has a convergent subsequence $((v_{n''})_t, (y_{n''})_t))_{n''}$, whose limit is (v_t, y_t). Replacing n by n'' in (5.3) and letting $n'' \to \infty$ proves existence.

Recall that by construction $(v_n(t,x), y_n(t,x)) \to (w(t,x), z(t,x))$ as $n \to \infty$ and $(v_n(t,x), y_n(t,x)) \in \Sigma_\epsilon$ for all n. Since Σ_ϵ is closed, it follows that $(w(t,x), z(t,x)) \in \Sigma_\epsilon$. Also, notice that $\|((v_{n''})_x, (y_{n''})_x))\|_{C(\Omega_{\tilde{T}})^2} \le N$ and $\|((v_{n''})_t, (y_{n''})_t))\|_{C(\Omega_{\tilde{T}})^2} \le N$ for all n'', and from these the C^0-estimates for the derivatives of (w, z) follow immediately by taking the limit $n'' \to \infty$. Uniqueness can be shown in a standard way. □

Theorem 5.4. *Let (w_1, z_1) and (w_2, z_2) be solutions of the quasilinear system (5.1) corresponding to the boundary data (h_{01}, h_1) and (h_{02}, h_2) in $B^{1,2}[T, M_3]$, respectively. Then there exists a constant C independent of (h_{01}, h_1) and (h_{02}, h_2) such that if $T \in (0, \tilde{T}]$ then for $x = 0, \ell$ we have*

$$\|(w_1(\cdot,x), z_1(\cdot,x)) - (w_2(\cdot,x), z_2(\cdot,x))\|_{C[0,T]^2} \le C\|(h_{01}, h_1) - (h_{02}, h_2)\|_{C[0,T]^2}.$$

Proof. Let $W = w_1 - w_2$, $Z = z_1 - z_2$, $\lambda_1 = \lambda(w_1, z_1)$ and $\mu_1 = \mu(w_1, z_1)$. Then W and Z satisfy the following system

$$\begin{cases} W_t + \lambda_1 W_x = f(t,x,w_1,z_1) - f(t,x,w_2,z_2) - (\lambda(w_1,z_1) - \lambda(w_2,z_2))w_{2x} \\ Z_t + \mu_1 Z_x = g(t,x,w_1,z_1) - g(t,x,w_2,z_2) - (\mu(w_1,z_1) - \mu(w_2,z_2))z_{2x} \\ Z(t,0) = G_0(t, h_{01}(t), w_1(t,0)) - G_0(t, h_{02}(t), w_2(t,0)) \\ W(t,\ell) = G(t, h_1(t), z_1(t,\ell)) - G(t, h_2(t), z_2(t,\ell)) \\ W(0,x) = 0, \qquad Z(0,x) = 0. \end{cases}$$

For each $t \in [0, T]$, define

$$\begin{aligned} \hat{Z}_L(t) &= \sup\{|Z(t,x)| : x \in [0, x_0(t)]\}, \\ \hat{Z}_C(t) &= \sup\{|Z(t,x)| : x \in [x_0(t), x_\ell(t)]\}, \\ \hat{W}(t) &= \sup\{|W(t,x)| : x \in [0, x_\ell(t)]\}. \end{aligned}$$

Let $\hat{Z}(t) = \max(\hat{Z}_L(t), \hat{Z}_C(t))$. Using the fact that μ, λ, f, g are Lipschitz continuous, $|w_{2x}| \leq N$ and $|z_{2x}| \leq N$ we obtain that for $(t,x) \in \Omega_T^L \cup \Omega_T^C$,

$$|W(t,x)| \leq C \int_0^t |W(\tau, x_{\lambda_1}(\tau))| + |Z(\tau, x_{\lambda_1}(\tau))| \, d\tau.$$

Thus,

$$\hat{W}(t) \leq C \int_0^t \hat{W}(\tau) + \hat{Z}(\tau) \, d\tau.$$

If $(t,x) \in \Omega_T^C$ then using the fact that $\hat{Z}_C(\tau) \leq \hat{Z}(\tau)$ we have

$$\hat{Z}_C(t) \leq C \int_0^t \hat{W}(\tau) + \hat{Z}(\tau) \, d\tau.$$

Now, the μ_1-characteristic at (t,x) intersects the left boundary at exactly one point with time coordinate $\zeta(t,x)$. Then it follows that for $(t,x) \in \Omega_T^L$

$$\begin{aligned} |Z(t,x)| \quad &\leq \quad C\|h_{01} - h_{02}\|_{C[0,T]} + C \int_0^{\zeta(t,x)} \hat{W}(\tau) + \hat{Z}(\tau) \, d\tau \\ &\quad + C \int_{\zeta(t,x)}^t |W(\tau, x_{\mu_1}(\tau))| + |Z(\tau, x_{\mu_1}(\tau))| \, d\tau. \end{aligned}$$

Hence,

$$\hat{Z}_L(t) \leq C\|h_{01} - h_{02}\|_{C[0,T]} + C \int_0^t \hat{W}(\tau) + \hat{Z}(\tau) \, d\tau,$$

and it follows that, by taking the maximum,

$$\hat{Z}(t) \leq C\|h_{01} - h_{02}\|_{C[0,T]} + C \int_0^t \hat{W}(\tau) + \hat{Z}(\tau) \, d\tau.$$

Adding our results gives us

$$\hat{W}(t) + \hat{Z}(t) \leq C\|h_{01} - h_{02}\|_{C[0,T]} + C \int_0^t \hat{W}(\tau) + \hat{Z}(\tau) \, d\tau$$

and using Gronwall's inequality we get $\hat{W}(t) + \hat{Z}(t) \leq Ce^{CT}\|h_{01} - h_{02}\|_{C[0,T]}$. Upon taking the supremum we have $\|(W,Z)\|_{C(\Omega_T^L \cup \Omega_T^C)} \leq Ce^{CT}\|h_{01} - h_{02}\|_{C[0,T]}$, and if we take $x = 0$ we obtain a part of the desired result. The other half can be also established in a similar manner. □

We also note that ran $(w,z) \subset \Sigma_\epsilon$ implies that $\|(w,z)\|_{C(\Omega_{\tilde{T}})^2} \leq \|(w_0, z_0)\|_{C[0,\ell]^2} + \epsilon$. From this remark, we now choose $M_1 = \|(w_0, z_0)\|_{C[0,\ell]^2} + \epsilon$. Now we can prove the main result of this paper.

Proof.[Proof of Theorem 1.1] The map $\mathfrak{S} : B^{0,4}[\tilde{T}, M_1] \to B^{0,4}[\tilde{T}, M_1]$ is well-defined from the previous section and Theorem 5.3. It remains to show that $\mathfrak{S}$ is contractive. For this purpose, let $\mathbf{v}^i = (\varphi_0^i, \theta_0^i, \varphi^i, \theta^i) \in B^{0,4}[\tilde{T}, M_1]$ for $i = 1, 2$. Then Theorem 2.2 and Theorem 5.4 imply that

$$\|\mathfrak{S}(\mathbf{v}^1) - \mathfrak{S}(\mathbf{v}^2)\|_{C[0,\tilde{T}]^4} \leq C\|\mathfrak{S}_1(\mathbf{v}^1) - \mathfrak{S}_1(\mathbf{v}^2)\|_{C[0,\tilde{T}]^2} \leq CL\tilde{T}\|\mathbf{v}^1 - \mathbf{v}^2\|_{C[0,\tilde{T}]^4}$$

and so $\mathfrak{G} : B^{0,4}[\breve{T}, M_1] \to B^{0,4}[\breve{T}, M_1]$ is a contraction where $0 < \breve{T} < \min(\tilde{T}, \frac{1}{CL})$. Therefore we obtain a classical solution $(w, z, h_0, h) \in C^1([0, \breve{T}] \times [0, \ell])^2 \times C^1[0, \breve{T}]^2$. Moreover, from (H2) it follows that $(h_0, h) \in C^2[0, \breve{T}]^2$. The uniqueness can be shown using similar arguments as those in Theorem 5.4. □

REFERENCES

[1] Borzì A, Propst G. Numerical investigations of the Liebau phenomenon. *Z. Angew. Math. Phys.* 2003; **54**:1050–1072.

[2] Čanić S, Kim E. Mathematical analysis of quasilinear effects in a hyperbolic model of blood flow through compliant axi-symmetric vessels. *Math. Methods Appl. Sci.* 2003; **26**:1161–1186.

[3] Cannon J, Ewing R. A coupled nonlinear hyperbolic-parabolic system. *J. Math. Anal. Appl.* 1977; **58**:665–686.

[4] Courant R, Hilbert D. *Methods of Mathematical Physics II.* 1962. John Wiley & Sons.

[5] Douglis A. Some existence theorems for hyperbolic systems of partial differential equations in two independent variables. *Comm. Pure Appl. Math.* 1952; **5**:119–154.

[6] Fernández M, Milišić V, Quarteroni A. Analysis of a geometrical multiscale blood flow model based on the coupling of ODEs and hyperbolic PDEs. *Multiscale Model. Simul. 2005*; **5**:215–236.

[7] Li TT. 1994. *Global Classical Solutions for Quasilinear Hyperbolic Systems.* Paris, Masson.

[8] Li TT, Yu W. *Boundary Value Problems for Quasilinear Hyperbolic Systems*, Duke University Mathematics Series V. 1985. Durham, NC, Duke University Mathematics Department.

[9] Ottesen JT. Valveless pumping in a fluid-filled closed elastic tube-system: one-dimensional theory with experimental validation. *J. Math. Biol.* 2003; **46**:309–332.

[10] Peralta G, Propst G. Local well-posedness of a class of hyperbolic PDE-ODE systems on a bounded interval. *J. Hyperbolic Diff. Eq.* 2014; **11**:705–747.

[11] Peralta G, Propst G. Stability and boundary controllability of a linearized model of flow in an elastic tube. *ESAIM: Control, Optimisation and Calculus of Variations* 2015; **21**:583–601.

[12] Ruan W. A coupled system of ODEs and quasilinear hyperbolic PDEs arising in a multiscale blood flow model. *J. Math. Anal. Appl.* 2008; **343**:778–798.

[13] Ruan W, Clark M, Zhao M, Curcio A. A quasilinear hyperbolic system that models blood flow in a network, *in Focus on Mathematical Physics Research (Ed. Charles V. Benton)* 2004; 203–230.

[14] Ruan W, Clark M, Zhao M, Curcio A. A hyperbolic system of equations of blood flow in an arterial network. *SIAM J. Appl. Math.* 2003; **64**:637–667.