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Local Well-posedness of a Class of Hyperbolic PDE-ODE Systems on a Bounded Interval

LOCAL WELL-POSEDNESS OF A CLASS OF HYPERBOLIC PDE-ODE SYSTEMS ON A BOUNDED INTERVAL

GILBERT PERALTA AND GEORG PROPST*

Abstract.
The well-posedness of a hyperbolic system of first order quasilinear PDEs with ODE boundary conditions on a bounded interval is discussed. Such systems occur in multiscale blood flow models, valveless pumping and fluid mechanics. The theory is presented in the setting of Sobolev spaces H^m, $m \geq 3$ is an integer, an appropriate set-up when it comes to proving existence of smooth solutions using energy estimates. A blow-up criterion will be shown stating that if the maximal time of existence is finite, then the state leaves every compact subset of the hyperbolicity region or its first order derivatives blow-up. Finally, we cite some examples which fit in the general framework presented.

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1. Introduction

The aim of this paper is to obtain a well-posedness result for a hyperbolic system of first order quasilinear partial differential equations in the bounded interval $\Omega = (0,1)$ with dynamic boundary conditions

$$\begin{cases} u_t(t,x) + A(u(t,x))u_x(t,x) = f(u(t,x)), & t>0,\ 0<x<1, \\ B_0 u(t,0) = b_0(p_0(t), h(t)), & t>0, \\ B_1 u(t,1) = b_1(p_1(t), h(t)), & t>0, \\ \dot h(t) = H(h(t), q(t), u(t,0), u(t,1)), & t>0, \\ u(0,x) = u_0(x), & 0<x<1, \\ h(0) = h_0. \end{cases} \tag{1.1}$$

The unknown state variables are $u : [0,T] \times [0,1] \to \mathbb{R}^n$ and $h : [0,T] \to \mathbb{R}^d$ that take values in the open and convex sets $\mathcal{U}$ and $\mathcal{H}$, respectively. We assume for simplicity that $0 \in \mathcal{U}$ and $0 \in \mathcal{H}$. This is not too restrictive since one can shift a general problem to this case. The coefficients appearing in (1.1) are assumed to have the following properties: The flux matrix $A : \mathcal{U} \to M_{n\times n}(\mathbb{R})$ and the source term $f : \mathcal{U} \to \mathbb{R}^n$ are both infinitely differentiable. The boundary matrices

$B_0 \in M_{p\times n}(\mathbb{R})$ and $B_1 \in M_{(p-n)\times n}(\mathbb{R})$ are of full rank, where p is the number of incoming characteristics from the left boundary, or equivalently, the number of positive eigenvalues of the flux matrix.

According to the diagonalizability assumption **(D)** below, $n-p$ is the number of incoming characteristics from the right boundary. This assumption further implies that we are in the non-characteristic case. It should be noted that unlike in multidimensions, cf. [2, Chapter 11], for which the boundary matrix should be of constant maximal rank along the boundary, in the case of one space dimension the boundary matrices can have different ranks. However, the sum of their ranks should be the same as the number of components of the state vector u. The boundary data p_0, p_1, and q are given by $p_0 : [0,T] \to \mathbb{R}^{n_0}$, $p_1 : [0,T] \to \mathbb{R}^{n_1}$, $q : [0,T] \to \mathbb{R}^{n_2}$, while $b_0 : \mathbb{R}^{n_0} \times \mathcal{H} \to \mathbb{R}^p$, $b_1 : \mathbb{R}^{n_1} \times \mathcal{H} \to \mathbb{R}^{n-p}$ and $H : \mathcal{H} \times \mathbb{R}^{n_2+2n} \to \mathbb{R}^d$. Again for simplicity we assume that b_0, b_1 and H are all infinitely differentiable.

If b_0 and b_1 are independent of h then (1.1) includes systems of balance laws that are decoupled from the h-dynamics. If H is independent of h then (1.1) includes balance laws with nonlocal boundary conditions of the form

$$B_y u(t,y) = b_y\left(p_y(t), \int_0^t H(q(s), u(s,0), u(s,1))\,\mathrm{d}s\right), \qquad 0<t<T,\ y=0,1.$$

Systems of the form (1.1) occur in multiscale blood flow models [6], [9], [19], [21] and in valveless pumping [5], [14], [17]. Our well-posedness results are based on Sobolev spaces. The motivation for studying the well-posedness in Sobolev spaces, rather than the spaces of continuously differentiable functions [9], [11], [12], lies in the later study of global-in-time existence of smooth solutions for which energy estimates formulated in Sobolev norms are used, see [16]. The presence of a damping term, the bounded space domain and the ODE boundary conditions will not cause much technical difficulty, we will address a way on how to treat them. Broadly speaking, we will follow the frameworks in Benzoni-Gavage and Serre [2] and Métivier [13] to prove our result.

However, there will be differences specially when it comes to the full nonlinear PDE-ODE system where an appropriate linearization and a modified a priori estimate will be used. Recent results regarding the mixing of conservation laws and balance laws with ODEs on the boundary, but with another notion of solutions and on a semi-infinite interval, are given in [3] and [4], respectively.

One possible generalization of (1.1) is to consider nonlinear boundary conditions, e.g. $B(u,h) = 0$ where B satisfies the condition $B(0) = 0$. To deal with the nonlinearity, one first study the linearized problem. The linearized boundary condition takes the form $\tilde{B}(v,g)u = \tilde{g}$ for which the boundary matrix $\tilde{B}$ depends on t through the frozen coefficients v and g. We shall not pursue this generalization and consider the simpler case where the boundary matrices are constant. Regarding time-dependent boundary matrices we refer to [2, Chapter 9]. We believe that the method applied here will work on these types of problems.

Aside from the assumptions that we have already mentioned, we further consider the following hypotheses.

(FS) *Friedrichs Symmetrizability.* The differential operator

$$L_w = \partial_t + A(w)\partial_x$$

is Friedrichs symmetrizable for all $w \in \mathcal{U}$, i.e., there exists a symmetric positive-definite matrix-valued function $S \in \mathscr{C}^\infty(\mathcal{U}; M_{n\times n}(\mathbb{R}))$, called the *Friedrichs symmetrizer*, that is bounded as well as its derivatives, $S(w)A(w)$ is symmetric for all $w \in \mathcal{U}$, and there exists $\alpha > 0$ such that $S(w) \geq \alpha I_n$ for all $w \in \mathcal{U}$.

(D) *Diagonalizability.* For each $w \in \mathcal{U}$, $A(w)$ is diagonalizable with p positive eigenvalues and $n - p$ negative eigenvalues. In particular, $A(w)$ is invertible and has n independent eigenvectors.

(UKL) *Uniform Kreiss-Lopatinskiĭ Condition.* There exists $C > 0$ such that for all $w \in \mathcal{U}$

$$\|V\| \leq C\|B_0 V\|, \qquad \text{for all } V \in E^u(A(w)),$$

and

$$\|V\| \leq C\|B_1 V\|, \qquad \text{for all } V \in E^s(A(w)),$$

where $E^u(A)$ and $E^s(A)$ denote the unstable and stable subspaces of a matrix A, respectively.

Friedrichs symmetrizability is used in deriving pointwise-in-time estimates. The diagonalizability assumption implies that we are in the non-characteristic case. Finally, the Uniform Kreiss-Lopatinskiĭ Condition tells us what forms of the boundary conditions are appropriate.

We also assume that $f(0) = 0$, $H(0) = 0$, and $b(0) = 0$. Again these are not restrictions since one may consider affine shifts of the state spaces. Other assumptions, for example on the initial and boundary data, will be stated later.

According to our hypotheses, we include the case of non-symmetric fluxes with symmetrizers. The diagonalizability assumption though, would give us a new diagonal system through a change of variables, and thus the flux matrix will be trivially symmetric. However, the cost of this diagonalization would be that the boundary matrices will be time-dependent. For this reason, we do not diagonalize the system.

To prove that (1.1) has a unique solution in appropriate function spaces we use the classical way of linearizing the system and proceeding in an iteration scheme. In principle there are various ways to linearize (1.1); the choice that we take is the following: for given functions v and g in appropriate function spaces called the *frozen coefficients*, we consider the linear system

$$\begin{cases} u_t(t,x) + A(v(t,x))u_x(t,x) = f(v(t,x)), & t > 0,\ 0 < x < 1, \\ B_0 u(t,0) = b_0(p_0(t), h(t)), & t > 0, \\ B_1 u(t,1) = b_1(p_1(t), h(t)), & t > 0, \\ \dot{h}(t) = H(g(t), q(t), v(t,0), v(t,1)), & t > 0, \\ u(0,x) = u_0(x), & 0 < x < 1, \\ h(0) = h_0. \end{cases} \tag{1.2}$$

The system (1.2) is now *weakly* coupled in u and h in the sense that only u depends on h and not the other way around. Thus to address the existence and uniqueness of solutions of (1.2) we only need to consider the PDE and ODE parts separately. The ODE part is easy since integration gives us immediately

$$h(t) = h_0 + \int_0^t H(g(s), q(s), v(s, 0), v(s, 1))\, \mathrm{d}s.$$

The PDE part is more involved and will be handled within the frameworks in [2], [13]. The linearization (1.2) is advantageous when deriving a priori estimates in connection with the nonlinear problem (1.1).

We developed well-posedness in the Sobolev space H^m, where $m \geq 3$ is an integer. This assumption is needed in applying commutator estimates, see Proposition 3.2 below. For first order equations in one space dimension, it seems desirable to provide well-posedness for $m = 2$, but we are not able to improve our results.

The structure of the paper is as follows. Section 2 is devoted to constructing boundary symmetrizers necessary for L^2 well-posedness of linear variable-coefficient hyperbolic PDEs on a bounded interval. In Section 3, we derive various a priori estimates in Sobolev spaces that will be used in Section 4 to prove additional regularity of solutions for the PDE part. The local existence, uniqueness and blow-up criterion for the nonlinear system (1.1) will be given in Section 5. Finally, in Section 6 we give some examples.

2. Symmetrizers and L^2 Well-Posedness of the Linear PDE Part

Most of the results in this section are parallel to those in multidimensions given in [2], and therefore, we only point at the deviating parts; those details that are the same or similar are referred to the said text. This will also serve as a venue to realize that the theory originally developed to treat multidimensional problems simplifies in the case of one space dimension.

Let us rewrite the boundary conditions in a single equation. Define $u_{|\partial\Omega}(t) = (u(t, 0), u(t, 1))$, $p = (p_0, p_1)$,

$$B = \begin{pmatrix} B_0 & O_{p\times n} \\ O_{(n-p)\times n} & B_1 \end{pmatrix} \in M_{n\times 2n}(\mathbb{R}), \qquad b(p, h) = \begin{pmatrix} b_0(p_0, h) \\ b_1(p_1, h) \end{pmatrix}.$$

Here $O_{p\times n}$ denotes the $p \times n$ zero matrix. The boundary conditions in (1.1) can now be written in a single equation $Bu_{|\partial\Omega} = b(p, h)$. Given $v = v(t, x)$ consider the first order linear differential operator $L_v = \partial_t + A(v)\partial_x$. As mentioned in the introduction, the first step requires a well-posedness theory for the initial boundary value problem (IBVP)

$$\begin{cases} L_v u = f, & 0 < t < T,\ 0 < x < 1, \\ Bu_{|\partial\Omega} = g, & 0 < t < T, \\ u_{|t=0} = u_0, & 0 < x < 1, \end{cases} \tag{2.1}$$

where f, g, u_0 are given data in appropriate function spaces and $T > 0$ is arbitrary. The existence and uniqueness of solutions of (2.1) follows in a classical way using energy and duality methods. First, one considers the pure boundary value problem

(BVP)

$$\begin{cases} L_v u = f, & t \in \mathbb{R},\ 0 < x < 1, \\ Bu_{|\partial\Omega} = g, & t \in \mathbb{R}, \end{cases} \tag{2.2}$$

then go to the case of homogenous IBVP, i.e. (2.1) with $u_0 = 0$, and finally the general IBVP (2.1).

Energy estimates for (2.2) can be obtained by *symmetrizing* the boundary conditions with the aid of a functional boundary symmetizer. Functional boundary symmetrizers can be obtained if the boundary conditions are strictly dissipative. However, there are boundary value problems that are not strictly dissipative, for instance, the examples we consider in this paper. For the case of smooth coefficients, functional boundary symmetrizers are derived using pseudo-differential calculus. For systems having coefficients that are only at least Lipschitz, para-differential calculus is the appropriate tool in constructing them. In the following, we recall the definition of the functional boundary symmetrizer on a bounded interval.

Definition 2.1. *A functional boundary symmetrizer for (L_v, B) is a two-parameter family of self-adjoint operators $\{R_v^\gamma(x) : \gamma \geq \gamma_0,\ x \in [0,1]\}$, where $\gamma_0 \geq 1$, such that*

(1) $R_v^\gamma \in W^{1,\infty}([0,1]; \mathcal{L}(L^2(\mathbb{R})))$ is uniformly bounded in $\gamma \geq \gamma_0$,

(2) there exists $C > 0$ such that for all $x \in [0,1]$ and $\gamma \geq \gamma_0$,

$$\mathrm{Re}(R_v^\gamma(x) T^\gamma_{\mathcal{A}_v(x)}) \geq C\gamma I_n$$

where $\mathrm{Re}\, A = \frac{1}{2}(A + A^*)$ is the real part of an operator A, $\mathcal{A}_v(x) = -(\gamma + i\delta)A(v(\cdot,x))^{-1}$, $\delta \in \mathbb{R}$, and $T^\gamma_{\mathcal{A}_v(x)}$ is the paradifferential operator associated with the symbol $\mathcal{A}_v(x) \in \Gamma_1^1$ with parameters $x \in [0,1]$ and γ,

(3) and there exist $\alpha, \beta > 0$ such that

$$-\nu(x)\langle R_v^\gamma(x)u, u\rangle_{L^2(\mathbb{R})} + \beta\|B_x u\|^2_{L^2(\mathbb{R})} \geq \alpha\|u\|^2_{L^2(\mathbb{R})}$$

for $x \in \{0,1\}$ and $u \in L^2(\mathbb{R})^n$, where $\nu(0) = -1$ and $\nu(1) = 1$.

For the definition of the class of symbols Γ_1^1 we refer to the Appendix in [2]. The above definition is adapted from the one given in [2, Definition 9.1] in the case of half-space. For general smooth domains, energy estimates can be obtained from the case of half-space and using coordinate patches. In the case of a bounded interval the situation is simpler since one can simultaneously include the conditions on the left and right boundaries.

The reason why we want functional boundary symmetrizers is that they naturally induce a priori estimates, see (2.3) below, necessary for well-posedness theory. All throughout this section we assume that $A \in \mathscr{C}^\infty(\mathcal{U}; M_{n\times n}(\mathbb{R}))$ is constant outside a compact set in $\mathcal{U}$ and $v \in W^{1,\infty}(\mathbb{R}\times\Omega)$ satisfies $\|v\|_{W^{1,\infty}(\mathbb{R}\times\Omega)} \leq K$ and ran $v \subset \mathcal{K}$, where $0 \in \mathcal{K}$ is a compact and convex subset of $\mathcal{U}$. Then it can be shown [2, Chapter 9] that there exist $c = c(K,\mathcal{K}) > 0$ and $\gamma_0 = \gamma_0(K,\mathcal{K}) \geq 1$, both independent of v and u, such that for all $\gamma \geq \gamma_0$ and $u \in \mathscr{D}(\mathbb{R}\times\overline{\Omega})$

$$\sqrt{\gamma}\|u\|_{L^2(\mathbb{R}\times\Omega)} + \|u_{|\partial\Omega}\|_{L^2(\mathbb{R})} \leq c\left(\frac{1}{\sqrt{\gamma}}\|L_v^\gamma u\|_{L^2(\mathbb{R}\times\Omega)} + \|Bu_{|\partial\Omega}\|_{L^2(\mathbb{R})}\right), \tag{2.3}$$

where $L_v^\gamma = L_v + \gamma I_n$, provided that there is a functional boundary symmetrizer for (L_v, B). Here, I_n is the $n \times n$ identity matrix.

Hence, one technical step is to prove the existence of functional boundary symmetrizers. This can be done using the so-called Kreiss symmetrizers which are first defined locally and then extended to a global one using compactness and homogeneity arguments. In the following, we let $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$, $\mathbb{P} = \mathbb{C}^+ \setminus \{0\}$ and define the time-space-frequency set $\mathbb{X} := \mathbb{R} \times [0,1] \times \mathbb{P}$. We denote by $\operatorname{ran} v$ the range of a function v.

Definition 2.2. *Let $A \in \mathscr{C}^\infty(\mathcal{U}; M_{n\times n}(\mathbb{R}))$, B_0 and B_1 be constant matrices and $v \in W^{1,\infty}(\mathbb{R} \times \Omega)$ such that $\operatorname{ran} v \subset \mathcal{K} \subset \mathcal{U}$. A local Kreiss symmetrizer for (A, v, B_0, B_1) at $\underline{X} = (\underline{t}, \underline{x}, \underline{\tau}) \in \mathbb{X}$ is a Hermitian matrix-valued function $r \in \mathscr{C}^\infty(\tilde{\mathcal{U}} \times \mathcal{O}; M_{n\times n}(\mathbb{C}))$, where $\tilde{\mathcal{U}} \times \mathcal{O}$ is open in $\mathcal{U} \times \mathbb{P}$ and $v(\mathcal{V}(\underline{t}, \underline{x})) \subset \tilde{\mathcal{U}}$ for some neighbourhood $\mathcal{V}(\underline{t}, \underline{x})$ of $(\underline{t}, \underline{x})$ in $\mathbb{R} \times [0,1]$, such that there exists an invertible matrix-valued function $T \in \mathscr{C}^\infty(\tilde{\mathcal{U}} \times \mathcal{O}; GL(n, \mathbb{C}))$ with the following properties*

(a) there exists $C > 0$ such that $\operatorname{Re}(r(X)T(X)^{-1}\mathcal{A}(X)T(X)) \geq (C \operatorname{Re}\tau) I_n$, where $\mathcal{A}(X) = -\tau A(v(t,x))^{-1}$, for all $X = (v(t,x), \tau)$ with $(t,x,\tau) \in \mathcal{V}(\underline{t}, \underline{x}) \times \mathcal{O}$

(b) and if in addition, $\underline{X} \in \mathbb{R} \times \{0,1\} \times \mathbb{P}$, then there exist $\alpha, \beta > 0$ such that for all $(t,x,\tau) \in \mathcal{V}(\underline{t}, \underline{x}) \times \mathcal{O}$ we have

$$-\nu(x)r(X) + \beta T(X)^* B_x^\top B_x T(X) \geq \alpha I_n$$

where $X = (v(t,x), \tau)$.

For general constantly hyperbolic systems in multidimensions, the construction of local Kreiss symmetrizers is long and technical. It utilizes tools in algebraic geometry and matrix analysis. However, for certain physical systems such as the Euler equations, the construction is relatively easier. The case of one space dimension is also easy for which the local Kreiss symmetrizers can be taken in diagonal form, thanks to our assumption **(D)**.

Now we show how to construct the local Kreiss symmetrizers. Using homogeneity and compactness arguments it is enough to construct local Kreiss symmetrizers at points on the compact set

$$\mathbb{X}_1 := [-M, M] \times [0,1] \times \{\tau \in \mathbb{C}^+ : |\tau| = 1\}$$

for $M > 0$ large enough. We start with the case where $\operatorname{Re}\tau > 0$. The matrix $\mathcal{A}(w,\tau) = -\tau A(w)^{-1}$ is hyperbolic for all $w \in \mathcal{U}$. Indeed, we have

$$E_-(w,\tau) := E^s(\mathcal{A}(w,\tau)) = E^u(A(w)), \quad E_+(w,\tau) := E^u(\mathcal{A}(w,\tau)) = E^s(A(w)).$$

These show that $E_-(w,\tau)$ and $E_+(w,\tau)$ are independent of τ as long as $\operatorname{Re}\tau > 0$.

Let $\underline{X} = (\underline{t}, \underline{x}, \underline{\tau}) \in \mathbb{X}_1$ be such that $\operatorname{Re}\underline{\tau} > 0$ and $\tilde{\mathcal{U}} \times \mathcal{O}$ be an open set in $\mathcal{U} \times \mathbb{P}$ containing $(v(\underline{t}, \underline{x}), \underline{\tau})$, where $\tilde{\mathcal{U}}$ and $\mathcal{O}$ are open sets in $\mathcal{U}$ and $\mathbb{P} \cap \{\operatorname{Re}\tau > 0\}$, respectively. By continuity of v, there exists an open set $\mathcal{V}(\underline{t}, \underline{x})$ in $\mathbb{R} \times [0,1]$ such that $v(\mathcal{V}(\underline{t}, \underline{x})) \subset \tilde{\mathcal{U}}$. For each $w \in \tilde{\mathcal{U}}$ we let $T_0(w) \in \mathscr{C}^\infty(\mathcal{U}; M_{n\times n}(\mathbb{C}))$ be the matrix consisting of the eigenvectors of $A(t,x)^{-1}$, arranged in such a way that the first p columns correspond to the p positive eigenvalues, and the rest correspond to the

$n-p$ negative eigenvalues. Then $A(w)^{-1}$ can be diagonalized as

$$T_0(w)^{-1}A(w)^{-1}T_0(w) = \begin{pmatrix} \Sigma^+(w) & O_{p\times(n-p)} \\ O_{(n-p)\times p} & \Sigma^-(w) \end{pmatrix}$$

where $\Sigma^+(w) = \operatorname{diag}(\lambda_1(w),\dots,\lambda_p(w))$ and $\Sigma^-(w) = \operatorname{diag}(\lambda_{p+1}(w),\dots,\lambda_n(w))$ are the diagonal matrices with the positive eigenvalues and negative eigenvalues of $A(w)^{-1}$ as entries, respectively. Define

$$T(w,\tau) = T_0(w)$$

for all $(w,\tau)\in\tilde{\mathcal{U}}\times\mathcal{O}$. Then we have

$$T(w,\tau)^{-1}\mathcal{A}(w,\tau)T(w,\tau) = \begin{pmatrix} -\tau\Sigma^+(w) & O_{p\times(n-p)} \\ O_{(n-p)\times p} & -\tau\Sigma^-(w) \end{pmatrix}.$$

Suppose $0<\underline{x}<1$. Then the Hermitian matrix

$$r(w,\tau) = \begin{pmatrix} -I_p & O_{p\times(n-p)} \\ O_{(n-p)\times p} & \mu I_{n-p} \end{pmatrix} \tag{2.4}$$

can be chosen as a local Kreiss symmetrizer at $\underline{X}$ for any $\mu\geq 1$ and T defined above is the associated invertible-matrix valued function.

If $\underline{x}=0$, then the same form of $r(w,\tau)$ given by (2.4) is possible for sufficiently large μ. This is the place where one requires the Kreiss-Lopantiskiĭ condition. Reducing $\tilde{\mathcal{U}}$ if necessary, we can assume without loss of generality that the spectral projections $P_-(w,\tau)$ and $P_+(w,\tau)$ onto $E_-(w,\tau)$ and $E_+(\tau,w)$, respectively, are well-defined. These projections can be written as Dunford-Taylor integrals and by a classical argument in [10], they can be chosen so that they are $\mathscr{C}^\infty$ in w and analytic in τ. Since $E_-(w,\tau)$ and $E_+(w,\tau)$ are independent of τ then $P_-(w,\tau)$ and $P_+(w,\tau)$ are also independent of τ. By **(UKL)**, for all $V\in\mathbb{C}^n$ and $(w,\tau)\in\tilde{\mathcal{U}}\times\mathcal{O}$ we have

$$\begin{aligned} \|P_-(w,\tau)V\| &\leq C\|B_0P_-(w,\tau)V\| = C\|B_0(V-P_+(w,\tau)V)\| \\ &\leq C_1(\|B_0V\|+\|P_+(w,\tau)V\|). \end{aligned} \tag{2.5}$$

With this estimate it can be shown, see [2, pp. 238–239], that for sufficiently large μ, r given by (2.4) is a local Kreiss symmetrizer at $\underline{X}$. If $\underline{x}=1$ then analogously one can choose

$$r(w,\tau) = \begin{pmatrix} -\mu I_p & O_{p\times(n-p)} \\ O_{(n-p)\times p} & I_{n-p} \end{pmatrix}$$

where μ is again sufficiently large.

The next step is to construct symmetrizers at points with $\operatorname{Re}\tau=0$ of the frequency set $\mathbb{P}\cap\{|\tau|=1\}=\{\pm i\}$. However, for nonzero real number δ, $E_-(w,i\delta)$ is not the stable subspace of $\mathcal{A}(w,i\delta)$ anymore. Note that $E_-(w,i\delta)$ is the zero subspace. Instead, we extend the definition of $E_-(w,\tau)$ by continuity, or equivalently, the definition of the spectral projections $P_-(w,\tau)$. For each $(w,\delta)\in\mathcal{U}\times(\mathbb{R}\setminus\{0\})$ we define

$$P_\pm(w,i\delta) = P_\pm(w,\sigma+i\delta)$$

where $\sigma>0$. This definition of $P_\pm$ is independent on σ as long as it is a positive real number. Moreover, one immediately have the continuity of the projections up to the boundary of the frequency set

$$\lim_{\mathcal{U}\times\mathbb{P}\ni(z,\tau)\to(w,i\delta)} P_\pm(z,\tau) = P_\pm(w,i\delta).$$

We define $E_\pm(w,\tau) := \operatorname{ran} P_\pm(w,\tau)$, for $\operatorname{Re}\tau = 0$.

Suppose that $\underline{X} = (\underline{t},\underline{x},\underline{\tau}) \in \mathbb{X}_1$ where $\operatorname{Re}\underline{\tau} = 0$. The neighborhoods $\tilde{\mathcal{U}}, \mathcal{O}$, and $\mathcal{V}$ along with matrices r and T are the same as in the construction above. If $0 < \underline{x} < 1$ then we choose r as in (2.4). If $\underline{x} = 0$, by passing to the limit of projections in (2.5), we still have the estimate

$$\|P_-(w,\tau)V\| \le C\|B_0V\| + \|P_+(w,\tau)V\|$$

for all $V \in \mathbb{C}^n$ and $(w,\tau) \in \tilde{\mathcal{U}} \times \mathcal{O}$. Once we have this estimate we can proceed to the same manner as before. The case $\underline{x} = 1$ is analogous.

With local Kreiss symmetrizers at every point on the compact set $\mathbb{X}_1$ in hand, one can then extend it to a global Kreiss symmetrizer using compactness and homogeneity arguments. In other words, there exists a function $\mathcal{R}_v : \mathbb{X} \to M_{n\times n}(\mathbb{C})$ such that $\mathcal{R}_v(x) := \mathcal{R}_v(\cdot, x, \cdot) \in \Gamma_1^0$ for $x \in [0,1]$ and for $\gamma \ge \gamma_0$ it holds that

$$\begin{aligned}
\operatorname{Re}(\mathcal{R}_v(t,x,\tau)\mathcal{A}(v(t,x),\tau)) &\ge C\gamma I_n, \\
-\nu(x)\mathcal{R}_v(t,x,\tau) + \beta B_x^\top B_x &\ge \alpha I_n, \qquad x \in \{0,1\},
\end{aligned}$$

for some constants $\alpha, \beta, C > 0$ and $\gamma_0 \ge 1$ depending only on K and $\mathcal{K}$. Since $\mathcal{R}_v(x) \in \Gamma_1^0$ for each $x \in \Omega$, it follows that $\{T^\gamma_{\mathcal{R}_v(x)} : \gamma \ge 1\}$ is a family of paradifferential operators of order 0. Thus, their operator norms in $\mathcal{L}(L^2(\mathbb{R}))$ are uniformly bounded in γ, and since the symbols are Lipschitz in the parameter x, they are also uniformly bounded in x. The desired functional boundary symmetrizer is given by

$$R_v^\gamma(x) := \frac{1}{2}(T^\gamma_{\mathcal{R}_v(x)} + (T^\gamma_{\mathcal{R}_v(x)})^*).$$

Refer to [2, pp. 248–250] for more technical details.

Remark 2.3. *We note that with our choice of the local Kreiss symmetrizers we have the following refined property for (a) in Definition 2.2*

$$r(X)T(X)^{-1}\mathcal{A}(X)T(X) = \tau\Delta(X)$$

with some diagonal matrix Δ satisfying $\Delta(X) \ge CI_n$ uniformly in X. This additional property can be used to prove (2) in Definition 2.1.

With the aid of the a priori estimates one can prove an existence and uniqueness result in L^2.

Theorem 2.4. *Suppose that* **(FS)**, **(D)** *and* **(UKL)** *hold and let $T > 0$ be arbitrary. For all $u_0 \in L^2(\Omega)$, $f \in L^2((0,T)\times\Omega)$, $g \in L^2(0,T)$ and for all $v \in W^{1,\infty}([0,T]\times\overline{\Omega})$ such that $\operatorname{ran} v \subset \mathcal{K} \subset \mathcal{U}$, where $0 \in \mathcal{K}$ is compact and convex, and $\|v\|_{W^{1,\infty}([0,T]\times\overline{\Omega})} \le K$, there exists a unique solution $u \in L^2((0,T)\times\Omega)$ in distributional sense of the linear system (2.1). Furthermore, $u_{|\partial\Omega} \in L^2(0,T)$, $u \in C([0,T];L^2(\Omega))$ and there exists $C = C(K,\mathcal{K}) > 0$ such that*

$$\|u\|_{C([0,T];L^2(\Omega))} + \frac{1}{\sqrt{T}}\|u\|_{L^2((0,T)\times\Omega)} + \|u_{|\partial\Omega}\|_{L^2(0,T)}$$

$$\leq C\left(\|u_0\|_{L^2(\Omega)} + \sqrt{T}\|f\|_{L^2((0,T)\times\Omega)} + \|g\|_{L^2(0,T)}\right).$$

Proof. The methods in [2, Chapter 9] or [13] can be adapted in the present case and for this reason we omit the details here. □

We also have a result regarding the intuitive idea that the more regular the data, the more regular the solutions are. We postpone the statement and proof of this and derive further a priori estimates required in the proof. These will be discussed in the next section.

Remark 2.5. *Using Theorem 2.4 and a fixed point argument, one can also show the L^2 well-posedness of a linear PDE-ODE system where the coefficients of the PDE are in $W^{1,\infty}$ and the coefficients of the ODE are in L^∞. In the constant coefficient case, the solution in L^2 coincides with the one given by $\mathcal{C}_0$-semigroup theory. However, the former method yields that the traces $u(\cdot,0)$ and $u(\cdot,1)$ are in $L^2(0,T)$, which cannot be obtained directly from semigroup methods, e.g. [15]. The regularity of the traces is sometimes called a hidden regularity property.*

3. A Priori Estimates in Sobolev Spaces

Given an open set $\mathcal{O}\subset\mathbb{R}^2=\{(t,x): t,x\in\mathbb{R}\}$, $\gamma\geq 1$ and a nonnegative integer m, the space $H^m_\gamma(\mathcal{O})$ is defined to be the usual Sobolev space with γ-depending norm

$$\|u\|_{H^m_\gamma(\mathcal{O})} := \sum_{|\alpha|\leq m}\gamma^{m-|\alpha|}\|\partial^\alpha u\|_{L^2(\mathcal{O})} < \infty.$$

It is not hard to see from the definition that

$$\gamma^{m-k}\|w\|_{H^k(\mathcal{O})} \leq \|w\|_{H^m_\gamma(\mathcal{O})}, \qquad 0\leq k\leq m,\ w\in H^m(\mathcal{O}). \tag{3.1}$$

It can be shown that there exist constants $0<c<C$ independent of both u and γ such that

$$c\sum_{|\alpha|\leq m}\gamma^{m-|\alpha|}\|e^{-\gamma t}\partial^\alpha u\|_{L^2(\mathcal{O})} \leq \|e^{-\gamma t}u\|_{H^m_\gamma(\mathcal{O})} \leq C\sum_{|\alpha|\leq m}\gamma^{m-|\alpha|}\|e^{-\gamma t}\partial^\alpha u\|_{L^2(\mathcal{O})}$$

whenever $e^{-\gamma t}u\in H^m(\mathcal{O})$. When $\mathcal{O}=\mathbb{R}^2$ then the norm $\|u\|_{H^m_\gamma(\mathbb{R}^2)}$ is equivalent to $\|\operatorname{Op}(\lambda^{m,\lambda})u\|_{L^2(\mathbb{R}^2)}$, where $\operatorname{Op}(\lambda^{m,\lambda})$ is the pseudo-differential operator with symbol $\lambda^{m,\gamma}(\delta,\xi)=(\gamma^2+\delta^2+\xi^2)^{m/2}$.

Let $CH^m([0,T]\times\Omega)=\bigcap_{p=0}^m C^p([0,T];H^{m-p}(\Omega))$, where m is a nonnegative integer, be equipped with the norm

$$\|u\|_{CH^m([0,T]\times\Omega)} = \left(\sum_{j=0}^m \sup_{\tau\in[0,T]}\|\partial_t^j u(\tau)\|^2_{H^{m-j}(\Omega)}\right)^{1/2}.$$

We write $CL^2([0,T]\times\Omega)$ instead of $CH^0([0,T]\times\Omega)$. The space $CH^m([0,T]\times\Omega)$ equipped with the norm $\|\cdot\|_{CH^m([0,T]\times\Omega)}$ is a Banach space.

3.1. Some Classical Sobolev Estimates.

In this section, we state various estimates in Sobolev spaces which can be used to derive a priori estimates.

Proposition 3.1. *Let Ω be an open cube or a strip in $\mathbb{R}^d$. For all real numbers $s,t \geq 0$ such that $s+t>0$, if $u \in H^s(\Omega)$ and $v \in H^t(\Omega)$ then $uv \in H^r(\Omega)$ for all $0 \leq r \leq \min(s,t)$ such that $r + d/2 < s+t$. Furthermore, there exists $C = C(r,s,t,\Omega) > 0$ such that*

$$\|uv\|_{H^r(\Omega)} \leq C\|u\|_{H^s(\Omega)}\|v\|_{H^t(\Omega)}.$$

In particular, $H^s(\Omega)$ is a Banach algebra for all $s > d/2$.

Proof. The proof follows from a well-known result in the case $\Omega = \mathbb{R}^d$, e.g. [2, Theorem C.10]. Indeed, we recall that given a real $q \geq 0$ there exists a continuous operator $E_q : H^q(\Omega) \to H^q(\mathbb{R}^d)$ such that $(E_q u)_{|\Omega} = u$ and

$$\|E_q u\|_{H^q(\mathbb{R}^d)} \leq C_q \|u\|_{H^q(\Omega)}$$

for some constant $C_q = C_q(\Omega) > 0$ independent of $u \in H^s(\Omega)$, see e.g. [1, p. 207–208] . Then $uv = (E_s u\, E_t v)_{|\Omega} \in H^r(\Omega)$ and

$$\|uv\|_{H^r(\Omega)} \leq \|E_s u\, E_t u\|_{H^r(\mathbb{R}^d)} \leq C\|E_s u\|_{H^s(\mathbb{R}^d)}\|E_t v\|_{H^t(\mathbb{R}^d)} \leq C\|u\|_{H^s(\Omega)}\|v\|_{H^t(\Omega)}.$$

This proves the proposition. □

By induction, if $s_1, \dots, s_N \geq 0$ are real numbers such that $s_1 + \cdots + s_N > 0$ and if $u_i \in H^{s_i}(\Omega)$ for all $1 \leq i \leq N$, then $u_1 \cdots u_N \in H^r(\Omega)$ whenever $0 \leq r \leq \min_{1\leq i \leq N} s_i$ and $r + d/2 < s_1 + \cdots + s_N$, and moreover, we have the estimate

$$\|u_1 \cdots u_N\|_{H^r(\Omega)} \leq C\|u_1\|_{H^{s_1}(\Omega)} \cdots \|u_N\|_{H^{s_N}(\Omega)} \tag{3.2}$$

for some $C > 0$ independent of u_i for $1 \leq i \leq N$.

In a similar way the following commutator estimate can be shown.

Proposition 3.2. *Let Ω be an open cube or a strip in $\mathbb{R}^d$, $s \geq [d/2]+2$, $a \in H^s(\Omega)$ and $u \in H^{s-1}(\Omega)$. Then for all $1 \leq |\alpha| \leq s$ we have*

$$\|[\partial^\alpha, a]u\|_{L^2(\Omega)} \leq C\|a\|_{H^s(\Omega)}\|u\|_{H^{|\alpha|-1}(\Omega)}.$$

Proposition 3.3. *Let Ω be an open cube or a strip in $\mathbb{R}^d$, $s > d/2$ and $F \in \mathscr{C}^\infty(\mathbb{R})$ such that $F(0) = 0$. If $u \in H^s(\Omega)$ then $F(u) \in H^s(\Omega)$ and there exists a continuous function $C : [0,\infty) \to [0,\infty)$ such that*

$$\|F(u)\|_{H^s(\Omega)} \leq C(\|u\|_{L^\infty(\Omega)})\|u\|_{H^s(\Omega)}.$$

Proof. The proof uses the same ideas as in the proof of the Proposition 3.1. We note that the extension operator $E_q : H^q(\Omega) \to H^q(\mathbb{R}^d)$ can be chosen, e.g. successive application of Seeley's reflection argument [1, p. 84], in such a way that $\|u\|_{L^\infty(\mathbb{R}^d)} \leq C(q,\Omega)\|u\|_{L^\infty(\Omega)}$. Using the same extension argument above and [2, Theorem C.12], one can prove the proposition. □

Similarly, using [2, Corollary C.3] one can prove the following.

Proposition 3.4. *Let Ω be an open cube or a strip in $\mathbb{R}^d$, $s > d/2$ and $F \in \mathscr{C}^\infty(\mathbb{R})$. Then there exists a continuous function $C : [0, \infty) \to (0, \infty)$ such that for all $u, v \in H^s(\Omega)$ we have*

$$\|F(u) - F(v)\|_{H^s(\Omega)} \le C(\max(\|u\|_{H^s(\Omega)}, \|v\|_{H^s(\Omega)}))\|u - v\|_{H^s(\Omega)}.$$

3.2. Sobolev Estimates with time interval $\mathbb{R}$.. Let $v \in H^m(\mathbb{R} \times \Omega)$ takes values on a compact set $\mathcal{K} \subset \mathcal{U}$, $\|v\|_{W^{1,\infty}(\mathbb{R}\times\Omega)} \le K$, $\|v\|_{H^m(\mathbb{R}\times\Omega)} \le R$ and $u \in \mathscr{D}(\mathbb{R} \times \overline{\Omega})$. First, we estimate in terms of the norm $\|\cdot\|_{H^m_\gamma}$, where $m \ge 3$ is an integer. We divide the derivation of the estimates into pure time derivatives and mixed derivatives.

3.2.1. Time-Derivatives. Applying the a priori estimate (2.3) to $w = \partial_t^\alpha u$ for $\alpha = 0, 1, \dots, m$ one obtains

$$\begin{aligned}\sqrt{\gamma}\|\partial_t^\alpha u\|_{L^2(\Omega;L^2(\mathbb{R}))} &+ \|(\partial_t^\alpha u)_{|\partial\Omega}\|_{L^2(\mathbb{R})}\\ &\le c\left(\frac{1}{\sqrt{\gamma}}\|L_v^\gamma \partial_t^\alpha u\|_{L^2(\Omega;L^2(\mathbb{R}))} + \|B(\partial_t^\alpha u)_{|\partial\Omega}\|_{L^2(\mathbb{R})}\right). \qquad (3.3)\end{aligned}$$

Since B is a constant matrix, the boundary terms on the right hand side of (3.3) are given by

$$\begin{aligned}\sum_{\alpha=0}^m \gamma^{m-\alpha}\|B(\partial_t^\alpha u)_{|\partial\Omega}\|_{L^2(\mathbb{R})} &= \sum_{\alpha=0}^m \gamma^{m-\alpha}\|\partial_t^\alpha (Bu_{|\partial\Omega})\|_{L^2(\mathbb{R})}\\ &= \|Bu_{|\partial\Omega}\|_{H^m_\gamma(\mathbb{R})}. \qquad (3.4)\end{aligned}$$

Here, the trace and the derivative commute since u is smooth. The term $L_v^\gamma \partial_t^\alpha u$ is more involved. We rewrite it as

$$L_v^\gamma \partial_t^\alpha u = A(v)\partial_t^\alpha (A(v)^{-1} f) + A(v)[A(v)^{-1}L_v^\gamma, \partial_t^\alpha]u \qquad (3.5)$$

where $f = L_v^\gamma u$.

For the first term on the right hand side of (3.5) we write

$$A(v)\partial_t^\alpha (A(v)^{-1} f) = A(v)\partial_t^\alpha (\mathcal{A}(v) f) + A(v)A(0)^{-1}\partial_t^\alpha f. \qquad (3.6)$$

where $\mathcal{A}(v) = A(v)^{-1} - A(0)^{-1}$ satisfies $\mathcal{A}(0) = 0$. Taking the L^2-norm in (3.6) and applying the triangle inequality

$$\|A(v)\partial_t^\alpha (A(v)^{-1} f)\|_{L^2(\mathbb{R}\times\Omega)} \le C\|\partial_t^\alpha (\mathcal{A}(v) f)\|_{L^2(\mathbb{R}\times\Omega)} + C\|f\|_{H^\alpha(\mathbb{R}\times\Omega)}. \qquad (3.7)$$

Here and below, C is a generic positive constant which depends only on m, $\mathcal{K}$ and K. Let us estimate the first term on the right hand side of (3.7). Since the case $\alpha = 0$ is nothing but the L^2-estimate (2.3) we only need to consider the case where $\alpha \ge 1$. If $\alpha = 1$ then $\partial_t(\mathcal{A}(v) f) = (\partial_t \mathcal{A}(v)) f + \mathcal{A}(v)\partial_t f$, which can be estimated immediately

$$\gamma^{m-1}\|\partial_t(\mathcal{A}(v) f)\|_{L^2(\mathbb{R}\times\Omega)} \le C\gamma^{m-1}\|f\|_{H^1(\mathbb{R}\times\Omega)} \le C\|f\|_{H^m_\gamma(\mathbb{R}\times\Omega)}.$$

Suppose that $\alpha \ge 2$. Then using Proposition 3.1 and (3.1)

$$\begin{aligned}\gamma^{m-\alpha}\|\partial_t^\alpha (\mathcal{A}(v) f)\|_{L^2(\mathbb{R}\times\Omega)} &\le C\gamma^{m-\alpha}\|v\|_{H^\alpha(\mathbb{R}\times\Omega)}\|f\|_{H^\alpha(\mathbb{R}\times\Omega)}\\ &\le C\|v\|_{H^\alpha(\mathbb{R}\times\Omega)}\|f\|_{H^\alpha_\gamma(\mathbb{R}\times\Omega)}.\end{aligned}$$

Therefore it holds that for all $\alpha = 0, 1, \dots, m$,

$$\gamma^{m-\alpha}\|A(v)\partial_t^\alpha(A(v)^{-1}f)\|_{L^2(\mathbb{R}\times\Omega)} \leq C(1 + \|v\|_{H^m(\mathbb{R}\times\Omega)})\|f\|_{H^m_\gamma(\mathbb{R}\times\Omega)}. \tag{3.8}$$

We can rewrite the commutator in (3.5) in terms of derivatives with respect to t only. Indeed, a straightforward computation gives us

$$A(v)[A(v)^{-1}L_v^\gamma, \partial_t^\alpha]u = A(v)[\partial_t^\alpha, A(v)^{-1}]\partial_t u + \gamma A(v)[\partial_t^\alpha, A(v)^{-1}]u. \tag{3.9}$$

Writing $A(v)^{-1} = (A(v)^{-1} - A(0)^{-1}) + A(0)^{-1}$, applying the commutator estimate Proposition 3.2 (this is the place where we need the assumption $m \geq 3$) in each term of (3.9) together with (3.1) and Propsition 3.3 we have

$$\gamma^{m-\alpha}\|A(v)[A(v)^{-1}L_v^\gamma, \partial_t^\alpha]u\|_{L^2(\mathbb{R}\times\Omega)} \leq C\|v\|_{H^m(\mathbb{R}\times\Omega)}\|u\|_{H^m_\gamma(\mathbb{R}\times\Omega)}. \tag{3.10}$$

Applying (3.8) and (3.10) in (3.5) and then taking the sum yield

$$\begin{aligned}&\sum_{\alpha=0}^{m}\gamma^{m-\alpha}\|L_v^\gamma\partial_t^\alpha u\|_{L^2(\Omega;L^2(\mathbb{R}))} \\ &\leq C(1 + \|v\|_{H^m(\mathbb{R}\times\Omega)})(\|L_v^\gamma u\|_{H^m_\gamma(\mathbb{R}\times\Omega)} + \|u\|_{H^m_\gamma(\mathbb{R}\times\Omega)}).\end{aligned} \tag{3.11}$$

Thus, according to (3.3), (3.4) and (3.11) we have the following estimate on the time derivatives

$$\begin{aligned}&\sqrt{\gamma}\|u\|_{L^2(\Omega;H^m_\gamma(\mathbb{R}))} + \|u_{|\partial\Omega}\|_{H^m_\gamma(\mathbb{R})} \\ &\leq \frac{C}{\sqrt{\gamma}}(1 + \|v\|_{H^m(\mathbb{R}\times\Omega)})\|L_v^\gamma u\|_{H^m_\gamma(\mathbb{R}\times\Omega)} + C\|Bu_{|\partial\Omega}\|_{H^m_\gamma(\mathbb{R})} \\ &\quad + \frac{C}{\sqrt{\gamma}}(1 + \|v\|_{H^m(\mathbb{R}\times\Omega)})\|u\|_{H^m_\gamma(\mathbb{R}\times\Omega)} =: CN(u, v).\end{aligned}$$

It is important to note that on the right hand side, the norms of v are independent of γ.

3.2.2. Spatial and Mixed Derivatives. To obtain estimates involving derivatives with respect to x we use the operator L_v^γ. We show by strong induction that

$$\gamma^{m-k-\alpha+1/2}\|\partial_x^k\partial_t^\alpha u\|_{L^2(\mathbb{R}\times\Omega)} \leq CN(u, v)$$

holds for all k and α such that $k + \alpha \leq m$. The case $k = 0$ only involves time-derivatives and hence the basis step was already established. Suppose we have shown that for all j and α such that $j = 0, \dots, k$ and $j + \alpha \leq m$ we have

$$\gamma^{m-(j-1)-\alpha-1/2}\|\partial_x^j\partial_t^\alpha u\|_{L^2(\mathbb{R}\times\Omega)} \leq CN(u, v). \tag{3.12}$$

We show that this also holds for $k + 1$ and α such that $k + 1 + \alpha \leq m$.

First, by applying $\partial_x^k\partial_t^\alpha$ to the equality

$$\partial_x u = \mathcal{A}(v)(f - \partial_t u - \gamma u) + A(0)^{-1}(f - \partial_t u - \gamma u), \tag{3.13}$$

one obtains

$$\begin{aligned}\partial_x^{k+1}\partial_t^\alpha u \;=\; &\partial_x^k\partial_t^\alpha[\mathcal{A}(v)(f - \partial_t u - \gamma u)] \\ &+ A(0)^{-1}(\partial_x^k\partial_t^\alpha f - \partial_x^k\partial_t^{\alpha+1}u - \gamma\partial_x^k\partial_t^\alpha u).\end{aligned} \tag{3.14}$$

The first term in (3.14) may be expanded using the Leibniz's rule as

$$\partial_x^k \partial_t^\alpha [\mathcal{A}(v)(f - \partial_t u - \gamma u)] = \sum_{j=0}^{k} \sum_{l=0}^{\alpha} c_{jl} \partial_x^{k-j} \partial_t^{\alpha-l} \mathcal{A}(v)\, \partial_x^j \partial_t^l (f - \partial_t u - \gamma u). \quad (3.15)$$

By the induction hypothesis (3.12), one has already an estimate for the second term in (3.14)

$$\gamma^{m-k-\alpha-1/2} \|A(0)^{-1}(\partial_x^k \partial_t^\alpha f - \partial_x^k \partial_t^{\alpha+1} u - \gamma \partial_x^k \partial_t^\alpha u)\|_{L^2(\mathbb{R}\times\Omega)} \le CN(u,v). \quad (3.16)$$

Next, we estimate the terms appearing in the sum (3.15) and for this we consider different cases.

Case 1. If $k-j+\alpha-l \le 1$ then one has the estimate $\|\partial_x^{k-j}\partial_t^{\alpha-l}\mathcal{A}(v)\|_{L^\infty(\mathbb{R}\times\Omega)} \le C$, while the terms $\gamma^{m-k-\alpha-1/2}\partial_x^j \partial_t^{l+1} u$ and $\gamma^{m-k-\alpha+1/2}\partial_x^j\partial_t^l u$ can be estimated using the induction hypothesis: Since $j \le k$ and $k+\alpha \ge j+l$

$$\begin{aligned}
&\gamma^{m-k-\alpha-1/2} \|\partial_x^{k-j}\partial_t^{\alpha-l}\mathcal{A}(v)\, \partial_x^j \partial_t^l (f - \partial_t u - \gamma u)\|_{L^2(\mathbb{R}\times\Omega)} \\
&\le C\gamma^{m-k-\alpha-1/2}(\|f\|_{H^{j+l}(\mathbb{R}\times\Omega)} + \|\partial_x^j\partial_t^{l+1}u\|_{L^2(\mathbb{R}\times\Omega)} + \gamma\|\partial_x^j\partial_t^l u\|_{L^2(\mathbb{R}\times\Omega)}) \\
&\le C\bigg(\frac{1}{\sqrt{\gamma}}\gamma^{m-(j+l)}\|f\|_{H^{j+l}(\mathbb{R}\times\Omega)} + \gamma^{m-(j-1)-(l+1)-1/2}\|\partial_x^j\partial_t^{l+1}u\|_{L^2(\mathbb{R}\times\Omega)} \\
&\quad + \gamma^{m-(j-1)-l-1/2}\|\partial_x^j\partial_t^l u\|_{L^2(\mathbb{R}\times\Omega)}\bigg) \le CN(u,v).
\end{aligned}$$

Case 2. If $k-j+\alpha-l = 2$ then we first estimate with respect to time and then integrate with respect to space. In the following, for simplicity we write u, v, f for $u(\cdot,x), v(\cdot,x), f(\cdot,x)$, respectively. Using an $L^2 - L^\infty$ estimate, the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and $\gamma \ge 1$ we have

$$\begin{aligned}
&\gamma^{m-k-\alpha-1/2}\|\partial_x^{k-j}\partial_t^{\alpha-l}\mathcal{A}(v)\, \partial_x^j\partial_t^l(f - \partial_t u - \gamma u)\|_{L^2(\mathbb{R})} \\
&\le C\|v\|_{H^2(\mathbb{R})}\gamma^{m-k-\alpha-1/2}(\|\partial_x^j\partial_t^l f\|_{H^1(\mathbb{R})} + \|\partial_x^j\partial_t^{l+1}u\|_{H^1(\mathbb{R})} + \gamma\|\partial_x^j\partial_t^l u\|_{H^1(\mathbb{R})}) \\
&\le \frac{C}{\sqrt{\gamma}}\|v\|_{H^2(\mathbb{R})}(\|f\|_{H^m_\gamma(\mathbb{R})} + \|u\|_{H^m_\gamma(\mathbb{R})}).
\end{aligned}$$

Integrating with respect to x over Ω and applying the embedding $H^3(\mathbb{R}\times\Omega) \hookrightarrow L^\infty(\Omega; H^2(\mathbb{R}))$ we obtain

$$\begin{aligned}
&\gamma^{m-k-\alpha-1/2}\|\partial_x^{k-j}\partial_t^{\alpha-l}\mathcal{A}(v)\, \partial_x^j\partial_t^l(f - \partial_t u - \gamma u)\|_{L^2(\mathbb{R}\times\Omega)} \\
&\le \frac{C}{\sqrt{\gamma}}\|v\|_{H^3(\mathbb{R}\times\Omega)}(\|f\|_{H^m_\gamma(\mathbb{R}\times\Omega)} + \|u\|_{H^m_\gamma(\mathbb{R}\times\Omega)}) \ \le\ CN(u,v).
\end{aligned}$$

Case 3. If $k-j+\alpha-l \ge 3$ then $j+l+3 \le k+\alpha \le m$ and we have

$$\begin{aligned}
&\gamma^{m-k-\alpha-1/2}\|\partial_x^{k-j}\partial_t^{\alpha-l}\mathcal{A}(v)\, \partial_x^j\partial_t^{l+1}u\|_{L^2(\mathbb{R}\times\Omega)} \\
&\le C\|v\|_{H^m(\mathbb{R}\times\Omega)}\gamma^{m-k-\alpha-1/2}\|\partial_x^j\partial_t^{l+1}u\|_{L^\infty(\mathbb{R}\times\Omega)} \\
&\le C\|v\|_{H^m(\mathbb{R}\times\Omega)}\gamma^{m-(k+\alpha)-1/2}\|u\|_{H^{j+l+3}(\mathbb{R}\times\Omega)} \\
&\le C\|v\|_{H^m(\mathbb{R}\times\Omega)}\gamma^{m-(j+l+3)-1/2}\|u\|_{H^{j+l+3}(\mathbb{R}\times\Omega)} \\
&\le \frac{C}{\sqrt{\gamma}}\|v\|_{H^m(\mathbb{R}\times\Omega)}\|u\|_{H^m_\gamma(\mathbb{R}\times\Omega)} \le \frac{C}{\sqrt{\gamma}}N(u,v),
\end{aligned}$$

and similar for the other terms $\partial_x^{k-j}\partial_t^{\alpha-l}\mathcal{A}(v)\,\partial_x^j\partial_t^l f$ and $\gamma\partial_x^{k-j}\partial_t^{\alpha-l}\mathcal{A}(v)\,\partial_x^j\partial_t^l u$.

Combining the three cases in (3.15) one has

$$\gamma^{m-k-\alpha-1/2}\|\partial_x^k\partial_t^\alpha[\mathcal{A}(v)(f-\partial_t u-\gamma u)]\|_{L^2(\mathbb{R}\times\Omega)} \le CN(u,v), \tag{3.17}$$

and taking the sum of (3.16) and (3.17) in (3.14) we have

$$\gamma^{m-k-\alpha-1/2}\|\partial_x^{k+1}\partial_t^\alpha u\|_{L^2(\mathbb{R}\times\Omega)} \le CN(u,v),$$

which establishes the induction step.

3.2.3. Weighted-in-Time Estimates.

The above estimates give us finally the following estimate

$$\begin{aligned}&\sqrt{\gamma}\|u\|_{H_\gamma^m(\mathbb{R}\times\Omega)} + \|u_{|\partial\Omega}\|_{H_\gamma^m(\mathbb{R})}\\ &\le C\left(\frac{1}{\sqrt{\gamma}}(1+\|v\|_{H^m(\mathbb{R}\times\Omega)})\|L_v^\gamma u\|_{H_\gamma^m(\mathbb{R}\times\Omega)} + \|Bu_{|\partial\Omega}\|_{H_\gamma^m(\mathbb{R})}\right)\\ &\quad + \frac{C}{\sqrt{\gamma}}(1+\|v\|_{H^m(\mathbb{R}\times\Omega)})\|u\|_{H_\gamma^m(\mathbb{R}\times\Omega)}\end{aligned} \tag{3.18}$$

for all $u \in \mathscr{D}(\mathbb{R}\times\overline{\Omega})$ where $C = C(\mathcal{K}, K) > 0$. Choosing γ large enough, the last term on the right hand side of (3.18) can be absorbed by the first term on the left hand side and therefore

$$\sqrt{\gamma}\|u\|_{H_\gamma^m(\mathbb{R}\times\Omega)} + \|u_{|\partial\Omega}\|_{H_\gamma^m(\mathbb{R})} \le C\left(\frac{1}{\sqrt{\gamma}}\|L_v^\gamma u\|_{H_\gamma^m(\mathbb{R}\times\Omega)} + \|Bu_{|\partial\Omega}\|_{H_\gamma^m(\mathbb{R})}\right) \tag{3.19}$$

where the constant $C > 0$ also depends only on the $W^{1,\infty}$-norm and H^m-norm of v and the compact set $\mathcal{K}$. The passage from (3.19) from (3.18) by absorption would not be possible if we have the H_γ^m-norm of v in (3.18) instead of its H^m-norm.

Replacing u by $e^{-\gamma t}u$, which is still in $\mathscr{D}(\mathbb{R}\times\overline{\Omega})$ provided that u is, noting that $L_v^\gamma(e^{-\gamma t}u) = e^{-\gamma t}L_v u$, and then applying a density argument, we have the following a priori estimate.

Theorem 3.5. *Let $v \in H^m(\mathbb{R}\times\Omega)$ taking values on a compact set $\mathcal{K} \subset \mathcal{U}$, $\|v\|_{W^{1,\infty}(\mathbb{R}\times\Omega)} \le K$ and $\|v\|_{H^m(\mathbb{R}\times\Omega)} \le R$. Then there exist $C_m = C_m(\mathcal{K}, K, R) > 0$ and $\gamma_m = \gamma_m(\mathcal{K}, K, R) \ge 1$ such that for every $\gamma \ge \gamma_m$ and for every $u \in e^{\gamma t}H^{m+1}(\mathbb{R}\times\Omega)$ it holds that*

$$\begin{aligned}&\sqrt{\gamma}\|e^{-\gamma t}u\|_{H_\gamma^m(\mathbb{R}\times\Omega)} + \|e^{-\gamma t}u_{|\partial\Omega}\|_{H_\gamma^m(\mathbb{R})}\\ &\le C_m\left(\frac{1}{\sqrt{\gamma}}\|e^{-\gamma t}L_v u\|_{H_\gamma^m(\mathbb{R}\times\Omega)} + \|e^{-\gamma t}Bu_{|\partial\Omega}\|_{H_\gamma^m(\mathbb{R})}\right).\end{aligned} \tag{3.20}$$

The proof of Theorem 3.5 given above follows the ideas given in the proof of Theorem 9.7 in [2]. However, we have a different estimate in (3.8). In [2, p. 252], the authors seem to use the estimate

$$\|vf\|_{L^2(\Omega;e^{\gamma t}H_\gamma^m(\mathbb{R}))} \le C\|v\|_{L^2(\Omega;H^m(\mathbb{R}))}\|f\|_{L^2(\Omega;e^{\gamma t}H_\gamma^m(\mathbb{R}))}$$

which does not hold in general. We resolved this by estimating in terms of the norm in $H_\gamma^m(\mathbb{R}\times\Omega)$.

3.3. Sobolev Estimates with Time Interval $(-\infty, T]$. Now suppose that $v \in H^m((-\infty,T]\times\Omega)$ and $u \in \mathscr{D}((-\infty,T]\times\overline{\Omega})$ with $u_{|t<0}=0$. Then thanks to **(FS)** the a priori estimate

$$\begin{aligned}&\|u(t)\|_{L^2(\Omega)} + \sqrt{\gamma}\|u\|_{L^2(\Omega;L^2(-\infty,T])} + \|u_{|\partial\Omega}\|_{L^2(-\infty,T]}\\ &\le C\left(\frac{1}{\sqrt{\gamma}}\|L_v^\gamma u\|_{L^2(\Omega;L^2(-\infty,T])} + \|Bu_{|\partial\Omega}\|_{L^2(-\infty,T]}\right)\end{aligned}$$

holds for all $\gamma \ge \gamma_0(\mathcal{K},K) \ge 1$. See [13] for a proof of this estimate. The same procedure as in Section 3.1 gives us the inequality

$$\begin{aligned}&\sum_{\alpha=0}^{m}\gamma^{m-\alpha}\|\partial_t^\alpha u(t)\|_{L^2(\Omega)} + \sqrt{\gamma}\|u\|_{H^m((-\infty,T]\times\Omega)} + \|u_{|\partial\Omega}\|_{H^m_\gamma(-\infty,T]}\\ &\le \frac{C}{\sqrt{\gamma}}(1+\|v\|_{H^m((-\infty,T]\times\Omega)})\|L_v^\gamma u\|_{H^m_\gamma((-\infty,T]\times\Omega)} + C\|Bu_{|\partial\Omega}\|_{H^m_\gamma(-\infty,T]}\\ &\quad + \frac{C}{\sqrt{\gamma}}(1+\|v\|_{H^m((-\infty,T]\times\Omega)})\|u\|_{H^m_\gamma((-\infty,T]\times\Omega)} \;=:\; CN(u,v).\end{aligned} \tag{3.21}$$

We proceed by induction for the pointwise in time estimates for the spatial derivatives. Assume that for k with $k+\alpha\le m$ we have already shown that (the basis step $k=0$ is nothing but the L^2-estimate, which is already given by (3.21))

$$\gamma^{m-k-\alpha}\|\partial_x^k\partial_t^\alpha u(t)\|_{L^2(\Omega)} \le CN(u,v), \qquad t\in(-\infty,T].$$

We show that this is true for $k+1$ when $k+1+\alpha\le m$. Recall our formula (3.14), and let J denote the second term, i.e., $J := \partial_x^k\partial_t^\alpha[\mathcal{A}(v)(f-\partial_t u-\gamma u)]$. The following weighted Sobolev estimate will be used.

Proposition 3.6. *For every $w\in H^1((-\infty,T]\times\Omega)$ and $\gamma>0$ we have*

$$\|w\|^2_{L^\infty((-\infty,T];L^2(\Omega))} \le \gamma\|w\|^2_{L^2((-\infty,T]\times\Omega)} + \frac{1}{\gamma}\|\partial_t w\|^2_{L^2((-\infty,T]\times\Omega)}. \tag{3.22}$$

Proof. By a standard density argument, we may suppose that $w\in\mathscr{D}((-\infty,T]\times\overline{\Omega})$. Let $R_0<0$ be such that w vanishes for all $t\le R_0$. For simplicity we assume that w is scalar-valued. Let $R\le 2R_0-T$ and $\frac{T+R}{2}\le\tau\le T$. Using Young's inequality

$$\begin{aligned}|w(\tau,x)|^2 &= \int_R^\tau \partial_t(|w(t,x)|^2)\,\mathrm{d}t\\ &= 2\int_R^\tau w(t,x)w_t(t,x)\,\mathrm{d}t\\ &\le \gamma\int_R^T |w(t,x)|^2\,\mathrm{d}t + \frac{1}{\gamma}\int_R^T|w_t(t,x)|^2\,\mathrm{d}t.\end{aligned}$$

Letting $R\to-\infty$ we have

$$|w(\tau,x)|^2 \le \gamma\int_{-\infty}^T |w(t,x)|^2\,\mathrm{d}t + \frac{1}{\gamma}\int_{-\infty}^T |w_t(t,x)|^2\,\mathrm{d}t$$

for all $\tau\in(-\infty,T]$ and $x\in\overline{\Omega}$. Integrating the previous inequality over Ω and taking the supremum over all $\tau\in(-\infty,T]$ give us (3.22). □

Using (3.22) together with the induction hypothesis yields an estimate for the second term in (3.14)

$$\gamma^{m-(k+1)-\alpha}\|\partial_x^k\partial_t^\alpha f(t)-\partial_x^k\partial_t^{\alpha+1}u(t)-\gamma\partial_x^k\partial_t^\alpha u(t)\|_{L^2(\Omega)}\le CN(u,v). \tag{3.23}$$

As in the computation of mixed derivatives, one obtains

$$\begin{aligned}\gamma^{m-k-\alpha-1/2}\|J\|_{L^2((-\infty,T]\times\Omega)} &\le CN(u,v)\\ \gamma^{m-(k+1)-\alpha-1/2}\|\partial_t J\|_{L^2((-\infty,T]\times\Omega)} &\le CN(u,v).\end{aligned}$$

Thus, by the weighted Sobolev estimate (3.22) we have the estimate

$$\begin{aligned}&\gamma^{m-(k+1)-\alpha}\|J(t)\|_{L^2(\Omega)}\\ &\le C\left(\gamma^{m-(k+1)-\alpha+1/2}\|J\|_{L^2((-\infty,T]\times\Omega)}+\gamma^{m-(k+1)-\alpha-1/2}\|\partial_t J\|_{L^2((-\infty,T]\times\Omega)}\right)\\ &\le CN(u,v).\end{aligned} \tag{3.24}$$

Combining (3.23) and (3.24) proves the induction step.

Therefore we have the full estimate

$$\begin{aligned}&\sum_{|\beta|\le m}\gamma^{m-|\beta|}\|\partial^\beta u(t)\|_{L^2(\Omega)}+\sqrt{\gamma}\|u\|_{H^m_\gamma((-\infty,T]\times\Omega)}+\|u_{|\partial\Omega}\|_{H^m_\gamma(-\infty,T]}\\ &\le\frac{C}{\sqrt{\gamma}}(1+\|v\|_{H^m((-\infty,T]\times\Omega)})\|L_v^\gamma u\|_{H^m_\gamma((-\infty,T]\times\Omega)}+C\|Bu_{|\partial\Omega}\|_{H^m_\gamma(-\infty,T]}\\ &\quad+\frac{C}{\sqrt{\gamma}}(1+\|v\|_{H^m((-\infty,T]\times\Omega)})\|u\|_{H^m_\gamma((-\infty,T]\times\Omega)}\end{aligned}$$

for all $t\in(-\infty,T]$. Now replace u by $e^{-\gamma t}u$, choose γ large enough, so that the last term on the right hand side can be absorbed by the second term on the left hand side, and use the norm-equivalence

$$\sum_{|\beta|\le m}\gamma^{m-|\beta|}\|\partial^\beta(e^{-\gamma t}u(t))\|_{L^2(\Omega)}\simeq\sum_{|\beta|\le m}\gamma^{m-|\beta|}e^{-\gamma t}\|\partial^\beta u(t)\|_{L^2(\Omega)}$$

we have the following a priori estimate.

Lemma 3.7. *Let $m\ge 3$ be an integer. For each $v\in H^m((-\infty,T)\times\Omega)$ satisfying $\operatorname{ran} v\subset\mathcal{K}$, $\|v\|_{W^{1,\infty}((-\infty,T]\times\Omega)}\le K$ and $\|v\|_{H^m((-\infty,T]\times\Omega)}\le R$ and for all $u\in H^{m+1}((0,T)\times\Omega)$ such that $u_{|t=0}=0$, there exist $C_m=C_m(\mathcal{K},K,R)>0$ and $\gamma_m(\mathcal{K},K,R)\ge 1$ such that for all $\gamma\ge\gamma_m$ and for all $\tau\in[0,T]$ the following a priori estimate holds*

$$\begin{aligned}&\sum_{|\alpha|\le m}\gamma^{m-|\alpha|}e^{-\gamma\tau}\|\partial^\alpha u(\tau)\|_{L^2(\Omega)}+\sqrt{\gamma}\|e^{-\gamma t}u\|_{H^m_\gamma((0,\tau)\times\Omega)}+\|u_{|\partial\Omega}\|_{H^m_\gamma(0,\tau)}\\ &\le C_m\left(\frac{1}{\sqrt{\gamma}}\|e^{-\gamma t}L_v u\|_{H^m_\gamma((0,\tau)\times\Omega)}+\|e^{-\gamma t}Bu_{|\partial\Omega}\|_{H^m_\gamma(0,\tau)}\right).\end{aligned} \tag{3.25}$$

The a priori estimate (3.25) is different from those in [2] and [13] because in (3.25) the constants C_m and γ_m depend only on the H^m-norm of v and not on its H^m_γ-norm.

3.4. Gagliardo-Nirenberg Type Estimates. For initial boundary value problems with zero initial conditions, the a priori estimate (3.25) will be used. The next step is to derive an a priori estimate which can be used for problems that are not starting initially from zero. In preparation we borrow the Gagliardo-Nirenberg type estimates in [13, pp. 69–71].

Theorem 3.8. *Let m be a positive integer and $T > 0$. Then there exists $C > 0$, independent of T, such that for all $u \in H^m((-\infty, T) \times \Omega)$ and $1 \le |\alpha| \le m$ we have*

$$\|\partial_x^\alpha u\|_{L^{2m/|\alpha|}((-\infty,T)\times\Omega)} \le C\|u\|^{1-|\alpha|/m}_{L^\infty((-\infty,T)\times\Omega)}\|u\|^{|\alpha|/m}_{H^m((-\infty,T)\times\Omega)}.$$

A similar estimate also holds for $u \in H^m(-\infty, T)$.

The following is a modification of Proposition 4.5.5 in [13].

Theorem 3.9. *For all $m \in \mathbb{N}$ there exists $C = C(m) > 0$ such that for all $T > 0$, $\psi \in H^m(0, T)$ and $1 \le j \le m$ we have*

$$\|\psi^{(j)}\|_{L^{2m/j}(0,T)} \le C(K_{m,T}(\psi)^{1-m/j}(\|\psi\|_{H^m(0,T)} + K_{m,T}(\psi))^{m/j} + K_{m,T}(\psi))$$

where

$$K_{m,T}(\psi) = \|\psi\|_{L^\infty(0,T)} + \sum_{i=0}^{m-1} |\psi^{(i)}(0)|.$$

In particular,

$$\|\psi^{(j)}\|_{L^{2m/j}(0,T)} \le C(\|\psi\|_{H^m(0,T)} + K_{m,T}(\psi)).$$

Proof. We adjust the proof in [13]. Given $\psi \in H^m(0, T)$, let $\psi_1 \in H^m(\mathbb{R})$ be such that $\psi_1^{(i)}(0) = \psi^{(i)}(0)$ for all $i = 0, \dots, m-1$ and using the fact that the trace operator has a continuous right inverse

$$\|\psi_1\|_{H^m(\mathbb{R})} \le C\sum_{i=0}^{m-1} |\psi_1^{(i)}(0)| = C\sum_{i=0}^{m-1} |\psi^{(i)}(0)|, \tag{3.26}$$

where $C > 0$ is independent of ψ. Let $\psi_2 = \psi - \psi_1 \in H^m(0, T)$. Then (3.26) and the Sobolev embedding theorem $H^m(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ imply

$$\begin{aligned} \|\psi_2\|_{L^\infty[0,T]} &\le \|\psi\|_{L^\infty[0,T]} + \|\psi_1\|_{L^\infty(\mathbb{R})} \\ &\le \|\psi\|_{L^\infty[0,T]} + C\|\psi_1\|_{H^m(\mathbb{R})} \le CK_{m,T}(\psi) \end{aligned} \tag{3.27}$$

and

$$\|\psi_2\|_{H^m(0,T)} \le \|\psi\|_{H^m(0,T)} + \|\psi_1\|_{H^m(0,T)} \le \|\psi\|_{H^m(0,T)} + CK_{m,T}(\psi). \tag{3.28}$$

By construction, it holds that $\psi_2^{(i)}(0) = 0$ for $i = 0, \dots, m-1$ and therefore extending ψ_2 by 0 for $t < 0$ we have $\psi_2 \in H^m(-\infty, T)$. By the Gagliardo-Nirenberg inequality

$$\|\psi_2^{(j)}\|_{L^{2m/j}(-\infty,T)} \le C\|\psi_2\|^{1-j/m}_{L^\infty[0,T]}\|\psi_2\|^{j/m}_{H^m(0,T)} \tag{3.29}$$

$$\|\psi_1^{(j)}\|_{L^{2m/j}(\mathbb{R})} \le C\|\psi_1\|^{1-j/m}_{L^\infty(\mathbb{R})}\|\psi_1\|^{j/m}_{H^m(\mathbb{R})} \le C\|\psi_1\|_{H^m(\mathbb{R})}. \tag{3.30}$$

Hence, (3.27)–(3.30) imply that

$$\|\psi^{(j)}\|_{L^{2m/j}(0,T)} \le \|\psi_1^{(j)}\|_{L^{2m/j}(0,T)} + \|\psi_2^{(j)}\|_{L^{2m/j}(0,T)}$$

$$\leq C\|\psi_2\|^{1-j/m}_{L^\infty[0,T]}\|\psi_2\|^{j/m}_{H^m(0,T)} + C\|\psi_1\|_{H^m(\mathbb{R})}$$
$$\leq C(K_{m,T}(\psi)^{1-j/m}(\|\psi\|_{H^m(0,T)} + K_{m,T}(\psi))^{j/m} + K_{m,T}(\psi)).$$

This proves the first part. The second part follows immediately using the elementary inequality $a^{1-r}(a+b)^r \leq a+b$ for $a, b \geq 0$ and $0 < r < 1$. □

Theorem 3.10. *For each positive integers m there exists $C = C(m) > 0$ such that for all $T > 0$ and $u \in H^m((0,T)\times\Omega)\cap L^\infty((0,T)\times\Omega)$ satisfying $\partial_t^j u_{|t=0} \in H^{m-j}(\Omega)$ for $0 \leq j \leq m-1$ we have*

$$\|\partial^\alpha u\|_{L^{2m/|\alpha|}((0,T)\times\Omega)}$$
$$\leq C(\tilde{K}_{m,T}(u)^{1-|\alpha|/m}(\|u\|_{H^m((0,T)\times\Omega)} + \tilde{K}_{m,T}(u))^{|\alpha|/m} + \tilde{K}_{m,T}(u))$$

for $1 \leq |\alpha| \leq m$ where

$$\tilde{K}_{m,T}(u) = \|u\|_{L^\infty((0,T)\times\Omega)} + \sum_{i=0}^{m-1} \|\partial_t^i u(0)\|_{H^{m-i}(\Omega)}.$$

In particular,

$$\|\partial^\alpha u\|_{L^{2m/|\alpha|}((0,T)\times\Omega)} \leq C(\|u\|_{H^m((0,T)\times\Omega)} + \tilde{K}_{m,T}(u)).$$

Proof. The proof is similar as in the previous theorem, see [13, Proposition 4.5.6] for the details. □

A function F is said to be a *nonlinear function of u of order k* if

$$F(u) = \sum_{l=1}^{N} \sum_{|\alpha_1|+\cdots+|\alpha_l|=k} F_{l,\alpha_1,\ldots,\alpha_l}(u)[\partial^{\alpha_1}u, \ldots, \partial^{\alpha_l}u]$$

where $\alpha_i \in \mathbb{N}_0^2$ and $F_{l,\alpha_1,\ldots,\alpha_l}$ are multilinear mappings depending smoothly on u and there exists $(\alpha_1, \ldots, \alpha_l)$ such that $|\alpha_1| + \cdots + |\alpha_l| = k$ and $F_{l,\alpha_1,\ldots,\alpha_l} \neq 0$.

Theorem 3.11. *Let m be a positive integer and F be a nonlinear function of order $k \leq m$. There exists $C > 0$ which depends continuously on its argument, such that for all $T > 0$ and $u \in H^m((0,T)\times\Omega)\cap L^\infty((0,T)\times\Omega)$ satisfying $\partial_t^j u_{|t=0} \in H^{m-j}(\Omega)$ for $0 \leq j \leq m-1$*

$$\|F(u)\|_{L^{2m/k}((0,T)\times\Omega)} \leq C(\tilde{K}_{m,T}(u))(\|u\|_{H^m((0,T)\times\Omega)} + \tilde{K}_{m,T}(u))^{k/m}.$$

In particular,

$$\|F(u)\|_{L^{2m/k}((0,T)\times\Omega)} \leq \tilde{C}(\tilde{K}_{m,T}(u))(\|u\|_{H^m((0,T)\times\Omega)} + 1)$$

where $\tilde{C} \geq 1$. A similar statement holds for $\psi \in H^m(0,T)$ where $m \in \mathbb{N}$.

Proof. For simplicity we assume that u is scalar valued. First, note that we have $\|F_{l,\alpha}(u)\|_{L^\infty((0,T)\times\Omega)} \leq C(\|u\|_{L^\infty((0,T)\times\Omega)})$. Suppose that $|\alpha_1| + \cdots + |\alpha_l| = k$. Define $p_i = \frac{2m}{|\alpha_i|}$, where we use the convention that $p_i = \infty$ if $\alpha_i = 0$. Then $\sum_{i=1}^{l} \frac{1}{p_i} = \sum_{i=1}^{l} \frac{|\alpha_i|}{2m} = \frac{k}{2m}$. By Hölder's inequality and Theorem 3.10

$$\|\partial^{\alpha_1}u\cdots\partial^{\alpha_l}u\|_{L^{2m/k}((0,T)\times\Omega)} \leq \|\partial^{\alpha_1}u\|_{L^{p_1}((0,T)\times\Omega)}\cdots\|\partial^{\alpha_l}u\|_{L^{p_l}((0,T)\times\Omega)}$$

$$
\begin{aligned}
&\le \prod_{i=1}^{l} C\tilde{K}_{k,T}(u)^{1-2/p_i}((\|u\|_{H^m((0,T)\times\Omega)} + \tilde{K}_{k,T}(u))^{2/p_i} + \tilde{K}_{k,T}(u)^{2/pi}) \\
&\le C^l \tilde{K}_{k,T}(u)^{l-k/m} \prod_{i=1}^{l}((\|u\|_{H^m((0,T)\times\Omega)} + \tilde{K}_{k,T}(u))^{2/p_i} + \tilde{K}_{k,T}(u)^{2/pi}) \\
&\le (2C)^l \tilde{K}_{k,T}(u)^{l-k/m} \prod_{i=1}^{l}(\|u\|_{H^m((0,T)\times\Omega)} + \tilde{K}_{k,T}(u))^{2/p_i} \\
&\le C(\tilde{K}_{k,T}(u))(\|u\|_{H^k((0,T)\times\Omega)} + \tilde{K}_{k,T}(u))^{k/m}.
\end{aligned}
$$

Taking the sum of all terms, we obtain the estimate of the theorem. □

Using classical Sobolev embedding theorems and the identity $u(t) = u(0) + \int_0^t u'(\tau)\,\mathrm{d}\tau$ for a.e. $t \in [0,T]$ and for $u \in W^{1,1}([0,T];X)$ where X is a Banach space, the following estimates can be shown by induction.

Theorem 3.12. *Let m be a nonnegative integer and $T > 0$. There exists a $C > 0$ independent of T such that for all $u \in H^{m+2}((0,T)\times\Omega)$ we have*

$$
\|u\|_{W^{m,\infty}((0,T)\times\Omega)} \le \sum_{k=0}^{m} \|\partial_t^k u_{|t=0}\|_{W^{m-k,\infty}(\Omega)} + C\sqrt{T}\|u\|_{H^{m+2}((0,T)\times\Omega)}.
$$

Theorem 3.13. *Let m be a positive integer. There exists $C > 0$ such that for all $T > 0$ and $u \in H^m(0,T)$ we have*

$$
\|u\|_{H^{m-1}(0,T)} \le C\left(\sum_{i=0}^{m-1} \sqrt{T}|u^{(i)}(0)| + T\|u\|_{H^m(0,T)}\right).
$$

Also, there exists $C > 0$ such that for all $T > 0$ and $u \in H^m((0,T)\times\Omega)$ we have

$$
\|u\|_{H^{m-1}((0,T)\times\Omega)} \le C\left(\sum_{i=0}^{m-1} \sqrt{T}\|\partial_t^i u_{|t=0}\|_{H^{m-i-1}(\Omega)} + T\|u\|_{H^m((0,T)\times\Omega)}\right).
$$

4. Well-Posedness of the Linear PDE Part in Sobolev Spaces

The first step is to prove additional time regularity in Theorem 2.4 in the homogeneous case under additional smoothness assumptions on the frozen coefficient v and on the data f and g. First, we have the following extension result.

Lemma 4.1. *Let $m \ge 3$ be a positive integer and $v \in H^m((0,T)\times\Omega)$ be such that $\|v\|_{H^m((0,T)\times\Omega)} \le R$, $\|v\|_{W^{1,\infty}((0,T)\times\Omega)} \le K$ and the range of v lies on a compact and convex set $\mathcal{K}$ containing 0. Then there exist $\breve{v} \in H^m(\mathbb{R}^2)$ and $(\breve{v}_\epsilon)_{\epsilon>0} \subset \mathscr{C}^\infty(\mathbb{R}^2)$ such that $\breve{v}_{|(0,T)\times\Omega} = v$, $\|\breve{v}_\epsilon - \breve{v}\|_{H^m(\mathbb{R}^2)} \to 0$ as $\epsilon \to 0^+$, and for every $\epsilon > 0$ sufficiently small we have $\|\breve{v}_\epsilon\|_{H^m(\mathbb{R}^2)} \le C(T,R)$, $\|\breve{v}_\epsilon\|_{W^{1,\infty}(\mathbb{R}^2)} \le C(K)$ and the range of $\breve{v}_\epsilon$ lies on a δ-neighborhood of $\mathcal{K}$, for a fixed $\delta > 0$.*

Proof. Let $\theta \in \mathscr{C}_0^\infty([0,\infty);[0,1])$ be such that $\theta(0) = 1$ and $\theta^{(j)}(0) = 0$ for every $1 \leq j \leq m-1$. For $a > 0$ define $\theta_a : \mathbb{R} \to [0,T]$ by

$$\theta_a(s) = \begin{cases} \theta(-s), & \text{if } s < 0, \\ 1, & \text{if } 0 \leq s \leq a, \\ \theta(s-a), & \text{if } s > a. \end{cases}$$

By construction $\theta_a \in H^m(\mathbb{R})$. Let $\tilde{v} \in H^m([-T,2T]\times[-1,2])$ be the extension of v using Seeley's reflection argument [1, p. 84]. The construction of $\tilde{v}$ implies that $\|\tilde{v}\|_{W^{1,\infty}((-T,2T)\times(-1,2))} \leq C(K)$. Define $\breve{v}(t,x) = \theta_T(t)\theta_1(x)\tilde{v}(t,x)$, where $\tilde{v}$ is extended by zero outside $[-T,2T]\times[-1,2]$. Reducing the support of θ if necessary, it can be shown that $\breve{v} \in H^m(\mathbb{R}^2)$ and the range of $\breve{v}$ lies on a $\delta/2$-neighborhood of $\mathcal{K}$. Let $\breve{v}_\epsilon = \rho_\epsilon \star \breve{v} \in \mathscr{C}^\infty(\mathbb{R}^2)$ where ρ_ϵ is a standard mollifier in the variable (t,x) and the star denotes convolution. By definition, $\breve{v} = v$ on $(0,T)\times\Omega$ and $\|\breve{v}_\epsilon - \breve{v}\|_{H^m(\mathbb{R}^2)} \to 0$ as $\epsilon \to 0^+$. The remaning properties can be easily checked using the Sobolev embedding theorem. □

Theorem 4.2. *In the framework of Theorem 2.4, suppose in addition that we have $v \in H^m((0,T)\times\Omega)$ for some integer $m \geq 3$ and $\|v\|_{H^m((0,T)\times\Omega)} \leq R$. If $f \in H^m((0,T)\times\Omega)$ and $g \in H^m(0,T)$ satisfy $(\partial_t^j f)_{|t=0} = 0$ and $(\partial_t^j g)_{|t=0} = 0$ for $0 \leq j \leq m-1$ then the solution u of the IBVP*

$$L_v u = f, \qquad Bu_{|\partial\Omega} = g, \qquad u_{|t=0} = 0 \tag{4.1}$$

lies in $CH^m([0,T]\times\Omega)$ with trace $u_{|\partial\Omega} \in H^m(0,T)$ and $(\partial_t^j u)_{|t=0} = 0$ for $0 \leq j \leq m-1$. Furthermore, there exist $C_m = C_m(\mathcal{K},K,R,T) > 0$ and $\gamma_m = \gamma_m(\mathcal{K},K,R,T) \geq 1$ such that for all $\gamma \geq \gamma_m$ and for all $\tau \in [0,T]$ we have

$$\begin{aligned} &\sum_{|\alpha|\leq m} \gamma^{m-|\alpha|} e^{-\gamma\tau}\|\partial^\alpha u(\tau)\|_{L^2(\Omega)} + \sqrt{\gamma}\|e^{-\gamma t}u\|_{H^m_\gamma((0,\tau)\times\Omega)} + \|u_{|\partial\Omega}\|_{H^m_\gamma(0,\tau)} \\ &\leq C_m\left(\frac{1}{\sqrt{\gamma}}\|e^{-\gamma t}f\|_{H^m_\gamma((0,\tau)\times\Omega)} + \|e^{-\gamma t}g\|_{H^m_\gamma(0,\tau)}\right). \end{aligned} \tag{4.2}$$

Proof. Let $\breve{f} \in H^m(\mathbb{R}\times\Omega)$ and $\breve{g} \in H^m(\mathbb{R})$ be extensions of f and g both vanishing for $t < 0$. Such extensions are possible due to the assumptions on f and g at $t = 0$. Let $\breve{u}$ be the solution of the pure boundary value problem

$$L_{\breve{v}}\breve{u} = \breve{f} \quad \text{in } \mathbb{R}\times\Omega, \qquad B\breve{u}_{|\partial\Omega} = \breve{g} \quad \text{in } \mathbb{R},$$

where $\breve{v}$ is the extension of v in Lemma 4.1. Using the a priori estimate (2.3), it can be shown that this BVP has a unique solution $\breve{u} \in L^2(\mathbb{R}\times\Omega)$ with trace $\breve{u}_{|\partial\Omega} \in L^2(\mathbb{R})$. Furthermore, $\breve{u} \in H^m(\mathbb{R}\times\Omega)$ and $\breve{u}_{|\partial\Omega} \in H^m(\mathbb{R})$. A proof similar to the proof of [2, Theorem 9.21] shows that $u := \breve{u}_{|[0,T]} \in H^m((0,T)\times\Omega)$ is the solution of the homogeneous IBVP (4.1) and it satisfies all the conclusions of the theorem except the energy estimate (4.2) and the additional regularity in time. To see this we use the usual *weak equals strong argument* as suggested in [2]. We will do this step because this will reveal some important remarks that are required in the proof of Theorem 4.5 below. Let ρ_ϵ be a standard mollifier with respect to t

chosen in such a way that $\rho_\epsilon \star \breve{u} =: u_\epsilon$ vanishes for $t < 0$. The notation $R_\epsilon u = \rho_\epsilon \star u$ will also be used. Then $u_\epsilon \in H^m(\Omega; H^{+\infty}(\mathbb{R}))$ where $H^{+\infty}(\mathbb{R}) = \bigcap_{m\in\mathbb{R}} H^m(\mathbb{R})$.

The next step is to show additional regularity in x. Note that

$$A_{\breve{v}}^{-1} L_{\breve{v}} \breve{u} = A_{\breve{v}}^{-1}\partial_t \breve{u} + \partial_x \breve{u} = A_{\breve{v}}^{-1} \breve{f}.$$

Let $\alpha \in \mathbb{N}_0^2$ be a multi-index with $|\alpha| \le m$. Applying ∂^α to both sides of the latter equality gives

$$A_{\breve{v}}^{-1}\partial_t(\partial^\alpha \breve{u}) + \partial_x(\partial^\alpha \breve{u}) = \partial^\alpha(A_{\breve{v}}^{-1}\breve{f}) + [A_{\breve{v}}^{-1}\partial_t, \partial^\alpha]\breve{u}. \tag{4.3}$$

Since the commutator $[A_{\breve{v}}^{-1}\partial_t, \partial^\alpha]$ is of order $|\alpha|$ and $\breve{u} \in H^m(\mathbb{R}\times\Omega)$, it follows that $[A_{\breve{v}}^{-1}\partial_t, \partial^\alpha]\breve{u} \in L^2(\mathbb{R}\times\Omega)$. Mollifying both sides of (4.3) with respect to time yields

$$A_{\breve{v}}^{-1}\partial_t(\partial^\alpha u_\epsilon) + \partial_x(\partial^\alpha u_\epsilon) = R_\epsilon(\partial^\alpha(A_{\breve{v}}^{-1}\breve{f}) + [A_{\breve{v}}^{-1}\partial_t, \partial^\alpha]\breve{u}) + [A_{\breve{v}}^{-1}\partial_t, R_\epsilon]\partial^\alpha \breve{u}. \tag{4.4}$$

Let F_ϵ be the right hand side of (4.4). Solving for $\partial_x(\partial^\alpha u_\epsilon)$ shows that $\partial_x(\partial^\alpha u_\epsilon) \in L^2(\mathbb{R}\times\Omega)$. Therefore, $u_\epsilon \in H^{m+1}(\Omega; H^{+\infty}(\mathbb{R})) \subset H^{m+1}(\mathbb{R}\times\Omega)$. In other words, mollification in time gives additional regularity in time, and together with the PDE one has additional regularity in space.

As $\epsilon \to 0$ it holds that

$$L_{\breve{v}}\partial^\alpha u_\epsilon \to L_{\breve{v}}\partial^\alpha \breve{u}, \qquad \text{in } L^2(\mathbb{R}\times\Omega). \tag{4.5}$$

Indeed, we have $R_\epsilon(\partial^\alpha(A_{\breve{v}}^{-1}\breve{f}) + [A_{\breve{v}}^{-1}\partial_t, \partial^\alpha]\breve{u}) \to \partial^\alpha(A_{\breve{v}}^{-1}\breve{f}) + [A_{\breve{v}}^{-1}\partial_t, \partial^\alpha]\breve{u}$ and $[A_{\breve{v}}^{-1}\partial_t, R_\epsilon]\partial^\alpha\breve{u} \to 0$ both in $L^2(\mathbb{R}\times\Omega)$, where we used the extension of Friedrichs theorem [2, Theorem C.14] for the latter. Now (4.5) follows from

$$[A_{\breve{v}}^{-1}\partial_t, \partial^\alpha]\breve{u} = [A_{\breve{v}}^{-1}L_{\breve{v}}, \partial^\alpha]\breve{u} = A_{\breve{v}}^{-1}L_{\breve{v}}\partial^\alpha\breve{u} - \partial^\alpha(A_{\breve{v}}^{-1}\breve{f})$$

since $[\partial_x, \partial^\alpha]\breve{u} = 0$ and $L_{\breve{v}}\breve{u} = \breve{f}$.

Applying the a priori estimate (3.25) to $u_\epsilon - u_{\epsilon'} \in H^{m+1}(\mathbb{R}\times\Omega)$ one obtains

$$\sum_{|\alpha|\le m} \gamma^{m-|\alpha|} e^{-\gamma T} \sup_{\tau\in[0,T]} \|\partial^\alpha(u_\epsilon - u_{\epsilon'})(\tau)\|_{L^2(\Omega)} + \|(u_\epsilon - u_{\epsilon'})_{|\partial\Omega}\|_{H_\gamma^m(0,T)}$$
$$\le C_m \left(\frac{1}{\sqrt{\gamma}} \|e^{-\gamma t} L_{\breve{v}}(u_\epsilon - u_{\epsilon'})\|_{H_\gamma^m((0,T)\times\Omega)} + \|e^{-\gamma t} B(u_\epsilon - u_{\epsilon'})_{|\partial\Omega}\|_{H_\gamma^m(0,T)} \right).$$

Since $g_\epsilon = R_\epsilon \breve{g}$ vanishes for $t < 0$ and $B(u_\epsilon)_{|\partial\Omega} = R_\epsilon(B\breve{u}_{|\partial\Omega}) = g_\epsilon$ we have

$$\|e^{-\gamma t} B(u_\epsilon - u_{\epsilon'})_{|\partial\Omega}\|_{H_\gamma^m(0,T)} \le \|e^{-\gamma t}(g_\epsilon - g_{\epsilon'})_{|\partial\Omega}\|_{H_\gamma^m(\mathbb{R})} \to 0$$

as $\epsilon, \epsilon' \to 0$. On the other hand, since $u_\epsilon - u_{\epsilon'}$ vanish for $t < 0$ and the function $t \mapsto e^{-\gamma t}$ is uniformly bounded on compact intervals we have

$$\|e^{-\gamma t} L_{\breve{v}}(u_\epsilon - u_{\epsilon'})\|_{H_\gamma^m((0,T)\times\Omega)} \le C\|A_{\breve{v}}\|_{H_\gamma^m(\mathbb{R}\times\Omega)} \|A_{\breve{v}}^{-1} L_{\breve{v}}(u_\epsilon - u_{\epsilon'})\|_{H_\gamma^m(\mathbb{R}\times\Omega)}.$$

Using commutators we can rewrite

$$\partial^\alpha(A_{\breve{v}}^{-1}L_{\breve{v}}(u_\epsilon - u_{\epsilon'})) = [\partial^\alpha, A_{\breve{v}}^{-1}L_{\breve{v}}](u_\epsilon - u_{\epsilon'}) - A_{\breve{v}}^{-1}L_{\breve{v}}\partial^\alpha(u_\epsilon - u_{\epsilon'}).$$

Since $u_\epsilon \to \breve{u}$ in $H^m(\mathbb{R}\times\Omega)$ and $[\partial^\alpha, A_{\breve{v}}^{-1}L_{\breve{v}}]$ is of order $|\alpha| \le m$, the commutator term on the right hand side tends to zero in $L^2(\mathbb{R}\times\Omega)$ as $\epsilon, \epsilon' \to 0$. On the other hand, the second term also tends to zero in $L^2(\mathbb{R}\times\Omega)$ according to (4.5). Therefore, from (4.6) we can see that $(u_{1/n})_n$ and $((u_{1/n})_{|\partial\Omega})_n$ are Cauchy sequences in $CH^m([0,T]\times\Omega)$ and $H^m(0,T)$, respectively. Their limits are u and $u_{|\partial\Omega}$ since

$u_{1/n} \to u$ in $CL^2([0,T]\times\Omega)$ and $(u_{1/n})_{|\partial\Omega} \to u_{|\partial\Omega}$ in $L^2(0,T)$ by a similar argument as in the proof of [2, Theorem 9.19].

It remains to establish the energy estimate (4.2). First, let us note that

$$\partial^\alpha L_{\breve{v}} u_\epsilon = [\partial^\alpha, L_{\breve{v}}]u_\epsilon + L_{\breve{v}}\partial^\alpha u_\epsilon \to [\partial^\alpha, L_{\breve{v}}]\breve{u} + L_{\breve{v}}\partial^\alpha \breve{u} = \partial^\alpha \breve{f} \tag{4.6}$$

in $L^2(\mathbb{R}\times\Omega)$. Thus $L_{\breve{v}}u_\epsilon \to \breve{f}$ in $H^m(\mathbb{R}\times\Omega)$. Applying the a priori estimate (3.25) to $u_{1/n} \in H^{m+1}(\mathbb{R}\times\Omega)$ and letting $n\to\infty$ proves (4.2). □

Now, we will consider the IBVP with nonzero initial condition. For this, one needs compatibility conditions which we are going to state. Given sufficiently smooth functions f and u_0 define recursively the functions $u_i : \Omega \to \mathbb{R}^n$ by

$$u_i(x) = \partial_t^{i-1} f(0,x) - \sum_{l=0}^{i-1} \binom{i-1}{l} \partial_t^l A(v(0,x))\partial_x u_{i-1-l}(x), \quad x\in\Omega. \tag{4.7}$$

The data (u_0, f, g) are said to be *compatible up to order p* if

$$Bu_{i|\partial\Omega} = \partial_t^i g(0), \qquad i = 0,\dots,p.$$

By the embedding

$$H^m((0,T)\times\Omega) \hookrightarrow H^{j+1}((0,T);H^{m-j-1}(\Omega)) \hookrightarrow C^j([0,T];H^{m-j-1}(\Omega))$$

for $0\le j\le m-1$, we have $\partial_t^j v_{|t=0} \in H^{m-j-1}(\Omega)$. However, stronger assumptions are needed for these traces in the general IBVP as included in the following theorem.

Theorem 4.3. *Consider the framework of Theorem 2.4 and suppose that v satisfies the conditions of Theorem 4.2. Suppose in addition that $\partial_t^j v_{|t=0} \in H^{m-j}(\Omega)$ for all $0\le j\le m-1$. If the data*

$$(u_0, f, g) \in H^{m+1/2}(\Omega) \times H^m((0,T)\times\Omega) \times H^m(0,T)$$

is compatible up to order $m-1$, then the initial boundary value problem

$$L_v u = f, \qquad Bu_{|\partial\Omega} = g, \qquad u_{|t=0} = u_0 \tag{4.8}$$

has a unique solution $u \in CH^m([0,T]\times\Omega)$ and $u_{|\partial\Omega} \in H^m(0,T)$.

Remark 4.4. *The proof of this theorem is contained in the second step of the proof of* [2, Theorem 9.22] *using an appropriate lifting result. This is where the additional regularity for u_0 is needed. The proof shows that the solution takes the form $u = u_{a|[0,T]} + u_h$ where $u_a \in H^{m+1}(\mathbb{R}\times\Omega)$ and u_h is a solution of an IBVP with zero initial data. Therefore, according to the proof of Theorem 4.2, there exists $(u_n)_n \subset H^{m+1}((0,T)\times\Omega)$ such that*

$$\begin{aligned} u_n &\to u, & &\text{in} \quad CH^m([0,T]\times\Omega),\\ (u_n)_{|\partial\Omega} &\to u_{|\partial\Omega}, & &\text{in} \quad H^m(0,T),\\ L_v u_n &\to L_v u, & &\text{in} \quad H^m(0,T). \end{aligned} \tag{4.9}$$

The extra regularity imposed on the data u_0 is not necessary since one can have the same result even when it is only in $H^m(\Omega)$. This is the content of the following theorem.

Theorem 4.5. *The conclusions of Theorem 4.3 still hold even for initial data $u_0 \in H^m(\Omega)$.*

To prove this theorem, one requires the following a priori estimate. This is similar to the one given in Lemma 3.7, but with additional terms for the nonzero initial condition.

Lemma 4.6. *For every $v \in H^m((0,T)\times\Omega)$ satisfying the conditions in Theorem 4.3 and for every $u \in H^{m+1}((0,T)\times\Omega)$ we have*

$$\|u\|_{CH^m([0,T]\times\Omega)} + \|u_{|\partial\Omega}\|_{H^m(0,T)} \\ \leq C\left(\|L_v u\|_{H^m((0,T)\times\Omega)} + \|Bu_{|\partial\Omega}\|_{H^m(0,T)} + \sum_{i=0}^{m}\|\partial_t^i u_{|t=0}\|_{H^{m-i}(\Omega)}\right),$$

where $C>0$ depends only on $T,\mathcal{K},K,R$ and $\|\partial_t^j v_{|t=0}\|_{H^{m-j}(\Omega)}$ for $0\le j\le m-1$.

Proof. In the following proof, $C>0$ will be a generic constant as in the statement of the lemma independent of $\tau\in[0,T]$. As before, let $f = L_v u$ and $g = Bu_{|\partial\Omega}$. We will use the following a priori estimate

$$\|w(\tau)\|_{L^2(\Omega)} + \frac{1}{\sqrt{\tau}}\|w\|_{L^2((0,\tau)\times\Omega)} + \|w_{|\partial\Omega}\|_{L^2(0,\tau)} \\ \leq C(\|w_{|t=0}\|_{L^2(\Omega)} + \sqrt{\tau}\|L_v w\|_{L^2((0,\tau)\times\Omega)} + \|Bw_{|\partial\Omega}\|_{L^2(0,\tau)}) \tag{4.10}$$

which holds for all $\tau\in(0,T]$ and for all $w\in H^1((0,T)\times\Omega)$, where $C = C(\mathcal{K},K) > 0$. By a standard density argument, it is enough to prove the a priori estimate for $u \in \mathscr{D}([0,T]\times\overline{\Omega})$. Applying ∂_t^j for $j=0,\dots,m$ to the equality $L_v u = f$, we obtain $L_v\partial_t^j u = f_j := A(v)\partial_t^j(A(v)^{-1}f) - A(v)[\partial_t^j, A(v)^{-1}L_v]u$ and similarly $B(\partial_t^j u)_{|\partial\Omega} = \partial_t^j g$ for $j=0,\dots,m$. Taking $w = \partial_t^j u$ in (4.10) we have

$$\|\partial_t^j u(\tau)\|_{L^2(\Omega)} + \frac{1}{\sqrt{\tau}}\|\partial_t^j u\|_{L^2((0,\tau)\times\Omega)} + \|\partial_t^j(u_{|\partial\Omega})\|_{L^2(0,\tau)} \\ \leq C(\|\partial_t^j u_{|t=0}\|_{L^2(\Omega)} + \sqrt{\tau}\|f_j\|_{L^2((0,\tau)\times\Omega)} + \|\partial_t^j g\|_{L^2(0,\tau)}). \tag{4.11}$$

We are going to estimate each term on the right hand side of this inequality. Expanding the commutator in f_j for $j\ge 1$ we have

$$A(v)[\partial_t^j, A(v)^{-1}L_v]u = A(v)\sum_{1\le l\le j} c_{ij}\partial_t^{l-1}(\mathrm{d}A(v)^{-1}\partial_t v)\partial_t^{j-l}(\partial_t u),$$

where $\mathrm{d}A$ is the first order differential of A and c_{ij} are constants. Let us estimate the L^2-norm of each term in the above sum. If $j=1$, then we immediately have the estimate $\|(\mathrm{d}A(v)^{-1}\partial_t v)\partial_t u\|_{L^2((0,\tau)\times\Omega)} \le C\|\partial_t u\|_{L^2((0,\tau)\times\Omega)}$. Suppose that $j\ge 2$. Then Hö's inequality implies that

$$\|\partial_t^{l-1}(\mathrm{d}A(v)^{-1}\partial_t v)\partial_t^{j-l}(\partial_t u)\|_{L^2((0,\tau)\times\Omega)} \\ \le \|\partial_t^{l-1}(\mathrm{d}A(v)^{-1}\partial_t v)\|_{L^{2(j-1)/(l-1)}((0,\tau)\times\Omega)}\|\partial_t^{j-l}(\partial_t u)\|_{L^{2(j-1)/(j-l)}((0,\tau)\times\Omega)}.$$

Since $\partial_t^{l-1}(\mathrm{d}A(v)^{-1}\partial_t v)$ is a nonlinear function of $\partial_t v$ with order $l-1$ the first factor can be estimated using Theorem 3.11 by

$$\|\partial_t^{l-1}(\mathrm{d}A(v)^{-1}\partial_t v)\|_{L^{2(j-1)/(l-1)}((0,\tau)\times\Omega)} \le C(\tilde{K}_{j-1,\tau}(\partial_t v))(\|\partial_t v\|_{H^{j-1}((0,\tau)\times\Omega)} + 1).$$

The term involving u can also be estimated using Theorem 3.10

$$\|\partial_t^{j-l}(\partial_t u)\|_{L^{2(j-1)/(j-l)}((0,\tau)\times\Omega)} \le C(\|\partial_t u\|_{H^{j-1}((0,\tau)\times\Omega)} + \tilde{K}_{j-1,\tau}(\partial_t u)).$$

Theorem 3.12 and the Sobolev embedding $H^{k+1}(\Omega) \hookrightarrow W^{k,\infty}(\Omega)$ imply

$$\tilde{K}_{j-1,\tau}(\partial_t u) \leq C\left(\sqrt{\tau}\|u\|_{H^3((0,\tau)\times\Omega)} + \sum_{i=0}^{m-1}\|\partial_t^i u_{|t=0}\|_{H^{m-i}(\Omega)}\right).$$

Furthermore, we have $\|A(v)\partial_t^j(A(v)^{-1}f)\|_{L^2((0,\tau)\times\Omega)} \leq C\|f\|_{H^m((0,T)\times\Omega)}$. Combining all our estimates and using $\tau \leq T$, we deduce that

$$\|f_j\|_{L^2((0,\tau)\times\Omega)} \leq C\left(\|f\|_{H^m((0,T)\times\Omega)} + \|u\|_{H^m((0,\tau)\times\Omega)} + \sum_{i=0}^{m}\|\partial_t^i u_{|t=0}\|_{H^{m-i}(\Omega)}\right).$$

Therefore we obtain the a priori estimate

$$\sum_{j=0}^{m}\|\partial_t^j u(\tau)\|_{L^2(\Omega)} + \|u_{|\partial\Omega}\|_{H^m(0,\tau)} \tag{4.12}$$
$$\leq C\left(\|f\|_{H^m((0,T)\times\Omega)} + \|g\|_{H^m(0,T)} + \sum_{i=0}^{m}\|\partial_t^i u_{|t=0}\|_{H^{m-i}(\Omega)} + \|u\|_{H^m((0,\tau)\times\Omega)}\right).$$

For convenience we denote by $N(u)$ the term on the right hand side of (4.12).

The next step is to estimate the mixed derivatives. We proceed by an induction argument to prove that

$$\|\partial_x^k\partial_t^j u(\tau)\|_{L^2(\Omega)} \leq N(u) \tag{4.13}$$

for all $k+j \leq m$. The basis step $k=0$ is given by (4.12). Before proceeding to the induction step, we prove the estimate in the separate case where $k=j=1$. The PDE gives us

$$\partial_x\partial_t u(\tau) = \partial_t(A(v(\tau))^{-1}f(\tau)) - \partial_t(A(v(\tau))^{-1})\partial_t u(\tau) - A(v(\tau))^{-1}\partial_t^2 u(\tau).$$

The estimates on time-derivatives we have shown above and the Sobolev embedding theorem imply

$$\|\partial_x\partial_t u(\tau)\|_{L^2(\Omega)} \leq N(u). \tag{4.14}$$

Now we go to the induction step. Suppose that (4.13) is true for k and j such that $k+j \leq m$. The PDE gives us

$$\partial_x^{k+1}\partial_t^j u = \partial_x^k\partial_t^j(A(v)^{-1}f) - \partial_x^k\partial_t^j(A(v)^{-1}\partial_t u)$$

for $k+1+j \leq m$ and $k \geq 0$. On one hand, by the Sobolev embedding theorem

$$\|\partial_x^k\partial_t^j(A(v(\tau))^{-1}f(\tau))\|_{L^2(\Omega)} \leq C\|f\|_{H^m((0,T)\times\Omega)}$$

for all $\tau \in [0,T]$. On the other hand, Leibniz's rule gives us

$$\|\partial_x^k\partial_t^j(A(v(\tau))^{-1}\partial_t u(\tau))\|_{L^2(\Omega)} \leq \sum_{l=0}^{k}\sum_{i=0}^{j} c_{li}\|\partial_x^{k-l}\partial_t^{j-i}A(v(\tau))^{-1}\partial_x^l\partial_t^{i+1}u(\tau)\|_{L^2(\Omega)}$$

for some constants c_{li}.

Let us consider separate cases. If $k-l+j-i \leq m-2$ then for all $\tau \in [0,T]$

$$\begin{aligned}
&\|\partial_x^{k-l}\partial_t^{j-i}A(v(\tau))^{-1}\partial_x^l\partial_t^{i+1}u(\tau)\|_{L^2(\Omega)}\\
&\quad\leq \|\partial_x^{k-l}\partial_t^{j-i}A(v(\tau))^{-1}\|_{L^\infty(\Omega)}\|\partial_x^l\partial_t^{i+1}u(\tau)\|_{L^2(\Omega)}\\
&\quad\leq C\|\partial_x^{k-l}\partial_t^{j-i}A(v)^{-1}\|_{H^2((0,T)\times\Omega)}\|\partial_x^l\partial_t^{i+1}u(\tau)\|_{L^2(\Omega)} \leq N(u)
\end{aligned}$$

where the last inequality is due to the induction hypothesis. If $k-l+j-i=m-1$ then $k+j=m-1$ and $i=l=0$ and therefore applying (4.14)

$$\begin{aligned}
&\|\partial_x^{k-l}\partial_t^{j-i}A(v(\tau))^{-1}\partial_x^l\partial_t^{i+1}u(\tau)\|_{L^2(\Omega)}\\
&\quad\le \|\partial_x^{k-l}\partial_t^{j-i}A(v(\tau))^{-1}\|_{L^2(\Omega)}\|\partial_t u(\tau)\|_{L^\infty(\Omega)}\\
&\quad\le C\|\partial_x^{k-l}\partial_t^{j-i}A(v)^{-1}\|_{H^1((0,T);L^2(\Omega))}(\|\partial_t u(\tau)\|_{L^2(\Omega)}+\|\partial_x\partial_t u(\tau)\|_{L^2(\Omega)})\\
&\quad\le N(u)
\end{aligned}$$

for all $\tau\in[0,T]$. Taking the sum completes the proof of the induction.

Combining the estimates for the time derivatives and the mixed derivatives gives

$$\sum_{|\beta|\le m}\|\partial^\beta u(\tau)\|_{L^2(\Omega)}+\|u_{|\partial\Omega}\|_{H^m(0,T)}\le N(u). \tag{4.15}$$

Squaring this inequality and applying Gronwall's inequality give the estimate stated in the lemma. □

Proof.[Proof of Theorem 4.5] It can be shown that there exists a sequence of more regular functions $(u_0^k)_k\subset H^{m+1/2}(\Omega)$ such that $u_0^k\to u_0$ in $H^m(\Omega)$ and the data (u_0^k,f,g) is still compatible up to order $m-1$ for all k, see for instance [18]. Let u_k be the solution of the corresponding initial boundary value problem with data (u_0^k,f,g) given by Theorem 4.3. Then the difference $w=u_k-u_j$ satisfies

$$L_v w=0 \quad\text{in } (0,T)\times\Omega,\qquad Bw_{|\partial\Omega}=0\quad\text{in } (0,T),\qquad w_{|t=0}=u_0^k-u_0^j\quad\text{in } \Omega.$$

According to Remark 4.4, there exists a sequence $w_n\in H^{m+1}((0,T)\times\Omega)$ such that $w_n\to w$ in $CH^m([0,T]\times\Omega)$, $L_vw_n\to 0$ in $H^m((0,T)\times\Omega)$ and $(w_n)_{|\partial\Omega}\to 0$ in $H^m(0,T)$. Thus, applying the a priori estimate in the previous lemma to w_n and passing to the limit $n\to\infty$, we have

$$\begin{aligned}
&\|u_k-u_j\|_{CH^m([0,T]\times\Omega)}+\|(u_k)_{|\partial\Omega}-(u_j)_{|\partial\Omega}\|_{H^m(0,T)}\\
&\quad\le C\sum_{i=0}^m\|\partial_t^iu_k(0)-\partial_t^iu_j(0)\|_{H^{m-i}(\Omega)}.
\end{aligned}$$

By recursion we have $\partial_t^iu_k(0)=u_{k,i}\to u_i$ in $H^{m-i}(\Omega)$, where $u_{k,i}$ are the functions defined recursively in (4.7) with u_0^k as the initial term. Thus, $(u_k)_k$ and $((u_k)_{|\partial\Omega})_k$ are Cauchy sequences in $CH^m([0,T]\times\Omega)$ and $H^m(0,T)$, respectively, and let u and $\tilde u$ be their limits. Since $u_k\to u$ in $H^1((0,T)\times\Omega)$, the continuity of the trace operator implies $(u_k)_{|\partial\Omega}\to u_{|\partial\Omega}$ in $L^2(0,T)$ and thus $\tilde u=u_{|\partial\Omega}$. Passing to the limit $k\to\infty$ in the IBVP satisfied by u_k, we can see that u is the required solution. □

Remark 4.7. *Given a positive integer k, using Remark 4.4, there exists a function $u_k^k\in H^{m+1}((0,T)\times\Omega)$ such that $\|u_k^k-u_k\|_{CH^m([0,T]\times\Omega)}<\frac{1}{k}$ and $\|(u_k^k)_{|\partial\Omega}-(u_k)_{|\partial\Omega}\|_{H^m(0,T)}<\frac{1}{k}$ where u_k is the solution corresponding to the initial data u_0^k in the proof of the previous theorem. By the triangle inequality we have $u_k^k\to u$ in $CH^m([0,T]\times\Omega)$ and $(u_k^k)_{|\partial\Omega}\to u_{|\partial\Omega}$ in $H^m(0,T)$. Moreover, since $L_vu_k^k-L_vu_j^j=F_k-F_j$ where $F_k\to f$ in $H^m((0,T)\times\Omega)$, see (4.6) for instance, it follows that $(L_vu_k^k)_k$ is a Cauchy sequence in $H^m((0,T)\times\Omega)$. Since $L_vu_k^k\to L_vu$ in $L^2((0,T)\times\Omega)$*

we have $L_v u_k^k \to L_v u$ in $H^m((0,T)\times\Omega)$. This implies that the a priori estimate in Lemma 4.6 holds for the solution u of the initial boundary value problem (4.8).

5. The Nonlinear System

The existence of smooth solutions requires and also implies *compatibility conditions* between the initial data and the boundary data. These are additional constraints for the initial and boundary data. The compatibility conditions are obtained by (a) formally differentiating the PDE with respect to time, (b) evaluate the time derivatives at $t = 0$ and use the initial data to compute the spatial derivatives and (c) differentiate the boundary conditions, use the information in (b) and evaluate them along the boundary. The result in (c) will be the compatibility conditions.

Suppose that u and h are C^p-functions satisfying $\partial_t u + A(u)\partial_x u = f(u)$ in $(t,x) \in (0,T)\times\Omega$ and $\dot{h} = H(h, q, u_{|\partial\Omega})$ in $t \in (0,T)$, respectively. Then by Leibniz's rule

$$\partial_t^i u = -\sum_{l=0}^{i-1} \binom{i-1}{l} \partial_t^l(A(u))\partial_x\partial_t^{i-1-l}u + \partial_t^{i-1}f(u), \qquad i = 1,\ldots,p.$$

The terms $\partial_t^l(A(u))$ and $\partial_t^{i-1}f(u)$ can be expanded with the aid of Faá di Bruno's formula. If u is C^i up to the boundary then we must have

$$B_y\partial_t^i u(0,y) = D_t^i b_y(p_y(t), h(t))_{|t=0}, \qquad y = 0,1.$$

We can use Faá di Bruno's formula to expand the right hand term and then use the ODE satisfied by h. Thus, we are led to the following definitions. Given a sufficiently smooth function $u_0 : \Omega \to \mathbb{R}^n$ with values in $\mathcal{U}$, recursively define the function $u_i : \Omega \to \mathbb{R}^n$ as

$$\begin{aligned} u_1 &= -A(u_0)\partial_x u_0 + f(u_0) \\ u_i &= -\sum_{l=0}^{i-1}\sum_{k=1}^{l}\sum_{l_1+\cdots+l_k=l}\binom{i-1}{l} c_{l_1,\ldots,l_k}(\mathrm{d}^k A)(u_0)[u_{l_1},\ldots,u_{l_k}]\partial_x u_{i-1-l} \\ &\quad -A(u_0)\partial_x u_{i-1} + \sum_{k=1}^{i-1}\sum_{l_1+\cdots+l_k=i-1} c_{l_1,\ldots,l_k}(\mathrm{d}^k f)(u_0)[u_{l_1},\ldots,u_{l_k}], \\ &\quad \text{for } i = 2,\ldots,p \end{aligned} \tag{5.1}$$

where $\mathrm{d}^k F$ denotes the kth order differential of a smooth function F viewed as a multilinear form. Here, $c_{l_1,\ldots,l_k}$ are nonnegative coefficients which depend only on i.

Given $h_0 \in \mathcal{H}$ define $\eta = (h_0, q(0), u_0(0), u_0(1))$,

$$\begin{aligned} h_1 &= H(\eta) \\ h_i &= \sum_{k=1}^{i-1}\sum_{l_1+\cdots+l_k=i-1} c_{l_1,\ldots,l_k}(\mathrm{d}^k H)(\eta)[z_{l_1},\ldots,z_{l_k}], \qquad \text{for } i = 2,\ldots,p-1, \end{aligned} \tag{5.2}$$

where $z_j = (h_j, q^{(j)}(0), u_j(0), u_j(1))^\top$ and the u_j are defined according to (5.1). For $y = 0,1$, define

$$C_{y,0} = b_y(p_y(0), h_0)$$

$$C_{y,i} \;=\; \sum_{k=1}^{i}\sum_{l_1+\cdots+l_k=i} c_{l_1,\ldots,l_k}(\mathrm{d}^m b_y)(p_y(0),h_0)[w_{l_1,y},\ldots,w_{l_k,y}]$$

where $w_{k,y} = (p_y^{(k)}(0), h_k)^\top$. With these notations, we are now in position to state the necessary compatibility conditions.

(CC$_m$) Let $m \geq 1$ be an integer and $T > 0$. The data

$$(u_0, h_0, p, q) \in H^m(0,1) \times \mathcal{H} \times H^m(0,T) \times H^m(0,T)$$

are said to be compatible up to order $m-1$ if $B_y u_i(y) = C_{y,i}$ for all $i = 0,\ldots,m-1$ and $y = 0,1$.

We are going to state the regularity properties of the functions u_i, $i = 1,\ldots,m$, defined in (5.1) for a given $u_0 \in H^m(\Omega)$.

Lemma 5.1. *Let $s \geq 1$ be an integer. Let $u_0 \in H^s(\Omega)$ with range lying in a compact subset $\mathcal{K}$ of $\mathcal{U}$ and $u_1,\ldots,u_s$ be defined as in (5.1). Then $u_i \in H^{s-i}(\Omega)$ for all $1 \leq i \leq s$. Moreover, there exist continuous functions $C_i : [0,\infty) \to [0,\infty)$ such that*

$$\|u_i\|_{H^{s-i}(\Omega)} \leq C_i(\|u_0\|_{H^s(\Omega)}), \qquad 1 \leq i \leq s. \tag{5.3}$$

Proof. We follow the proof in [2, pp. 322–323] and proceed by strong induction on i. In this proof, all Sobolev spaces are defined in $\Omega = (0,1)$. By redefining A and f in (1.1) outside a neighborhood of $\mathcal{K}$, one can assume without loss of generality that A and f are $\mathscr{C}^\infty$ in $\mathbb{R}^n$. From the assumption that $f(0) = 0$, we have $f(u_0) \in H^s$ by Proposition 3.3. We rewrite

$$A(u_0)\partial_x u_0 = (A(u_0) - A(0))\partial_x u_0 + A(0)\partial_x u_0.$$

Propostion 3.3 can now be applied so that $A(u_0) - A(0) \in H^s$, since Ω is bounded. Thus, $(A(u_0) - A(0))\partial_x u_0 \in H^{s-1}$ by Proposition 3.1. Moreover we have

$$\begin{aligned}
\|A(u_0)\partial_x u_0\|_{H^{s-1}} &\leq C\|A(u_0) - A(0)\|_{H^s}\|\partial_x u_0\|_{H^{s-1}} + |A(0)|\|\partial_x u_0\|_{H^{s-1}} \\
&\leq C(\|u_0\|_{L^\infty})\|u_0\|_{H^s}\|\partial_x u_0\|_{H^{s-1}} + |A(0)|\|\partial_x u_0\|_{H^{s-1}} \\
&\leq C(\|u_0\|_{H^s})
\end{aligned}$$

by the Sobolev embedding $H^s \hookrightarrow L^\infty$. The H^{s-1}-norm of $f(u_0)$ can be estimated similarly. Thus, $u_1 \in H^{s-1}$ and (3.2) holds for $i = 1$.

Suppose that for $1 \leq i \leq s$ we have $u_k \in H^{s-k}$ and $\|u_k\|_{H^{s-k}} \leq C_k(\|u_0\|_{H^s})$ holds for $k = 0,1,\ldots,i-1$. We show that $u_i \in H^{s-i}$ and (5.3) holds. A similar argument as above yields $A(u_0)\partial_x u_{i-1} \in H^{s-i}$. The triple sum in u_i contains terms of the form

$$\varrho(u_0)u_{l_1,j_1}\cdots u_{l_k,j_k}\partial_x u_{i-1-l,\sigma} \tag{5.4}$$

where $l_1 + \cdots + l_k = l$ for $k = 1,\ldots,l$, with $l = 1,\ldots,i-1$ and for some $\varrho \in \mathscr{C}^\infty$. Here u_{l_1,j_1} denotes the j_1th component of the vector u_{l_1}. By the induction hypothesis $u_{l_1,j_1} \in H^{s-l_1},\ldots,u_{l_k,j_k} \in H^{s-l_k}$, $\partial_x u_{i-1-l,\sigma} \in H^{s-(i-1-l)} \subset H^{s-i+l}$ and $\varrho(u_0) \in H^s$. Since

$$\min(s, s-l_1,\ldots,s-l_k, s-i+l) \geq \min(s-l, s-i+1) = s-i+1$$

and since $ks \geq s > 1/2$

$$s + (s-l_1) + \cdots + (s-l_k) + (s-i+l) = (k+2)s - i > s - i + 1/2$$

it follows from the remark succeeding Proposition 3.1 that (5.4) lies in H^{s-i}.

Similarly, the double sum in u_i contains terms of the from

$$\vartheta(u_0)u_{l_1,j_1}\cdots u_{l_k,j_k} \tag{5.5}$$

where $l_1+\cdots+l_k = i-1$ for some $\vartheta \in \mathscr{C}^\infty$. Because

$$\min(s, s-l_1, \ldots, s-l_k) \geq s-(i-1) = s-i+1$$

and

$$s+(s-l_1)+\cdots+(s-l_k) = (k+1)s-(i-1) > s-i+1/2$$

the terms of the form (5.5) belong to H^{s-i}. By collecting all our observations, we obtain that $u_i \in H^{s-i}$. The estimate $\|u_i\|_{H^{s-i}(\Omega)} \leq C_i(\|u_0\|_{H^s(\Omega)})$ can be shown from the definition of u_i, the induction hypothesis, and (3.2). □

Theorem 5.2. *Let $m \geq 3$, $T_0 > 0$ and $(u_0, h_0, p, q) \in H^m(\Omega) \times \mathcal{H} \times H^m(0,T_0) \times H^m(0,T_0)$. Assume that the range of u_0 lies in a compact and convex set $\mathcal{K}_0 \subset \mathcal{K}_1 \subset \mathcal{U}$, $h_0 \in \mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{H}$ where $\mathcal{K}_1$ and $\mathcal{G}_1$ are also compact and convex sets containing neighborhoods of $\mathcal{K}_0$ and $\mathcal{G}_0$, respectively, and moreover $\|u_0\|_{H^m(\Omega)} \leq M$. Suppose that* **(FS)**, **(D)**, **(UKL)** *and* **(CC$_m$)** *hold. Then there exists $T \in (0, T_0)$ depending only on $(\mathcal{K}_1, \mathcal{G}_1, M)$ such that the nonlinear system (1.1) has a unique solution $(u,h) \in CH^m([0,T]\times\Omega) \times H^m(0,T)$. Furthermore, $u_{|\partial\Omega} \in H^m(0,T)$ and consequently $h \in H^{m+1}(0,T)$.*

Proof. The proof is a Picard iteration scheme using the linear well-posedness theory of Section 4.

Step 1. Existence of initial functions for the iteration scheme. In this step, we find $v \in CH^m([0,T_0]\times\Omega)$ such that $\partial_t^j v(0) = u_j$ for all $0 \leq j \leq m-1$. The following construction is inspired by [8], [22]. Let $g \in H^m(0,T_0)$ be such that $\partial_t^j g(0) = h_j$ for all $0 \leq j \leq m-1$ where h_j are the constants defined from (5.2) and $\|g\|_{H^m(0,T_0)} \leq C\sum_{j=0}^{m-1}|h_j|$. This is possible by the trace theorem. Consider the initial boundary value problem

$$v_t + A(u_0)v_x = f(u_0) + G, \qquad Bv_{|\partial\Omega} = b(p,g), \qquad v(0) = u_0, \tag{5.6}$$

for some function $G \in H^m((0,T_0)\times\Omega)$ to be specified below.

The existence result Theorem 4.5 for linear systems shows that the system (5.6) has a unique solution $v \in CH^m([0,T_0]\times\Omega)$ with $v_{|\partial\Omega} \in H^m(0,T_0)$ *provided* that the data $(u_0, f(u_0)+G, b(p,g))$ is compatible up to order $m-1$ for the linear system (5.6). To ensure this, let v_j for $0 \leq j \leq m-1$ be $\partial_t^j v_{|t=0}$ that is obtained from (5.6) by formal differentiation. Similarly, let $\tilde{v}_j$ be $\partial_t^j \tilde{v}_{|t=0}$ that is obtained from

$$\tilde{v}_t + A(u_0)\tilde{v}_x = f(u_0), \qquad \tilde{v}(0) = u_0, \tag{5.7}$$

by differentiating formally. The equation $v_j = u_j$ holds if

$$\partial_t^j G(0) = u_j - \tilde{v}_j \in H^{m-j}(\Omega) \subset H^{m-1-j+1/2}(\Omega), \qquad 0 \leq j \leq m-1. \tag{5.8}$$

By the trace theorem there exists $G \in H^m((0,T_0)\times\Omega)$ such that (5.8) holds and

$$\|G\|_{H^m((0,T_0)\times\Omega)} \leq C(\|u_0\|_{H^m(\Omega)}) \tag{5.9}$$

for some continuous function $C : [0,\infty) \to [0,\infty)$. This estimate follows from the trace theorem and a result similar to Lemma 5.1 applied to the PDEs (5.6) and

(5.7). Since $B_y v_j(0,y) = B_y u_j(0,y) = C_{y,j}$ for $y = 0,1$ and $0 \le j \le m-1$, due to the compatibility condition for the nonlinear system, it follows that the data $(u_0, f(u_0)+G, b(p,g))$ is compatible up to order $m-1$ for the linear system (5.6).

Step 2. An invariant set. Let $R,K,T > 0$. Define $V^m_{T,K,R}$ to be a subset of $CH^m([0,T]\times\Omega)\times H^m(0,T)$ such that $(v,g)\in V^m_{T,K,R}$ if and only if

- (V1) *Compatibility*: $\partial_t^j v_{|t=0} = u_j$ for all $0\le j\le m-1$ and $\partial_t^j g(0) = h_j$ for all $0\le j\le m-1$ where u_j and h_j are defined by (5.1) and (5.2)
- (V2) *Range condition*: $\mathrm{ran}(v,g)\subset \mathcal{K}_1\times\mathcal{G}_1$
- (V3) $W^{1,\infty}$*-bound*: $\|v\|_{W^{1,\infty}((0,T)\times\Omega)} + \|g\|_{W^{1,\infty}(0,T)} \le K$
- (V4) H^m*-bound*: $\|v\|_{H^m((0,T)\times\Omega)} + \|v_{|\partial\Omega}\|_{H^m(0,T)} + \|g\|_{H^m(0,T)} \le R$.

Consider the function $(v,g)\in CH^m([0,T_0]\times\Omega)\times H^m(0,T_0)$ constructed in the previous step. By construction of g, we already know that $\|g\|_{H^m(0,T_0)}\le C(\mathcal{G}_1,M)$. According to Remark 4.7

$$\|v\|_{CH^m([0,T_0]\times\Omega)} + \|v_{|\partial\Omega}\|_{H^m(0,T_0)}$$
$$\le C\left(\|f(u_0)+G\|_{H^m((0,T_0)\times\Omega)} + \|b(p,g)\|_{H^m(0,T_0)} + \sum_{i=0}^m \|\partial_t^i v_{|t=0}\|_{H^{m-i}(\Omega)}\right)$$

where C depends on the range of u_0, which lies in $\mathcal{K}_0$, and on $\|u_0\|_{H^m((0,T_0)\times\Omega)} \le C(T_0,M)$. From this, it can be seen that

$$\|v\|_{H^m((0,T_0)\times\Omega)} + \|v_{|\partial\Omega}\|_{H^m(0,T_0)} \le C(\mathcal{K}_1,\mathcal{G}_1,M) =: R_1$$

where we removed the explicit dependence of C on T_0 since it is fixed from the beginning. By Theorem 3.12 and the PDE (5.6)

$$\|v\|_{W^{1,\infty}((0,T_0)\times\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + \|f(u_0)+G(0,\cdot)-A(u_0)\partial_x u_0\|_{L^\infty(\Omega)} + \sqrt{T_0}R_1.$$

Applying the Sobolev embedding theorem and (5.9), we have $\|v\|_{W^{1,\infty}((0,T_0)\times\Omega)} \le C(R_1,M)$. One can do the same procedure for the $W^{1,\infty}$-norm of g. Hence,

$$\|v\|_{W^{1,\infty}((0,T_0)\times\Omega)} + \|g\|_{W^{1,\infty}(0,T_0)} \le C(\mathcal{K}_1,\mathcal{G}_1,M) =: K_1.$$

Finally, for the range condition, Theorem 3.12 and $v_{|t=0} = u_0$ imply that $\|v-u_0\|_{L^\infty((0,T)\times\Omega)} \le TR_1$. Therefore, there exists $T_1 = T_1(R_1) > 0$ such that the range of v lies in $\mathcal{K}_1$ for all $T\in(0,T_1]$. Using the same argument, it can be shown that the range of g also lies in $\mathcal{G}_1$ for all $T\in(0,T_1]$ by reducing T_1 if necessary. Hence, $V^m_{T,K,R}$ is nonempty for all $K\ge K_1$, $R\ge R_1$ and for $T\in(0,T_1]$ for some $T_1 = T_1(\mathcal{K}_1,\mathcal{G}_1,M) > 0$.

We will show that there exist $K > K_1$, $R = R(K) > R_1$ and $T = T(R) > 0$ such that given $(v,g)\in V^m_{T,K,R}$ the solution of the linear system

$$\begin{cases} u_t + A(v)u_x = f(v), & t>0,\ 0<x<1,\\ Bu_{|\partial\Omega} = b(p,h), & t>0,\\ \dot h = H(g,q,v_{|\partial\Omega}), & t>0,\\ u_{|t=0} = u_0, & 0<x<1,\\ h(0) = h_0, & \end{cases} \tag{5.10}$$

satisfies $(u,h) \in V^m_{T,K,R}$. Let us verify the regularity of (u,h). Note that $\partial_t^j v \in CH^{m-j}([0,T]\times\Omega)$ and so, $\partial_t^j v \in C^{m-j-1}([0,T]\times\Omega) \subset C([0,T]\times[0,1])$ for all $0 \le j \le m-1$. Therefore

$$\partial_t^j (v_{|\partial\Omega})_{|t=0} = (\partial_t^j v)_{|\{t=0\}\times\partial\Omega} = (\partial_t^j v_{|t=0})_{|\partial\Omega} = u_{j|\partial\Omega}, \qquad 0 \le j \le m-1.$$

Together with (V1), it can be shown that the compatibility conditions are satisfied by (u,h). Since

$$h(t) = h_0 + \int_0^t H(g(s), q(s), v_{|\partial\Omega}(s))\, \mathrm{d}s$$

we have $h \in H^{m+1}(0,T)$ and therefore $u \in CH^m([0,T]\times\Omega)$ with $u_{|\partial\Omega} \in H^m(0,T)$ according to Theorem 4.5 . Furthermore, u and h satisfy (V1) since v and g satisfy the same property. Thus, by Theorem 3.12

$$\|u\|_{W^{1,\infty}([0,T]\times\Omega)} + \|h\|_{W^{1,\infty}(0,T)} \le C(\mathcal{K}_1, M) + R\sqrt{T}.$$

Take $K = 2\max(K_1, C(M,\mathcal{K}_1))$. Letting $T = T(R,\mathcal{K}_1,\mathcal{G}_1,M) > 0$ small enough, condition (V3) is satisfied by (u,h).

A similar argument using the same Theorem 3.12 implies that (u,h) satisfies (V2) by reducing T if necessary. It remains to prove that (u,h) also satisfies (V4). Indeed, as in [13], one can prove the following additional a priori estimate

$$\|u\|_{H^m([0,T]\times\Omega)} + \|u_{|\partial\Omega}\|_{H^m(0,T)} + \|h\|_{H^m(0,T)} \le R \tag{5.11}$$

for some $R = R(K) > R_1$. The proof of this estimate is straightforward but lengthy. For this reason, we postpone its proof. In summary, $V^m_{T,K,R}$ is invariant under the map $(v,g) \mapsto (u,h)$ where (u,h) solves (5.10) for some $T, K, R > 0$.

Step 3. Existence and higher regularity. Let $V = V^m_{T,K,R}$ where the parameters T, K, and R are those given in the previous step. Let $(u^0, h^0) \in V$ be given and for each nonnegative integer k, define (u^{k+1}, h^{k+1}) recursively to be the solution of

$$\begin{cases} u_t^{k+1} + A(u^k)u_x^{k+1} = f(u^k), & t>0,\ 0<x<1, \\ Bu^{k+1} = b(p, h^{k+1}), & t>0, \\ \dot{h}^{k+1} = H(h^k, q, u^k_{|\partial\Omega}), & t>0, \\ u^{k+1}_{|t=0} = u_0, & 0<x<1, \\ h^{k+1}(0) = h_0, & \end{cases} \tag{5.12}$$

Note that the boundary condition in (5.12) depends on h^{k+1}, which is possible because h^{k+1} does not depend on u^{k+1} and at the same time couples the PDE to the ODE. Then according to Step 2, $(u^{k+1}, h^{k+1}) \in V$ for all $k = 1, 2, \ldots$ Thus, $(u^k, (u^k)_{|\partial\Omega}, h^k)$ is bounded in $H^m((0,T)\times\Omega)\times H^m(0,T)\times H^m(0,T)$ and one can extract a weakly convergent subsequence. By compact embedding and by extracting an appropriate subsequence, $(u^k, (u^k)_{|\partial\Omega}, h^k)$ converges in $L^2((0,T)\times\Omega)\times L^2(0,T)\times L^2(0,T)$ and let $(u, \tilde{u}, h)$ be the limit. The limit is necessarily in $H^m((0,T)\times\Omega)\times H^m(0,T)\times H^m(0,T)$.

Since $(u^k, (u^k)_{|\partial\Omega}, h^k)$ is bounded in $H^m((0,T)\times\Omega)\times H^m(0,T)\times H^m(0,T)$, by interpolation theory for Sobolev spaces, $(u^k, (u^k)_{|\partial\Omega}, h^k) \to (u, \tilde{u}, h)$ in $H^s((0,T)\times\Omega)\times H^s(0,T)\times H^s(0,T)$ for all $s \in [0,m)$. The continuity of the trace operator implies that $(u^k)_{|\partial\Omega} \to u_{|\partial\Omega}$ in $L^2(0,T)$ and therefore, $u_{|\partial\Omega} = \tilde{u}$ by uniqueness of limits in $L^2(0,T)$. By passing to the L^2-limit in the system satisfied by (u^k, h^k), we can see that the pair (u,h) satisfies the nonlinear system (1.1). Note that $\partial_t^j u_{|t=0} = u_j \in H^{m-j}(\Omega)$ for $0 \le j \le m-1$ from Lemma 5.1. Finally, Theorem 4.5 implies the additional regularity $u \in CH^m([0,T]\times\Omega)$.

Step 4. Uniqueness. Let (u_1,h_1) and (u_2,h_2) be two solutions of the system (1.1) on the time interval $[0,T]$. Introducing the variables $w = u_1 - u_2$ and $\eta = h_1 - h_2$, we have the system

$$\begin{cases} L_{u_1} w = f(u_1) - f(u_2) - (A(u_1) - A(u_2))\partial_x u_2, & 0<t<T,\ 0<x<1 \\ Bw_{|\partial\Omega} = b(p,h_1) - b(p,h_2), & 0<t<T, \\ \dot\eta = H(h_1,q,u_{1|\partial\Omega}) - H(h_2,q,u_{2|\partial\Omega}), & 0<t<T, \\ w_{|t=0} = 0, & 0<x<1, \\ \eta_{|t=0} = 0. \end{cases}$$

Let $\mathcal{K}\times\mathcal{G} \subset \mathcal{U}\times\mathcal{H}$ be a compact set both containing the ranges of (u_1,h_1) and (u_2,h_2), and let $K>0$ be such that the $W^{1,\infty}$-norms of (u_1,h_1) and (u_2,h_2) are bounded above by K. According to Theorem 2.4, there exists $C = C(\mathcal{K},K) > 0$ such that for all $0<\tau\le T$

$$\begin{aligned} \|w\|^2_{CL^2([0,\tau]\times\Omega)} + \|w_{|\partial\Omega}\|^2_{L^2(0,\tau)} \ &\le\ C\tau\|f(u_1) - f(u_2)\|^2_{L^2((0,\tau)\times\Omega)} \\ + C\tau\|(A(u_1) - A(u_2))\partial_x u_2\|^2_{L^2((0,\tau)\times\Omega)} \ &+\ C\|b(p,h_2) - b(p,h_1)\|^2_{L^2(0,\tau)}. \end{aligned} \tag{5.13}$$

By the mean value theorem

$$\|b(p,h_1) - b(p,h_2)\|^2_{L^2(0,\tau)} \le C\|\eta\|^2_{L^2(0,\tau)}. \tag{5.14}$$

A similar argument proves that

$$\begin{aligned} &\|f(u_1) - f(u_2)\|^2_{L^2((0,\tau)\times\Omega)} + \|(A(u_1) - A(u_2))\partial_x u_2\|^2_{L^2((0,\tau)\times\Omega)} \\ &\le C\|w\|^2_{L^2((0,\tau)\times\Omega)} \le C\tau\|w\|^2_{CL^2([0,\tau]\times\Omega)}. \end{aligned} \tag{5.15}$$

The differential equation for η gives us the following pointwise estimate

$$|\eta(t)|^2 \le C\tau(\|\eta\|^2_{L^2(0,\tau)} + \|w_{|\partial\Omega}\|^2_{L^2(0,\tau)}), \qquad t\in[0,\tau].$$

Integrating the last inequality and choosing $\tau = \tau(\mathcal{K},K) > 0$ small enough

$$\|\eta\|^2_{L^2(0,\tau)} \le \frac{C\tau^2}{1 - C\tau^2}\|w_{|\partial\Omega}\|^2_{L^2(0,\tau)}. \tag{5.16}$$

From (5.13)–(5.14) and reducing $\tau>0$ if necessary, it can be seen that $w=0$ on $[0,\tau]$ and from (5.16) $\eta = 0$ as well on $[0,\tau]$. Repeating the process on intervals of the form $[k\tau,(k+1)\tau]$ for positive integers k shows that $w=0$ and $\eta=0$ on $[0,T]$ and therefore the uniqueness of solutions. □

Now, we prove the estimate (5.11) used in the third step of the proof of the previous theorem. The proof of this estimate is similar to the proof of Lemma 4.6. However, the difference is that the source terms appearing on the PDE and the boundary condition now depend on the frozen coefficients v and g. From the proof of Lemma 4.6, we already have the estimate

$$\frac{1}{\sqrt{T}}\|u\|_{L^2(\Omega;H^m(0,T))} + \|u_{|\partial\Omega}\|_{H^m(0,T)}$$

$$\leq C\left(\sum_{j=1}^{m}\|\partial_t^j u_{|t=0}\|_{L^2(\Omega)}^2+\sqrt{T}\sum_{j=1}^{m}\|f_j\|_{L^2((0,T)\times\Omega)}+\|b(p,h)\|_{H^m(0,T)}\right) \tag{5.17}$$

for all $T\in(0,T_0]$, where $f_j=A(v)\partial_t^j(A(v)^{-1}f(v))-A(v)[\partial_t^j,A(v)^{-1}L_v]u$. For the rest of the proof C will denote a positive constant depending only on $T_0,K,\mathcal{K}_1,\mathcal{G}_1,M$ $\|p\|_{H^m(0,T_0)}$, $\|q\|_{H^m(0,T_0)}$, and is independent on R and T. The commutator has been estimated uniformly in T in the proof of Lemma 4.6. Let us consider the first term of f_j. Note that it is a nonlinear function of order at most m and thus, by Theorem 3.11 we have

$$\|A(v)\partial_t^j(A(v)^{-1}f(v))\|_{L^2((0,T)\times\Omega)}\leq C(\|v\|_{H^m((0,T)\times\Omega)}+1).$$

Since $(u,h)\in V_{T,K,R}^m$ we have $\partial_t^j u_{|t=0}=u_j$ for all $0\leq j\leq m-1$. Using this in (5.17) and recalling Lemma 5.1, we have

$$\begin{aligned}&\frac{1}{\sqrt{T}}\|u\|_{L^2(\Omega;H^m(0,T))}+\|u_{|\partial\Omega}\|_{H^m(0,T)} \\ &\leq C(1+\sqrt{T}\|v\|_{H^m((0,T)\times\Omega)}+\sqrt{T}(1+R)\|u\|_{H^m((0,T)\times\Omega)}+\|b(p,h)\|_{H^m(0,T)})\end{aligned} \tag{5.18}$$

where R is a positive constant to be chosen below.

The next step is to estimate the boundary terms on the right hand side of (5.17). By Theorem 3.11 once more, we obtain

$$\|b(p,h)\|_{H^m(0,T)}\leq C(K_{m,T}(p,h))(\|p\|_{H^m(0,T)}+\|h\|_{H^m(0,T)}+1).$$

The fact that $(u,h)\in V$ implies that $h^{(j)}(0)=h_j$ for all $0\leq j\leq m-1$. The differential equation $\dot{h}=H(q,g,v_{|\partial\Omega})$ for h gives us the estimate

$$\|h\|_{H^m(0,T)}\leq C(K_{m-1,T}(q,g,v_{|\partial\Omega}))(\|q\|_{H^{m-1}(0,T)}+\|g\|_{H^{m-1}(0,T)}+1).$$

With these together with Theorem 3.13 we have

$$\|b(p,h)\|_{H^m(0,T)}\leq C(T\|g\|_{H^m(0,T)}+1). \tag{5.19}$$

Using (5.19) in (5.18), we have

$$\begin{aligned}&\frac{1}{\sqrt{T}}\|u\|_{L^2(\Omega;H^m(0,T))}+\|u_{|\partial\Omega}\|_{H^m(0,T)} \\ &\leq C(1+\sqrt{T}\|v\|_{H^m((0,T)\times\Omega)}+\sqrt{T}(1+R)\|u\|_{H^m((0,T)\times\Omega)}+T\|g\|_{H^m(0,T)}).\end{aligned} \tag{5.20}$$

It remains to estimate the mixed derivatives. As usual we proceed by an induction argument. Suppose that $\|\partial_x^l\partial_t^j u\|_{L^2((0,T)\times\Omega)}\leq N(u)$ for all $l=0,1\ldots,k-1$ and j such that $l+j\leq m$, where $N(u)$ is the right hand side of (5.20). Let k and j be integers such that $k+j\leq m$. The PDE implies that

$$\partial_x^k\partial_t^j u=\partial_x^{k-1}\partial_t^j(A(v)^{-1}f(v))-\partial_x^{k-1}\partial_t^j(A(v)^{-1}\partial_t u).$$

The first term on the right hand side is a nonlinear function of v of order at most $m-1$, and therefore using Theorem 3.11, Theorem 3.13 and (V1) we have

$$\|\partial_x^{k-1}\partial_t^j(A(v)^{-1}f(v))\|_{L^2((0,T)\times\Omega)}\leq C(T\|v\|_{H^m((0,T)\times\Omega)}+1).$$

We can expand the second term using Leibniz's rule and estimate each term in the sum. Let $0\leq l\leq k-1$ and $0\leq j\leq i$. If $l+i\leq m-3$ then Theorem 3.11 implies

$$\|\partial_x^{k-1-l}\partial_t^{j-i}(A(v)^{-1})\partial_x^l\partial_t^{i+1}u\|_{L^2((0,T)\times\Omega)}$$

$$\leq \|\partial_x^{k-1-l}\partial_t^{j-i}(A(v)^{-1})\|_{L^2((0,T)\times\Omega)}\|\partial_x^l\partial_t^{i+1}u\|_{L^\infty((0,T)\times\Omega)}$$
$$\leq C(1+\|v\|_{H^{m-1}((0,T)\times\Omega)})\|u\|_{W^{m-2}((0,T)\times\Omega)}.$$

According to Theorem 3.13 we have

$$\begin{aligned}\|u\|_{W^{m-2}((0,T)\times\Omega)} &\leq \sum_{k=0}^{m-2}\|\partial_t^k u_{|t=0}\|_{W^{m-2-k}(\Omega)} + C\sqrt{T}\|u\|_{H^m((0,T)\times\Omega)}\\ &\leq C\sum_{k=0}^{m-2}\|\partial_t^k u_{|t=0}\|_{H^{m-k-1}(\Omega)} + C\sqrt{T}\|u\|_{H^m((0,T)\times\Omega)}.\end{aligned}$$

Thus, $\|\partial_x^{k-1-l}\partial_t^{j-i}(A(v)^{-1})\partial_x^l\partial_t^{i+1}u\|_{L^2((0,T)\times\Omega)} \leq N(u)$.

Suppose that $l+i = m-2, m-1$ then $k-1-l+j-i = 1, 0$. Applying a standard $L^\infty - L^2$ estimate yields

$$\|\partial_x^{k-1-l}\partial_t^{j-i}(A(v)^{-1})\partial_x^l\partial_t^{i+1}u\|_{L^2((0,T)\times\Omega)} \leq C\|\partial_x^l\partial_t^{i+1}u\|_{L^2((0,T)\times\Omega)} \leq N(u)$$

where the last inequality is due to the induction hypothesis. This completes the proof of the induction step. Therefore we have

$$\left(\frac{1}{\sqrt{T}} - C\sqrt{T}(1+R)\right)\|u\|_{H^m((0,T)\times\Omega)} + \|u_{|\partial\Omega}\|_{H^m(0,T)}$$
$$\leq C(1+\sqrt{T}\|v\|_{H^m((0,T)\times\Omega)} + \sqrt{T}\|g\|_{H^m(0,T)}) \leq C(1+\sqrt{T}R).$$

Choosing $R = \max(5C, R_1)$ where C is the constant in the last inequality and choosing $T = T(R) > 0$ small enough so that $\frac{1}{\sqrt{T}} - C\sqrt{T}(1+R) > \frac{1}{2}$ and $\sqrt{T}R < 1$ finally proves (5.11).

To close this section, we prove the following standard blow-up criterion for first order quasilinear PDEs. The idea of the proof is the following: Boundedness in $W^{1,\infty}$ of the local solution implies boundedness in H^m, which can be further improved to show boundedness in CH^m. If this is known, then a standard argument shows that the solution can be extended.

Theorem 5.3. *Let $(u,h) \in CH^m([0,T]\times\Omega)\times H^m(0,T)$ be a solution of (1.1) having a trace $u_{|\partial\Omega} \in H^m(0,T)$, where $m \geq 3$ is an integer, and T^* be the maximal time of existence. If $T^* < \infty$ then the range of (u,h) on $[0,T]\times[0,1]$ leaves every compact subset of $\mathcal{U}\times\mathcal{H}$ as $T \to T^*$, i.e. for every compact set $\mathcal{K}\times\mathcal{G}$ in $\mathcal{U}\times\mathcal{H}$ there exists $\epsilon > 0$ and $(t,x) \in (0, T^*-\epsilon]\times[0,1]$ such that $(u(t,x), h(t)) \notin \mathcal{K}\times\mathcal{G}$, or*

$$\limsup_{t\uparrow T^*}\|\partial_x u(t)\|_{L^\infty[0,1]} = \infty.$$

Proof. Suppose that the range of (u,h) on $[0,T]\times\Omega$ lies in a compact subset $\mathcal{K}_0\times\mathcal{G}_0$ of $\mathcal{U}\times\mathcal{H}$, $\|u\|_{W^{1,\infty}([0,T]\times[0,1])} \leq K_0$ for some constant $K_0 > 0$ and $(u,h) \in CH^m([0,T]\times\Omega)\times H^m(0,T)$ for all $T \in (0,T^*)$. We show that there exists a $\tau > 0$ such that the solution can be extended to a solution $(u,h) \in CH^m([0,T^*+\tau]\times\Omega)\times H^m(0,T^*+\tau)$ satisfying $u_{|\partial\Omega} \in H^m(0,T^*+\tau)$.

Step 1. Uniform boundedness in $CH^m \times H^m$. The following estimates are again in the same spirit as before, but now, the frozen coefficients are the solutions of the

PDE. For completeness, we include their proof. We will use the following a priori estimate, see [2, p. 280] for example, for all $u \in H^1((0,T)\times\Omega)$ and for all $\gamma \geq \gamma_0$

$$\begin{aligned}&\sqrt{\gamma}\|u\|_{L^2((0,T)\times\Omega)} + \|u_{|\partial\Omega}\|_{L^2(0,T)}\\ &\leq C\left(\frac{1}{\sqrt{\gamma}}\|L_u u\|_{L^2((0,T)\times\Omega)} + \|Bu_{|\partial\Omega}\|_{L^2(0,T)} + \|u_{|t=0}\|_{L^2(\Omega)}\right)\end{aligned}$$

for some constants $C > 0$ and $\gamma_0 \geq 1$ depending only on $(\mathcal{K}_0, \mathcal{G}_0, K_0)$. Applying this estimate to $\partial_t^j u$, for $j = 0, 1, \ldots, j$ where $k = 0, 1, \ldots, m$ we have

$$\begin{aligned}&\sqrt{\gamma}\|u\|_{L^2(\Omega;H^k(0,T))} + \|u_{|\partial\Omega}\|_{H^k(0,T)}\\ &\leq C\left(\frac{1}{\sqrt{\gamma}}\sum_{j=0}^{k}\|f_j\|_{L^2((0,T)\times\Omega)} + \|b(p,h)\|_{H^k(0,T)} + 1\right)\end{aligned}$$

where $f_j = A(u)\partial_t^j(A(u)^{-1}f(u)) - A(u)[\partial_t^j, A(u)^{-1}L_u]u$.

For $j \geq 1$, f_j is a nonlinear function of $\partial_t u$ of order at most $j-1$. Thus, using Theorem 3.11 we have

$$\|f_j\|_{L^2((0,T)\times\Omega)} \leq C(\|\partial_t u\|_{H^{j-1}((0,T)\times\Omega)} + 1) \leq C(\|u\|_{H^j((0,T)\times\Omega)} + 1).$$

The case of $f_0 = f(u)$ can be done merely by the mean-value theorem. On the other hand, by a similar argument we also have $\|b(p,h)\|_{H^k(0,T)} \leq C(\|h\|_{H^k(0,T)} + 1)$. The differential equation for h gives us $\|h\|_{L^2(0,T)} \leq C$ and $\|h\|_{H^k(0,T)} \leq C(\|h\|_{H^{k-1}(0,T)} + \|u_{|\partial\Omega}\|_{H^{k-1}(0,T)} + 1)$ for $1 \leq k \leq m$. Combining all of these in a recursive manner, we obtain

$$\sqrt{\gamma}\|u\|_{L^2(\Omega;H^m(0,T))} + \|u_{|\partial\Omega}\|_{H^m(0,T)} + \|h\|_{H^m(0,T)} \leq C\left(\frac{1}{\sqrt{\gamma}}\|u\|_{H^m((0,T)\times\Omega)} + 1\right).$$

From the PDE, we note that $\partial_x u = A(u)^{-1}f(u) - A(u)^{-1}\partial_t u$. Therefore, $\partial_x^j\partial_t^k u$ can be written in terms of derivatives of u with respect to t only, and is a nonlinear function of u of order at most $k+j$. Fixing $x \in \Omega$, we apply Theorem 3.11 to the function $u(\cdot,x) \in H^m(0,T)$ to obtain

$$\|\partial_x^j\partial_t^k u(\cdot,x)\|_{L^2(0,T)} \leq C(\|u(\cdot,x)\|_{H^m(0,T)} + 1).$$

Integrating over the bounded domain Ω yields

$$\|\partial_x^j\partial_t^k u\|_{L^2((0,T)\times\Omega)} \leq C(\|u\|_{L^2(\Omega;H^m(0,T))} + 1).$$

Combining this with our estimates above and choosing γ large enough we have

$$\|u\|_{H^m((0,T)\times\Omega)} + \|h\|_{H^m(0,T)} \leq C, \qquad \text{for all } 0 < T < T^*. \tag{5.21}$$

for some constant $C > 0$ independent of $T \in (0,T^*)$.

Let $\varphi \in \mathscr{D}(\mathbb{R})$ be a cut-off function such that $\varphi(t) = 0$ if $t \leq T^*/4$ and $\varphi(t) = 1$ if $t \geq T^*/2$. Multiplying the system (1.1) by this cut-off function, we have the new homogeneous system for $w = \varphi u$ and $g = \varphi h$

$$\begin{cases} w_t + A(u)w_x = \varphi f(u) + \dot{\varphi} u, & 0 < t < T,\ 0 < x < 1, \\ Bw_{|\partial\Omega} = \varphi b(p,h), & 0 < t < T, \\ \dot{g} = \varphi H(h,q,u_{|\partial\Omega}) + \dot{\varphi} h, & 0 < t < T, \\ w_{|t=0} = 0, & 0 < x < 1, \\ g_{|t=0} = 0. \end{cases} \tag{5.22}$$

Applying the energy estimates for the initial boundary value problem with homogeneous data (4.2), together with the previous result (5.21) shows that there exists an $M > 0$ independent of T such that

$$\|u\|_{CH^m([0,T]\times\Omega)} + \|h\|_{H^m(0,T)} \le M, \qquad \text{for all } 0 < T < T^*.$$

Step 2. Extension. According to the previous step there exist an $M > 0$ and a sequence $(t_n)_n \subset (0,T)$ such that $t_n \to T^*$ and $\|u(t_n)\|_{H^m} + |h(t_n)| \le M$ for all n. Consider the initial boundary value problem

$$\begin{cases} v_t + A(v)v_x = f(v), & t > 0,\ 0 < x < 1, \\ Bv_{|\partial\Omega} = b(p,g), & t > 0, \\ \dot{g} = H(g,q,v_{|\partial\Omega}), & t > 0, \\ v_{|t=0} = u(t_n), & 0 < x < 1, \\ g_{|t=0} = h(t_n). \end{cases} \tag{5.23}$$

The local existence result Theorem 5.2 implies that there exists $\tau > 0$, depending only on M and in some neighborhoods of $\mathcal{K}_0$ and $\mathcal{G}_0$, but independent of n, such that (5.23) has a unique solution on $[0,\tau]$. Choose n large enough so that $t_n + \tau > T^*$. Then the pair of functions (w,η) defined by

$$(w,\eta)(t) = \begin{cases} (u,h)(t), & 0 \le t \le t_n, \\ (v,g)(t-t_n), & t_n \le t \le t_n + \tau, \end{cases}$$

lies in $CH^m([0,t_n+\tau]\times\Omega) \times H^m(0,t_n+\tau)$ since (u,h) and (v,g) must coincide in $[t_n,(t_n+T^*)/2]$ by uniqueness. Thus, (w,η) satisfies (1.1). Therefore, the solution (u,h) can be extended up to the time $t_n + \tau > T^*$. This completes the proof of the theorem. □

6. Examples

In this section we cite some examples that fit in the general system (1.1).

6.1. Flow in an elastic tube. Consider the following system modelling the velocity v of an incompressible fluid contained in an elastic tube of length ℓ, cross-section a and is connected to a tank at each end having cross-section a_T and level heights h_0, h_ℓ, respectively,

$$\begin{cases} a_t(t,x) + v(t,x)a_x(t,x) + a(t,x)v_x(t,x) = 0, \quad 0 < t < T,\ 0 < x < \ell, \\ v_t(t,x) + \dfrac{\kappa^2 a_x(t,x)}{\sqrt{a(t,x)}} + v(t,x)v_x(t,x) = -\beta v(t,x), \quad 0 < t < T,\ 0 < x < \ell, \\ a_T \dot{h}_0(t) = -a(t,0)v(t,0), \quad 0 < t < T, \\ a_T \dot{h}_\ell(t) = a(t,\ell)v(t,\ell), \quad 0 < t < T, \\ a(t,0) = a_0(1 + p_0(t) + bh_0(t))^2, \quad 0 < t < T, \\ a(t,\ell) = a_0(1 + p_\ell(t) + bh_\ell(t))^2, \quad 0 < t < T, \end{cases} \tag{6.1}$$

see [5], [14], [17]. Here, a_0 is the rest cross-sectional area of the tube, $b, \kappa > 0$ are parameters incorporating the material properties of the tube and $\beta \geq 0$ is a parameter modeling linear tube friction. The tanks are subjected from above by external forcing pressures represented by p_0 and p_ℓ. Letting $u = (u_1, u_2) := (a, v)$, $h = (h_1, h_2) := (h_0, h_\ell)$, and $p = (p_1, p_2) := (p_0, p_\ell)$ we can transform (6.1) into (1.1) with

$$A(u) = \begin{pmatrix} u_2 & u_1 \\ \kappa^2 u_1^{-\frac{1}{2}} & u_2 \end{pmatrix}, \quad f(u) = \begin{pmatrix} 0 \\ -\beta u \end{pmatrix}, \quad B_0 = B_\ell = (\,1 \ \ 0\,),$$

$$b(p,h) = \begin{pmatrix} a_0(1 + p_1 + bh_1)^2 \\ a_0(1 + p_2 + bh_2)^2 \end{pmatrix}, \quad H(h,u,w) = \begin{pmatrix} -\frac{1}{a_T} u_1 u_2 \\ \frac{1}{a_T} w_1 w_2 \end{pmatrix}.$$

The eigenvalues of the flux matrix $A(u)$ are given by $\lambda(u) = u_2 - \kappa u_1^{\frac{1}{4}}$ and $\mu(u) = u_2 + \kappa u_1^{\frac{1}{4}}$ with corresponding eigenvectors

$$e_\lambda(u) = \begin{pmatrix} u_1 \\ -\kappa u_1^{\frac{1}{4}} \end{pmatrix}, \quad e_\mu(u) = \begin{pmatrix} u_1 \\ \kappa u_1^{\frac{1}{4}} \end{pmatrix},$$

respectively. Let $\tilde{\mathcal{U}} = \{(u_1, u_2) \in \mathbb{R}^2 : u_1 > 0, |u_2| < \kappa u_1^{\frac{1}{4}}\}$. It follows that $A(w)$ has one negative and one positive eigenvalue for every $w \in \tilde{\mathcal{U}}$. Thus, $E^s(A(w)) = \operatorname{span}\{e_\lambda(w)\}$ and $E^u(A(w)) = \operatorname{span}\{e_\mu(w)\}$. The estimate $\|e_\mu(w)\| \leq C\|B_0 e_\mu(w)\|$ is equivalent to

$$u_1 \leq \kappa^{-4}(C^2 - 1)^2 u_1^4. \tag{6.2}$$

Let $\tilde{\mathcal{U}}_\epsilon = \{w \in \tilde{\mathcal{U}} : \operatorname{dist}(w, \partial\tilde{\mathcal{U}}) > \epsilon\}$ for $\epsilon > 0$. By continuity it can be seen from (6.2) that there exists $C_\epsilon > 1$ such that $\|e_\mu(w)\| \leq C_\epsilon \|B_0 e_\mu(w)\|$ for all $w \in \tilde{\mathcal{U}}_\epsilon$. By positive homogeneity of the norm it follows that $\|V\| \leq C_\epsilon \|B_0 V\|$ for all $V \in E^u(A(w))$ and for all $w \in \tilde{\mathcal{U}}_\epsilon$. Similarly, $\|V\| \leq C_\epsilon \|B_\ell V\|$ for all $V \in E^s(A(w))$ whenever $w \in \tilde{\mathcal{U}}_\epsilon$. Therefore, the uniform Kreiss-Lopatinskiĭ condition holds in $\tilde{\mathcal{U}}_\epsilon$.

It remains to verify Friedrichs symmetrizability. It can be easily seen that the matrix

$$S(w) = \begin{pmatrix} \kappa^2 u_1^{-\frac{3}{2}} & 0 \\ 0 & 1 \end{pmatrix}$$

is a Friedrichs symmetrizer of the system. For $R > 0$ define $\mathcal{U} = \{w \in \tilde{\mathcal{U}}_\epsilon : \|w\| < R\}$. It is clear that there exists $\alpha = \alpha(\epsilon, R) > 0$ such that $S(w) \geq \alpha I_2$ for all $w \in \mathcal{U}$. Therefore, if the the initial data for the system (6.1) and the boundary data p

satisfy the conditions of Theorem 5.2 then (6.1) has a unique solution $(a, v, h_0, h_\ell) \in CH^m([0,T] \times [0,\ell])^2 \times H^{m+1}(0,T)^2$ for some $T > 0$. Moreover, if the maximal time $T^* > 0$ of existence is finite then either the range of (a, v, h_0, h_ℓ) leaves every compact set of $\mathcal{U} \times \mathbb{R}^2$ or

$$\limsup_{t \uparrow T^*} (\|\partial_x a(t)\|_{L^\infty[0,\ell]} + \|\partial_x v(t)\|_{L^\infty[0,\ell]}) = \infty.$$

6.2. Multiscale blood flow model. Consider the following system [9], [20]

$$\begin{cases} a_t(t,x) + q_x(t,x) = 0 \\ q_t(t,x) + \left(\dfrac{q(t,x)^2}{a(t,x)}\right)_x + \dfrac{1}{\rho} a(t,x) p_x(t,x) = -8\pi\rho\nu \dfrac{q(t,x)}{a(t,x)} \end{cases} \tag{6.3}$$

with $0 < t < T$ and $0 < x < \ell$. This models the flow rate q of the blood in a vessel of cross-section a and length ℓ. The pressure p is given by the constitutive law

$$p = \frac{\sqrt{\pi} h E}{a_0(1-\sigma^2)}(\sqrt{a} - \sqrt{a_0}). \tag{6.4}$$

All the parameters are positive and they represent various physical quantities depicting the properties of the blood and the vessel. Here, a_0, E, h, σ denote the rest cross-section, Young's modulus, thickness and Poisson coefficient of the vessel wall, respectively, whereas ρ is the blood density and ν is the kinematic blood viscosity.

To have a more realistic description of the cardiovascular system, lumped parameter models based on ordinary differential equations were introduced. These ODEs can be derived by linearizing and integrating the hyperbolic models with respect to space. Following [9] we have

$$\dot{y}_0(t) = A_0 y_0(t) + r_{H0}(t, y_0(t)) + s_0(t, y_0(t)) \tag{6.5}$$

$$\dot{y}_\ell(t) = A_\ell y_\ell(t) + r_{H\ell}(t, y_\ell(t)) + s_\ell(t, y_\ell(t)) \tag{6.6}$$

where $y_0(t), y_\ell(t) \in \mathbb{R}^m$, A_0, A_ℓ are $m \times m$ matrices and $r_{H0}, r_{H\ell}, s_0, s_\ell$ are source terms. The coupling of the hyperbolic PDE (6.3) and the ODEs (6.5) and (6.6) is done by imposing the pressure at the boundaries to be equal to a specific entry of the ODE, i.e.,

$$p(t,0) = y_{0i}(t), \qquad p(t,\ell) = y_{\ell j}(t) \tag{6.7}$$

for some $1 \le i, j \le m$. Writing the system in terms of a and q only by using the constitutive law (6.4), it can be shown as in the previous example that (6.3)–(6.6) can be written in the form (1.1) and satisfies **(FS)**, **(D)** and **(UKL)** with appropriate $\mathcal{U}$. Alternatively, one can diagonalize the system as in [9], and thus, Friedrichs symmetrizability is easily checked. The boundary matrices will be transformed, however, the UKL condition is preserved. This can be verified in the same manner as in the previous example and for this reason we omit the details.

6.3. 1-Tank model. Consider a 1-D tank of length ℓ filled with inviscid incompressible irrotational fluid which is subjected by a horizontal force. Then using the

Saint-Venant equation, one can derive the following system [7]

$$\begin{cases} H_t(t,x) + v(t,x)H_x(t,x) + H(t,x)v_x(t,x) = 0, \quad 0 < t < T,\ 0 < x < \ell, \\ v_t(t,x) + gH_x(t,x) + v(t,x)v_x(t,x) = -u(t), \quad 0 < t < T,\ 0 < x < \ell, \\ v(t,0) = v(t,L) = 0, \quad 0 < t < T, \\ \dot{s}(t) = u(t), \quad 0 < t < T, \\ \dot{D}(t) = s(t), \quad 0 < t < T, \end{cases} \tag{6.8}$$

where g is the gravitational force, H is the height of the fluid in the tank, v is the referential horizontal velocity of water, s is the horizontal velocity of the tank, D is the horizontal displacement of the tank and u is the horizontal acceleration of the tank in the absolute referential and is viewed as the control.

Note that the PDE part is not of the same form as the PDE part in (1.1), but instead, it is of the form

$$u_t(t,x) + A(u(t,x))u_x(t,x) = F(t,x).$$

The results given in the previous sections extend to the case where there is an extra source term F on the right hand side of the PDE part.

References

[1] R. A. Adams, *Sobolev Spaces*, Pure and Applied Mathematics (Academic Press, New York, 1975).

[2] S. Benzoni-Gavage and D. Serre, *Multidimensional Hyperbolic Partial Differential Equations*, Oxford Mathematical Monographs (Oxford University Press, 2007).

[3] R. Borsche, R. M. Colombo, and M. Garavello, On the coupling of hyperbolic conservation laws with ordinary differential equations, *Nonlinearity* **23** (2010) 2749–2770.

[4] R. Borsche, R. M. Colombo, and M. Garavello, Mixed systems: ODEs - balance laws, *Journal of Differential Equations* **252** (2012) 2311–2338.

[5] A. Borzì and G. Propst, Numerical investigation of the Liebau phenomenon, *Z. Angew. Math. Phys.* **54** (2003) 1050–1072.

[6] S. Čanić and E. H. Kim, Mathematical analysis of quasilinear effects in a hyperbolic model of blood flow through compliant axi-symmetric vessels, *Math. Methods Appl. Sci.* **26** (2003) 1161–1186 .

[7] J. M. Coron, *Control and Nonlinearity*, Mathematical Surveys and Monographs, Vol. 136 (American Mathematical Society, Providence, 2007).

[8] H. B. da Veiga, Perturbation theory and well-posedness in Hadamard's sense of hyperbolic initial-boundary value problems, *Nonlinear Analysis, Theory, Methods and Applications* **22** (1994) 1285–1308.

[9] M. Fernández, V. Milišić, and A. Quarteroni, Analysis of a geometrical multiscale blood flow model based on the coupling of ODEs and hyperbolic PDEs, *Multiscale Model. Simul.* **4** (2005) 215–236.

[10] T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed., Classics in Mathematics (Springer-Verlag, New York, 1980).

[11] T.-T. Li, *Global Classical Solutions for Quasilinear Hyperbolic Systems*, Wiley-Masson Series Research in Applied Mathematics (Masson, J. Wiley, Paris, 1994).

[12] T.-T. Li and W. C. Yu, *Boundary Value Problems for Quasilinear Hyperbolic Systems* (Duke University Mathematics Series V, Duke University Mathematics Department, Durham, NC, 1985).

[13] G. Métivier, Stability of multidimensional shocks, in *Advances in the Theory of Shock Waves* (ed. H. Freistühler and A. Szepessy, Birkhäuser, Boston, 2003), pp. 25–103.

[14] J. T. Ottesen, Valveless pumping in a fluid-filled closed elastic tube-system: one-dimensional theory with experimental validation, *J. Math. Biol.* **46** (2003) 309–332.

[15] G. Peralta and G. Propst, Stability and boundary controllability of a linearized model of flow in an elastic tube, to appear in ESAIM: COCV.

[16] G. Peralta and G. Propst, Global smooth solution to a hyperbolic system on an interval with dynamic boundary conditions, submitted.

[17] H. J. Rath and I. Teipel, Der Fördereffekt in ventillosen, elastischen Leitungen, *Z. Angew. Math. Phys.* **29** (1978) 123–133.

[18] J. B. Rauch and F. J. Massey, Differentiability of solutions to hyperbolic initial-boundary value problems, *Trans. Amer. Math. Soc.* **189** (1974) 303–318.

[19] W. Ruan, A coupled system of ODEs and quasilinear hyperbolic PDEs arising in a multiscale blood flow model, *J. Math. Anal. Appl.* **343** (2008) 778–798.

[20] W. H. Ruan, M. E. Clark, M. Zhao and A. Curcio, A hyperbolic system of equations of blood flow in an arterial network, *SIAM J. Appl. Math.* **64** (2003) 637–667.

[21] W. H. Ruan, M. E. Clark, M. Zhao, and A. Curcio, Global solution to a hyperbolic problem arising in the modeling of blood flow in circulatory systems, *J. Math. Anal. Appl.* **331** (2007) 1068–1092.

[22] S. Schochet, The compressible Euler equations in a bounded domain : existence of solutions and the incompressible limit, *Commun. Math. Phys.* **104** (1986) 49–75.